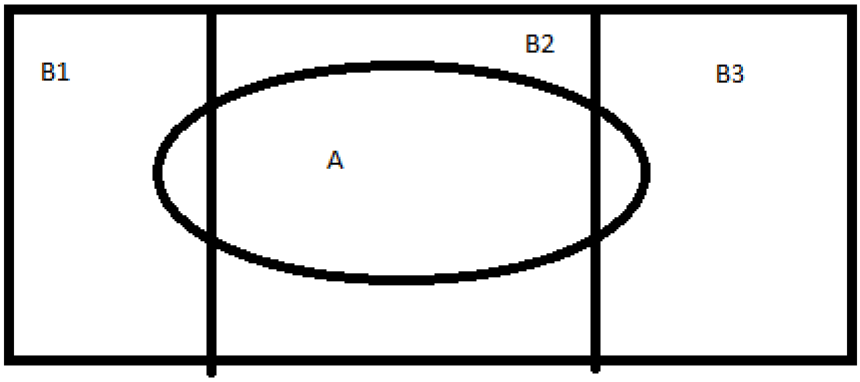
Week 5

Bayes Theorem

Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events whose union is the sample space S of an experiment. Let A be an arbitrary event of S such that P(A)$\neq$ 0. Then,

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_n \cap A)} = \frac{P(B_1 \cap A)}{\sum_{i=1}^{n} P(B_i \cap A)}$$



Partitioning where n=3, $S = B_1 \cup B_2 \cup B_3$, $A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$. Then,

$P(A) = P(B_1 \cap A) + (B_2 \cap A) + P(B_3 \cap A)$.

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)} = \frac{P(B_1 \cap A)}{P(B_1 \cap A) + (B_2 \cap A) + P(B_3 \cap A)}$$

Ex: There exist two jars. While the first one contains 4 black and 6 red balls, the second one contains 3 black and 2 red balls. A jar is randomly drawn and a ball is chosen from this jar.

a. What is the probability of drawing a black ball?

b. Given that a black ball is drawn, what is the probability that that black ball is drawn from the first jar?

A: the first jar, B: the second jar, C: Choosing black ball

a.$P(C) = \frac{1}{2}\frac{4}{10} + \frac{1}{2}\frac{3}{5} = \frac{4+6}{20} = \frac{1}{2}$

b.$P(A|C) = \frac{P(A \cap C)}{P(A)} = \frac{4/20}{1/2} = \frac{8}{20}$

Ex: In a factory, machine A produces 30 percent of the output, machine B produces 25 percent of the output and machine C produces of the remaining 45 percent. One percent of the output of machine A is defective, as is 1.2 percent of Output, and 2 percent of C’s. In any given day, the three machines produce 10,000 items. An item drawn at random from a day's output is defective. What is the probability that it is produced by machine A?

E: defective item, B1: item produced by machine A, B2: item produced by machine B, B3: item produced by machine C

$$P(B_1) = 0.30, P(B_2) = 0.25, P(B_3) = 0.45$$

$$P(E|B_1) = 0.010, P(E|B_2) = 0.012, P(E|B_2) = 0.020,$$

$$P(B_1 \cap E) = P(B_1)P(E|B_1) = (0.30)(0.010) = 0.003$$

$$P(B_2 \cap E) = P(B_2)P(E|B_2) = (0.25)(0.012) = 0.003$$

$$P(B_3 \cap E) = P(B_2)P(E|B_3) = (0.45)(0.020) = 0.009$$

$$P(E) = 0.003 + 0.003 + 0.009 = 0.015$$

$$P(B_1|E) = \frac{P(B_1 \cap E)}{P(E)} = \frac{0.003}{0.015} = 0.20$$

Numbers Determined by Experiments: Random Variables

Random variables and their probability functions

Two important concepts will be studied, which are random variable and probability function

Definition (Random Variable): A variable whose value is a number determined by the outcome of an experiment is called a random variable. The random variable is denoted by capital letter and its numeric value is denoted by small letter. For example, while random variable is denoted by X, its corresponding value is denoted by x.

Some examples related to random variable

1.The number of children in each family

2.The number of cars sold in a month

3.The portion of chemical in a pain killer

Definition (Discrete Random Variable): Let X be a random variable and its corresponding values be x1, x2,…, xn, respectively. The values of random variable either takes finite number of values or takes countably infinite number of values. For example, two coins are tossed. Let X be the number of Heads observed. Then, sample space S={(H,H),(H,T),(T,H),(T,T)}.

