

1 Basic notions

1.2 Graph isomorphism

Rem 1.8. Isomorphism is an equivalence relation of graphs. (reflexive, symmetric, transitive)

1.3 The adjacency and incidence matrices

1.4 Degree

Fact 1. For any G on the vx set $[n]$ with adjacency and incidence matrices A and B , we have $BB^T = D + A$, where

$$D = \begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix}$$

Prop 1.22. For every $G = (V, E)$, $\sum_{v \in G} d(G) = 2|E|$

Cor 1.23. Every $gr.$ has an even $\#$ vx s of odd degree.

1.5 Subgraphs

1.7 Walks, paths and cycles

Prop 1.32. Every walk from u to v in G contains a path between u and v .

Prop 1.33. Every G with minimum degree $\delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

Rem 1.34. Note that we have also proved that a $gr.$ with minimum degree $\delta \geq 2$ contains cycles of at least $\delta - 1$ different lengths. This fact, and the statement of Proposition 1.32, are both tight, to see this, consider the complete $gr.$ $G = K_{\delta+1}$.

1.8 Connectivity

Prop 1.39. A $gr.$ with n vx s and m edges has at least $n - m$ $conn.$ components.

1.9 Graph operations and parameters

Not 1.44. Let $\omega(G)$ denote the $\#$ vx s in a maximum-size clique in G , let $\alpha(G)$ denote the $\#$ vx s in a maximum-size independent set in G .

Claim 1.45. A vx set $U \subseteq V(G)$ is a clique iff $U \subseteq V(\overline{G})$ is an independent set.

Cor 1.46. We have $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.

2 Trees

2.1 Trees

Lem 2.3. Every finite tree with at least two vx s has at least two leaves. Deleting a leaf from an n - vx tree produces a tree with $n - 1$ vx s.

2.2 Equivalent definitions of trees

Thm 2.4. For an n - vx simple G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vx s).
(a) G is $conn.$ and has no cycles.
(b) G is $conn.$ and has $n - 1$ edges.
(c) G has $n - 1$ edges and no cycles.
(d) For every pair $u, v \in V(G)$, there is exactly one u, v -path in G .

Lem 2.6. An edge contained in a cycle is not a cut-edge.

Cor 2.8. • Every $conn.$ $gr.$ on n vx s has at least $n - 1$ edges and contains a spanning tree,
• Every edge of a tree is a cut-edge,
• Adding an edge to a tree creates exactly one cycle.

2.3 Cayley's formula

Thm 2.11 (Cayley's Formula). There are n^{n-2} trees with vx set $[n]$.

Prop 2.14. For an ordered n -element set S , the Prüfer code f is a bijection between the trees with vx set S and the sequences in S^{n-2} .

3 Connectivity

3.1 Vertex connectivity

Prop 3.3. For every G , $\kappa(G) \leq \delta(G)$.

Rem 3.4. High minimum degree does not imply connectivity. Consider two disjoint copies of K_n .

Thm 3.5 (Mader 1972). Every $gr.$ of average degree at least $4k$ has a k - $conn.$ subgraph.

3.2 Edge connectivity

Rem 3.8. An edge cut is a disconnecting set but not the other way around. However, every minimal disconnecting set is a cut.

Thm 3.9. $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

3.3 Blocks

Rem 3.12. If a block B has at least three vx s, then B is 2- $conn.$ If an edge is a block of G then it is a cut-edge of G .

Prop 3.13. Two blocks in a $gr.$ share at most one vx .

Prop 3.16. The block $gr.$ of a $conn.$ $gr.$ is a tree.

3.4 2-conn. graphs

Thm 3.18 (Whitney 1932). A G having at least three vx s is 2- $conn.$ if and only if each pair $u, v \in V(G)$ is $conn.$ by a pair of internally disjoint u, v -paths in G .

Cor 3.19. G is 2- $conn.$ and $|G| \geq 3$ iff every two vx s in G lie on a common cycle.

3.5 Menger's Thm

Thm 3.21 (Menger 1927). Let $G = (V, E)$ be a $gr.$ and let $S, T \subseteq V$. Then the maximum $\#$ vx -disjoint $S-T$ paths is equal to the minimum size of an $S-T$ separating vx set.

Cor 3.22. For $S \subseteq V$ and $v \in V \setminus S$, the minimum $\#$ vx s distinct from v separating v from S in G is equal to the maximum $\#$ paths forming an $v-S$ fan in G . (that is, the maximum $\#$ $\{v\}-S$ paths which are disjoint except at v).

Cor 3.25. Let u and v be two distinct vx s of G .

1. If $(u, v) \notin E$, then the minimum $\#$ vx s different from u, v separating u from v in G is equal to the

maximum # internally vx -disjoint $u - v$ paths in G .

