Computation & optimization for Lasso - part 2

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Overview

1. Coordinate Descent
2. Least Angle Regression
3. Comparison of Optimization Methods
4. Recall: Duality
5. ADMM
6. Screening Rules

Coordinate Descent Algorithm

What is the Coordinate Descent (CD) Algorithm?

$$\beta_k^{t+1} = \underset{\beta_k}{\operatorname{argmin}}\ f(\beta_1^t, \beta_2^t, \ldots, \beta_k, \beta_{k+1}^t, \ldots, \beta_p^t)$$

and $\beta_j^{t+1} = \beta_j^t$ for $j \neq k$

- An iterative algorithm that updates from β^t to β^{t+1} by choosing a single coordinate, and minimizing over this coordinate.

Separability Condition

Motivation

Does CD procedure converge to the global minimum of a convex function?

- **Sufficient Condition:** the function is continuously differentiable and strictly convex in each coordinate.

$\Rightarrow$ restrictive

Separability Condition

Suppose the cost function f has the additive decomposition:

$$f(\beta_1, ..., \beta_p) = g(\beta_1, ..., \beta_p) + \sum_{j=1}^{p} h_j(\beta_j)$$

where $g : \mathbb{R}^p \to \mathbb{R}$ is differentiable and convex, and the univariate functions $h_j : \mathbb{R} \to \mathbb{R}$ are convex.

- Lasso: $g(\beta) = \frac{1}{2N}\|\mathbf{y} - \mathbf{X}\beta\|_2^2$ and $h_j(\beta_j) = \lambda|\beta_j|$ satisfies the condition

Separability Condition: Example

An Example of failure of Coordinate Descent

$$\underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \ \frac{1}{2N} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \ \lambda_1 \sum_{j=1}^{p} |\beta_j| + \ \lambda_2 \sum_{j=2}^{p} |\beta_j - \beta_{j-1}|$$

- $h(\beta)$ is not separable
- Fused Lasso: coordinate descent procedure is not guaranteed to find the global minimum

Separability Condition: Example

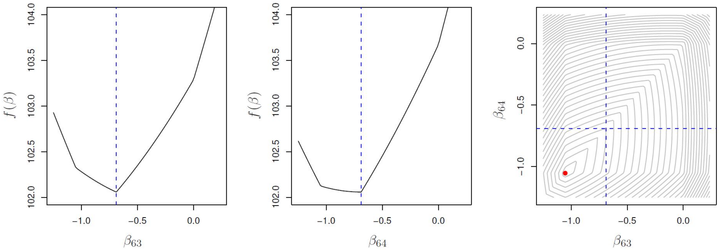


Figure: Fused Lasso: CD fails to reach the global minimum from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p111.

Lasso & Coordinate Descent

Optimality Condition:

$$-\frac{1}{2N}\sum_{i=1}^{N}(y_i - \beta_0 - \sum_{k=1}^{p} x_{ik}\beta_k)x_{ij} + \lambda s_j = 0$$

where $s_j \in sign(\beta_j)$ for $j = 1, 2, ..., p$

- Define the **partial residual**: $r_i^{(j)} = y_i - \sum_{k \neq j} x_{ik}\hat{\beta}_k$
- Then the solution for $\hat{\beta}_j$ satisfies:

$$\hat{\beta}_j = \frac{S_\lambda(\frac{1}{N}\sum_{i=1}^{N} r_i^{(j)} x_{ij})}{\frac{1}{N}\sum_{i=1}^{N} x_{ij}^2}$$

where $S_\lambda(\theta) = sign(\theta)(|\theta| - \lambda)_+$

Lasso & Coordinate Descent

- Illustration of Coordinate Descent in R

Strategies to make the operation efficient:

Naive Updating

$r_i^{(j)} = y_i - \sum_{k \neq j} x_{ik} \hat{\beta}_k = r_i + x_{ij} \hat{\beta}_j$

$\frac{1}{N} \sum_{i=1}^{N} x_{ij} r_i^{(j)} = \frac{1}{N} \sum_{i=1}^{N} x_{ij} r_i + \hat{\beta}_j$

Covariance Updating

$\sum_{i=1}^{N} x_{ij} r_i = \langle x_j, y \rangle - \sum_k \langle x_j, x_k \rangle \beta_{\hat{k}}$

Warm Starts: For a decreasing sequence of values $\{\lambda_0^L\}$, $\hat{\beta}(\lambda_l)$ is typically a very good warm start for the solution $\hat{\beta}(\lambda_{l+1})$. We set $\lambda_0 = \frac{1}{N} max |\langle x_j, y \rangle|$ and $\lambda_L \approx 0$.

