Cavendish Experiment

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***Abstract*—We have conducted Cavendish experiment to measure the Newtonian Gravitational Constant G. We used a torsion balance, like Cavendish did, and we have tried to measure G with a reasonable precision. Since the gravitational force is the weakest force in the nature, G is hard to measure precisely. We have found the constant to be $G = 9.78 \times 10^{-11} \pm 1.18 \times 10^{-11} m^3 kg^{-1} s^{-2}$ which is 2.63σ away from the recommended value. Also, we have calculated the mass of the Earth and we found it to be $M_{Earth} = 4.08 \times 10^{24} \pm 5.10^{23} kg$ which is 3.80σ away from the recommended value.**

I. Introduction

The Cavendish experiment is performed by an English scientist Henry Cavendish. He attempted to measure the relative density of the Earth using the Newton's law of gravitation and he succeeded[1]. Since the gravitational force is the weakest force in the nature, it is significant to measure the Newtonian Gravitational constant G accurately which is also small. In the experiment, like Cavendish did, we have used a torsion balance with small and big balls to find G. Using the constant G, we have calculated the mass of the Earth. We have tested the following theory:

Newton's Gravitation Law:

$$F = \frac{GmM}{b^2} \tag{1}$$

where, M and m are masses of big and small balls

b is the radius between big and small balls.

The aim is to use a torsion balance which applies a torque to the system and balance it with the gravitational force.

Torque:

$$\tau = \kappa * \Delta\theta \tag{2}$$

where, $\Delta\theta$ is deviation of angle from the equiilbrium position and κ is the spring constant.

Balancing F and τ:

$$\tau = \kappa * \Delta\theta = 2Fr \tag{3}$$

where, r is the radius between small ball and rotation axis

Balanced Torque and Force:

$$\kappa\Delta\theta = 2\frac{GmM}{b^2} r \Rightarrow G = \frac{\kappa b^2 \Delta\theta}{2mMr} \tag{4}$$

We have obtained a systematic method to measure G with the equation 4. However, we need to determine the unknowns which are κ and $\Delta\theta$. To do that, we should investigate the behavior of the system when it is at the resonance mode. We achieve this by arranging the positions of big balls at rise and fall times.

Equation of Motion:

$$I\ddot{\theta} + \beta\dot{\theta} + \kappa\theta = 0 \tag{6}$$

where, I is moment of inertia and β is damping constant

In Resonance:

$$\ddot{\theta} + w_r^2\theta = 0 \tag{8}$$

where, $w_r = \sqrt{k/I}$ is resonance frequency.

Solving the Equation:

$$\theta(t) = A + Bsin(w_r t + \phi) \tag{9}$$

where, A and B are constants, ϕ is the phase angle

With equation 9, we can determine the spring constant κ; however, we need to determine the $\Delta\theta$ value. To do that, we need to investigate the nature of the system when we damp the system by stop arranging the big balls and leave them next to an arbitrary small ball.

Solution to the Damped System:

$$\theta(t) = A' + B' sin(w_d t + \phi')e^{-\beta t} \tag{10}$$

where, w_d is damping frequency,

A' , B' are constants, β is the decay constant.

and ϕ' is the phase angle.

The angles are not directly measured, but they are measured by a angle sensor which gives voltage outputs. We know that the sensor obeys the linear equation:

$$V(\theta) = a\theta + b \tag{11}$$

We will find "a" in the Equation 11 by the difference of the angle $\Delta\Theta$ at peak points of resonance mode. Then, we can calculate the angle $\Delta\Theta$ by using a laser, ruler and trigonometry with a small angle approximation.

$$a = \frac{\Delta\Theta}{2B} \tag{12}$$

$$\Delta\Theta = \frac{\Delta S}{2L} \tag{13}$$

where, L is the distance between ruler and apparatus

ΔS is the difference of peaks on the ruler

Combining Eq. 12 and 13:

$$a = \frac{\Delta S}{4LB} \tag{14}$$

Using Eq. 14:

$$\Delta\theta = a|A' - A| \Rightarrow \Delta\theta = \frac{|A' - A|\Delta S}{4LB} \tag{15}$$

We have obtained every unknown parameters for calculating G. If we consider the attraction between a mass on the Earth, knowing the constant G, we can calculate the mass of the Earth.

