Computation & optimization for Lasso - part 2

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ETH Zürich

22 October 2018

2018-10-02

Computation & optimization

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Overview

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1. Coordinate Descent
2. A Simulation Study
3. Least Angle Regression
4. ADMM
5. Minor-Max Algorithms
6. Alternating Minimizations
7. Screening Rules

Alternating Direction Method of Multipliers (ADMM)

Problem

$$\underset{\beta\in\mathbb{R}^m,\theta\in\mathbb{R}^n}{\text{minimize}}\, f(\beta)+g(\theta) \quad \text{subject to } \mathbf{A}\beta+\mathbf{B}\theta=c$$

Lagrangian

$$f(\beta)+g(\theta)+\rho||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

Augmented Lagrangian

$$L_\rho(\beta,\theta,\mu) := f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle+\frac{\rho}{2}||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

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Augmented: scalar product with μ gets added

Dual variable update

$$\beta^{t+1} = \arg\min_{\beta \in \mathbb{R}^m} L_\rho(\beta, \theta^t, \mu^t)$$

$$\theta^{t+1} = \arg\min_{\theta \in \mathbb{R}^m} L_\rho(\beta^{t+1}, \theta, \mu^t)$$

$$\mu^{t+1} = \mu^t + \rho(\mathbf{A}\beta^{t+1} + \mathbf{B}\theta^{t+1} - c)$$

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ADMM - Why?

- convex problems with nondifferentiable constraints
- blockwise computation
  - sample blocks
  - feature blocks

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ADMM for the Lasso

Problem in Lagrangian form

$$\underset{\beta \in \mathbb{R}^p, \theta \in \mathbb{R}^p}{\text{minimize}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\theta\|_1 \right\} \quad \text{such that } \beta - \theta = 0$$

Update

$$\begin{aligned} \beta^{t+1} &= (\mathbf{X}^T\mathbf{X} + \rho\mathbf{I})^{-1}(\mathbf{X}^T\mathbf{y} + \rho\theta^t - \mu^t) \\ \theta^{t+1} &= \mathcal{S}_{\lambda/\rho}(\beta^{t+1} + \mu^t/\rho) \\ \mu^{t+1} &= \mu^t + \rho(\beta^{t+1} - \theta^{t+1}) \end{aligned}$$

where $\mathcal{S}_{\lambda/\rho}(z) = \text{sign}(z)(|z| - \frac{\lambda}{\rho})_+$.

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Computational cost: Initially $\mathcal{O}(p^3)$, which is a lot, for the SVD(singular value decomposition of $\mathbf{X}$), after that comparable to coordinate descent or composite gradient from earlier

Minorization-Maximization Algorithms (MMA)

- Problem: minimize $f(\beta)$ over $\beta \in \mathbb{R}^p$ for f possibly non-convex
- Introduce additional variable θ
- Use θ to majorize (bound from above) the objective function to be minimized

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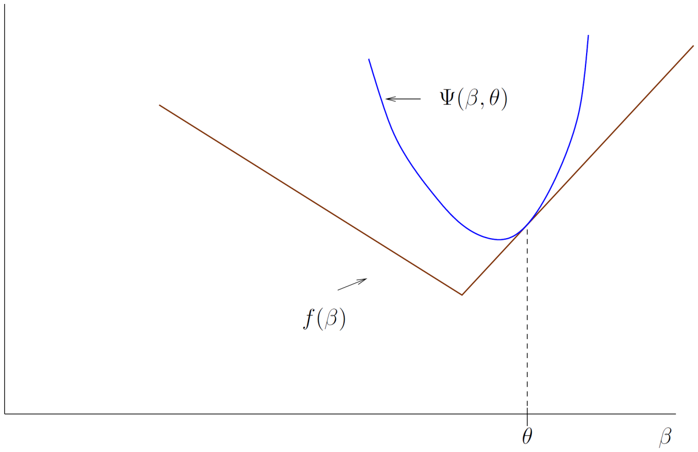
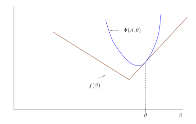


Figure: Figure 5.10 from [Hastie et al., 2015]


MMA visually

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MMA analytically

Dual Polytope Projection (DPP)

Suppose we want to calculate a lasso solution at $\lambda < \lambda_{\text{max}}$. The DPP rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda_{\text{max}} - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\text{max}} - \lambda}{\lambda}$$

Sequential DPP rule

Suppose we have the lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < \lambda' - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\text{max}} - \lambda}{\lambda}$$

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Global Strong Rule

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$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda - (\lambda_{\text{max}} - \lambda) = 2\lambda - \lambda_{\text{max}}$$

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Paragraphs of Text

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Bullet Points

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Blocks of Highlighted Text

Block 1

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Block 3

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Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

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Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

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Theorem (Mass–energy equivalence)
$E = mc^2$

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Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
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Figure

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An example of the `\cite` command to cite within the presentation:

This statement requires citation [Hastie et al., 2015].

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References

Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015)
Statistical learning with sparsity: the Lasso and generalizations
CRC Press; Boca Raton, FL

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The End

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The End