Week 6

Definition (Cumulative Distribution Function or Distribution Function for Discrete Case): Let X be a random variable. The cumulative distribution function or distribution function is denoted by F(x) and is defined by

$$F(x) = P(X \leq x) = \sum_{X_i \leq x} P(X \leq x_i)$$

Ex: Let X be a discrete random variable whose probability distribution is defined by

X=x	-1	0	1	3
P(X=x)	1/6	2/6	2/6	1/6

Find F(x).

$$P(X < -1) = 0$$

$$P(X \leq -1) = \frac{1}{6}$$

$$P(X \leq 0) = P(X = -1) + P(X = 0) = \frac{1}{6} + \frac{2}{6} = \frac{3}{6}$$

$$P(X \leq 1) = P(X = -1) + P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{2}{6} + \frac{2}{6} = \frac{5}{6}$$

Instead of writing $P(X \leq 3) = \frac{1}{6} + \frac{2}{6} + \frac{2}{6} + \frac{1}{6} = \frac{6}{6} = 1$

$$P(X \geq 3) = 1$$

So, $X \leq -1 \ \rightarrow -1 \leq x < 0, \quad X \leq 0 \ \rightarrow 0 \leq x < 1, \quad X \leq 1 \ \rightarrow 1 \leq x < 3$

Then,

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{6}, & -1 \leq x < 0 \\ \frac{3}{6}, & 0 \leq x < 1 \\ \frac{5}{6}, & 1 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Ex: let X be a discrete random variable whose distribution function is defined by

$$F(x) = \begin{cases} 0, & x < -3 \\ 0.2, & -3 \le x < 0 \\ 0.3, & 0 \le x < 2 \\ 0.7, & 2 \le x < 3 \\ 1. & x \ge 3 \end{cases}$$

Find probability distribution of X.

$$P(X < -3) = 0, \qquad P(-3 \le x < 0) = 0.2, \qquad P(0 \le x < 2) = 0.3,$$
$$P(2 \le x < 3) = 0.7, \qquad P(x \ge 3) = 1$$

$$P(-3 \le x < 0) = 0.2 = P(X = -3)$$
$$P(0 \le x < 2) = 0.3 = P(X = 0) - P(X = -3) = 0.3 - 0.2 = 0.1$$
$$P(2 \le x < 3) = 0.7 = P(X = 2) - P(X = 0) = 0.7 - 0.3 = 0.4$$
$$P(X \ge 3) = 1 - P(X < 3) = 1 - 0.7 = 0.3$$

X=x	-3	0	2	3
F(x)=P(X≤)	0.2	0.1	0.4	0.3

Definition (Cumulative Distribution Function or Distribution Function for Continuous Case): Let X be a continuous random variable whose probability density function is called f(x). The cumulative distribution function or distribution function is called F(x) and is defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

Ex: let X be a continuous random variable whose probability density function is defined by

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find F(x)

$$F(x) = P(X \le x) = \int_{0}^{x} 2xdx = x^2|_0^x = x^2 - 0 = x^2$$

$$F(x) = \begin{cases} x^2, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

Remark: If X is a continuous random variable, then F(x) is a continuous function for all x values.

Theorem: a. F(x) is a non-decreasing function, namely, $x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$

b. $\lim_{x \to -\infty} F(x) = 0 \; and \; \lim_{x \to \infty} F(x) = 1$

Theorem: a. Let X ne a continuous random variable and whose probability density function is denoted by f(x). Then its distribution function is denoted by F(x) that is differentiated by all x values in the domain.

$$f(x) = \frac{d}{dx}(F(x))$$

Also, $P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b) = F(b) - F(a)$

b.Let X be discrete random variable taking values of $x_1 < x_2 < \cdots < x_{n-1} < x_n < \cdots$, Let F(x) be distribution function.

$$P(X = x_i) = P(X = x_i) - P(X = x_{i-1})$$

Ex: Let X be a discrete random variable whose distribution function is denoted by

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x \ge 10 \end{cases}$$

a.$P(2 < X \le 6) =?$

b. P(X=4)=?

c. Write down the probability function.

a. $P(2 < X \le 6) = F(6) - F(2) = F(6) - F(1) = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}$

b. P(X=4)=F(4)-F(1)=1/2-1/3=1/6

c.

X=x	1	4	6	10
P(X=x)	1/3	1/6	1/3	1/6

Ex: Let X be a continuous random variable whose probability density function is defined by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & otherwise \end{cases}$$

Find F(x)

$$F(x) = P(X \leq 1) + P(X \leq 2) = \int_0^x xdx + \int_1^x 2 - xdx = \frac{x^2}{2} + \frac{4x - x^2 - 3}{2}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 < x < 1 \\ \frac{4x - x^2 - 3}{2}, & 1 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

Expected Value of a Random Variable

Definition: Let X be either a discrete or a continuous random variable corresponding to P(X=x) probability function or f(x) probability density function. The expected value of X denoted by E(X) is defined by

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i) \;\; for\ discrete\ case$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad for\ continuous\ case$$

The expected value is a called the mena value or average value of the random variable.

Ex: Let X be discrete random variable whose probability function is defined by

X=x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Find E(x)

$$E(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 3/2$$

Remark: Expected value takes any values in $(-\infty, \infty)$.

Ex: Let X be a continuous random variable whose probability density function is defined by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find E(x)

$$E(x) = \int_0^\infty xe^{-x}dx = -e^{-x}|_0^\infty = 0 - (-1) = 1$$

Properties of Expected Value

Definition: When Y=g(x) is defined, E(Y) could be computed for both discrete and continuous cases as follows:

$$E(Y) = E(g(x_i)) = \sum_{i=1}^{n} g(x_i)P(X = x_i)$$

$$E(Y) = E\big(g(x)\big) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Ex:Let X be discrete random variable whose probability function is defined by

X=x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Find E(2X-1)

Y=2X-1.

So, for x=0,1,2,3→ 2x-1 would take -1,1,3, 5 with same probabilities

Then,

$$E(Y) = E(2X - 1) = (-1)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (3)\left(\frac{3}{8}\right) + (5)\left(\frac{1}{8}\right) = \frac{16}{8} = 2$$

Or, We know that E(X)=3/2 then E(2X-1)=2E(X)-1=(2)(3/2)-1=2

Ex: Let X be a continuous random variable whose probability density function is defined by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find E(2X)

Y=2X

$$E(Y) = E(2X) = \int_0^\infty 2xe^{-x}dx = 2[-xe^{-\infty}|_0^\infty + \int_0^\infty e^{-\infty}dx] = 2$$

Or, we know that E(X)=1 then E(2X)=2E(X)=2.1=2