2. The minimum # edges separating u from v in G is equal to the maximum # edge-disjoint $u - v$ paths in G .

Thm 3.26. (Global Version of Menger's Theorem)

1. A gr. is k -conn. iff it contains k internally vx -disjoint paths between any two vxs .
2. A gr. is k -edge-conn. iff it contains k edge-disjoint paths between any two vxs .

4 Eu. & Ha. cyc.

4.1 Eul. trails & tours

Thm 4.5. A conn. (multi)gr. has an Eulerian tour iff each vx has even degree.

Lem 4.6. Every maximal trail in an even gr. (i.e., a gr. where all the vxs have even degree) is a closed trail.

Cor 4.7. A conn. multi G has an Eulerian trail iff it has either 0 or 2 vxs of odd degree.

4.2 Hamilton paths and cycles

Prop 4.10. If G is Hamiltonian then for any set $S \subseteq V$ the gr. $G \setminus S$ has at most $|S|$ conn. components.

Cor 4.11. If a conn. bipartite gr. $G = (V, E)$ with bipartition $V = A \cup B$ is Hamiltonian then $|A| = |B|$.

Thm 4.13 (Dirac 1952). If G is a simple gr. with $n \geq 3$ vxs and if $\delta(G) \geq n/2$, then G is Hamiltonian.

Thm 4.15 (Ore 1960). If G is a simple gr. with $n \geq 3$ vxs such that for every pair of non-adjacent vxs u, v of G we have $d(u) + d(v) \geq |G|$, then G is Hamiltonian.

5 Matchings

Rem 5.3. A matching in a G corresponds to an independent set in the line gr. $L(G)$.

Prop 5.6. $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

5.2 Hall's Theorem

Thm 5.7 (Hall 1935). A bipartite gr. $G = (V, E)$ with bipartition $V = A \cup B$ has a matching covering A iff $|N(S)| \geq |S| \forall S \subseteq A$

Cor 5.8. If in a bipartite gr. $G = (A \cup B, E)$ we have $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed $d \in \mathbb{N}$, then G contains a matching of cardinality $|A| - d$.

Cor 5.9. If a bipartite gr. $G = (A \cup B, E)$ is k -regular with $k \geq 1$, then G has a perfect matching.

Cor 5.10. Every regular gr. of positive even degree has a 2-factor (a spanning 2-regular subgraph).

Rem 5.11. A 2-factor is a disjoint union of cycles covering all the vxs of a graph

Cor 5.13. A collection A_1, \dots, A_n has an SDR iff for all $I \subseteq [n]$ we have $|\bigcup_{i \in I} A_i| \geq |I|$.

Thm 5.15 (König 1931). If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vx cover of G .

5.3 Matchings in general graphs: Tutte's Theorem

Given a G , let $q(G)$ denote the # its odd components, i.e. the ones of odd order. If G has a perfect matching then clearly $q(G \setminus S) \leq |S| \forall S \subseteq V(G)$ since every odd component of $G \setminus S$ will send an edge of the matching to S , and each such edge covers a different vx in S .

Thm 5.16 (Tutte 1947). A G has a perfect matching iff $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

Cor 5.17 (Petersen 1891). Every 3-regular gr. with no cut-edge has a perfect matching.

Cor 5.19 (Berge 1958). The largest matching in an n - vx G covers $n + \min_{S \subseteq V(G)} (|S| - q(G \setminus S))$ vxs .

6 Planar Graphs

Thm 6.5 (Jordan curve theorem). A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.

Rem 6.6. This is not true in three dimensions. In \mathbb{R} there is a surface called the Möbius band which has only one side.

Rem 6.7. The faces of G are pairwise disjoint (they are separated by the edges of G). Two points are in the same face iff there is a polygonal path between them which does not cross an edge of G . Also, note that a finite gr. has a single unbounded face

(the area "outside" of the graph).

Prop 6.8. A plane forest has exactly one face.

Prop 6.11. If $l(f_i)$ denotes the length of a face f_i in a plane G , then $2e(G) = \sum l(f_i)$.

Thm 6.12 (Euler's formula 1758). If a conn. plane G has exactly n vxs , e edges and f faces, then $n - e + f = 2$.

Thm 6.14. If G is a planar gr. with at least three vxs , then $e(G) \leq 3|G| - 6$. If G is also triangle-free, then $e(G) \leq 2|G| - 4$.

Cor 6.15. If G is a planar bipartite n - vx gr. with $n \geq 3$ vxs then G has at most $2n - 4$ edges.

Cor 6.16. K_5 and $K_{3,3}$ are not planar.