$$mg = \frac{GmM_{Earth}}{R_{Earth}^2} \Rightarrow M_{Earth} = \frac{gR_{Earth}^2}{G} \tag{16}$$

where, R_{Earth} is the radius of the Earth

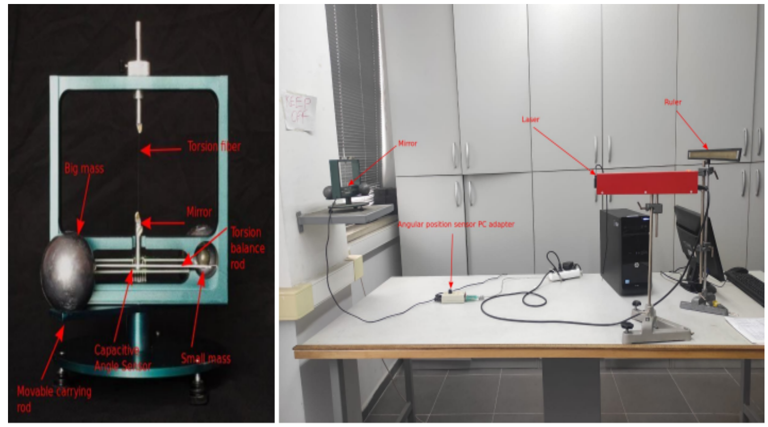
g is the acceleration of the gravity.

II. The Experimental Setup

We have used a torsion balance which has a fine fiber attached to a rod. The rod has two small balls on its both ends. We then introduce 2 big balls to the system, by the Newton's Gravitation Law, there is a attraction between small and big balls. The attraction occures as a deviation of the equilibrium angle.

We then change position of the big massses so that we can create a resonance. After keeping the system at resonance for a while, we place the big balls near to the small balls which will damp the system. we then detect the deviation to find G as shown in the introduction. The method we have used is called "dynamical method".

- Movable carrying rod (for big masses)
- Torsion balance fiber
- Torsion balance rod
- Mirror
- Big masses
- Small masses
- Angular position sensor
- Laser
- Ruler
- Angular position sensor PC adapter



(a) Torsion balance (b) Optical measurement system and adapter

Fig. 1. Apparatus. [2]

Fig. 2. Cavendish's Torsion Balance Apparatus [3]

III. The Analysis

Known and Measured parameters are listed below.

TABLE I
KNOWN PARAMETERS

Parameters	Value	Relative Errors
I	1.43 $\times 10^{-4} kgm^2$	6.17 $\times 10^{-3} kgm^2$
b	4.61025 $\times 10^{-2} m$	3.42 $\times 10^{3} m$
r	6.6653 $\times 10^{-2} m$	5.57 $\times 10^{4} m$
M	1.0385 kg	9.63 $\times 10^{-4} kg$
m	1.4573 $\times 10^{-2} kg$	6.86 $\times 10^{-5} kg$

where, I is the moment of inertia. b is the radius between small and big masses. r is the radius between small masses and center of the rod. M is the mass of big ball and m is the mass of small ball.

TABLE II
MEASURED PARAMETERS

Parameters	Value	Errors
L	1.55m	0.01 m
Δ S	0.025 m	0.003

where L is the length between the apparatus and the ruler. ΔS is the difference on the ruler between peaks of the resonance mode.

Using the data we have obtained from the angle sensor, we have drawn the following plots. As can be seen at Fig.3, there is a sinusoidal behavior at t = [2150, 2900]. We have fitted a sinusoidal function as Eq.9 suggests (See. Fig4). At t = [3150-3900] we can see the damping behavior of the system as Eq.10 suggests, we have fitted damping sinusoidal function (See. Fig5).

We have done the fits with Root's built-in function. The parameters have been found by root are showed on the legends of the figures.

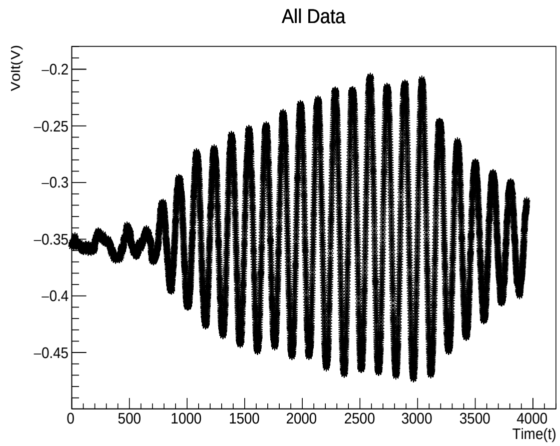


Fig. 3. Plot of All Data. Volt vs Time.

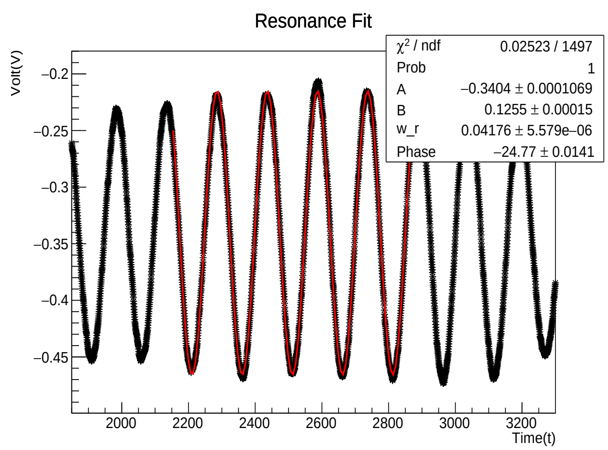


Fig. 4. Plot of Resonance. Volt vs Time.
Used Eq.9 for fit.

