COM 205 - Digital Logic Design
Boolean Algebra and Logic Gates

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ALKÜ

Last Week

- Arithmetic Operations
- Binary Codes
- Registers

Boolean Algebra

- A set of elements : any collection of objects having common property A={1,2,3,4}
- A set of operators: a binary operator defined on a set S of elements is a rule that assigns to each pair of elements from S, a unique element from S.
- A number of unproved axioms or postulates

Boolean Algebra

- Example:
  - a*b=c
    - * is a Binary operator;
      - if it specifies a rule for finding c from the pair (a,b) and
      - if a,b,c $\in$ S
    - However, * is not a Binary operator if a,b $\in$ S is c$\notin$S.

Boolean Algebra

- Postulates of a mathematical system form the basic assumptions from which it is possible to deduce the rules, theorems and properties of the system.
- Most common postulates:
    1. Closure: A set S is closed wrt. a Binary operator if, for every pair of elements of S, the binary operator specifies a rule for obtaining a unique element of S.
        - Ex: N (set of Natural numbers) is closed wrt. Binary operatör (+)
        - N (set of Natural numbers) is not closed wrt. Binary operatör (-)
    2. Associative Law: A Binary operator * on a set S is said to be associative whenever $(x*y)*z=x*(y*z)$ for all $x,y,z \in S$.
    3. Commutative Law: A Binary operator * on a set S is said to be commutative whenever $x*y=y*x$ for all $x,y \in S$.

Boolean Algebra

- Postulates..

4. Identity Element: A set S is said to have an identity element wrt. A Binary operation * on S if there exists an element e Є S with the property that e*x=x*e=x for every x Є S.

   Ex: 0 is an identity element wrt. the binary operator + on the set of integers I={…, -3,-2,-1,0,1,2,3,…} since x+0=0+x=x for any x Є I.

5. Inverse: A set S having the identity element e wrt. A Binary operator * is said to have an inverse whenever, for every x Є S there exists an element y Є S s.t x*y=e.

   Ex: set of integers I, and the operatör + with e=0, the inverse of an element a is (-a) since a+(-a)=0.

6. Disributive Law: If * and · are two Binary operators on a set S, * is said to be distributive over · whenever

   x*(y · z)=(x*y) · (x*z)

Boolean Algebra

1.Closure
2.Associative
3.Commutative
4.Identity element
5.Inverse
6.Distributive

- A field is an example of an algebraic structure
  - A set of elements
  - Two Binary operators, having properties 1-5 and both operators combine to give property 6.

Ex: A set of real numbers, together with Binary operators + and · forms the field of real numbers.

The operators and the postulates have the following meanings:

The Binary operatör + defines addition
The additive identity is 0.
The additive inverse defines subtraction.
The Binary operator · defines multiplication.
The multiplicative identity is 1.
For $a \neq 0$ the multiplicative inverse of $a = \frac{1}{a}$ defines division (ie. a · $\frac{1}{a}$ = 1)
The only distributive law applicable is that of · over +:

a · (b+c) = (a · b) + (a · c)

Axiomatic Definition of Boolean Algebra

- In 1847, George Boole developed Boolean Algebra.
- In 1938, Shannon introduced a two-valued Boolean algebra called switching algebra which represents the properties of bistable electrical switching circuits.
- In 1904 Huntington defined the postulates for Boolean Algebra.

Boolean Algebra

- Boolean Algebra is an algebraic structure defined by a set of elements B, together with two Binary operators + and · , provided that the following (Huntington) postulates are satisfied:
  1. (a) Structure is closed wrt. +
     (b)Structure is closed wrt. ·
  2. (a) The element 0 is an identity element wrt. +; x+0=0+x=x
     (b) The element 1 is an identity element wrt. · ; x · 1= 1 · x=x
  3. (a) The structure is commutative wrt. +; x+y = y+x
     (b) The structure is commutative wrt. · ; $\mathrm{x} \cdot \mathrm{y} = \mathrm{y} \cdot \mathrm{x}$
  4. (a) The operator · is distributive over +; x· $(\mathrm{y} + \mathrm{z}) = (\mathrm{x} \cdot \mathrm{y}) + (\mathrm{x} \cdot \mathrm{z})$
     (b) The operator + is disrtibutive over · ; $\mathrm{x} + (\mathrm{y} \cdot \mathrm{z}) = (\mathrm{x} + \mathrm{y}) \cdot (\mathrm{x} + \mathrm{z})$
  5. For every element $x \in B$,there exits an element $x' \in B$(complement of x) s.t
     (a) x+x'=1
     (b) x · x'=0
  6. There exits at least two elements x,y ∈ B s.t $x \neq y$

