Week 1

Sample Space, Sample Points, and Events

Definition (experiment): Any process or a series of trials resulting in outcomes with their corresponding probabilities is called an experiment

Some experiments are as follows:

1. Toss a coin and observe the number of heads appearing

2. A jar contains 5 black and 3 red balls. Drawing a ball from the jar is called an experiment

3. A machine located in a mill produces some defective products in a given time interval. The number of defective products is called an experiment.

Definition (Sample Space): The set that contains all possible outcomes of a given experiment is called a sample space and is denoted by S. Each element in S is called a sample point.

Ex: Toss a die. All possible outcomes of tossing a die are denoted by

$$S = \{1,2,3,4,5,6\}$$

Ex: Toss a coin. All possible outcomes of tossing a coin are denoted by

$$S = \{H, T\}$$

Ex: Selection of a male student among a group of male students in a class and measuring his height is an experiment denoted by

$$S = \{x | x > 0\}$$

Any measurement of a male student whose height is greater than 0 is just one of the possible outcomes.

Ex: Toss three coins whose sample space is denoted by

$$S = \{(H,H,H),(H,T,H),(T,H,H),(H,H,T),(T,T,H),(T,H,T),(H,T,T),(T,T,T)\}$$

Definition (Event): Any subset of a sample space denoted by S is called an event. The empty set and sample space itself are subsets of sample space S. Hence, they are also called events.

Ex: Toss a die. An event is defined as the die showing 4 or greater than 4

S={1,2,3,4,5,6}, A={4,5,6}

Ex: Toss a die. An event is defined as the die showing a number divided by 2 with an integer number as an outcome.

S={1,2,3,4,5,6}, B={2,4,6}

Ex: Toss a pair of dice. An event is defined as the total number of the two dice less than 6.

S={(m, n):$1 \leq m, n \leq 6$}

A={(1,1),(1,2),(2,1),(1,3),(3,1),(2,2),(2,3),(3,2),(4,1),(1,4)}

Definition (Impossible and Certain Events): Any event is a subset of sample space denoted by S. Hence, both empty set and sample space are subsets of sample space, which are denoted by (Ø) and S, respectively. While a certain event is an event observed each time, an impossible event is an event that cannot be observed at any time.

Ex: Toss three coins. Event A is defined as showing three heads.

$$S = \{(H,H,H),(H,T,H),(T,H,H),(H,H,T),(T,T,H),(T,H,T),(H,T,T),(T,T,T)\}$$

A={(H,H,H)}

Ex: Cast a pair of dice. An event is defined as showing the total of two dice 3 or less than 3.

S={(m, n): $1 \leq m, n \leq 6$} olup,

A={(1,1),(1,2),(2,1)}

Definition (Random Event): The occurrence of any event depending on randomness is called a random event.

Definition (Disjoint Events): If $A \cap B$=Ø then A and B are called disjoint.

Ex: Cast a die. A is defined as showing odd numbers. B is defined as showing even numbers.

A={1,3,5} B={2,4,6}. $A \cap B$=Ø. A and B are called disjoint events.

The rules for counting sample points

Definition (Addition Rule): Suppose we have two operations. Both operations are disjoint. The first operation could be executed with N_1 in different ways. The second operation could be executed with N_2 in different ways. Either the first or the second operation could be executed with N_1+N_2 in different ways.

Ex: Cast a pair of dice. The total number of two dice is either 4 or 7.

A: That the total of two dice is 4 is called event A.

B: That the total of two dice is 7 is called event B.

A= {(1,3), (3,1), (2,2)}

B={(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)}

3+6=9 different ways.

Ex: In a classroom containing 15 female and 24 male students, In how many different ways could a representative person for this classroom be chosen?

15+24=39

Theorem (The Extension of Addition Rule): Suppose we have K operations. The first operation could be executed with N_1 in different ways. The second operation could be executed with N_2 in different ways, and keep going with this manner, the kth operation could be executed with N_k in different ways. Assuming that two operations could not be executed at the same time. One of the K operations could be executed by the $N_1+N_2+N_3+\ldots+N_k$ in different ways.

