# 1 Basic notions

## 1.2 Graph isomorphism

*Rem* 1.8. Isomorphism is an equivalence relation of graphs. (reflexive, symmetric, transitive)

## 1.3 The adjacency and incidence matrices

## 1.4 Degree

*Fact* 1. For any $G$ on the vx set $[n]$ with adjacency and incidence matrices $A$ and $B$, we have $BB^T = D + A$, where

$$D = \begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix}$$

**Prop 1.22.** *For every $G = (V, E)$, $\sum_{v \in G} d(G) = 2|E|$*

**Cor 1.23.** *Every gr. has an even # vxs of odd degree.*

## 1.5 Subgraphs

## 1.7 Walks, paths and cycles

**Prop 1.32.** *Every walk from $u$ to $v$ in $G$ contains a path between $u$ and $v$.*

**Prop 1.33.** *Every $G$ with minimum degree $\delta \geq 2$ contains a path of length $\delta$ and a cycle of length at least $\delta + 1$.*

*Rem* 1.34. Note that we have also proved that a gr. with minimum degree $\delta \geq 2$ contains cycles of at least $\delta - 1$ different lengths. This fact, and the statement of Proposition 1.32, are both tight, to see this, consider the complete gr. $G = K_{\delta+1}$.

## 1.8 Connectivity

**Prop 1.39.** *A gr. with $n$ vxs and $m$ edges has at least $n - m$ conn. components.*

## 1.9 Graph operations and parameters

*Not* 1.44. Let $\omega(G)$ denote the # vxs in a maximum-size clique in $G$, let $\alpha(G)$ denote the # vxs in a maximum-size independent set in $G$.

*Claim* 1.45. A vx set $U \subseteq V(G)$ is a clique iff $U \subseteq V(\overline{G})$ is an independent set.

**Cor 1.46.** *We have $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.*

# 2 Trees

## 2.1 Trees

**Lem 2.3.** *Every finite tree with at least two vxs has at least two leaves. Deleting a leaf from an $n$-vx tree produces a tree with $n - 1$ vxs.*

## 2.2 Equivalent definitions of trees

**Thm 2.4.** *For an $n$-vx simple $G$ (with $n \geq 1$), the following are equivalent (and characterize the trees with $n$ vxs). (a) $G$ is conn. and has no cycles. (b) $G$ is conn. and has $n - 1$ edges. (c) $G$ has $n - 1$ edges and no cycles. (d) For every pair $u, v \in V(G)$, there is exactly one $u, v$-path in $G$.*

**Lem 2.6.** *An edge contained in a cycle is not a cut-edge.*

**Cor 2.8.**
- *Every conn. gr. on $n$ vxs has at least $n$ - 1 edges and contains a spanning tree,*
- *Every edge of a tree is a cut-edge,*
- *Adding an edge to a tree creates exactly one cycle.*

## 2.3 Cayley's formula

**Thm 2.11** (Cayley's Formula)**.** *There are $n^{n-2}$ trees with vx set $[n]$.*

**Prop 2.14.** *For an ordered $n$-element set $S$, the Prüfer code $f$ is a bijection between the trees with vx set $S$ and the sequences in $S^{n-2}$.*

# 3 Connectivity

## 3.1 Vertex connectivity

**Prop 3.3.** *For every $G$, $\kappa(G) \leq \delta(G)$.*

*Rem* 3.4. High minimum degree does not imply connectivity. Consider two disjoint copies of $K_n$.

**Thm 3.5** (Mader 1972)**.** *Every gr. of average degree at least $4k$ has a $k$-conn. subgraph.*

## 3.2 Edge connectivity

*Rem* 3.8. An edge cut is a disconnecting set but not the other way around. However, every minimal disconnecting set is a cut.

**Thm 3.9.** $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

## 3.3 Blocks

*Rem* 3.12. If a block $B$ has at least three vxs, then $B$ is 2-conn. If an edge is a block of $G$ then it is a cut-edge of $G$.

**Prop 3.13.** *Two blocks in a gr. share at most one vx.*

**Prop 3.16.** *The block gr. of a conn. gr. is a tree.*

## 3.4 2-conn. graphs

**Thm 3.18** (Whitney 1932)**.** *A $G$ having at least three vxs is 2-conn. if and only if each pair $u, v \in V(G)$ is conn. by a pair of internally disjoint $u, v$-paths in $G$.*

**Cor 3.19.** *$G$ is 2-conn. and $|G| \geq 3$ iff every two vxs in $G$ lie on a common cycle.*

## 3.5 Menger's Thm

**Thm 3.21** (Menger 1927)**.** *Let $G = (V, E)$ be a gr. and let $S, T \subseteq V$ . Then the maximum # vx-disjoint $S$–$T$ paths is equal to the minimum size of an $S - T$ separating vx set.*

**Cor 3.22.** *For $S \subseteq V$ and $v \in V \setminus S$, the minimum # vxs distinct from $v$ separating $v$ from $S$ in $G$ is equal to the maximum # paths forming an $v - S$ fan in $G$. (that is, the maximum # $\{v\} - S$ paths which are disjoint except at $v$).*