Lasso & Coordinate Descent

Active-set Convergence: Define the active set A and iterate the algorithm using only variables in A.

Strong-set Convergence: Define the strong set S and iterate the algorithm using only variables in S.

Sparsity: Sparsity of the design matrix X makes the operation of inner product efficient.

Details in page 113 and page 114 of Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015) Statistical learning with sparsity: the Lasso and generalizations.

Elastic Net & Coordinate Descent

$$minimize_{\beta_0,\beta} \frac{1}{2}\sum_{i=1}^{N}(y_i-\beta_0-x_i^T\beta)^2+\lambda\left[\frac{1}{2}(1-\alpha)||\beta||_2^2+\alpha||\beta||_2\right]_1$$

- Combination of L1 and L2 penalty
- Satisfy the separability condition
- The solution satisfies:

$$\hat{\beta}_j = \frac{S_{\alpha\lambda}(\frac{1}{N}\sum_{i=1}^{N} r_i^{(j)} x_{ij})}{\frac{1}{N}\sum_{i=1}^{N} x_{ij}^2+(1-\alpha)\lambda}$$

Logistic Regression & Coordinate Descent

Background

- **Class Label G**: Take values 1 and -1

  Denote $p(x_i; \beta_0, \beta) = Pr(G = 1|x_i)$

  Define log odds: $\log \frac{Pr(G=-1|x)}{Pr(G=1|x)} = \beta_0 + x^T\beta$

- **Maximize penalized log-likelihood:**

  $\frac{1}{N}\sum_{i=1}^{N}\{I(g_i = 1) \cdot \log(p(x_i; \beta_0, \beta))$
  $+I(g_i = -1) \cdot \log(1 - p(x_i; \beta_0, \beta))\} - \lambda||\beta||_1$

  Denote $y_i = I(g_i = -1)$

Explicit form of log likelihood (without penalty):

$L(\beta_0, \beta) = \frac{1}{N}\sum_{i=1}^{N}\left[y_i \cdot (\beta_0 + x_i^T\beta) - \log(1 + e^{\beta_0 + x_i^T\beta})\right]$

Logistic Regression & Coordinate Descent

Background

- Form a **quadratic objective function** using Taylor expansion about current estimates $(\tilde{\beta_0}, \tilde{\beta})$: Idea of Newton method, Iterated Weighted Least Square problem

  $L_Q(\beta_0, \beta) = -\frac{1}{2N} \sum_{i=1}^{N} w_i(z_i - \beta_0 - {x_i}^T \beta)^2 + C(\tilde{\beta_0}, \tilde{\beta})$

- Use Coordinate Descent to solve the problem

  $minimize_{(\beta_0,\beta)}\{ - L_Q(\beta_0, \beta) + \lambda||\beta||_1\}$

Logistic Regression & Coordinate Descent

Algorithm

OUTER LOOP: Decrement λ

MIDDLE LOOP: Update the **quadratic approximation** L_Q using the current parameters $(\tilde{\beta}_0, \tilde{\beta})$

INNER LOOP: Run the coordinate descent algorithm on the penalized weighted least squares problem

Least Angle Regression

Introduction

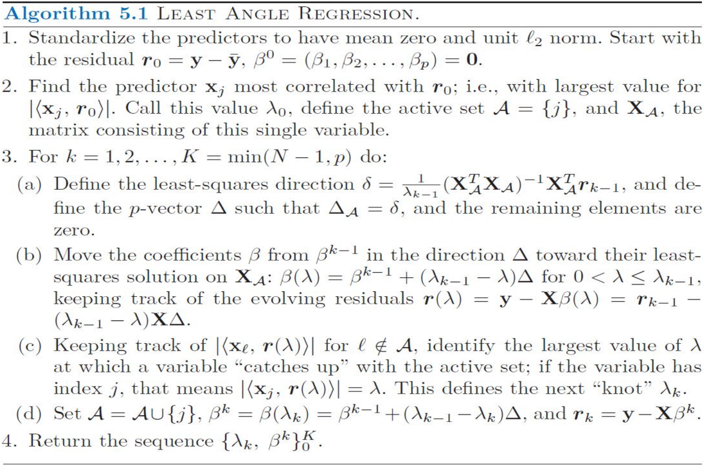
- Relates to Forward Selection method
- Relates to Lasso method
- Able to deliver the entire solution path of the Lasso problem with squared-error loss as a function of the regularization parameter λ

