On the computation of errors for $N_{\rm f} = 2 + 1$ renormalized masses

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1 Procedure to obtain renormalised $\overline{\rm MS}$ quark masses

Our determination of renormalized quark masses proceeds as follows:

1. Bare quark masses are determined in large volume simulations for several ensembles. They are renormalized in the SF scheme at the hadronic reference scale $\mu_{\rm had}$, using the renormalization constants obtained in [2]. This allows to perform chiral-continuum extrapolations and end up with a value of $\overline{m}_{\rm SF}(\mu_{\rm had})$.

2. From this quantity one can obtain the RGI mass, by multiplying times the running factor $M/\overline{m}_{\rm SF}(\mu_{\rm had})$ quoted in [2].

3. Finally, one may want to obtain the value of the renormalised mass in the $\overline{\rm MS}$ scheme at some scale μ. In order to do that, one has to solve the RG equations in the form

$$\frac{\mu}{\Lambda} = \left[b_0\bar{g}^2(\mu)\right]^{\frac{b_1}{2b_0^2}}\; e^{\frac{1}{2b_0\bar{g}^2(\mu)}}\; \exp\left\{\int_0^{\bar{g}(\mu)} {\rm d}g\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}, \tag{1.1}$$

$$\frac{\overline{m}_{\overline{\rm MS}}(\mu)}{M} = \left[2b_0\bar{g}^2(\mu)\right]^{\frac{d_0}{2b_0}} \exp\left\{\int_0^{\bar{g}(\mu)} {\rm d}g\left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}. \tag{1.2}$$

   In practice, one inputs $\Lambda_{\overline{\rm MS}}$ from [1] in Eq. (1.1), and solves it at some fixed perturbative order to obtain $\bar{g}(\mu)$, which comes with an error induced by the uncertainty on $\Lambda_{\overline{\rm MS}}$. This is then fed to Eq. (1.2), which yields $\overline{m}_{\overline{\rm MS}}(\mu)/M$ — again with an error that comes from the one on $\Lambda_{\overline{\rm MS}}$. It is then possible to compute

$$\overline{m}_{\overline{\rm MS}}(\mu) = M\,\frac{\overline{m}_{\overline{\rm MS}}(\mu)}{M}\,. \tag{1.3}$$

2 Error propagation

If we fully split Eq. (1.3) into the factors that are calculated separately, we have

$$\begin{aligned}\overline{m}_{\overline{\rm MS}}(\mu) &= \overline{m}_{\rm SF}(\mu_{\rm had})\,\frac{M}{\overline{m}_{\rm SF}(\mu_0/2)}\,\frac{\overline{m}_{\rm SF}(\mu_0/2)}{\overline{m}_{\rm SF}(\mu_{\rm had})}\,\frac{\overline{m}_{\overline{\rm MS}}(\mu)}{M} \\ &= F_1\,F_2\,F_3\,F_4\,.\end{aligned} \tag{2.1}$$

The two factors $F_{2,3}$ in which the total running $M/\overline{m}_{\rm SF}(\mu_{\rm had})$ is split correspond to the high- and low-energy regimes of the running, which are dealt with separately. The key observation is that F_3 is correlated with $\Lambda_{\overline{\rm MS}}$, since they are computed on the same set of ensembles. Since the uncertainty of F_4 does also come from the one on $\Lambda_{\overline{\rm MS}}$, this then means that there is a correlation between F_3 and F_4.

The numerical input to put all pieces together comes from [1, 2, 3], and is the following:

$$\Lambda_{\overline{\rm MS}} = 341(12)\ \text{MeV}\,; \tag{2.2}$$

$$F_2 = 1.7505(89)\,; \tag{2.3}$$

$$F_3 = 0.5226(43)\,; \tag{2.4}$$

$$\text{cov}(\Lambda_{\overline{\rm MS}}, F_3) = \begin{pmatrix} 150.7862014579557\ \text{MeV}^2 & -0.01716315895017156\ \text{MeV} \\ -0.01716315895017156\ \text{MeV} & 0.00001830314004469306 \end{pmatrix}. \tag{2.5}$$

On the other hand, F_1 and F_2 are completely uncorrelated between them and with the other two factors. Therefore, the full error propagation only needs to take into account the uncertainties on F_1 and F_2, and the covariance matrix $\text{cov}(F_3, F_4)$, obtained from $\text{cov}(\Lambda_{\overline{\rm MS}}, F_3)$ quoted above and $\text{cov}(\Lambda_{\overline{\rm MS}}, F_4)$ obtained from the perturbative computation described in the third item of the first section.

3 `ADerrors` implementation

The error propagation described above can be performed by hand easily. However, the integration of Eqs. (1.1,1.2) has to be done numerically beyond two loops. It is thus convenient to integrate the procedure within `ADerrors`. This is achieved by inputting $\Lambda_{\overline{\rm MS}}$, $F_{1,2,3}$ and $\text{cov}(\Lambda_{\overline{\rm MS}}, F_3)$, coding a function to perform the numerical integration of the RG equations, and tagging the items appropriate to signal the non-trivial covariances.

References

[1] M. Bruno *et al.* [ALPHA], Phys. Rev. Lett. **119** (2017) no.10, 102001 doi:10.1103/PhysRevLett.119.102001 [arXiv:1706.03821 [hep-lat]].

[2] I. Campos *et al.* [ALPHA], Eur. Phys. J. C **78** (2018) no.5, 387 doi:10.1140/epjc/s10052-018-5870-5 [arXiv:1802.05243 [hep-lat]].

[3] A. Ramos, private communication.