Comparison of Boolean Algebra with Ordinary Algebra

1. Huntington postulates do not include the associative law. However, this law holds for Boolean algebra and can be derived from other postulates.
2. The distributive law of + over · $(x+(y \cdot z)=(x+y) \cdot (x+z))$ is valid for Boolean algebra, but not for ordinary algebra.
3. Boolean Algebra does not have additive or multiplicative inverse; therefor there are no subtraction or division operations.
4. Postulate 5 defines an operatör called the complement that is not available in ordinary algebra.
5. Ordinary algebra deals with the real numbers. Boolean algebra deals with a set of elements B, which has not been defined. We will deal with two-valued Boolean Algebra whose set of elements B is defined as {0,1}.

Boolean Algebra

- +, · symbols for operators are intentionally chosed due to the similarity between ordinary algebra and boolean algebra.
- One should be careful about using the rules of boolean algebra.
- In order to be able to have a Boolean Algebra, one must Show:
    1. The elements of the set B,
    2. The rules of operation for the Binary operators,
    3. The set of elements B, together with the two operations satisfy the six Huntington postulates.
- We will deal with the application of Boolean algebra to the gate type circuits.

Two-Valued Boolean Algebra

- Defined on a set of two elements B={0,1} with rules for the two Binary operators + and · as follows:

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

AND

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

OR

x	x'
0	1
1	0

NOT

Two-Valued Boolean Algebra

- We must show that the Huntington postulates are valid for the set B={0,1} and the two Binary operators + and ·.
  1. Closure is obviously seen from the table. As result of each operation is either 0 or 1. 1,0 ∈B.
  2. From the tables:

     (a) 0+0=0   0+1=1+0=1

     (b) 1 · 1 =1   1 · 0 = 0 · 1 = 0
  3. The commutative laws are obvious from the symmetry of the Binary operator tables.

The identity element 0 for +
identity element 1 for · by postulate 2.

Two-Valued Boolean Algebra

4. (a) The distributive law x · (y+z) = (x ·y)+(x ·z) can be shown to hold from the operator tables by forming a truth table.

x	y	z	y+z	x· (y+z)	x· y	x· z	(x· y)+(x· z)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

- (B) The distributive law $x + (y \cdot z) = (x + y) \cdot (x + z)$ can be shown to hold from the operator tables by forming a truth table.

x	y	z	y·z	X+ (y·z)	x+y	x+z	(x+ y) ·(x+z)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Two-Values Boolean Algebra

4.(b) The distributive law of + over $\cdot$can be shown to hold by means of a truth table similar to part (a).

5. From the complement table, it is easily shown that

    (a)$x+x'=1$ since $0+0'=0+1=1$ and $1+1'=1+0=1$

    (b)$x \cdot x'=0$ since $0 \cdot 0'= 0 \cdot 1=0$

    Postulate 1 is verified.

6. Postulate 6 is satisfied since the two-valued Boolean Algebra has two elements 1 and 0 with $1\neq 0$.

We have established a two-valued Boolean Algebra havins a set of two elements, 1 and 0, with two Binary opeators with rules equivalent to AND, OR, complement operaors. Boolean Algebra is defined mathematically and shown that it is very similar to the Binary logic.

Basic Theorems and Properties of Boolean Algebra

- Duality: Huntington postulates were listed in pairs (a & b). One part may be obtained from the other if the Binary operators and the identity elements are interchanged.