**Cor 3.25.** *Let $u$ and $v$ be two distinct vxs of $G$.*

1. *If $(u, v) \notin E$, then the minimum # vxs different from $u, v$ separating $u$ from $v$ in $G$ is equal to the*

*maximum # internally vx-disjoint $u-v$ paths in G.*

2. *The minimum # edges separating u from v in G is equal to the maximum # edge-disjoint $u-v$ paths in G.*

**Thm 3.26.** *(Global Version of Menger's Theorem)*

1. *A gr. is k-conn. iff it contains k internally vx-disjoint paths between any two vxs.*
2. *A gr. is k-edge-conn. iff it contains k edge-disjoint paths between any two vxs.*

# 4 Eu. & Ha. cyc.

## 4.1 Eul. trails & tours

**Thm 4.5.** *A conn. (multi)gr. has an Eulerian tour iff each vx has even degree.*

**Lem 4.6.** *Every maximal trail in an even gr. (i.e., a gr. where all the vxs have even degree) is a closed trail.*

**Cor 4.7.** *A conn. multiG has an Eulerian trail iff it has either 0 or 2 vxs of odd degree.*

## 4.2 Hamilton paths and cycles

**Prop 4.10.** *If $G$ is Hamiltonian then for any set $S \subseteq V$ the gr. $G \setminus S$ has at most $|S|$ conn. components.*

**Cor 4.11.** *If a conn. bipartite gr. $G = (V, E)$ with bipartition $V = A \cup B$ is Hamiltonian then $|A| = |B|$.*

**Thm 4.13** (Dirac 1952)**.** *If $G$ is a simple gr. with $n \geq 3$ vxs and if $\delta(G) \geq n/2$, then $G$ is Hamiltonian.*

**Thm 4.15** (Ore 1960)**.** *If $G$ is a simple gr. with $n \geq 3$ vxs such that for every pair of non-adjacent vxs $u, v$ of $G$ we have $d(u) + d(v) \geq |G|$, then $G$ is Hamiltonian.*

# 5 Matchings

*Rem* 5.3. A matching in a $G$ corresponds to an independent set in the line gr. $L(G)$.

**Prop 5.6.** $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

## 5.2 Hall's Theorem

**Thm 5.7** (Hall 1935)**.** *A bipartite gr. $G = (V, E)$ with bipartition $V = A \cup B$ has a matching covering $A$ iff $|N(S)| \geq |S| \forall S \subseteq A$*

**Cor 5.8.** *If in a bipartite gr. $G = (A \cup B, E)$ we have $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed $d \in \mathbb{N}$, then $G$ contains a matching of cardinality $|A| - d$.*

**Cor 5.9.** *If a bipartite gr. $G = (A \cup B, E)$ is k-regular with $k \geq 1$, then $G$ has a perfect matching.*

**Cor 5.10.** *Every regular gr. of positive even degree has a 2-factor (a spanning 2-regular subgraph).*

*Rem* 5.11. A 2-factor is a disjoint union of cycles covering all the vxs of a graph

**Cor 5.13.** *A collection $A_1, \ldots, A_n$ has an SDR iff for all $I \subseteq [n]$ we have $|\bigcup_{i \in I} A_i| \geq |I|$.*

**Thm 5.15** (König 1931)**.** *If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in $G$ equals the minimum size of a vx cover of $G$.*

## 5.3 Matchings in general graphs: Tutte's Theorem

Given a $G$, let $q(G)$ denote the # its odd components, i.e. the ones of odd order. If G has a perfect matching then clearly $q(G \setminus S) \leq |S| \forall S \subseteq V(G)$ since every odd component of $G \setminus S$ will send an edge of the matching to $S$, and each such edge covers a different vx in $S$.

**Thm 5.16** (Tutte 1947)**.** *$G$ has a perfect matching iff $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.*

**Cor 5.17** (Petersen 1891)**.** *Every 3-regular gr. with no cut-edge has a perfect matching.*

**Cor 5.19** (Berge 1958)**.** *The largest matching in an n-vx $G$ covers $n + min_{S \subseteq V(G)}(|S| - q(G \setminus S))$ vxs.*

# 6 Planar Graphs

**Thm 6.5** (Jordan curve theorem)**.** *A simple closed polygonal curve $C$ consisting of finitely many segments partitions the plane into exactly two faces, each having $C$ as boundary.*

*Rem* 6.6. This is not true in three dimensions. In $\mathbb{R}$ there is a surface called the Möbius band which has only one side.

*Rem* 6.7. The faces of $G$ are pairwise disjoint (they are separated by the edges of G). Two points are in the same face iff there is a polygonal path between them which does not cross an edge of $G$. Also, note that a finite gr. has a single unbounded face (the area "outside" of the graph).

