1. 设 A,B 是随机事件， A 与 B 互不相容， $P(A)=\frac{1}{2},P(B)=\frac{1}{3}$,则 $P(A-B|A\cup B)=$

3/5__.

解：$\because A$、B互不相容

$$\therefore P(A-B|A\cup B)=P(A|A\cup B)=\frac{P\left(A\cap\left(A\cup B\right)\right)}{P\left(A\cup B\right)}=\frac{P\left(A\right)}{P\left(A\cup B\right)}=\frac{1/2}{1/2+1/3}=\frac{3}{5}$$

2. 设随机变量 X 的分布函数为

$$F(x)=\begin{cases}0, & x<0,\\ 1/2, & x=0,\\ kx+b, & 0<x<2,\\ 1, & x\geq 2.\end{cases}$$

，$F(x)$在 x=2 处左连续，则 k=_

1/4__.

解：$\because F\left(x\right)$在任意点x右连续$\therefore F\left(x+0\right)=F\left(0\right)$，则：

$$F\left(x+0\right)=\lim_{x\to 0^+}F\left(x\right)=\lim_{x\to 0^+}\left(kx+b\right)=b=F\left(0\right)=\frac{1}{2}$$

而$F\left(x\right)$又在x=2处左连续，$\therefore F\left(2\text{-}0\right)=F\left(2\right)$，则：

$$F\left(2\text{-}0\right)=\lim_{x\to 2^-}F\left(x\right)=\lim_{x\to 2^-}\left(kx+b\right)=\lim_{x\to 2^-}\left(kx+\frac{1}{2}\right)=F\left(2\right)=1$$

即： $2k+\frac{1}{2}=1,\therefore k=\frac{1}{4}$

3. 设 $X\sim N(0,1)$，$Y=X^2$，则 $\rho_{XY}=$__0______.

解：$\because \rho_{XY}=\dfrac{Cov\left(X,Y\right)}{\sqrt{D\left(X\right)}\cdot\sqrt{D\left(Y\right)}}$，$X\sim N\left(0,1\right)$，$Y=X^2$

且：$Cov\left(X,Y\right)=E\left(X-E\left(X\right)\right)\left(Y-E\left(Y\right)\right)$

$\because Y=X^2$，则：$E\left(Y\right)=E\left(X^2\right)=D\left(Y\right)=1$

$\therefore Cov\left(X,Y\right)=E\left(X-E\left(X\right)\right)\left(Y-E\left(Y\right)\right)=E\left(X\left(X^2-1\right)\right)=0$

$\therefore \rho_{XY}=0$

4. 设二维随机变量 (X,Y) 服从 $N(0,0,1,4,0.5)$ ，则 $P\{2X<Y\}=$__1/2________,

$Cov(X+Y,X-2Y)=$___-8_____.

解：$\because (X,Y)\sim N(0,0,1,4,0.5)$，令：$Z=2X-Y$，则$Z$也服从正态分布。

$$\therefore D(Z)=D(2X-Y)=4D(X)+D(Y)-4E(XY)=4D(X)+D(Y)-4Cov(X,Y)$$
$$=4D(X)+D(Y)-4\rho_{XY}\sqrt{D(X)}\cdot\sqrt{D(Y)}=4\times1+4-4\times0.5\times2=4$$

$\therefore D(Z)\sim N(0,4)$，其中$Cov(X,Y)=\rho_{XY}\sqrt{D(X)}\cdot\sqrt{D(Y)}=1$

$$\therefore P(2X<Y)=P(Z<0)=\int_{-\infty}^{0} f_Z(z)dz=\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=\frac{1}{2}$$

且：$Cov(X+Y,X-2Y)=Cov(X,X)\text{-}2Cov(X,Y)+2Cov(Y,X)\text{-}2Cov(Y,Y)$
$$=D(X)\text{-}Cov(X,Y)\text{-}2D(Y)=1\text{-}1\text{-}2\times4=\text{-}8$$

设 (X,Y) 的概率密度为 $f(x,y)=\begin{cases}\dfrac{1}{1-x},0<x<y<1,\\ 0,\ \text{otherwise},\end{cases}$，则 $P\{X+Y\le1\}=$____________；

$P\{X\le0.5\,|\,Y=0.5\}=$______1______.

解：$P\{X+Y\le1\}=\iint\limits_{x+y\le1} f(x,y)dxdy=\int_{0}^{1/2}dx\int_{x}^{1-x}\frac{1}{1-x}dy=\int_{0}^{1/2}\frac{1-2x}{1-x}dx=1-\ln2$

而：$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$

$\therefore$ 当$0<y<1$时，$f_Y(y)=\int_0^y\frac{1}{1-x}dx=\text{-}\ln(1-y)$

$$\therefore f_Y(y)=\begin{cases}\text{-}\ln(1-y) & 0<y<1\\ 0 & \text{其它}\end{cases}$$

$\therefore$ 当$0<y<1$时，$f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=\begin{cases}-\frac{1}{(1-x)\ln(1-y)} & 0<x<y\\ 0 & \text{其它}\end{cases}$

$$\therefore f_{X|Y=0.5}(x|0.5)=\frac{f(x,0.5)}{f_Y(0.5)}=\begin{cases}\frac{1}{(1-x)\ln2} & 0<x<0.5\\ 0 & \text{其它}\end{cases}$$

$$\therefore P\{X\le0.5\,|\,Y=0.5\}=\int_{-\infty}^{0.5}f_{X|Y=0.5}(x|0.5)dx=1$$

5. **已知随机变量X服从均值为0.5的指数分布，$Y=1\text{-}e^{-2X}$,则Y的概率密度函数为**

$$\begin{cases}1 & 0<y<1\\ 0 & \text{其它}\end{cases}$$

解:$\because X$服从均值为0.5的指数分布,$\therefore f(x)=\begin{cases}\frac{1}{\theta}e^{-\frac{x}{\theta}} & x>0\\ 0 & \text{其它}\end{cases}$