1. (a) Structure is closed wrt. +
   (b)Structure is closed wrt. $\cdot$
2. (a) The element 0 is an identity element wrt. +; x+0=0+x=x
   (b) The element 1 is an identity element wrt. $\cdot$; $\mathrm{x} \cdot 1 = 1 \cdot \mathrm{x} = \mathrm{x}$
3. (a) The structure is commutative wrt. +; x+y = y+x
   (b) The structure is commutative wrt. $\cdot$; $\mathrm{x} \cdot \mathrm{y} = \mathrm{y} \cdot \mathrm{x}$
4. (a) The operator $\cdot$ is distributive over +; $\mathrm{x} \cdot (\mathrm{y} + \mathrm{z}) = (\mathrm{x} \cdot \mathrm{y}) + (\mathrm{x} \cdot \mathrm{z})$
   (b) The operator + is disrtibutive over $\cdot$; $\mathrm{x} + (\mathrm{y} \cdot \mathrm{z}) = (\mathrm{x} + \mathrm{y}) \cdot (\mathrm{x} + \mathrm{z})$
5. For every element $x \in B$,there exits an element $x' \in B$(complement of x) s.t
   (a) x+x'=1
   (b) $\mathrm{x} \cdot \mathrm{x}' = 0$
6. There exits at least two elements x,y $\in B$ s.t $x \neq y$

- If the dual of an algebraic expression is desired, we interchange OR and AND operators and replace 1's by 0's and 0's by 1's.

Basic Theorems and Properties of Boolean Algebra

- Basic Theorems:
  - Postulates and Theorems of Boolean Algebra:

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x+x'=1$	(b) $x \cdot x'=0$
Theorem 1	(a) $x+x=x$	(b) $x \cdot x = x$
Theorem 2	(a) $x+1=x$	(b) $X \cdot 0 = 0$
Theorem 3 (involution)	$(x')'=x$	
Postulate 3, commutative	(a) $x+y=y+x$	(b) $x \cdot y = y \cdot x$
Theorem 4 , associative	(a) $x+(y+z)=(x+y)+z$	(b) $x \cdot (y \cdot z)= (x \cdot y) \cdot z$
Postulate 4, distributive	(a) $x \cdot (y+z)=(x \cdot y)+(x \cdot z)$	(b) $x+(y \cdot z)=(x+y) \cdot (x+z)$
Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x \cdot y=x$	(b) $x \cdot (x+y)=x$

Basic Theorems and Properties of Boolean Algebra

- Postulates are basic axioms of the algebraic structure, and need no proof.
- The theorems must be proven from the postulates.

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x+x'=1$	(b) $x \cdot x'=0$
Theorem 1	(a) $x+x=x$	(b) $x \cdot x = x$
Theorem 2	(a) $x+1=x$	(b) $X \cdot 0 = 0$
Theorem 3 (involution)	$(x')'=x$	
Postulate 3, commutative	(a) $x+y=y+x$	(b) $x \cdot y = y \cdot x$
Theorem 4 , associative	(a) $x+(y+z)=(x+y)+z$	(b) $x \cdot (y \cdot z)= (x \cdot y) \cdot z$
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Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x+x'=1$	(b) $x \cdot x'=0$
Theorem 1	(a) $x+x=x$	(b) $x \cdot x = x$
Theorem 2	(a) $x+1=x$	(b) $X \cdot 0 = 0$
Theorem 3 (involution)	$(x')'=x$	
Postulate 3, commutative	(a) $x+y=y+x$	(b) $x \cdot y = y \cdot x$
Theorem 4 , associative	(a) $x+(y+z)=(x+y)+z$	(b) $x \cdot (y \cdot z)= (x \cdot y) \cdot z$
Postulate 4, distributive	(a) $x \cdot (y+z)=(x \cdot y)+(x \cdot z)$	(b) $x+(y \cdot z)=(x+y) \cdot (x+z)$
Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x \cdot y=x$	(b) $x \cdot (x+y)=x$