Ex: A case contains 4 white, 6 black, and 8 red balls. In how many ways could 1 white ball or 1 black ball or 1 red ball be drawn?

$N_1+N_2+N_3=4+6+8=18$

Ex: Assuming that a racetrack has 3 different flags and those flags are used to communicate with racers. In how many ways could at least 2 flags be used to communicate with racers? (Flags are denoted by F1, F2, and F3, respectively).

F1-F2/F1-F3/F2-F3/F2-F1/F3-F1/F3-F2 (6 different ways when 2 flags are used)

F1-F2-F3/F1-F3-F2/F2-F1-F3/F2-F3-F1/F3-F2-F1/F3-F1-F2 (6 different ways when 3 flags are used)

$N_1+N_2=6+6=12$

Ex: A cafeteria serves 4 different teas and 3 different coffees. Any customer entering this cafeteria should choose either tea or coffee. In how many ways could a customer choose either tea or coffee?

$N_1=4$ and $N_2=3$

$N_1+N_2=7$

Definition (Multiplication Rule): Suppose that we have two operations. The first operations could be executed with N_1 in different ways. Whenever the first operation is completed using one of the N_1 different ways. The second operation could be executed with N_2 in different ways. These two operations could be executed together by $N_1.N_2$ in different ways.

Ex: Toss a pair of dice. How many different outcomes should be obtained?

$S=\{(m, n)|\ m=1,2,3,4,5,6,\ n=1,2,3,4,5,6\}$

$N_1.N_2=6.6=36$

Ex: Any person traveling from a city called A to a city C passes through a city called B. There exist 4 different roads from A to B. There exist 2 different roads from B to C. Any person traveling from A to C passing through B. In how many ways could a person travel from A to C?

$N_1.N_2=4.2=8$

Theorem (Extension Multiplication Rule): Suppose we have K operations. The first operation could be executed with N_1 in different ways. Whenever the first operation is completed using one of the N_1 different ways. The second operation could be executed with N_2 in different ways. By keeping going with this manner, Kth operation could be operated with N_K in different ways. Then, all K operations could be executed with $N_1.N_2.N_3\ldots N_k$ different ways.

Ex: Toss a coin. Cast a die and Draw a card from a deck of cards. In how many ways could this selection occur?

$N_1.N_2.N_3=2.6.52=624$

Ex: The first two places are allocated to letters and the last three places are allocated to numbers. In how many ways could this selection occur? (Hint: there are no restrictions on how many times letters or numbers may be used)

29.29.10.10.10

Ex: A restaurant serves 3 types of soups, 2 main dishes, 3 types of salads, 5 types of beverages, and 3 types of desserts. A customer must select 1 main dish and 1 beverage. Then he affords to select just one type from the rest. In how many ways could he choose?

2.5.3+2.5.3+2.5.3=90

Ex: An experimenter partitions a field into 64 same lots. The experimenter uses 5 different seeds, 3 different pesticides, and 4 different fertilizers. Is it enough to conduct all trials using all 64 lots?

5.3.4=60<64.

Permutation and Combination

Definition: Positive integers from 1 to n could be multiplied and denoted by

$$n! = 1.2.3 \dots n = n.(n-1).(n-2) \dots 3.2.1$$

$$n! = n.(n-1)!$$

and this multiplication is called n factorial.

Remark: $0! = 1!$ =1

Ex: 5!=5.4.3.2.1=120

Ex: $\frac{10!}{6!} = \frac{10.9.8.7.6!}{6!} = 10.9.8.7 = 5040$

Ex: 8.7.6=$\frac{8!}{5!}$=336

Ex: given that n!=(2n+4)(n-2)! What is the value of n?

n(n-1)(n-2)…1=(2n+4)(n-2)(n-3)…1

n(n-1)=(2n+4)

$n^2 - 3n - 4 = 0$

(n-4)(n+1)=0

n=4 or n=-1. n must be positive. Then, n=4.

Definition (Permutation): A permutation of several objects is any arrangement of these objects in a definite order.

Ex: 4 persons in a cafeteria line in a queue. In how many ways could this queue occur?

$$4! = 4.3.2.1 = 24$$

Theorem: Permutation of using all objects could be done n! different ways denoted by P_n^n.