Least Angle Regression: Algorithm

- Start with all coefficients β_j equal to zero
- Find the predictor X_j **most correlated** with y
- **Increase** the coefficient β_j in the direction of the **sign** of its **correlation** with y
- Take **residuals** $r = y - \hat{y}$ along the way; Stop when some other predictor X_k has **as much correlation** with r as X_j has
- **Increase** β_j, β_k in their **joint least squares direction**, until some other predictor has as much correlation with the residual r

Continue until: all predictors are in the model

Least Angle Regression: Algorithm



```
Algorithm 5.1 Least Angle Regression.
1. Standardize the predictors to have mean zero and unit $\ell_2$ norm. Start with
   the residual $\boldsymbol{r}_0 = \mathbf{y} - \bar{\mathbf{y}}$, $\beta^0 = (\beta_1, \beta_2, \ldots, \beta_p) = \mathbf{0}$.
2. Find the predictor $\mathbf{x}_j$ most correlated with $\boldsymbol{r}_0$; i.e., with largest value for
   $|\langle \mathbf{x}_j, \boldsymbol{r}_0 \rangle|$. Call this value $\lambda_0$, define the active set $\mathcal{A} = \{j\}$, and $\mathbf{X}_{\mathcal{A}}$, the
   matrix consisting of this single variable.
3. For $k = 1, 2, \ldots, K = \min(N-1, p)$ do:
   (a) Define the least-squares direction $\delta = \frac{1}{\lambda_{k-1}}(\mathbf{X}_{\mathcal{A}}^T \mathbf{X}_{\mathcal{A}})^{-1} \mathbf{X}_{\mathcal{A}}^T \boldsymbol{r}_{k-1}$, and define
       the $p$-vector $\Delta$ such that $\Delta_{\mathcal{A}} = \delta$, and the remaining elements are zero.
   (b) Move the coefficients $\beta$ from $\beta^{k-1}$ in the direction $\Delta$ toward their least-squares
       solution on $\mathbf{X}_{\mathcal{A}}$: $\beta(\lambda) = \beta^{k-1} + (\lambda_{k-1} - \lambda)\Delta$ for $0 < \lambda \leq \lambda_{k-1}$,
       keeping track of the evolving residuals $\boldsymbol{r}(\lambda) = \mathbf{y} - \mathbf{X}\beta(\lambda) = \boldsymbol{r}_{k-1} - (\lambda_{k-1} - \lambda)\mathbf{X}\Delta$.
   (c) Keeping track of $|\langle \mathbf{x}_\ell, \boldsymbol{r}(\lambda) \rangle|$ for $\ell \notin \mathcal{A}$, identify the largest value of $\lambda$
       at which a variable "catches up" with the active set; if the variable has
       index $j$, that means $|\langle \mathbf{x}_j, \boldsymbol{r}(\lambda) \rangle| = \lambda$. This defines the next "knot" $\lambda_k$.
   (d) Set $\mathcal{A} = \mathcal{A} \cup \{j\}$, $\beta^k = \beta(\lambda_k) = \beta^{k-1} + (\lambda_{k-1} - \lambda_k)\Delta$, and $\boldsymbol{r}_k = \mathbf{y} - \mathbf{X}\beta^k$.
4. Return the sequence $\{\lambda_k, \beta^k\}_0^K$.
```

Figure: From Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015) Statistical learning with sparsity: the Lasso and generalizations, p119, Algorithm 5.1.