**Prop 6.8.** *A plane forest has exactly one face.*

**Prop 6.11.** *If $l(f_i)$ denotes the length of a face $f_i$ in a plane $G$, then $2e(G) = \sum l(f_i)$.*

**Thm 6.12** (Euler's formula 1758)**.** *If a conn. plane $G$ has exactly $n$ vxs, $e$ edges and $f$ faces, then $n - e + f = 2$.*

**Thm 6.14.** *If $G$ is a planar gr. with at least three vxs, then $e(G) \leq 3|G| - 6$. If $G$ is also triangle-free, then $e(G) \leq 2|G| - 4$.*

**Cor 6.15.** *If $G$ is a planar bipartite n-vx gr. with $n \geq 3$ vxs then $G$ has at most $2n - 4$ edges.*

**Cor 6.16.** *$K_5$ and $K_{3,3}$ are not planar.*

*Rem* 6.17 (Maximal planar graphs / triangulations). The proof of Theorem 6.14 shows that having 3n - 6 edges in a simple n-vx planar gr. requires 2e = 3f, meaning that every face is a triangle. If G has some face that is not a triangle, then we can add an edge between non-adjacent vxs on the boundary of this face to obtain a larger plane graph. Hence the simple plane graphs with 3n - 6 edges, the triangulations, and the maximal plane graphs are all the same family.

### 6.1 Platonic Solids

**Cor 6.21.** *If $K$ is a convex polytope with $v$ vxs, $e$ edges and $f$ faces then $v - e + f = 2$.*

# 7 Graph colouring

### 7.1 Vertex colouring

*Rem* 7.3. The vxs having a given colour in a proper colouring must form an independent set, so $\chi(G)$ is the minimum # independent sets needed to cover $V(G)$. Hence $G$ is $k$-colourable iff $G$ is $k$-partite. Multiple edges do not affect chromatic number. Although we define $k$-colouring using numbers from $\{1, \ldots, k\}$ as labels, the numerical values are usually unimportant, and we may use any set of size $k$ as labels.

### 7.3 Bounds on $\chi$

*Claim* 7.6. If $H$ is a subgr. of $G$ then $\chi(H) \leq \chi(G)$.

**Cor 7.7.** $\chi(G) \geq \omega(G)$

**Prop 7.9.** $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$

*Claim* 7.10. For any gr. $G = (V, E)$ and any $U \subseteq V$ we have $\chi(G) \leq \chi(G[U]) + \chi(G[V \setminus U])$.

*Claim* 7.11. For any graphs $G_1$ and $G_2$ on the same vx set, $\chi(G_1 \cup G_2) \leq \chi(G1)\chi(G2)$.

**Prop 7.12.** *(i)* $\chi(G)\chi(\overline{G}) \geq |G|$
*(ii)* $\chi(G) + \chi(\overline{G}) \leq |G| + 1$

### 7.4 Greedy colouring

**Prop 7.16.** *$G$ is $k$-degenerate iff there is an ordering $v_1, \ldots, v_n$ of the vxs of $G$ such that each $v_i$ has at most $k$ neighbours among the vxs $v_1, \ldots, v_{i-1}$.*

*Rem* 7.18. $\delta(G) \leq dg(G) \leq \Delta(G)$.

**Thm 7.19.** $\chi(G) \leq 1 + dg(G)$

**Cor 7.20.** $\chi(G) \leq \Delta(G) + 1$.

*Rem* 7.21. This bound is tight if $G = K_n$ or if $G$ is an odd cycle.

**Thm 7.22** (Brooks 1941)**.** *If $G$ is a conn. gr. other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

### 7.5 Colouring planar graphs

*Claim* 7.23. A (simple) planar $G$ contains a vx $v$ of degree at most 5.

**Cor 7.24.** *A planar $G$ is 5-degenerate and thus 6-colourable.*

**Thm 7.25** (5 colour theorem, Heawood 1890)**.** *Every planar $G$ is 5-colourable.*

**Thm 7.26** (Appel-Haken 1977, conjectured by Guthrie in 1852)**.** *Every planar gr. is 4-colourable. (the countries of every plane map can be 4-coloured so that neighbouring countries get distinct colours).*

### 7.6 Art gallery thm

**Thm 7.28.** *For any museum with $n$ walls, $\lfloor n = 3 \rfloor$ guards suffice.*

# 8 Col. results

**Thm 8.1** (Gallai, Roy)**.** *If $D$ is an orientation of $G$ with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of $G$.*

### 8.1 Large girth and large $\chi$

The bound $\chi(G) \geq \omega(G)$ can be tight, but (surprisingly) it can also be arbitrarily bad. There are graphs having arbitrarily large chromatic number, even though they do not contain $K_3$. Many constructions of such graphs are known, though none are trivial. We give one here.

**Thm 8.3.** *Mycielski's construction produces a $(k + 1)$-chromatic triangle-free gr. from a $k$-chromatic triangle-free graph.*

**Thm 8.5** (Erdos 1959)**.** *Given $k \geq 3$ and $g \geq 3$, there exists a gr. with girth at least $g$ and chromatic number at least $k$.*

**Thm 8.8.** *There is a tournament on $n$ vxs where any $\frac{\log_2(n)}{2}$ vxs are beaten by some other vx.*

### 8.2 $\chi$ and clique minors

**Thm 8.12** (Mader)**.** *If the average degree of $G$ is at least $2t-2$ then $G$ has a $K_t$ minor.*

*Rem* 8.13. It is known that $\overline{d}(G) \geq ct\sqrt{log(t)}$ already implies the existence of a $K_t$ minor in $G$, for some constant $c > 0$.