- Theorem 1(a) $x + x=x$

$x + x$	$= (x + x) \cdot 1$	Postulate 2b
	$= (x + x) \cdot (x + x')$	Postulate 5a
	$= x + (x \cdot x')$	Postulate 4b
	$= x + 0$	Postulate 5b
	$= x$	Postulate 2a

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b)$x \cdot 1 = x$
Postulate 5	(a) $x+x'=1$	(b)$x \cdot x'=0$
Theorem 1	(a) $x+x=x$	(b) $x \cdot x = x$
Theorem 2	(a) $x+1=x$	(b) $X \cdot 0 = 0$
Theorem 3 (involution)	$(x')'=x$	
Postulate 3, commutative	(a) $x+y=y+x$	(b) $x \cdot y = y \cdot x$
Theorem 4 , associative	(a)$x+(y+z)=(x+y)+z$	(b) $x \cdot (y \cdot z)= (x \cdot y) \cdot z$
Postulate 4, distributive	(a) $x \cdot (y+z)=(x\cdot y)+(x\cdot z)$	(b) $x+(y\cdot z)=(x+y) \cdot(x+z)$
Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x\cdot y=x$	(b) $x \cdot (x+y)=x$

- Theorem 1(b) $x\cdot x=x$

$x \cdot x$	$= x \cdot x + 0$	Postulate 2a
	$= x \cdot x + x\cdot x'$	Postulate 5b
	$= x \cdot (x+x')$	Postulate 4a
	$= x \cdot 1$	Postulate 5a
	$= x$	Postulate 2b

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x+x'=1$	(b) $x \cdot x'=0$
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Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x \cdot y=x$	(b) $x \cdot (x+y)=x$

- Theorem 2 (a) $x + 1= 1$

$x+1$	$= 1 \cdot (x+1)$	Postulate 2b
	$= (x+x') \cdot (x+1)$	Postulate 5a
	$= x + x' \cdot 1$	Postulate 4b
	$= x + x'$	Postulate 2b
	$= 1$	Postulate 5a

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
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Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x\cdot y=x$	(b) $x \cdot (x+y)=x$

- Theorem 2(b) $x \cdot 0 = 0$ (duality)

$x \cdot 0$	$= 0 + x \cdot 0$	Postulate 2a
	$= xx' + x0$	Postulate 5b
	$= x \cdot (x'+0)$	Postulate 4a
	$=xx'$	Postulate 2a
	$=0$	Postulate 5b

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
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- Theorem 3 $(x')'=x$

From postulate 5 $x+x'=1$ and $xx'=0$

(These two define the complement of x)

The complement of x' is x and is also $(x')'$

Since the complement is unique we have $(x')'=x$

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
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- Theorem 6(a) $x+x \cdot y = x$

$$
\begin{aligned}
x + x \cdot y &= x \cdot 1 + x \cdot y && \text{Postulate 2b} \\
&= x \cdot (1+y) && \text{Postulate 4a} \\
&= x \cdot (y +1) && \text{Postulate 3a} \\
&= x \cdot 1 && \text{Theorem 2a} \\
&= x && \text{Postulate 2b}
\end{aligned}
$$

Proofs of Theorems

Postulate 2	(a) $x+0=x$	(b) $x \cdot 1 = x$
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Theorem 5, DeMorgan	(a) $(x+y)'=x' \cdot y'$	(b) $(xy)'=x'+y'$
Theorem 6, Absorption	(a) $x+x\cdot y=x$	(b) $x \cdot (x+y)=x$

- Theorem 6 (b) $x \cdot (x+y) = x$ by duality

$x \cdot (x+y)$	$= (x+0) \cdot (x+y)$	Postulate 2a
	$= x + (0 \cdot y)$	Postulate 4b
	$=x + (y \cdot 0)$	Postulate 3b
	$= x + 0$	Theorem 2b
	$= x$	Postulate 2a

Proofs of Theorems

- The theorems of Boolean algebra can be proven by means of truth tables:
- Ex: $x + x \cdot y = x$

x	y	x·y	x + x·y
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Proofs of Theorems

- Ex: $(x+y)' = x' \cdot y'$

x	y	x+y	(x+y)’	x’	y’	x’y’
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0