Least Angle Regression: Geometric Representation

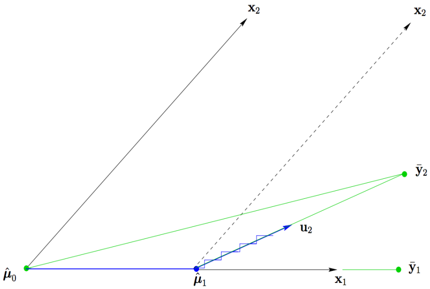


Figure: From Bradely Efron,Trevor Hastie, Iain Johnstone And Robert Tibshirani (2004), Least Angle Regression, The Annual of Statistics, p412, Figure 2.

Connection between LAR and Lasso

LAR

Let $\mathbb{A}$ be the active set of variables for the LAR algorithm. $\mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\beta(\lambda)) = \lambda \cdot s_j,\ \forall j \in \mathbb{A}$ where s_j is the sign of inner product λ.

LASSO

Let $\mathbb{B}$ be the active set of variables in the solution for a given value of λ.
$R(\beta) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\beta||_2^2 + \lambda||\beta||_1$

For differentiable $R(\beta)$, the stationary conditions give:
$\mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\beta) = \lambda \cdot sign(\beta_j), \forall j \in \mathbb{B}$

If sign (β_j) matches s_j, the coefficient would be identical.

Connection between LAR and Lasso

- R Example
- LAR algorithm explains that the coefficient paths for the Lasso are **piecewise linear**
- Coefficient paths differ if sign(β_j) is different from s_j
- **Modification** of LAR for computing Lasso solution:
  If a **nonzero** coefficient **crosses zero** before the next variable enters, **drop** it from $\mathbb{A}$ and recompute the current joint least squares direction.

Connection between LAR and Lasso

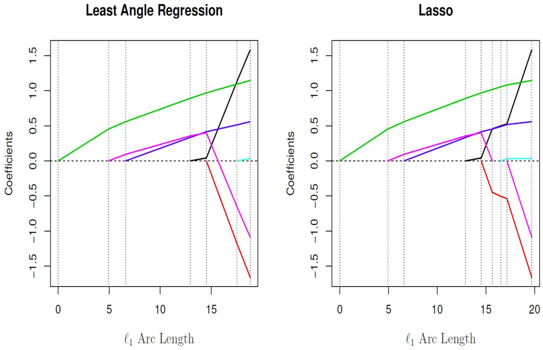


Figure: Cases where signs of λ and β disagree, from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015) Statistical learning with sparsity: the Lasso and generalizations, p120, Figure 5.9.

Algorithm Performance

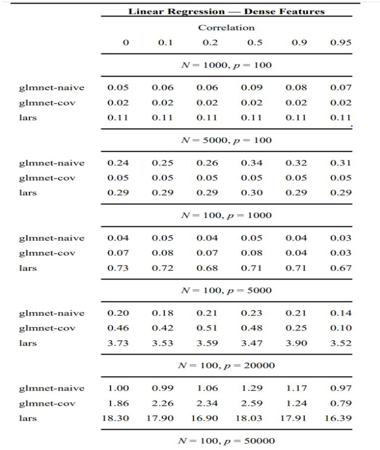
Simulation: Comparison of computation efficiency between CD and LAR

Set Up: [1]

- Generate Gaussian data with N observations and p predictors, with each pair of predictors X_j, X_k having the same population correlation ρ.
- Try different combination of N and p; Range ρ from 0 to 0.95.
  $Y = \sum_{j=1}^{p} X_j \beta_j + kZ$ where
  $\beta_j = (-1)^j exp(\frac{-2(j-1)}{20}), Z \sim N(0,1)$ and k is a constant.

[1] Jerome Friedman, Trevor Hastie, and Rob Tibshirani (2010) Regularization Paths for Generalized Linear Models via Coordinate Descent, p12.