X: The number of Heads observed. Then, X=0,1,2

{(H,H)}=2, {(H,T),(T,H)}=1, {(T,T)}=0

This is the case of X consisting of finite number of values. On the other hand, suppose that, we deal with observing the first H appearing in a trial. The sample space could be written as follows:

S={H, TH, TTH, TTTH,…..}

This is the case of X consisting of infinite number of values. Then, X=1,2,3,……

Definition (Continuous random variable): Let X be a random variable. If X takes values in a given interval or in intervals more than two, then it is called continuous random variable. For example, let X be a measurement of chemical in a substance. Then $X \in [0.01, 0.03]$.

The Distribution of a random Variable

Suppose that sample space S={(H,H),(H,T),(T,H),(T,T)}.

X: The number of Heads observed. Then, X=0,1,2

{(H,H)}=2, {(H,T),(T,H)}=1, {(T,T)}=0

Sample Space Points	(T,T)	(T,H)	(H,T)	(H,H)
X	0	1	1	2
P(X=x)	1/4	1/4	1/4	1/4

Then we can construct the probability distribution of X as follows:

X	0	1	2
P(X=x)	1/4	2/4	1/4

Definition (Probability Distribution): Let X be a random variable with possible values of x1, x2,…,xn and their associated probabilities of P(X1=x1)=f(x1), P(X2=x2)=f(x2),…,P(Xn=xn)=f(xn). Then he set f whose elements are ordered pairs

$$\left(x_i, f(x_i)\right), i = 1,2,..,n$$

is called a probability function of X.

A probability function satisfies two conditions expressed as follows:

1.$P(X_i = x_i) \geq 0 \; for \; all \; i = 1,2,\ldots,n$

2.$\sum_{i=1}^{n} P(X_i = x_i) = 1$

Ex: Let X be a discrete random variable whose probability distribution is given below as follows:

X	-3	0	2	3
P(X=x)	0.2	0.1	0.4	c

a. What is the value of c?

b. For which value of X is the largest probability attained?

c. P(X>0)=?

d. P(X=-2)=?

Answers

a.P(X=-3)+P(X=0)+P(X=2)+P(X=3)=1

0.2+0.1+0.4+c=1

c=0.3

b.P(X=2)=0.4

c.P(X>0)=P(X=2)+P(X=3)=0.4+0.3=0.7

d.P(X=-2)=0 since there exists no such an X value

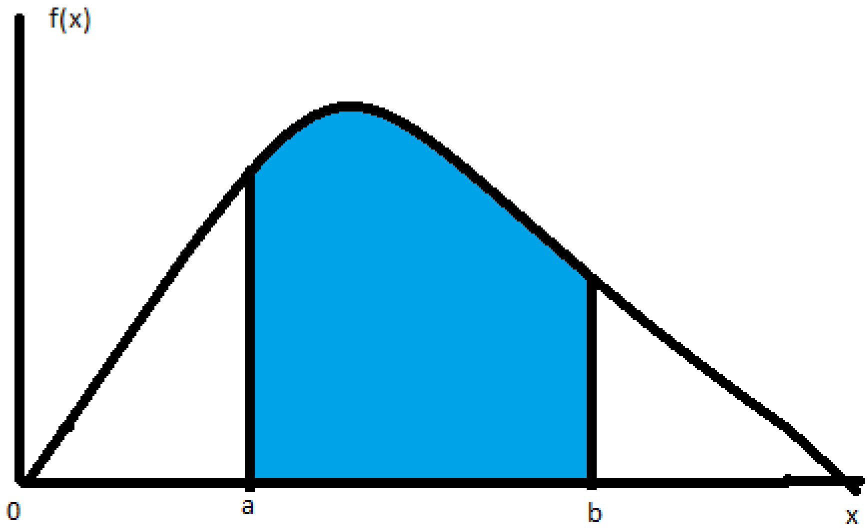
Definition (probability Density Function): Let X be a random variable take values in $(-\infty,\infty)$. Let f(x) be a function defined on this open interval. If two conditions given below are satisfied, then it is called a probability density function.

1.$f(x) \geq 0, for\ (-\infty,\infty)$

2.$\int_{-\infty}^{\infty} f(x)dx = 1$

For any area under f(x) is called a probability that is represented by

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f(x)dx$$



Ex: $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$

a.$P(0.25 < X < 0.75) =?$

b.$P(X > 0.25) =?$

a.$P(02.5 < X < 0.75) = \int_{0.25}^{0.75} 1dx = 0.50$

b. $P(X > 0.25) = \int_{0.25}^{1} 1dx = 0.75$

Ex: Let X be a random variable representing the duration of an electric bulb until it fails. Its probability distribution is defined by

$$f(x) = \begin{cases} \frac{a}{x^3}, & 1500 \leq x \leq 2500 \\ 0, & otherwise \end{cases}$$

It is known that f(x) is a probability density function. What is the value of a?

$$\int_{1500}^{2500} \frac{a}{x^3} dx = 1$$

$$\frac{-a}{2x^2} |_{1500}^{2500} = 1$$

a=7031250