### 8.3 Edge-colourings

*Rem* 8.15.
(i) An edge-colouring of a $G$ is the same as a vx-colouring of its line gr. $L(G)$.
(ii) A $G$ with maximum degree $\Delta$ has $\chi'(G) \geq \Delta$ since the edges incident to a vx of degree $\Delta$ must have different colours.
(iii) If $G$ has maximum degree $\Delta$ then $L(G)$ has maximum degree at most $2(\Delta - 1)$. $\Rightarrow \chi'(G) \leq 2\Delta - 1$

**Thm 8.16** (König 1916)**.** *If $G$ is a bipartite multigraph, then $\chi'(G) = \Delta(G)$.*

**Thm 8.17** (Vizing)**.** *Let $G$ be a simple gr. with maximum degree $\Delta$. Then $\Delta(G) \leq \chi 0(G) \leq \Delta(G) + 1$.*

### 8.4 List colouring

**Thm 8.20** (Erdos, Rubin, Taylor 1979)**.** *If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not $k$-choosable.*
).

**Thm 8.23** (Galvin 1995)**.** $\chi'_l(K_{n,n}) = n$
.

**Lem 8.25.** *If $D$ is a kernel-perfect orientation of $G$ and $f(x) = d^-_D(x)$ for all $x \in$*

*$V(G)$, then $G$ is $(1+f)$-choosable.*

# 9 The Matrix Tree Theorem

**Thm 9.1** (Cayley's formula)**.** *There are $n^{n-2}$ labeled trees on $n$ vxs.*

Now consider an arbitrary conn. simple $G$ on vx set $[n]$, and denote the # spanning trees by $t(G)$. The following celebrated result is Kirchhoff's matrix tree theorem. To formulate it, consider the incidence matrix $B$ of $G$ (as in Def 1.13), and replace one of the two 1's by -1 in an arbitrary manner to obtain the matrix $C$ (we say $C$ is the incidence matrix of an orientation of $G$). $M = CC^T$ is then a symmetric $n \times n$ matrix, which is

$$\begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix} - A_G$$

**Thm 9.2** (Matrix tree theorem)**.** *We have $t(G) = detM_{ii}$ for all $i = 1, \ldots, n$, where $M_{ii}$ results from $M$ by deleting the $i$-th row and the $i$-th column.*

**Thm 9.3** (Binet, Cauchy)**.** *If $P$ is an $r \times s$ matrix and $Q$ is an $s \times r$ matrix with $r \leq s$, then*

$$det(PQ) = \sum_Z (detP_Z)(detQ_Z)$$

*where $P_Z$ is the $r \times r$ submatrix of $P$ with column set $Z$, and $Q_Z$ is the $r \times r$ submatrix of $Q$ with the corresponding rows $Z$, and the sum is over all $r$-sets $Z \subseteq [s]$.*

# 10 More Thms on Hamiltonicity

**Thm 10.2** (Bondy Chvátal 1976)**.** *A simple $n$-vx gr. is Hamiltonian iff its closure is Hamiltonian.*

**Thm 10.3** (Chvátal 1972)**.** *Suppose $G$ has vx degrees $d_1 \leq \ldots d_n$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \geq n-i$, then $G$ is Hamiltonian.*

**Thm 10.4** (Chvátal-Erdos 1972)**.** *If $\kappa(G) \geq \alpha(G)$, then $G$ has a Hamiltonian cycle (unless $G = K_2$).*

## 10.1 Pósa's Lemma

Let $P$ be a path in a $G$, say from $u$ to $v$. Given a vx $x \in P$, we write $x^-$ for the vx preceding $x$ on $P$, and $x^+$ for the vx following $x$ on $P$ (whenever these exist). Similarly, for $X \subseteq V(P)$ we put $X^\pm := \{x^\pm : x \in X\}$

If $x \in P \setminus u$ is a neighbour of $u$ in $G$, then $P \cup \{(u,x)\} \setminus \{(x,x^-)\}$ (which is a path in $G$ with vx set $V(P)$) is said to have been obtained from $P$ by a rotation fixing $v$. A path obtained from $P$ by a (possibly empty) sequence of rotations fixing $v$ is a path derived from $P$. The set of starting vxs of paths derived from $P$, including $u$, will be denoted by $S(P)$. As all paths derived from $P$ have the same vx set as $P$, we have $S(P) \subseteq V(P)$.

*Rem* 10.5. If some sequence of rotations can delete the edge $(x,x^-)$, call this edge a broken edge. Note that every interval of the original path not containing broken edges is traversed by all derived paths as a whole piece (however, the direction can change).

**Lem 10.7.** *Let $G$ be a graph, let $P = u \ldots v$ be a longest path in $G$, and put $S := S(P)$. Then $\partial S \subseteq S^- \cup S^+$.*

**Lem 10.8.** *Let $G$ be a graph, let $P = u \ldots v$ be a longest path in $G$, and put $S := S(P)$. If $deg(u) \geq 2$ then $G$ has a cycle containing $S \cup \partial S$.*

**Cor 10.9.** *Fix $k \geq 2$ and let $G$ be a gr. such that for all $S \subseteq V(G)$ with $|S| \leq k$, we have $|\partial S| \geq |2S|$. Then $G$ has a cycle of length at least $3k$.*

## 10.2 Tournaments

**Thm 10.11.** *Every tournament has a Hamilton path.*

**Thm 10.13.** *A tournament $T$ is strongly conn. iff it has a Hamilton cycle.*

# 11 Kuratowski's Theorem

*Rem* 11.3. If $G$ contains a subdivision of $H$, it also contains an $H$-minor.