Rem 6.17 (Maximal planar graphs / triangulations). The proof of Theorem 6.14 shows that having $3n - 6$ edges in a simple n - vx planar gr. requires $2e = 3f$, meaning that every face is a triangle. If G has some face that is not a triangle, then we can

add an edge between non-adjacent vxs on the boundary of this face to obtain a larger plane graph. Hence the simple plane graphs with $3n - 6$ edges, the triangulations, and the maximal plane graphs are all the same family.

6.1 Platonic Solids

Cor 6.21. *If K is a convex polytope with v vxs, e edges and f faces then $v - e + f = 2$.*

7 Graph colouring

7.1 Vertex colouring

Rem 7.3. The vxs having a given colour in a proper colouring must form an independent set, so $\chi(G)$ is the minimum # independent sets needed to cover $V(G)$. Hence G is k -colourable iff G is k -partite. Multiple edges do not affect chromatic number. Although we define k -colouring using numbers from $\{1, \dots, k\}$ as labels, the numerical values are usually unimportant, and we may use any set of size k as labels.

7.3 Bounds on χ

Claim 7.6. If H is a subgr. of G then $\chi(H) \leq \chi(G)$.

Cor 7.7. $\chi(G) \geq \omega(G)$

Prop 7.9. $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$

Claim 7.10. For any gr. $G = (V, E)$ and any $U \subseteq V$ we have $\chi(G) \leq \chi(G[U]) + \chi(G[V \setminus U])$.

Claim 7.11. For any graphs G_1 and G_2 on the same vx set, $\chi(G_1 \cup G_2) \leq \chi(G_1)\chi(G_2)$.

Prop 7.12 (i) $\chi(G)\chi(\overline{G}) \geq |G|$

(ii) $\chi(G) + \chi(\overline{G}) \leq |G| + 1$

7.4 Greedy colouring

Prop 7.16. G is k -degenerate iff there is an ordering v_1, \dots, v_n of the vxs of G such that each v_i has at most k neighbours among the vxs v_1, \dots, v_{i-1} .

Rem 7.18. $\delta(G) \leq dg(G) \leq \Delta(G)$.

Thm 7.19. $\chi(G) \leq 1 + dg(G)$

Cor 7.20. $\chi(G) \leq \Delta(G) + 1$.

Rem 7.21. This bound is tight if $G = K_n$ or if G is an odd cycle.

Thm 7.22 (Brooks 1941). *If G is a conn. gr. other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

7.5 Colouring planar graphs

Claim 7.23. A (simple) planar G contains a vx v of degree at most 5.

Cor 7.24. *A planar G is 5-degenerate and thus 6-colourable.*

Thm 7.25 (5 colour theorem, Heawood 1890). *Every planar G is 5-colourable.*

Thm 7.26 (Appel-Haken 1977, conjectured by Guthrie in 1852). *Every planar gr. is 4-colourable. (the countries of every plane map can be 4-coloured so that neighbouring countries get distinct colours).*

7.6 Art gallery thm

Thm 7.28. *For any museum with n walls, $\lfloor n/3 \rfloor$ guards suffice.*

8 Col. results

Thm 8.1 (Gallai, Roy). *If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of G .*

8.1 Large girth and large χ

The bound $\chi(G) \geq \omega(G)$ can be tight, but (surpris-

ingly) it can also be arbitrarily bad. There are graphs having arbitrarily large chromatic number, even though they do not contain K_3 . Many constructions of such graphs are known, though none are trivial. We give one here.

Thm 8.3. *Mycielski's construction produces a $(k + 1)$ -chromatic triangle-free gr. from a k -chromatic triangle-free graph.*

Thm 8.5 (Erdos 1959). *Given $k \geq 3$ and $g \geq 3$, there exists a gr. with girth at least g and chromatic number at least k .*

Thm 8.8. *There is a tournament on n vxs where any $\frac{\log_2(n)}{2}$ vxs are beaten by some other vx.*

8.2 χ and clique minors

Thm 8.12 (Mader). *If the average degree of G is at least $2t - 2$ then G has a K_t minor.*

Rem 8.13. It is known that $\bar{d}(G) \geq ct\sqrt{\log(t)}$ already implies the existence of a K_t minor in G , for some constant $c > 0$.

8.3 Edge-colourings

Rem 8.15. (i) An edge-colouring of a G is the same as a vx-colouring of its line gr. $L(G)$.

(ii) A G with maximum degree Δ has $\chi'(G) \geq \Delta$ since the edges incident to a vx of degree Δ must have different colours.