Algorithm Performance



Linear Regression — Dense Features

	Correlation					
	0	0.1	0.2	0.5	0.9	0.95
			$N = 1000, p = 100$			
glmnet-naive	0.05	0.06	0.06	0.09	0.08	0.07
glmnet-cov	0.02	0.02	0.02	0.02	0.02	0.02
lars	0.11	0.11	0.11	0.11	0.11	0.11
			$N = 5000, p = 100$			
glmnet-naive	0.24	0.25	0.26	0.34	0.32	0.31
glmnet-cov	0.05	0.05	0.05	0.05	0.05	0.05
lars	0.29	0.29	0.29	0.30	0.29	0.29
			$N = 100, p = 1000$			
glmnet-naive	0.04	0.05	0.04	0.05	0.04	0.03
glmnet-cov	0.07	0.08	0.07	0.08	0.04	0.03
lars	0.73	0.72	0.68	0.71	0.71	0.67
			$N = 100, p = 5000$			
glmnet-naive	0.20	0.18	0.21	0.23	0.21	0.14
glmnet-cov	0.46	0.42	0.51	0.48	0.25	0.10
lars	3.73	3.53	3.59	3.47	3.90	3.52
			$N = 100, p = 20000$			
glmnet-naive	1.00	0.99	1.06	1.29	1.17	0.97
glmnet-cov	1.86	2.26	2.34	2.59	1.24	0.79
lars	18.30	17.90	16.90	18.03	17.91	16.39
			$N = 100, p = 50000$			

Figure: Comparison of computing time, from Jerome Friedman, Trevor Hastie, and Rob Tibshirani (2010) Regularization Paths for Generalized Linear Models via Coordinate Descent, Appendix 1.

Algorithm Performance

Simulation: Comparison of computation efficiency between Coordinate Descent, Proximal Gradient Descent and Nesterov Method

Set Up: [2]

- Generated an $N \times p$ predictor matrix X with standard Gaussian entries and pairwise correlation 0 or 0.5 between the features.
- $|\beta_j| = exp\big[-0.5(u(j-1))^2\big]$ and $u = \sqrt{\frac{\pi}{20}}$ and alternating signs -1,+1,-1...

[2] Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015) Statistical learning with sparsity: the Lasso and generalizations, p117.

Algorithm Performance

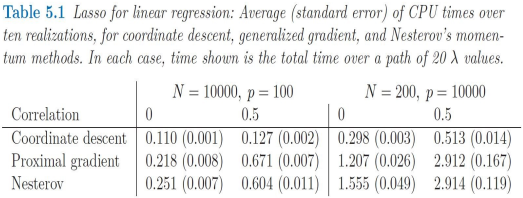


Table 5.1 *Lasso for linear regression: Average (standard error) of CPU times over ten realizations, for coordinate descent, generalized gradient, and Nesterov's momentum methods. In each case, time shown is the total time over a path of 20 λ values.*

	$N = 10000,\ p = 100$		$N = 200,\ p = 10000$	
Correlation	0	0.5	0	0.5
Coordinate descent	0.110 (0.001)	0.127 (0.002)	0.298 (0.003)	0.513 (0.014)
Proximal gradient	0.218 (0.008)	0.671 (0.007)	1.207 (0.026)	2.912 (0.167)
Nesterov	0.251 (0.007)	0.604 (0.011)	1.555 (0.049)	2.914 (0.119)

Figure: Comparison of computing efficiency between 3 methods, from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015) Statistical learning with sparsity: the Lasso and generalizations, p117, Table 5.1.

Recall: Duality in optimization

Primal	
Optimize	$\min f(x)$
Constraints	$g_i(x) \leq 0, h_j(x) = 0, x \in X$
Function	$L(x, \lambda, \mu) := f(x) + \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$
Dual	
Function	$q(\lambda, \mu) = \inf_{x \in X} L(x, \lambda, \mu)$
Constraints	$\lambda \geq 0$
Optimize	$\max_{\lambda \geq 0, \mu} q(\lambda, \mu)$

Why though? - **Dual problem is always convex!**

Alternating Direction Method of Multipliers (ADMM)

Problem - decomposable !

$$\underset{\beta\in\mathbb{R}^m,\theta\in\mathbb{R}^n}{\text{minimize}}\, f(\beta)+g(\theta) \quad \text{subject to } \mathbf{A}\beta+\mathbf{B}\theta-c=0$$

Lagrangian - decomposable !

$$f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle$$

Augmented Lagrangian - NOT decomposable !

$$L_\rho(\beta,\theta,\mu):=f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle+\frac{\rho}{2}||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

