Computation & optimization for Lasso - part 2

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ETH Zürich

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Overview

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1. Coordinate Descent
2. A Simulation Study
3. Least Angle Regression
4. ADMM
5. Minor-Max Algorithms
6. Alternating Minimizations
7. Screening Rules

Alternating Direction Method of Multipliers (ADMM)

Problem

$$\underset{\beta\in\mathbb{R}^m,\theta\in\mathbb{R}^n}{\text{minimize}}\, f(\beta)+g(\theta) \quad \text{subject to } \mathbf{A}\beta+\mathbf{B}\theta=c$$

Lagrangian

$$f(\beta)+g(\theta)+\rho||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

Augmented Lagrangian

$$L_\rho(\beta,\theta,\mu) := f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle+\frac{\rho}{2}||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

Augmented: scalar product with μ gets added

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Dual variable update

$$\beta^{t+1} = \arg\min_{\beta \in \mathbb{R}^m} L_\rho(\beta, \theta^t, \mu^t)$$
$$\theta^{t+1} = \arg\min_{\theta \in \mathbb{R}^m} L_\rho(\beta^{t+1}, \theta, \mu^t)$$
$$\mu^{t+1} = \mu^t + \rho(\mathbf{A}\beta^{t+1} + \mathbf{B}\theta^{t+1} - c)$$

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ADMM - Why?

- convex problems with nondifferentiable constraints
- blockwise computation
  - sample blocks
  - feature blocks

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Detailsfor blockwise computation in Exercise 5.12.

ADMM for the Lasso

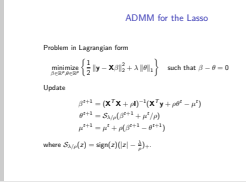
Problem in Lagrangian form

$$\underset{\beta\in\mathbb{R}^p,\theta\in\mathbb{R}^p}{\text{minimize}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\theta\|_1 \right\} \quad \text{such that } \beta - \theta = 0$$

Update

$$\begin{aligned}
\beta^{t+1} &= (\mathbf{X}^T\mathbf{X} + \rho\mathbf{I})^{-1}(\mathbf{X}^T\mathbf{y} + \rho\theta^t - \mu^t) \\
\theta^{t+1} &= \mathcal{S}_{\lambda/\rho}(\beta^{t+1} + \mu^t/\rho) \\
\mu^{t+1} &= \mu^t + \rho(\beta^{t+1} - \theta^{t+1})
\end{aligned}$$

where $\mathcal{S}_{\lambda/\rho}(z) = \text{sign}(z)(|z| - \frac{\lambda}{\rho})_+$.



Computational cost: Initially $\mathcal{O}(p^3)$, which is a lot, for the SVD(singular value decomposition of $\mathbf{X}$), after that comparable to coordinate descent or composite gradient from earlier

Minorization-Maximization Algorithms (MMA)

- Problem: minimize $f(\beta)$ over $\beta \in \mathbb{R}^p$ for f possibly non-convex
- Introduce additional variable θ
- Use θ to majorize (bound from above) the objective function to be minimized

Majorization-Minimization Algorithms work analoguosly.

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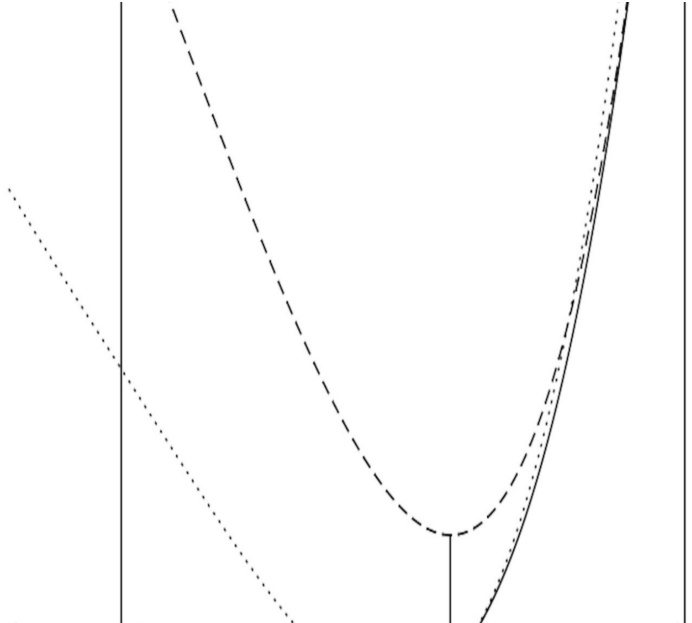
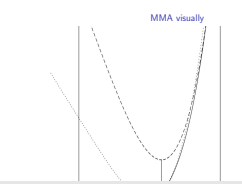
MMA visually





MMA analytically I

Def. $\Psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ majorizes f at $\beta \in \mathbb{R}^p$ if

$$\forall \theta \in \mathbb{R}^p \quad \Psi(\beta, \theta) \geq f(\beta)$$

with equality for $\theta = \beta$.

Minor-Max algorithm

- initialize β^0
- update with $\beta^{t+1} = \underset{\beta \in \mathbb{R}^p}{\arg\min}\, \Psi(\beta, \beta^t)$

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MMA analytically II

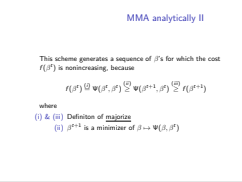
This scheme generates a sequence of β's for which the cost $f(\beta^t)$ is nonincreasing, because

$$f(\beta^t) \overset{(i)}{=} \Psi(\beta^t, \beta^t) \overset{(ii)}{\geq} \Psi(\beta^{t+1}, \beta^t) \overset{(iii)}{\geq} f(\beta^{t+1})$$

where

(i) & (iii) Definiton of majorize

(ii) β^{t+1} is a minimizer of $\beta \mapsto \Psi(\beta, \beta^t)$



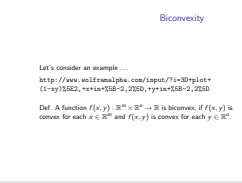
for inequalities: show previous slide

Biconvexity

Let's consider an example ...

`http://www.wolframalpha.com/input/?i=3D+plot+(1-xy)%5E2,+x+in+%5B-2,2%5D,+y+in+%5B-2,2%5D`

Def. A function $f(x, y) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is biconvex, if $f(x, y)$ is convex for each $x \in \mathbb{R}^m$ and $f(x, y)$ is convex for each $y \in \mathbb{R}^n$.



Mathematica: `3D plot (1-xy)^2, x in [-2,2], y in [-2,2]`

Dual Polytope Projection (DPP)

Suppose we want to calculate a lasso solution at $\lambda < \lambda_{\max}$. The DPP rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda_{\max} - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\max} - \lambda}{\lambda}$$

Sequential DPP rule

Suppose we have the lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < \lambda' - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\max} - \lambda}{\lambda}$$

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Global Strong Rule

Suppose we want to calculate a lasso solution at $\lambda < \lambda_{\max}$. The global strong rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda - (\lambda_{\max} - \lambda) = 2\lambda - \lambda_{\max}$$

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References

Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015)
Statistical learning with sparsity: the Lasso and generalizations
CRC Press; Boca Raton, FL

Jan De Leeuw (2015)
Block Relaxation Methods in Statistics
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Comments . . .

Questions . . .

Suggestions . . .

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Thanks for listening.

Fill out your feedback sheets!

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