(iii) If G has maximum degree Δ then $L(G)$ has maximum degree at most $2(\Delta - 1)$. $\Rightarrow \chi'(G) \leq 2\Delta - 1$

Thm 8.16 (König 1916). *If G is a bipartite multigraph, then $\chi'(G) = \Delta(G)$.*

Thm 8.17 (Vizing). *Let G be a simple gr. with maximum degree Δ . Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

8.4 List colouring

Thm 8.20 (Erdos, Rubin, Taylor 1979). *If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not k -choosable.*

Thm 8.23 (Galvin 1995). $\chi'_l(K_{n,n}) = n$

Lem 8.25. *If D is a kernel-perfect orientation of G and $f(x) = d_D^-(x)$ for all $x \in V(G)$, then G is $(1 + f)$ -choosable.*

9 The Matrix Tree Theorem

Thm 9.1 (Cayley's formula). *There are n^{n-2} labeled trees on n vxs.*

Now consider an arbitrary conn. simple G on vx set $[n]$, and denote the # spanning trees by $t(G)$. The following celebrated result is Kirchhoff's matrix tree theorem. To formulate it, consider the incidence matrix B of G (as in Def 1.13), and replace one of the two 1's by -1 in an arbitrary manner to obtain the matrix C (we say C is the incidence matrix of an orientation of G). $M = CC^T$ is then a symmetric $n \times n$ matrix, which is

$$\begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix} - A_G$$

Thm 9.2 (Matrix tree theorem). *We have $t(G) = \det M_{ii}$ for all $i = 1, \dots, n$, where M_{ii} results from M by deleting the i -th row and the i -th column.*

Thm 9.3 (Binet, Cauchy). *If P is an $r \times s$ matrix and Q*

is an $s \times r$ matrix with $r \leq s$, then

$\det(PQ) = \sum_Z (\det P_Z)(\det Q_Z)$ where P_Z is the $r \times r$ submatrix of P with column set Z , and Q_Z is the $r \times r$ submatrix of Q with the corresponding rows Z , and the sum is over all r -sets $Z \subseteq [s]$.

10 More Thms on Hamiltonicity

Thm 10.2 (Bondy Chvátal 1976). *A simple n -vx gr. is Hamiltonian iff its closure is Hamiltonian.*

Thm 10.3 (Chvátal 1972). *Suppose G has vx degrees $d_1 \leq \dots \leq d_n$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \geq n-i$, then G is Hamiltonian.*

Thm 10.4 (Chvátal-Erdos 1972). *If $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle (unless $G = K_2$).*

10.1 Pósa's Lemma

Let P be a path in a G , say from u to v . Given a vx $x \in P$, we write x^- for the vx preceding x on P , and x^+ for the vx following x on P (whenever these exist). Similarly, for $X \subseteq V(P)$ we put $X^\pm := \{x^\pm : x \in X\}$

If $x \in P \setminus u$ is a neighbour of u in G , then $P \cup \{(u, x)\} \setminus \{(x, x^-)\}$ (which is a path in G with vx set $V(P)$) is said to have been obtained from P by a rotation fixing v . A path obtained from P by a (possibly empty) sequence of rotations fixing v is a path derived from P . The set of starting vxs of paths derived from P , including u , will be denoted by $S(P)$. As all paths derived from P have the same vx set as P , we have $S(P) \subseteq V(P)$.

Rem 10.5. If some sequence of rotations can delete the edge (x, x^-) , call this edge a broken edge. Note that every interval of the original path not containing broken edges is traversed by all derived paths as a whole piece (however, the direction can change).

Lem 10.7. *Let G be a graph, let $P = u \dots v$ be a longest path in G , and put $S := S(P)$. Then $\partial S \subseteq S^- \cup S^+$.*

Lem 10.8. *Let G be a graph, let $P = u \dots v$ be a longest path in G , and put $S := S(P)$. If $\deg(u) \geq 2$ then G has a cycle containing $S \cup \partial S$.*

Cor 10.9. *Fix $k \geq 2$ and let G be a gr. such that for all $S \subseteq V(G)$ with $|S| \leq k$, we have $|\partial S| \geq |2S|$. Then G has a cycle of length at least $3k$.*

10.2 Tournaments

Thm 10.11. *Every tournament has a Hamilton path.*

Thm 10.13. *A tournament T is strongly conn. iff it has a Hamilton cycle.*

11 Kuratowski's Theorem

Rem 11.3. If G contains a subdivision of H , it also contains an H -minor.

Thm 11.5 (Kuratowski 1930). *A gr. is planar iff it has no Kuratowski subgraph.*

Thm 11.7. *If G is a gr. with no Kuratowski subgr. then G has a straightline drawing in the plane.*

11.1 Convex drawings of 3-conn. graphs

Thm 11.9 (Tutte 1960). *If G is a 3-conn. gr. which has no Kuratowski subgraphs then G has a convex drawing in the plane with no three vxs on a line.*

Lem 11.10 (Thomassen 1980). *Every 3-conn. G with at least five vxs has an edge e such that G/e is 3-conn.*

Lem 11.11. *If G has no Kuratowski subgraphs, then G/e has no Kuratowski subgraph, for any edge $e \in E(G)$.*

11.2 Reducing the general case to the 3-conn. case

Fact 2. We make three observations.