**Thm 11.5** (Kuratowski 1930)**.** *A gr. is planar iff it has no Kuratowski subgraph.*

**Thm 11.7.** *If $G$ is a gr. with no Kuratowski subgr. then $G$ has a straightline drawing in the plane.*

## 11.1 Convex drawings of 3-conn. graphs

**Thm 11.9** (Tutte 1960)**.** *If $G$ is a 3-conn. gr. which has no Kuratowski subgraphs then $G$ has a convex drawing in the plane with no three vxs on a line.*

**Lem 11.10** (Thomassen 1980)**.** *Every 3-conn. $G$ with at least five vxs has an edge $e$ such that $G/e$ is 3-conn.*

**Lem 11.11.** *If $G$ has no Kuratowski subgraphs, then $G/e$ has no Kuratowski subgraph, for any edge $e \in E(G)$.*

## 11.2 Reducing the general case to the 3-conn. case

*Fact* 2. We make three observations.

1. In a Kuratowski subgraph, there are three internally vx-disjoint paths connecting any two branch vxs. For $K_5$-subdivisions, we even have four such paths.
2. In a Kuratowski subgraph, there are four internally vx-disjoint paths between any two pairs of branch vxs.
3. Any cycle in a subdivision contains at least three branch vxs.

**Prop 11.14.** *Let $G$ be a gr. with at least 4 vxs which has no Kuratowski subgraph, and suppose that adding an* *edge-joining any pair of non-adjacent vxs creates a Kuratowski subgraph. Then G is 3-conn.*

# 12 Ramsey Theory

**Prop 12.1.** *Among six people it is possible to find three mutual acquaintances or three mutual non-acquaintances.*

As we shall see, given a natural number $s$, there is an integer $R$ such that if $n \geq R$ then every colouring of the edges of $K_n$ with red and blue contains either a red $K_s$ or a blue $K_s$. More generally, we define the Ramsey number $R(s,t)$ as the smallest value of $N$ for which every red-blue colouring of $K_N$ yields a red $K_s$ or a blue $K_t$. In particular, $R(s,t) = \infty$ if there is no such $N$ such that in every red-blue colouring of $K_N$ there is a red $K_s$ or a blue $K_t$. It is obvious that $R(s,t) = R(t,s)$ for every $s,t \geq 2$ and $R(s,2) = R(2,s) = s$.

**Thm 12.2** (Erdös, Szekeres). *The function $R(s,t)$ is finite for all $s,t \geq 2$. Quantitatively, if $s > 2$ and $t > 2$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$ and $R(s,t) \leq \binom{s+t-2}{s-1}$.*

**Thm 12.3.** *Given $k$ and $s_1, s_2, \ldots, s_k$, if $N$ is sufficiently large, then every colouring of $K_N$ with $k$ colours is such that for some $i, 1 \leq i \leq k$, there is a $K_{s_i}$ coloured with the i-th colour. The minimal value of $N$ for which this holds is usually denoted by $R_k(s_1, \ldots, s_k)$ , and it satisfies $R_k(s_1, \ldots, s_k) \leq R_{k-1}(R(s_1,s_2), s_3, \ldots, s_k)$.*

**Thm 12.4.** *Let $\min\{s,t\} > 3$. Then $R^{(3)}(s,t) \leq R(R^{(3)}(s-1,t), R^{(3)}(s,t-1)) + 1$*

## 12.1 Applications

**Thm 12.5** (Erdos-Szekeres 1935). *Given an integer $m$, there exists a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane, with no three collinear, contains an $m$-subset forming a convex $m$-gon.*

## 12.2 Bounds on Ramsey numbers

**Thm 12.6** (Erdös 1947). *For $p \geq 3$, we have $R(p,p) > 2^{p/2}$.*

**Thm 12.7.** *We have $R_k(3) \overset{def}{=} R_k(3,\ldots,3) \leq \lfloor e \cdot k! \rfloor + 1$.*

## 12.3 Ramsey theory for integers

**Thm 12.8** (Schur 1916). *For every $k \geq 1$ there is an integer $m$ such that every $k$-colouring of $[m]$ contains integers $x,y,z$ of the same colour such that $x+y=z$.*

## 12.4 Graph Ramsey numbers

*Rem* 12.10. Note that $R(G1,G2) \leq R(|G1|,|G2|)$.

**Thm 12.11** (Chvatal 1977). *If $T$ is any $m$-vx tree, then $R(T,K_n) = (m-1)(n-1)+1$*

# 13 Extremal problems

## 13.1 Turán's theorem

**Thm 13.5** (Turan 1941). *Among all the $n$-vx simple graphs with no $(r+1)$-clique, $T_{n,r}$ is the unique gr. having the maximum # edges.*

*Question* 13.6. Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be vectors such that $|a_i| \geq 1$ for each $i \in [n]$. What is the maximum # pairs satisfying $|a_i + a_j| < 1$?