Dual Variable Update

Alternating Direction Method of Multipliers

$$\beta^{t+1} = \underset{\beta \in \mathbb{R}^m}{\arg\min}\, L_\rho(\beta, \theta^t, \mu^t)$$

$$\theta^{t+1} = \underset{\theta \in \mathbb{R}^m}{\arg\min}\, L_\rho(\beta^{t+1}, \theta, \mu^t)$$

$$\mu^{t+1} = \mu^t + \rho(\mathbf{A}\beta^{t+1} + \mathbf{B}\theta^{t+1} - c)$$

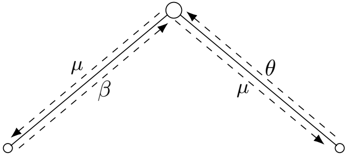


Figure: Own illustration of the dual ascent step in the ADMM algorithm utilising dual decomposition based on Geoff Gordon and Ryan Tibshirani (2012), Uses of Duality, `https://www.cs.cmu.edu/~ggordon/10725-F12/slides/18-dual-uses.pdf`.

ADMM - Why?

- convex problems with nondifferentiable constraints
- blockwise computation
  - sample blocks
  - feature blocks

ADMM for the Lasso - Problem

Problem in Lagrangian form

$$\underset{\beta\in\mathbb{R}^p,\theta\in\mathbb{R}^p}{\text{minimize}}\left\{\frac{1}{2}\left\|\mathbf{y}-\mathbf{X}\beta\right\|_2^2+\lambda\left\|\theta\right\|_1\right\}\quad\text{such that } \beta-\theta=0$$

Augmented Lagrangian

$$L_\rho(\beta,\theta,\mu):=\left\{\frac{1}{2}\left\|\mathbf{y}-\mathbf{X}\beta\right\|_2^2+\lambda\left\|\theta\right\|_1\right\}+\langle\mu,\beta-\theta\rangle+\frac{\rho}{2}||\beta-\theta||_2^2$$

ADMM for the Lasso - Update

Update

$$\begin{aligned}\beta^{t+1} &= (\mathbf{X}^T\mathbf{X} + \rho\mathbf{I})^{-1}(\mathbf{X}^T\mathbf{y} + \rho\theta^t - \mu^t)\\ \theta^{t+1} &= \mathcal{S}_{\lambda/\rho}(\beta^{t+1} + \mu^t/\rho)\\ \mu^{t+1} &= \mu^t + \rho(\beta^{t+1} - \theta^{t+1})\end{aligned}$$

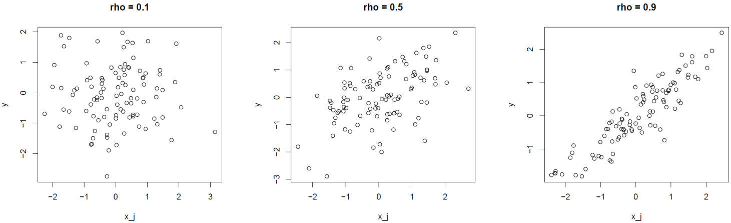
where $\mathcal{S}_{\lambda/\rho}(z) = \text{sign}(z)(|z| - \frac{\lambda}{\rho})_+$.

Screening Rules

- Pre-processing to eliminate features
- Very big data set, esp. large number of predictors
- Maybe too big to load into memory
- Screening rules eliminate predictors with minor calculation
- And very high / safe certainty (i.e. eliminated predictors would not show up in Lasso model based on full data)

They achieve a reduction in the number of variables, typically by an order of magnitude

What is a good predictor?



correlation is an inner product

high absolute correlation ($=$ large absolute inner product)

$>$ high predictive power (compare plots)

$>$ x_j with largest inner product has highest predictive power

$>$ thus for that j we are most willing to accept some penalty from λ

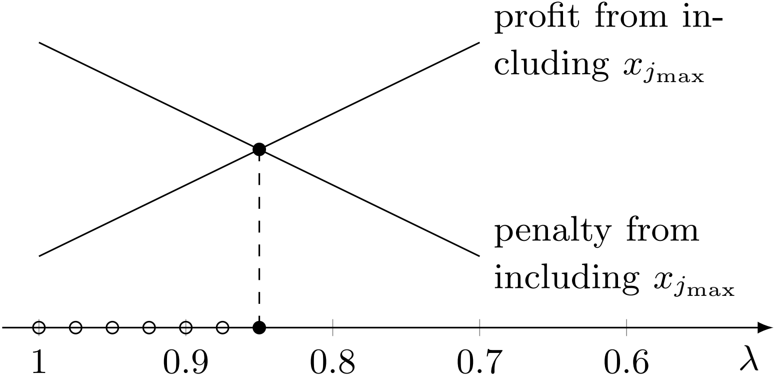
Lasso - an iterative algorithm

Let $\mathcal{A}$ be the active set of predictors. Let λ take values on a decreasing sequence.