1. In a Kuratowski subgraph, there are three internally vx-disjoint paths connecting any two branch vxs. For K_5 -subdivisions, we even have four such paths.
2. In a Kuratowski subgraph, there are four internally vx-disjoint paths between any two pairs of branch vxs.
3. Any cycle in a subdivision contains at least three branch vxs.

Prop 11.14. *Let G be a gr. with at least 4 vxs which has no Kuratowski subgraph, and suppose that adding an edge-joining any pair of non-adjacent vxs creates a Kuratowski subgraph. Then G is 3-conn.*

12 Ramsey Theory

Prop 12.1. Among six people it is possible to find three mutual acquaintances or three mutual non-acquaintances.

As we shall see, given a natural number s , there is an integer R such that if $n \geq R$ then every colouring of the edges of K_n with red and blue contains either a red K_s or a blue K_s . More generally, we define the Ramsey number $R(s, t)$ as the smallest value of N for which every red-blue colouring of K_N yields a red K_s or a blue K_t . In particular, $R(s, t) = \infty$ if there is no such N such that in every red-blue colouring of K_N there is a red K_s or a blue K_t . It is obvious that $R(s, t) = R(t, s)$ for every $s, t \geq 2$ and $R(s, 2) = R(2, s) = s$.

Thm 12.2 (Erdős, Szekeres). The function $R(s, t)$ is finite for all $s, t \geq 2$. Quantitatively, if $s > 2$ and $t > 2$ then $R(s, t) \leq R(s-1, t) + R(s, t-1)$ and $R(s, t) \leq \binom{s+t-2}{s-1}$.

Thm 12.3. Given k and s_1, s_2, \dots, s_k , if N is suf-

ficiently large, then every colouring of K_N with k colours is such that for some $i, 1 \leq i \leq k$, there is a K_{s_i} coloured with the i -th colour. The minimal value of N for which this holds is usually denoted by $R_k(s_1, \dots, s_k)$, and it satisfies $R_k(s_1, \dots, s_k) \leq R_{k-1}(R(s_1, s_2), s_3, \dots, s_k)$.

Thm 12.4. Let $\min\{s, t\} > 3$. Then $R^{(3)}(s, t) \leq R(R^{(3)}(s-1, t), R^{(3)}(s, t-1)) + 1$

12.1 Applications

Thm 12.5 (Erdős-Szekeres 1935). Given an integer m , there exists a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane, with no three collinear, contains an m -subset forming a convex m -gon.

12.2 Bounds on Ramsey numbers

Thm 12.6 (Erdős 1947). For $p \geq 3$, we have $R(p, p) > 2^{p/2}$.

Thm 12.7. We have $R_k(3) \stackrel{\text{def}}{=} R_k(3, \dots, 3) \leq [e \cdot k!] + 1$.

12.3 Ramsey theory for integers

Thm 12.8 (Schur 1916). For every $k \geq 1$ there is an

integer m such that every k -colouring of $[m]$ contains integers x, y, z of the same colour such that $x + y = z$.

12.4 Graph Ramsey numbers

Rem 12.10. Note that $R(G_1, G_2) \leq R(|G_1|, |G_2|)$.

Thm 12.11 (Chvatal 1977). If T is any m -vx tree, then $R(T, K_n) = (m-1)(n-1) + 1$

13 Extremal problems

13.1 Turán's theorem

Thm 13.5 (Turán 1941). Among all the n -vx simple graphs with no $(r+1)$ -clique, $T_{n,r}$ is the unique gr. having the maximum # edges.

Question 13.6. Let $a_1, \dots, a_n \in \mathbb{R}^d$ be vectors such that $|a_i| \geq 1$ for each $i \in [n]$. What is the maximum # pairs satisfying $|a_i + a_j| < 1$?

Claim 13.7. There are at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ such pairs.

Thm 13.9 (Erdős-Stone). Let H be a gr. of chromatic number $\chi(H) = r + 1$. Then for every $\varepsilon > 0$ and large enough n , $(1 - \frac{1}{r}) \frac{n^2}{2} \leq ex(n, H) \leq (1 - \frac{1}{r}) \frac{n^2}{2} + n^2 \varepsilon$

13.2 Bipartite Turán Theorems

Thm 13.11. If a G on n vxs contains no 4-cycles, then $e(G) \leq \left\lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \right\rfloor$.