*Claim* 13.7. There are at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ such pairs.

**Thm 13.9** (Erdos-Stone). *Let $H$ be a gr. of chromatic number $\chi(H) = r+1$. Then for every $\varepsilon > 0$ and large enough $n$, $\left(1-\frac{1}{r}\right)\frac{n^2}{2} \leq ex(n,H) \leq \left(1-\frac{1}{r}\right)\frac{n^2}{2} + n^2\varepsilon$*

## 13.2 Bipartite Turán Theorems

**Thm 13.11.** *If a $G$ on $n$ vxs contains no 4-cycles, then $e(G) \leq \left\lfloor \frac{n}{4}(1+\sqrt{4n-3}) \right\rfloor$.*

**Thm 13.13** (Kovári-Sós-Turán). *For any integers $r \leq s$, there is a constant $c$ such that every $K_{r,s}$-free gr. on $n$ vxs contains at most $cn^{1-\frac{1}{r}}$ edges. In other words, $ex(n,K_{r,s}) \leq cn^{1-\frac{1}{r}}$*

**Thm 13.14.** *There is $c$ depending on $k$ such that if $G$ is a gr. on $n$ vxs that contains no copy of $C_{2k}$, then $G$ has at most $cn^{1+\frac{1}{k}}$ edges.*

*Question* 13.15. Given $n$ points in the plane, how many pairs can be at distance 1?

**Thm 13.16** (Erdos). *There are at most $cn^{3/2}$ pairs.*

# 14 Exercises

## 14.1 Assignment 1

*P* 3. Prove that if a $G$ is not conn. then its complement $\overline{G}$ is conn. Converse is not true.

*P* 4. Show that every gr. on at least two vxs contains two vxs of equal degree.

*P* 5. Prove that every gr. with $n \geq 7$ vxs and at least $5n-14$ edges contains a subgr. with minimum degree at least 6.

*P* 6. Show that in a conn. gr. any two paths of maximum length share at least one vx.

*P* 7. Prove that a gr. is bipartite iff it contains no cycle of odd length.

## 14.2 Assignment 2

*P* 1. Show that in a tree containing an even # edges, there is at least one vx with even degree.

*P* 2. Given a $G$ and a vx $v \in V(G)$, $G - v$ denotes the subgr. of $G$ induced by the vx set $V(G) \setminus \{v\}$. Show that every conn. $G$ of order at least two contains vxs $x$ and $y$ such that both $G-x$ and $G-y$ are conn.

*P* 3. Let $T$ be an $n$-vx tree with exactly $2k$ odd-degree vxs. Prove that $T$ decomposes into $k$ paths (i.e. its edge-set is the disjoint union of $k$ paths).

$P$ 4. Prove that a conn. $G$ is a tree iff any family of pairwise (vx-)intersecting paths $P_1, \ldots, P_k$ in $G$ have a common vx.

$P$ 5(a) Prüfer codes corresponding to stars (i.e. to trees isomorphic to $K_{1,n-1}$) = 1 value.

(b) Prüfer codes containing exactly 2 different values = 2 connected stars

$P$ 6. Let $T$ be a forest on vx set $[n]$ with components $T_1, \ldots, T_r$. Prove, by induction on $r$, that the # spanning trees on $[n]$ containing $T$ is $n^{r-2} \prod_{i=1}^{r} |T_i|$. Deduce Cayley's formula.

## 14.3 Assignment 3

$P$ 1. Prove that a conn. $G$ is $k$-edge-conn. iff each block of $G$ is $k$-edge-conn.

$P$ 2. Let $G$ be a gr. and suppose some two vxs $u, v \in V(G)$ are separated by $X \subseteq V(G) \backslash \{u, v\}$. Show that $X$ is a minimal separating set (i.e. there is no proper subset $Y \subsetneq X$ that separates $u$ and $v$) iff every vx in $X$ has a neighbor in the component of $G - X$ containing $u$ and another in the component containing $v$.

$P$ 3. Show that if $G$ is a gr. with $|V(G)| = n \geq k+1$ and $\delta(G) \geq (n+k-2)/2$ then $G$ is $k$-conn.

$P$ 4. Prove that a $G$ with at least 3 vxs is 2-conn. iff for any three vxs $x, y, z$ there is a path from $x$ to $z$ containing $y$.

$P$ 5. Let $G$ be a $k$-conn. graph, where $k \geq 2$. Show that if $|V(G)| \geq 2k$ then $G$ contains a cycle of length at least $2k$.

## 14.4 Assignment 4

$P$ 1. Show that if $k > 0$ then the edge set of any conn. gr. with $2k$ vxs of odd degree can be split into $k$ trails.

$P$ 2. Let $G$ be a conn. gr. that has an Euler tour. T / F?

