Week 2

Question: A box contains 12 balls that consist of 5 black, 4 white, and 3 red. In how many ways could 6 balls be chosen with a constraint that at least one ball for each color should be in. (Hint: 2 possible solutions exist)

Theorem (Pascal Rule): $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \quad 1 \leq r \leq n$

Question: Given a set A={1,2,3,4,5}

a. How many subsets do A have (Hint: all possible subsets)

b. How many subsets do not contain both 1 and 2

c. How many subsets have at least 1 or 2.

Ex: There are 4 males and 2 females. A committee could be set up. In this committee consisting of at least 1 female and the number of males being at least two times, the number of females is required. How many committees could be set?

$\binom{2}{1}\binom{4}{2} + \binom{2}{1}\binom{4}{3} + \binom{2}{1}\binom{4}{4}+\binom{2}{2}\binom{4}{4}$

Ex: Given $2\binom{n}{4} = 3[\binom{n}{3} + \binom{n}{2}]$, Find n. Then, How many subsets composing of two elements at most could be set up?

$$\frac{2n!}{4!\ (n-4)!} = \frac{3n!}{3!\ (n-3)!} + \frac{3n!}{2!\ (n-2)!}$$

$$\frac{2n!}{(n-4)!\,4!} = \frac{2n(n-1)(n-2(n-3)(n-4)!}{(n-4)!\ 24} = \frac{n(n-1)(n-2)(n-3)}{12}$$

$$\frac{3n!}{(n-3)!\,3!} = \frac{3n(n-1)(n-2)(n-3)!}{(n-3)!\ 6} = \frac{n(n-1)(n-2)}{2}$$

$$\frac{3n!}{(n-2)!\,2!} = \frac{3n(n-1)(n-2)!}{(n-2)!\ 2} = \frac{3n(n-1)}{2}$$

We deal with the left-hand side of the equation first

$$\frac{n^3-3n^2+2n}{12}+\frac{3n^2-3n}{2}=\frac{n^3-n}{2}$$

Then the right-hand side is treated

$$\frac{n(n-1)(n-2)(n-3)}{12}=\frac{n^4-6n^3+11n^2-6n}{12}$$

When equating both sides,

$$\frac{n^4-6n^3+11n^2-6n}{12}=\frac{6n^3-6n}{12}$$

Thus,

$$n^4-12n^3+11n^2=0$$

$$n^2(n^2-12n+11)=0$$

$n^2=0$ or $n^2-12n+11=0$

n=0 or n=1 or n=11. Since n is defined for positive numbers n=0 is excluded n=1 could be but is not suitable for the question since we deal with subsets composed of 2 elements. Thus, n=11.

$$\binom{11}{0}+\binom{11}{1}+\binom{11}{2}=67$$

Ex: An instructor gives an exam consisting of 10 ten questions and asks them to answer 7 out of 10.

a. How many ways could a student answer those questions?

b. Instructor asks students to answer one of the two questions when questions are grouped as 1-2, 3-4, 5-6, 7-8, and 9-10 consecutively. How many ways could a student answer those questions?

a.$\binom{10}{7}=120$

b. $\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{5}{2}=320$

Ex: Given $\binom{n}{2}=45$ what is the value of n?

$$\frac{n!}{(n-2)!\,2!}=45$$

$$\frac{n(n-1)}{2}=45$$

$$n(n-1)=90$$

n=10

Ex: Given $\frac{P_4^n}{\binom{n-1}{3}} = 60$, Find n.

$$\frac{\frac{n!}{(n-4)!}}{\frac{(n-1)!}{(n-4)!\,3!}} = 60$$

$$\frac{n(n-1)(n-2)(n-3)}{\frac{(n-1)(n-2)(n-3)}{6}} = 60$$

$$\frac{6n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)} = 60$$

n=10

Remark 1: The order is taken into account in Permutation

Remark 2: The order is not taken into account in Combination

Combinations with Repetitions

Theorem: Suppose that n number of the objects that are all different and can be repeated without any conditions and constraints by taking into account the order of objects. The number of k-wise permutations is equal to n^k .