Thm 13.13 (Kovári-Sós-Turán). For any integers $r \leq s$, there is a constant c such that every $K_{r,s}$ -free gr. on n vxs contains at most $cn^{1-\frac{1}{r}}$ edges. In other words, $ex(n, K_{r,s}) \leq cn^{1-\frac{1}{r}}$

Thm 13.14. There is c depending on k such that if G is a gr. on n vxs that contains no copy of C_{2k} , then G has at most $cn^{1+\frac{1}{k}}$ edges.

Question 13.15. Given n points in the plane, how many pairs can be at distance 1?

Thm 13.16 (Erdős). There are at most $cn^{3/2}$ pairs.

14 Exercises

14.1 Assignment 1

P 3. Prove that if a G is not conn. then its complement \overline{G} is conn. Converse is not true.

P 4. Show that every gr. on at least two vxs contains two vxs of equal degree.

P 5. Prove that every gr. with $n \geq 7$ vxs and at

least $5n - 14$ edges contains a subgr. with minimum degree at least 6.

P 6. Show that in a conn. gr. any two paths of maximum length share at least one vx.

P 7. Prove that a gr. is bipartite iff it contains no cycle of odd length.

14.2 Assignment 2

P 1. Show that in a tree containing an even # edges, there is at least one vx with even degree.

P 2. Given a G and a vx $v \in V(G)$, $G - v$ denotes the subgr. of G induced by the vx set $V(G) \setminus \{v\}$. Show that every conn. G of order at least two contains vxs x and y such that both $G - x$ and $G - y$ are conn.

P 3. Let T be an n -vx tree with exactly $2k$ odd-degree vxs. Prove that T decomposes into k paths (i.e. its edge-set is the disjoint union of k paths).

P 4. Prove that a conn. G is a tree iff any family of pairwise (vx-)intersecting paths P_1, \dots, P_k in G have a common vx.

- P* 5(a) Prüfer codes corresponding to stars (i.e. to trees isomorphic to $K_{1,n-1}$) = 1 value.
- (b) Prüfer codes containing exactly 2 different values = 2 connected stars
- P* 6. Let T be a forest on vx set $[n]$ with components T_1, \dots, T_r . Prove, by induction on r , that the # spanning trees on $[n]$ containing T is $n^{r-2} \prod_{i=1}^r |T_i|$. Deduce Cayley's formula.
- ### 14.3 Assignment 3
- P* 1. Prove that a conn. G is k -edge-conn. iff each block of G is k -edge-conn.
- P* 2. Let G be a gr. and suppose some two vxs $u, v \in V(G)$ are separated by $X \subseteq V(G) \setminus \{u, v\}$. Show that X is a minimal separating set (i.e. there is no proper subset Y of X that separates u and v) iff every vx in X has a neighbor in the component of $G - X$ containing u and another in the component containing v .
- P* 3. Show that if G is a gr. with $|V(G)| = n \geq k + 1$ and $\delta(G) \geq (n + k - 2)/2$ then G is k -conn.
- P* 4. Prove that a G with at least 3 vxs is 2-conn. iff for any three vxs x, y, z there is a path from x to z containing y .
- P* 5. Let G be a k -conn. graph, where $k \geq 2$. Show that if $|V(G)| \geq 2k$ then G contains a cycle of length at least $2k$.
- ### 14.4 Assignment 4
- P* 1. Show that if $k > 0$ then the edge set of any conn. gr. with $2k$ vxs of odd degree can be split into k trails.
- P* 2. Let G be a conn. gr. that has an Euler tour. T / F?
- (a) If G is bipartite then it has an even # edges. T
- (b) If G has an even # vxs then it has an even # edges. F
- (c) For edges e and f sharing a vx, G has an Euler tour in which e and f appear consecutively. F
- P* 3. Let G be a conn. gr. on n vxs with minimum degree δ . Show that
- (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ , and
- (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.
- P* 4. Show that the maximum # edges in a non-Hamiltonian gr. on $n \geq 3$ vxs is $\binom{n-1}{2} + 1$.
- ### 14.5 Assignment 5
- P* 1. Let G be a conn. gr. on more than 2 vxs such that every edge is contained in some perfect matching of G . Show that G is 2-edge-conn.
- P* 2(a) Let G be a gr. on $2n$ vxs that has exactly one perfect matching. Show that G has at most n^2 edges.
- (b) Construct such a G containing exactly n^2 edges for any $n \in \mathbb{N}$.
- P* 3. Let A be a finite set with subsets A_1, \dots, A_n , and let d_1, \dots, d_n be positive integers. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \in [n]$ if and only if $|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} d_i$.
- P* 4. Suppose M is a matching in a bipartite gr. $G = (A \cup B, E)$. We say that a path $P = a_1 b_1 \dots a_k b_k$ is an augmenting path in G if $b_i a_{i+1} \in M$ for all $i \in [k-1]$ and a_1 and b_k are not covered by M . The name comes from the fact that the size of M can be increased by flipping the edges along P (in other words, taking the symmetric difference of M and P): by deleting the edges $b_i a_{i+1}$ from M and adding the edges $a_i b_i$ instead.
- (a) Prove Hall's theorem by showing that if Hall's condition is satisfied and M does not cover A , then there is an augmenting path in G .
- (b) Show that if M is not a maximum matching (i.e. there is a larger matching in G) then the gr. contains an augmenting path. Is this true for non-bipartite graphs as well? Y
- P* 5. Show that for $k \geq 1$, every k -regular $(k-1)$ -edge-conn. gr. on an even # vxs contains a perfect matching.
- ### 14.6 Assignment 6
- P* 2(a) Show that every planar gr. has a vx of degree at most 5. Is there a planar gr. with minimum degree 5? Y
- (b) Show that any planar bipartite gr. has vx of degree at most 3. Is there a planar bipartite gr. with minimum degree 3? Y
- P* 3. Show that a conn. plane G is bipartite iff all its faces have even length.
- P* 4. Let G be a gr. on $n \geq 3$ vxs and $3n - 6 + k$ edges for some $k > 0$. Show that any drawing of G in the plane contains at least k crossing pairs of edges.
- P* 5. Let G be a plane gr. with triangular faces and suppose the vxs are colored arbitrarily with three colors. Prove that there is an even # faces that get all three colors.
- P* 6. Let S be a set of $n \geq 3$ points in the plane such that any two of them have distance at least 1. Show that there are at most $3n - 6$ pairs of distance exactly 1.
- ### 14.7 Assignment 7
- P* 1. T / F?
- (a) If G and H are graphs on the same vx set, then $dg(G \cup H) \leq dg(G) + dg(H)$. F
- (b) If G and H are graphs on the same vx set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$. F
- (c) Every G has a $\chi(G)$ -coloring where $\alpha(G)$ vxs