(a) If $G$ is bipartite then it has an even # edges. T

(b) If $G$ has an even # vxs then it has an even # edges. F

(c) For edges $e$ and $f$ sharing a vx, $G$ has an Euler tour in which $e$ and $f$ appear consecutively. F

$P$ 3. Let $G$ be a conn. gr. on n vxs with minimum degree $\delta$. Show that

(a) if $\delta \leq \frac{n-1}{2}$ then $G$ contains a path of length $2\delta$, and

(b) if $\delta \geq \frac{n-1}{2}$ then $G$ contains a Hamiltonian path.

$P$ 4. Show that the maximum # edges in a non-Hamiltonian gr. on $n \geq 3$ vxs is $\binom{n-1}{2} + 1$.

## 14.5 Assignment 5

$P$ 1. Let $G$ be a conn. gr. on more than 2 vxs such that every edge is contained in some perfect matching of $G$. Show that $G$ is 2-edge-conn.

$P$ 2(a) Let $G$ be a gr. on $2n$ vxs that has exactly one perfect matching. Show that $G$ has at most $n^2$ edges.

(b) Construct such a $G$ containing exactly $n^2$ edges for any $n \in N$.

$P$ 3. Let $A$ be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n$ be positive integers. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \in [n]$ if and only if $|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} d_i$.

$P$ 4. Suppose $M$ is a matching in a bipartite gr. $G = (A \cup B, E)$. We say that a path $P = a_1 b_1 \ldots a_k b_k$ is an augmenting path in $G$ if $b_i a_{i+1} \in M$ for all $i \in [k-1]$ and $a_1$ and $b_k$ are not covered by $M$. The name comes from the fact that the size of $M$ can be increased by flipping the edges along $P$ (in other words, taking the symmetric difference of $M$ and $P$): by deleting the edges $b_i a_{i+1}$ from $M$ and adding the edges $a_i b_i$ instead.

(a) Prove Hall's theorem by showing that if Hall's condition is satisfied and M does not cover $A$, then there is an augmenting path in $G$.

(b) Show that if $M$ is not a maximum matching (i.e. there is a larger matching in $G$) then the gr. contains an augmenting path. Is this true for non-bipartite graphs as well? Y

$P$ 5. Show that for $k \geq 1$, every $k$-regular $(k-1)$-edge-conn. gr. on an even # vxs contains a perfect matching.

## 14.6 Assignment 6

$P$ 2(a) Show that every planar gr. has a vx of degree at most 5. Is there a planar gr. with minimum degree 5? Y

(b) Show that any planar bipartite gr. has vx of degree at most 3. Is there a planar bipartite gr. with minimum degree 3? Y

$P$ 3. Show that a conn. plane $G$ is bipartite iff all its faces have even length.

$P$ 4. Let $G$ be a gr. on $n \geq 3$ vxs and $3n - 6 + k$ edges for some $k > 0$. Show that any drawing of $G$ in the plane contains at least $k$ crossing pairs of edges.

$P$ 5. Let $G$ be a plane gr. with triangular faces and suppose the vxs are colored arbitrarily with three colors. Prove that there is an even # faces that get all three colors.

$P$ 6. Let $S$ be a set of $n \geq 3$ points in the plane such that any two of them have distance at least 1. Show that there are at most $3n-6$ pairs of distance exactly 1.

## 14.7 Assignment 7

$P$ 1. T / F?

(a) If $G$ and $H$ are graphs on the same vx set, then $dg(G \cup H) \leq dg(G) + dg(H)$. F

(b) If $G$ and $H$ are graphs on the same vx set, then

$\chi(G \cup H) \le \chi(G) + \chi(H)$. F

(c) Every $G$ has a $\chi(G)$-coloring where $\alpha(G)$ vxs get the same color. F

*P* 2. $G$ has the property that any two odd cycles in it intersect (they share at least one vx in common). Prove that $\chi(G) \le 5$.

*P* 3. For a vx $v$ in a conn. $G$, let $G_r$ be the subgr. of $G$ induced by the vxs at distance $r$ from $v$. Show that $\chi(G) \le max_{0 \le r \le n} \chi(G_r) + \chi(G_{r+1})$.

*P* 4. Let $l$ be the length of the longest path in a $G$. Prove $\chi(G) \le l + 1$ using the fact that if a gr. is not $d$-degenerate then it contains a subgr. of minimum degree at least $d + 1$.

*P* 5. Suppose the complement of $G$ is bipartite. Show that $\chi(G) = \omega(G)$.

### 14.8 Assignment 8

*P* 1. For a given natural number $n$, let $G_n$ be the following gr. with $\binom{n}{2}$ vxs and $\binom{n}{3}$ edges: the vxs are the pairs $(x, y)$ of integers with $1 \le x < y \le n$, and for each triple $(x, y, z)$ with $1 \le x < y < z \le n$, there is an edge joining vx $(x, y)$ to vx $(y, z)$. Show that for any natural number $k$, the gr. $G_n$ is triangle-free and has chromatic number $\chi(G_n) > k$ provided $n > 2k$.

*P* 2. Show that the theorem of Mader implies the following weakening of Hadwiger's conjecture: Any $G$ with $\chi(G) \ge 2^{t-2} + 1$ has a $K_t$-minor.