Ex: Given A={1,2,3,4}, by using all the elements in A, how many permutations could be set up using 2 wise, 3 wise, and 4 wise

2wise:{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4) }

$$4^2 = 16$$

Similarly, 3-wise and 4-wise, respectively

$$4^3 = 81 \; ve \; 4^4 = 256$$

Theorem: Suppose that n number of the objects that are all different and can be repeated without any conditions and constraints by not taking into account the order of objects. The number of k- wise combinations is denoted by

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Permutations of objects that are not all different

Theorem: Given a set of n objects having r_1 elements alike of one kind, and r_2 elements alike of the second kind and …, r_k elements of the kth kind; then the number of permutations of the n objects, taken all together, is denoted

$$\binom{n}{r_1, r_2, r_3, \dots, r_k} = \frac{n!}{r_1!\, r_2!\, r_3! \dots r_k!}$$

Ex: Utilizing the letters of A-L-L-A, how many words either meaningful or meaningless could be generated?

4!=4.3.2.1=24.

However, there are the same letters. So, repetitions should be disregarded from the total permutations. To show repetitions, one A and L are denoted by bold cases., 24 different permutations are presented in Table 1:

Table 1: 24 different permutations of A-L-L-A

AA**L**L	A**AL**L	**L**L**A**A	L**LA**A
AAL**L**	A**A**L**L**	**L**LA**A**	L**L**A**A**
ALAL	A**LA**L	**LA**LA	L**A**A**L**
ALLA	A**L**L**A**	**LA**AL	L**AL**A
ALA**L**	AL**LA**	**L**AL**A**	LA**LA**
AL**L**A	AL**AL**	**L**A**A**L	LA**AL**
3	0	3	0

$$\frac{4!}{2!\,2!} = 6$$

Ex: 2 red, 3 black, and 5 white beads are used to be ordered in a row. The same color beads are in the same magnitude and shape. How many different ways could this arrangement occur?

$$\binom{10}{2,3,5} = \frac{10!}{2!\,3!\,5!}$$

Ex: 4 red, 3 white, and 1 blue flag are used to arrange an arrow. How many different ways could this arrangement occur?

$$\frac{8!}{4!\,3!\,1!} = 280$$

Ex: By using the word called KARAKAYA, How many different ways could words be generated?

$$n = 8, r_K = 2, r_A = 4, r_R = 1, r_Y = 1$$

$$\frac{8!}{4!\,2!\,1!\,1!} = 840$$

Ordered and Unordered Partitions

Ordered **Partitions**

Theorem: let A be a set consisting of n different objects denoted by (A_1, A_2, …, A_k), and the number of elements in each partition is denoted by (r_1, r_2,…,r_k) where $n= r_1+r_2+…,+r_k$.

The number of ordered partitions is computed by

$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$

Ex: 9 different toys are shared among 4 siblings. While the youngest one will have 3 out of 9, the rest will have an equal number of toys. How many different ways could this arrangement occur?

$$\frac{9!}{3!\,2!\,2!\,2!} = 7560$$

Unordered Partitions

In an unordered partition, the total number of objects is split into subsets and the order does not count.

Ex: 12 students are distributed into 3 activity clubs. Each club must have 4 students. Let clubs be denoted by A, B, and C. How many different ways could this arrangement be occurred?

$$\binom{12}{4}\binom{8}{4}\binom{4}{4}\frac{1}{3!} = 5775$$

Ex: 10 persons are used to set up two groups. Each group must have at least one person. How many different ways could this arrangement occur?

$$\binom{10}{1}\binom{9}{9} + \binom{10}{2}\binom{8}{8} + \binom{10}{3}\binom{7}{7} + \cdots + \binom{10}{9}\binom{1}{1}$$

$$\binom{10}{1}.1+\binom{10}{2}.1+\binom{10}{3}.1+\cdots+\binom{10}{9}.1=2^{10}-\binom{10}{0}-\binom{10}{10}=2^{10}-2$$

Since it is an unordered partition, repetitions should be disregarded.

$$\frac{2^{10}-2}{2!}$$

Ex:8 friends took a short trip and they planned to stay at a hotel one night. The hotel has 3 vacancies whose 2 beds, 3 beds, and 3 beds in 3 rooms respectively, 2 out of 8 persons could not share the same room. How many different ways could this arrangement occur?

For these two persons, the arrangements should be as follows:

One of the two persons chooses one room then the second one is assigned to one of the other two rooms.