iterate

1. order predictors x_j not in $\mathcal{A}$ by their "effectiveness" using $\left|x_j^T y\right|$ or better $\left|x_j^T (y - \hat{y}_\lambda)\right|$, call the best predictor $x_{j_{\max}}$
2. move λ such that the positive effect from the best predictor $x_{j_{\max}}$ compensates the penalty by λ
3. calculate solution for chosen λ

Lasso - a visual intepretation


profit from including $x_{j_{\max}}$
penalty from including $x_{j_{\max}}$
1
0.9
0.8
0.7
0.6
λ

Back to screening rules

Let λ take values on a decreasing sequence. Let $\lambda_{\max}$ be the λ where the first predictor has a non-zero coefficient.

$$\lambda_{\max} = \max_j \left| x_j^T y \right|$$

Let $\mathcal{A}$ be the active set of predictors.

$$\forall j \in \mathcal{A} \;\; \lambda = \left| x_j^T (y - \hat{y}) \right|$$

$$\forall j \notin \mathcal{A} \;\; \lambda > \left| x_j^T (y - \hat{y}) \right|$$

$$\forall j \notin \mathcal{A} \;\; \lambda > \left| x_j^T (y - \hat{y}) \right|$$

R example

Global vs. Sequential

Global (one-time screening):
Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\text{max}}$.

Sequential (iterative screening):
Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$.

Dual Polytope Projection (DPP)

Global DPP Rule

Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\text{max}}$. The DPP rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda_{\text{max}} - \|\mathbf{x}_j\|_2 \, \|\mathbf{y}\|_2 \, \frac{\lambda_{\text{max}} - \lambda}{\lambda}$$

Sequential DPP rule

Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < \lambda' - \|\mathbf{x}_j\|_2 \, \|\mathbf{y}\|_2 \, \frac{\lambda_{\text{max}} - \lambda}{\lambda}$$

Strong Rule

Global Strong Rule

Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\text{max}}$. The global strong rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda - (\lambda_{\text{max}} - \lambda) = 2\lambda - \lambda_{\text{max}}$$

Sequential Strong Rule

Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < 2\lambda - \lambda'$$

Screening Rules - Example Setup

- simulated dataset
- $N = 200, p = 5000$ uncorrelated Gaussian predictor
- $1/4$ true non-zero coefficients
- 100 decreassing lambda values equally spaced on the log-scale
- Compare Global DPP, Global Strong, Sequential DDP, Sequential Strong
- no violations for either of the strong rules

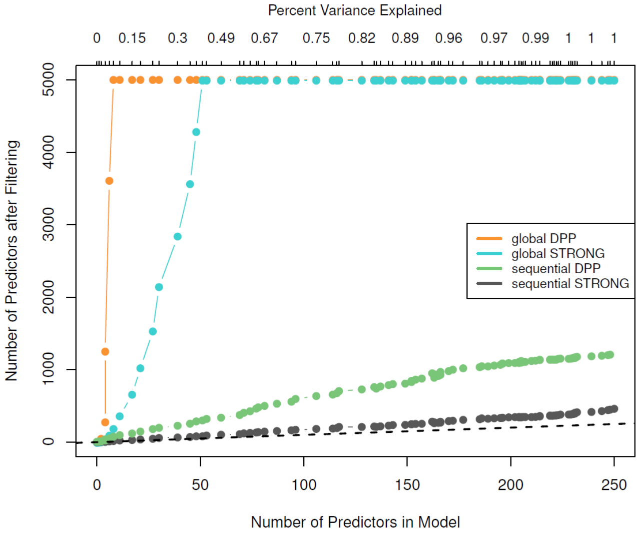


Figure: From Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, *CRC Press;* Boca Raton, FL, 12(3), 45 – 678.