*P* 3. Find the edge-chromatic # $K_n$ (don't use Vizing's theorem). $n$ for $n$ odd, $n - 1$ for $n$ even

*P* 4. Let $G$ be a conn. $k$-regular bipartite gr. with $k \ge 2$. Show, using König's theorem, that $G$ is 2-conn.

### 14.9 Assignment 9

*P* 1. Prove that every $G$ of maximum degree $\Delta$ has an equitable $(\Delta + 1)$-edge-coloring, i.e. one where each color class contains $\lfloor e = (\Delta + 1) \rfloor$ or $\lfloor e = (\Delta + 1) \rfloor$ edges, where $e$ is the # edges in $G$.

*P* 2. The cartesian product $H \times G$ of graphs $H$ and $G$ is the gr. with vx set $V(H) \times V(G)$, with an edge between $(v, u)$ and $(v', u')$ if $v = v'$ and $u$ is adjacent to $u'$ in $G$, or if $u = u'$ and $v$ is adjacent to $v'$ in $H$. Prove that if $\chi'(H) = \Delta(H)$ then $\chi'(H \times G) = \Delta(H \times G)$

*P* 3. Show that $\chi(C_n) = \chi_l(C_n)$ for any $n \ge 3$.

*P* 4. Let $G$ be a bipartite gr. on $n$ vxs. Prove that $\chi_l(G) \le 1 + \log_2(n)$ using the probabilistic method.

*P* 5. Let $G$ be a complete $r$-partite gr. with all parts of size 2. (In other words, $G$ is $K_{2r}$ minus a perfect matching.) Show, using a combination of induction and Hall's theorem, that $\chi_l(G) = r$.

### 14.10 Assignment 10

*P* 1. How many spanning trees does $K_{r,s}$ have? $r^{s-1} s^{r-1}$

*P* 2. Find the # spanning trees of $K_n - e$ (the complete gr. on n vxs with one edge removed): $(n - 2)n^{n-3}$

*P* 3. In this exercise we prove the following alternative form of the matrix-tree theorem. For an $n$-vx conn. $G$, the # spanning trees in $G$ is equal to the product of the nonzero eigenvalues of the Laplacian matrix $M$ of $G$, divided by $n$. (This matrix $M$ is as in the lecture notes).

*P* 4(a) Prove that any $n$-by-$n$ bipartite gr. with minimum degree $\delta > n/2$ contains a Hamilton cycle.

(b) Show that this is not necessarily the case if $\delta \le n/2$.

### 14.11 Assignment 11

*P* 1. T / F?

(a) If every vx of a tournament has positive in- and out-degree, then the tournament contains a directed Hamilton cycle. F

(b) If a tournament has a directed cycle, then it has a directed triangle. T

*P* 2. Let $G$ be a gr. on $n \ge 3$ vxs with at least $\alpha(G)$ vxs of degree $n - 1$. Show that $G$ is Hamiltonian.

*P* 3. Suppose $G$ is a gr. on $n$ vxs where all the degrees are at least $\frac{n+q}{2}$. Show that any set $F$ of $q$ independent edges is contained in a Hamiltonian cycle.

### 14.12 Assignment 12

*P* 1. The lower bound for $R(p, p)$ that you learn in the lectures is not a constructive proof: it merely shows the existence of a red-blue coloring not containing any monochromatic copy of $K_p$ by bounding the # bad graphs. Give an explicit coloring on $K_{(p-1)^2}$ that proves $R(p, p) > (p - 1)^2$.

*P* 2. Prove that for every fixed positive integer $r$, there is an $n$ such that any coloring of all the subsets of $[n]$ using $r$ colors contains two non-empty disjoint sets $X$ and $Y$ such that $X, Y$ and $X \cup Y$ have the same color.

*P* 3. Prove that for every $k \ge 2$ there exists an integer $N$ such that every coloring of $[N]$ with $k$ colors contains three distinct numbers $a, b, c$ satisfying $ab = c$ that have the same color.

*P* 4. For every $k \ge 2$ there is an $N$ such that any $k$-coloring of $[N]$ contains three distinct integers $a, b, c$ of the same color satisfying $a + b = c$.

*P* 5(a) Let $n \ge 1$ be an integer. Show that any sequence of $N \ge R(n, n)$ distinct numbers, $a_1, \dots, a_N$ contains a monotone (increasing or decreasing) subsequence of length $n$.

(b) Let $k, l \geq 1$ be integers and show that any sequence of $kl+1$ distinct numbers $a_1, \ldots, a_{kl+1}$ contains a monotone increasing subsequence of length $k+1$ or a monotone decreasing subsequence of length $l+1$.

## 14.13 Assignment 13

$P$ 1. Let $H$ be an arbitrary fixed gr. and prove that the sequence $ex(n,H)/\binom{n}{2}$ is (not necessarily strictly) monotone decreasing in $n$.

$P$ 2. Among all the $n$-vx $K_{r+1}$-free graphs, the Turan gr. $T_{n,r}$ contains the maximum # triangles (for any $r, n \geq 1$).

$P$ 3. Let $X$ be a set of $n$ points in the plane with no two points of distance greater than 1. Show that there are at most $\frac{n^2}{3}$ pairs of points in $X$ that have distance greater than $\frac{1}{\sqrt{2}}$.