Suppose that the first person decides to stay in a room having two beds. Then, another person is assigned to one of the other rooms:

$$\binom{1}{1}\binom{6}{1}\binom{6}{3}\binom{3}{3}=1.6.20.1=120$$

Suppose that the first person decides to choose the first room that has 3 beds. Then the second person should choose to stay in either a room having 2 beds or a room having 3 beds:

$\binom{1}{1}\binom{6}{2}\binom{5}{2}\binom{3}{3}+\binom{1}{1}\binom{6}{2}\binom{5}{3}\binom{2}{2}$=150

Suppose that the first person decides to choose the second room that has 3 beds. Then the second person should choose to stay in either a room having 2 beds or a room having 3 beds:

$\binom{1}{1}\binom{6}{2}\binom{5}{2}\binom{3}{3}+\binom{1}{1}\binom{6}{2}\binom{5}{3}\binom{2}{2}$=150

Then the total number of arrangements will be

120+150+150=420

Alternative solution

If no constraints,

$$\binom{8}{2}\binom{6}{3}\binom{3}{3}=560$$

Assuming that these two persons stay in the same room. Then 3 alternatives would be constructed as follows:

The room has 2 beds

$$\binom{2}{2}\binom{6}{3}\binom{3}{3} = 20$$

The room has 3 beds

$$\binom{2}{2}\binom{6}{1}\binom{5}{3}\binom{2}{2} = 60$$

The room has 3 beds

$$\binom{2}{2}\binom{6}{1}\binom{5}{3}\binom{2}{2} = 60$$

560-(20+60+60)=420

Binomial Theorem

Theorem: n positive integer

$(a+x)^n = \binom{n}{0}a^n x^0 + \binom{n}{1}a^{n-1}x^1 + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r}a^r x^{n-r} + \cdots + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{k}a^{n-r}x^r$

$\binom{n}{r}$ is called binomial coefficients

Ex:$(2x+3)^{20}$ is given. What is the coefficient of the 12$^{\text{th}}$ element in the binomial expansion?

$$\binom{20}{11}(2x)^{11}(3)^9 = \binom{20}{11}(2)^{11}(3)^9$$

The probability of an Event and Axioms of probability

Definition: If an experiment results in N different outcomes and M out of N is related to an event called A. then the probability of A denoted by P(A) is defined by

$$P(A) = \frac{M}{N}$$

Ex: Toss a die. Event A is defined as being an even number.

$S = \{1,2,3,4,5,6\}$ and A={2,4,6}. To compute the probability of A denoted by P(A),

$$P(A) = \frac{3}{6}$$

Ex: By using the numbers 1-2-3-4-5, three digits numbers would be generated without repetitions.

a. What is the probability of generating an even number?

b. What is the probability of generating a number that is a multiple of 5?

a. The total number of 3 digits numbers=N=5.4.3=60

a. The total number of 3-digit even numbers=M=4.3.2=24

$$P(A) = \frac{24}{60}$$

b. Event B: the total number of numbers that is multiple of t 5=K=4.3.1=12

$$P(B) = \frac{12}{60}$$

Ex: A sack contains 10 balls numbered from 1 to 10. 2 balls are drawn randomly. What is the probability of drawing the balls numbered 3 and 7?

The total number of drawing 2 balls =N=$\binom{10}{2}$=45

A: the total number of drawing both 3 and 7 at the same time=M=1

$$P(3\ ve\ 7) = P(3 \cap 7) = \frac{1}{45}$$

Axioms of Probability

Let D be an experiment and S be a sample space. The probability of any event A in S is denoted by P(A). The axioms are stated as follows:

a.$P(A) \geq 0$

b.$P(S) = 1$

c. the Either finite or infinite number of events belong to S. Suppose that those events are pairwise disjoint. Then,

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$ for finite cases

$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$. for infinite cases

Ex: Toss a pair of dice. What is the probability of getting a summation of 8?

A=(2,6),(6,2),(3,5),(5,3),(4,4)}, the total number of sample points satisfy the summation of 8, which is M=5

S={(1,1),…,(6,6)}=6^2, N=36=6^2.

$$P(A) = \frac{5}{36}$$

Some rules

Theorem: Let A_1 and A_2 be in S with $A_1 \subset A_2$ then, $P(A_1) \leq P(A_2)$