Summary I

Coordinate Descent

- An efficient algorithm implemented in glmnet but requires separability condition
- Application: Ridge, Lasso, Elastic Net, Logistic Regression, etc.; Failure: Fused Lasso

Least Angle Regression

- Similar to the idea of Forward Selection
- Computationally efficient but does not scale well to large problems

Connection between LASSO and LAR

- LAR could be modified to obtain Lasso solution
- Explains the fact that Lasso coefficient solution path is piece-wise linear

Summary II

ADMM

- Use duality to your advantage
- Limitations in speed for Lasso, but useful in more complex settings

Screening Rules

- Promising for very large *p*s
- Difficult to find best rule, field in development

Additional slides: Minorization-Maximization Algorithms (MMA)

- Problem: minimize $f(\beta)$ over $\beta \in \mathbb{R}^p$ for f possibly non-convex
- Introduce additional variable θ
- Use θ to majorize (bound from above) the objective function to be minimized

Majorization-Minimization Algorithms work analoguosly.

MMA visually

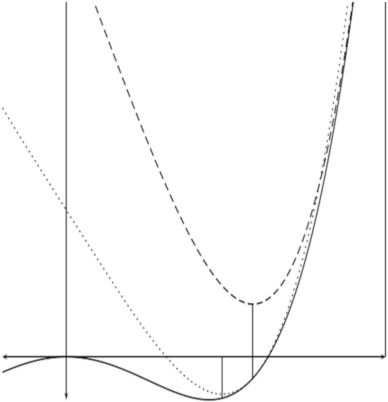


Figure: Figure from Jan De Leeuw (2015), Block Relaxation Methods in Statistics, `doi.org/10.13140/RG.2.1.3101.9607` (last accessed: 02.10.18)

MMA analytically I

Def. $\Psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ majorizes f at $\beta \in \mathbb{R}^p$ if

$$\forall \theta \in \mathbb{R}^p \quad \Psi(\beta, \theta) \geq f(\beta)$$

with equality for $\theta = \beta$.

Minor-Max algorithm

- initialize β^0
- update with $\beta^{t+1} = \arg\min_{\beta \in \mathbb{R}^p} \Psi(\beta, \beta^t)$

MMA analytically II

This scheme generates a sequence of βs for which the cost $f(\beta^t)$ is nonincreasing, because

$$f(\beta^t) \overset{(i)}{=} \Psi(\beta^t, \beta^t) \overset{(ii)}{\geq} \Psi(\beta^{t+1}, \beta^t) \overset{(iii)}{\geq} f(\beta^{t+1})$$

where

(i) & (iii) Definiton of majorize

(ii) β^{t+1} is a minimizer of $\beta \mapsto \Psi(\beta, \beta^t)$

Biconvexity

Let's consider an example ...

$f(\alpha, \beta) = (1 - \alpha\beta)^2$

Def. A function $f(\alpha, \beta) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is biconvex, if for each $\alpha \in \mathbb{R}^m$ the function $\alpha \mapsto f(\alpha, \beta)$ is convex and for each $\beta \in \mathbb{R}^n$ the function $\beta \mapsto f(\alpha, \beta)$ is convex. Analoguosly, a set $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$, for $\mathcal{A}, \mathcal{B}$ convex sets, is called biconvex, if it is convex in both directions.

Alternate Convex Search

Block coordinate descent applied to α and β blocks

1. Initialize (α^0, β^0) at some point in the biconvex set to minimize over
2. For $t = 0, 1, 2, \ldots$
   - (i) Fix $\beta = \beta^t$ and update $\alpha^{t+1} \in \underset{\alpha \in \mathcal{C}_{\beta^t}}{\arg\min} f(\alpha, \beta^t)$
   - (ii) Fix $\alpha = \alpha^{t+1}$ and update $\beta^{t+1} \in \underset{\alpha \in \mathcal{C}_{\alpha^{t+1}}}{\arg\min} f(\alpha^{t+1}, \beta)$

For a function bounded from below, the algorithm converges to a partial optimum (i.e. as biconvexity, only optimal in one coordinate if the other coordinate is fixed).

Comments . . .
Questions . . .
Suggestions . . .

That's it.

Thanks for listening.