Ex: Suppose that 8 glasses are arranged in the cupboard having spaces for 4 glasses. In how many ways could it be done?

$$P_4^8 = 8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!} = 1680$$

Definition: When using n objects and choosing r out of n at a time, the number of different permutations could be denoted by P_r^n with r<n and it is defined by

$$P_r^n = \frac{n!}{(n-r)!}$$

Ex: Calculate $P_2^5 - P_2^4$

$$P_2^5 = \frac{5!}{3!} = 20$$

$$P_2^4 = \frac{4!}{2!} = 12$$

$$P_2^5 - P_3^4 = 20 - 12 = 8$$

Theorem: Permuting r out of n at a time without repetition is defined by

$$P_r^n = \frac{n!}{(n-r)!}$$

Ex: By using the numbers 1-2-3-4-5, How many four-digit numbers could be generated?

a. Repetition is not allowed

b. Repetition is allowed

c. Without repetition, how many odd numbers could be generated?

a.5.4.3.2=120

b.5.5.5.5=625

c.4.3.2.3=72

Ex: Any document generated by a university is assigned to a reference code of 5 digits that consists of letters and numbers. a. the reference code of any document cannot begin with 0 (zero), In how many ways could a reference code be generated without repetition of just using numbers? b. If the first place is allocated to letters and the second place is not allowed to use zero, In how many ways could a reference be generated?

a.9.9.8.7.6

b.29.9.9.8.7

Ex: 6 females and 3 males sit in a row,

a. Given that 3 males are sitting next to each other, In how many ways could this arrangement be occurred?

b. Given that the last chair is reserved for a female and it is not allowed for males to sit next to each other, In how many ways could this arrangement be occurred?

a.7!.3!

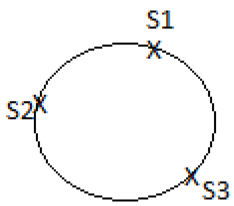
b.120.6!

Ex: A man having 6 pairs of gloves wants to pair a left one with a right one that does not belong to its pair. In how many ways could he arrange it?

6.5=30=$P_2^6 = \frac{6!}{(6-2)!}$

Theorem: If n objects would be arranged on a circle, this arrangement could be done with (n-1)! .

Ex: A dining table is ready for three persons called A, B, and C waiting for being seated. In how many ways could this arrangement be done?



Chair						
C1	A	A	B	B	C	C
C2	B	C	A	C	A	B
C3	C	B	C	A	B	A
Total	1	1	0	0	0	0

Either A or B or C could be seated in one of the chairs called C1 or C2 or C3. When anyone of the three persons is seated. The rest could be seated with 2! Then, the total number of permutations is as follows:

(3-1)!=2!

Ex: 5 persons meet at a table.

a. In how many ways could this meeting occurred?

b. If the president and vice-president sit next to each other, In how many ways could this meeting occur?

a.(5-1)!=4!=24

b.3!2!

Combinations

Definition (Combination): A combination is a selection of objects considered without regard to their order. A subset of r objects selected without regard to their order from a set of n different objects is called a combination of the n objects, taken r at a time. The total number of such combinations is denoted by C_r^n or $\binom{n}{r}$.

$$C_r^n = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

In other words,

$r!\,C_r^n = r!\binom{n}{r} = P_r^n = r!.\frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!}$

Result:

$$\binom{n}{n-r} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Ex: A deck of 52 cars is used to select 13 cards, In how many ways could this arrangement occur?

$$C_{13}^{52} = \binom{52}{13} = \frac{52!}{(52-13)!\,13!} = \frac{52!}{39!\,13!}$$

Ex:4 married couples are used to set up a committee consisting of 3 persons. In how many ways could a committee be set up?

a. If there is no constraint on the construction of a committee.

b. If a committee must contain 2 ladies and one gentleman.

c. A couple could not belong to the committee.

a. $\binom{8}{3}$=56

b. $\binom{4}{2}\binom{4}{1}$=24

c. $\binom{8}{3} - \binom{4}{1}\binom{6}{1}$=32