get the same color. F

P 2. G has the property that any two odd cycles in it intersect (they share at least one vx in common). Prove that $\chi(G) \leq 5$.

P 3. For a vx v in a conn. G , let G_r be the subgr. of G induced by the vxs at distance r from v . Show that $\chi(G) \leq \max_{0 \leq r \leq n} \chi(G_r) + \chi(G_{r+1})$.

P 4. Let l be the length of the longest path in a G . Prove $\chi(G) \leq l + 1$ using the fact that if a gr. is not d -degenerate then it contains a subgr. of minimum degree at least $d + 1$.

P 5. Suppose the complement of G is bipartite. Show that $\chi(G) = \omega(G)$.

14.8 Assignment 8

P 1. For a given natural number n , let G_n be the following gr. with $\binom{n}{2}$ vxs and $\binom{n}{3}$ edges: the vxs are the pairs (x, y) of integers with $1 \leq x < y \leq n$, and for each triple (x, y, z) with $1 \leq x < y < z \leq n$, there is an edge joining vx (x, y) to vx (y, z) . Show that for any natural number k , the gr. G_n is triangle-free and has chromatic number $\chi(G_n) > k$ provided $n > 2k$.

matic number $\chi(G_n) > k$ provided $n > 2k$.

P 2. Show that the theorem of Mader implies the following weakening of Hadwiger's conjecture: Any G with $\chi(G) \geq 2^{t-2} + 1$ has a K_t -minor.

P 3. Find the edge-chromatic # K_n (don't use Vizing's theorem). n for n odd, $n - 1$ for n even

P 4. Let G be a conn. k -regular bipartite gr. with $k \geq 2$. Show, using König's theorem, that G is 2-conn.

14.9 Assignment 9

P 1. Prove that every G of maximum degree Δ has an equitable $(\Delta + 1)$ -edge-coloring, i.e. one where each color class contains $\lfloor e/(\Delta + 1) \rfloor$ or $\lceil e/(\Delta + 1) \rceil$ edges, where e is the # edges in G .

P 2. The cartesian product $H \times G$ of graphs H and G is the gr. with vx set $V(H) \times V(G)$, with an edge between (v, u) and (v', u') if $v = v'$ and u is adjacent to u' in G , or if $u = u'$ and v is adjacent to v' in H . Prove

that if $\chi'(H) = \Delta(H)$ then $\chi'(H \times G) = \Delta(H \times G)$

P 3. Show that $\chi(C_n) = \chi_l(C_n)$ for any $n \geq 3$.

P 4. Let G be a bipartite gr. on n vxs. Prove that $\chi_l(G) \leq 1 + \log_2(n)$ using the probabilistic method.