$Y=1-e^{-2X}$,则$y=1-e^{-2x}$，$x=-\dfrac{1}{2}ln(1-y)$，其中$0<y<1$

则:$\dfrac{dx}{dy}=\dfrac{1}{2}\cdot\dfrac{1}{1-y}(0<y<1)$

$$\therefore f_Y(y)=\begin{cases}\frac{1}{0.5}e^{-\frac{1}{0.5}\times\left(-\frac{1}{2}\right)ln(1-y)}\frac{1}{2}\frac{1}{1-y} & 0<y<1\\ 0 & \text{其它}\end{cases}=\begin{cases}1 & 0<y<1\\ 0 & \text{其它}\end{cases}$$

6. 若X,Y独立，均服从标准正态分布. $U=X^2+Y^2$ 与$V=X/Y$ 的联合概率密度为

$$\begin{cases}\frac{1}{2\pi}e^{-\frac{u}{2}}\cdot\frac{1}{2\left(v^2+1\right)} & u\ge0,-\infty<v<+\infty\\ 0 & \text{否则}\end{cases}$$

解:$\because$若X,Y独立，均服从标准正态分布

$$\therefore f(x,y)=\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

$$\text{令:}\begin{cases}U=X^2+Y^2\\ V=X/Y\end{cases}\Rightarrow\begin{cases}x=\pm y\sqrt{\frac{u}{1+v^2}} & u\ge0\\ y=\pm\sqrt{\frac{u}{1+v^2}} & -\infty<v<+\infty\end{cases}$$

$$J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & 2y\\ 1/y & -\frac{x}{y^2}\end{vmatrix}=-\frac{2x^2}{y^2}-2=-2\left(v^2+1\right)$$

$$\therefore f(u,v)=\begin{cases}f(x,y)\cdot\frac{1}{|J|} & u\ge0,-\infty<v<+\infty\\ 0 & \text{否则}\end{cases}$$

$$=\begin{cases}\frac{1}{2\pi}e^{-\frac{u}{2}}\cdot\frac{1}{2\left(v^2+1\right)} & u\ge0,-\infty<v<+\infty\\ 0 & \text{否则}\end{cases}$$

7. (X,Y)的分布密度为$f(x,y)=\begin{cases}\lambda^2 e^{-\lambda x}, 0<y<x\\ 0,其他\end{cases}$，则$E[X\mid Y=y](y>0)$=$y+\dfrac{1}{\lambda}$

解：$\because (X,Y)$的分布密度为$f(x,y)=\begin{cases}\lambda^2 e^{-\lambda x}, 0<y<x\\ 0,其他\end{cases}$，则：

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)dx$$

$\therefore$ 当$y>0$时，$f_Y(y)=\int_{y}^{+\infty}\lambda^2 e^{-\lambda x}dx=\lambda e^{-\lambda y}$

则：$f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=\dfrac{\lambda^2 e^{-\lambda x}}{\lambda e^{-\lambda y}}=\lambda e^{-\lambda(x-y)}(x>y)$, 则:

当$y>0$时，$E[X\mid Y=y]=\int_{-\infty}^{+\infty} x f_{X|Y}(x|y)dx=\int_{y}^{+\infty} x\lambda e^{-\lambda(x-y)}dx=y+\dfrac{1}{\lambda}$

设随机变量 X 的特征函数为$g(t)=\cos^2 t$，则 X 的分布函数为___________。

解：$\because g(t)=\cos^2 t=\dfrac{1+\cos 2t}{2}=\dfrac{1}{2}+\dfrac{1}{2}\cdot\dfrac{e^{2it}+e^{-2it}}{2}$，即：

X	0	2	-2
P	1/2	1/4	1/4

$$\therefore F(x)=\begin{cases}0 & x<-2\\ 1/4 & -2<x<0\\ 3/4 & 0<x<2\\ 1 & x\ge 2\end{cases}$$

8. 若X,Y独立，均服从标准正态分布,则$E[\max(X,Y)]$=$\dfrac{1}{\sqrt{\pi}}$ 。

解：利用随机变量函数的数学期望计算公式，有：

$$E\left[\max\left(X,Y\right)\right]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\max\left(X,Y\right)f\left(x,y\right)dxdy\left(令：x=r\cos\theta，y=r\sin\theta\right)$$

$$=\int_{0}^{+\infty}dr\int_{-\pi}^{+\pi}\max\left(r\cos\theta,r\sin\theta\right)\frac{1}{2\pi}e^{-\frac{r^2}{2}}rd\theta$$

$$=\int_{0}^{+\infty}dr\int_{-3\pi/4}^{\pi/4}r\cos\theta\frac{1}{2\pi}e^{-\frac{r^2}{2}}rd\theta+\int_{0}^{+\infty}dr\int_{\pi/4}^{5\pi/4}r\sin\theta\frac{1}{2\pi}e^{-\frac{r^2}{2}}rd\theta$$

$$=\int_{0}^{+\infty}\frac{1}{\sqrt{2\pi}}r^2e^{-\frac{r^2}{2}}dr+\int_{0}^{+\infty}\frac{1}{\sqrt{2\pi}}r^2e^{-\frac{r^2}{2}}dr=\frac{1}{\sqrt{\pi}}$$