P 5. Let G be a complete r -partite gr. with all parts of size 2. (In other words, G is K_{2r} minus a perfect matching.) Show, using a combination of induction and Hall's theorem, that $\chi_l(G) = r$.

14.10 Assignment 10

P 1. How many spanning trees does $K_{r,s}$ have? $r^{s-1}s^{r-1}$

P 2. Find the # spanning trees of $K_n - e$ (the complete gr. on n vxs with one edge removed): $(n - 2)n^{n-3}$

P 3. In this exercise we prove the following alternative form of the matrix-tree theorem. For an n -vx conn. G , the # spanning trees in G is equal to the product of the nonzero eigenvalues of the Laplacian matrix M of G , divided by n . (This matrix M is as in the lecture notes).

P 4(a) Prove that any n -by- n bipartite gr. with minimum degree $\delta > n/2$ contains a Hamilton cycle.

(b) Show that this is not necessarily the case if $\delta \leq n/2$.

14.11 Assignment 11

P 1. T / F?

(a) If every vx of a tournament has positive in- and out-degree, then the tournament contains a directed Hamilton cycle. F

(b) If a tournament has a directed cycle, then it has a directed triangle. T

P 2. Let G be a gr. on $n \geq 3$ vxs with at least $\alpha(G)$ vxs of degree $n - 1$. Show that G is Hamiltonian.

P 3. Suppose G is a gr. on n vxs where all the degrees are at least $\frac{n+q}{2}$. Show that any set F of q independent edges is contained in a Hamiltonian cycle.

14.12 Assignment 12

P 1. The lower bound for $R(p, p)$ that you learn in the lectures is not a constructive proof: it merely shows the existence of a red-blue coloring not containing any monochromatic copy of

K_p by bounding the # bad graphs. Give an explicit coloring on $K_{(p-1)^2}$ that proves $R(p, p) > (p - 1)^2$.

P 2. Prove that for every fixed positive integer r , there is an n such that any coloring of all the subsets of $[n]$ using r colors contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same color.

P 3. Prove that for every $k \geq 2$ there exists an integer N such that every coloring of $[N]$ with k colors contains three distinct numbers a, b, c satisfying $ab = c$ that have the same color.

P 4. For every $k \geq 2$ there is an N such that any k -coloring of $[N]$ contains three distinct integers a, b, c of the same color satisfying $a + b = c$.

P 5(a) Let $n \geq 1$ be an integer. Show that any sequence of $N \geq R(n, n)$ distinct numbers, a_1, \dots, a_N contains a monotone (increasing or decreasing) subsequence of length n .

(b) Let $k, l \geq 1$ be integers and show that any sequence of $kl + 1$ distinct

numbers a_1, \dots, a_{kl+1} contains a monotone increasing subsequence of length $k + 1$ or a monotone decreasing subsequence of length $l + 1$.

14.13 Assignment 13

P 1. Let H be an arbitrary fixed gr. and prove that the sequence $ex(n, H)/\binom{n}{2}$ is (not necessarily strictly) monotone decreasing in n .

P 2. Among all the n -vx K_{r+1} -free graphs, the Turan gr. $T_{n,r}$ contains the maximum # triangles (for any $r, n \geq 1$).

P 3. Let X be a set of n points in the plane with no two points of distance greater than 1. Show that there are at most $\frac{n^2}{3}$ pairs of points in X that have distance greater than $\frac{1}{\sqrt{2}}$.

15 Notation

$V(G)$	vxes of G
$E(G)$	edges of G
$e(G)$	# edges
$A(G)$	adjacency matrix
$B(G)$	incidence matrix
$N(v)$	neighbourhood
$d(v)$	degree
$\delta(G)$	min $d(v)$
$\Delta(G)$	max $d(v)$
$\bar{d}(G)$	average $d(v)$

\bar{G}	complement of G	$dg(G)$	min degeneracy
C_k	complete gr on k vxes	G/e	e contracted
$C_{r,s}$	compl. bipatite gr.	$G - e$	e deleted
$\omega(G)$	max clique	$\xi'(G)$	edge chrom nr
$\alpha(G)$	max ind. set	$L(v)$	list of col avail
$\kappa(G)$	vx-connectivity	$\xi_l(G)$	list chrom nr
$\kappa'(G)$	edge-conn.	$R(s, t)$	Ramsey nr
$d(u, v)$	distance	$R(G_1, G_2)$	graph Ramsey nr
$L(G)$	line graph	$ex(n, H)$	extremal number
$\nu(G)$	max matching	$T_{n,r}$	Turan graph
$\tau(G)$	min vx cover	$\pi(H)$	$\lim_{n \rightarrow \infty} ex(n, H)/\binom{n}{2}$
$\xi(G)$	chrom nr		