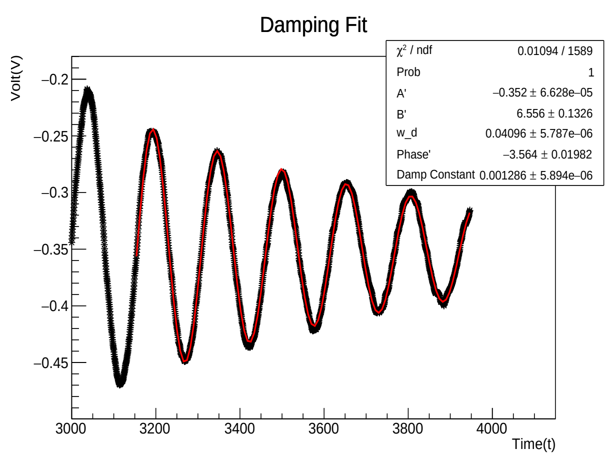
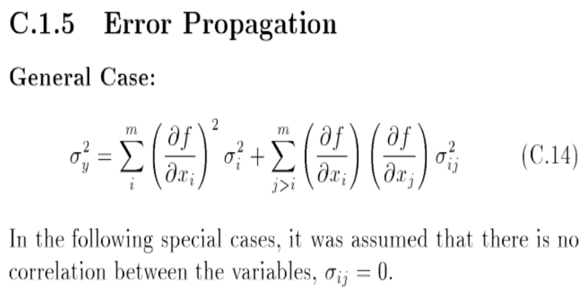


Fig. 5. Plot of All Damping. Volt vs Time.
Used Eq.10 for fit

From Fig 4-5, we see that the χ^2/ndf values are sufficient. Therefore, we have concluded that the fit functions and the data we have fitted are consistent with the theory. From the fits, we have obtained every parameter we have needed. Using the parameters and their errors, we have calculated the constant G and the error according to the following equation.



C.1.5 Error Propagation

General Case:

$$\sigma_y^2 = \sum_i^m \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2 + \sum_{j>i}^m \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)\sigma_{ij}^2 \qquad \text{(C.14)}$$

In the following special cases, it was assumed that there is no correlation between the variables, $\sigma_{ij} = 0$.

Fig. 6. Error Propagation Formula.[4]

We have found the parameters κ, $\Delta\theta$ and their uncertainties. Therefore we have calculated the G and its uncertainty from the Eq.4. Lastly, we have calculated the mass of the Earth by the Eq.16. We have taken $g = 9.81\frac{m}{s^2}$ and $R_{Earth} = 6.3781 \times 10^6 m$ to be exact.

TABLE III
CALCULATED PARAMETERS

Parameters	Value	Errors
κ	2.49 $\times 10^{-7} m^2 kgs^{-2}$	2. $\times 10^{-9} m^2 kgs^{-2}$
$\Delta\theta$	3.72 $\times 10^{-4} rad$	4. $\times 10^{-6} rad$
G	9.78 $\times 10^{-11} m^3 kg^{-1} s^{-2}$	1.18 $\times 10^{-11} m^3 kg^{-1} s^{-2}$
M_{Earth}	4.08 $\times 10^{24} kg$	5. $\times 10^{23} kg$

where, κ is the spring constant. $\Delta\theta$ is the deviation of the angle. We have found the κ value using the Eq. 8 and the resonance frequency obtained by resonance fit (See. Fig.4). We have found $\Delta\theta$ using the Eq.15, A' and A values of the damping fit and resonance fit (See. Fig.4-5).

IV. The Conclusion

We have found the Newtonian Gravitational Constant to be $G = 9.78 \times 10^{-11} \pm 1.18 \times 10^{-11} m^3 kg^{-1} s^{-2}$. The CODATA recommended value for G is $6.67430 \times 10^{-11} \pm 0.00015 \times 10^{-11} m^3 kg^{-1} s^{-2}$. We are 2.63σ away from the recommended value. The result is acceptable; however, we have too big uncertainty compared to the CODATA value.[5]

Using both CODATA value and our value, we have calculated the mass of the Earth. We have found it to be $M_{Earth} = 4.08 \times 10^{24} \pm 5.10^{23} kg$ and the CODATA value is $M_{Earth} = 5.97924 \times 10^{24} \pm 1.34379 \times 10^{20} kg$. We are 3.80σ away from the recommended value. This result is due to the fact that we have taken the gravitational acceleration and radius of Earth to be exact in our calculations.

We have managed to measure the Newtonian Gravitational Constant and we have "weighted the Earth". We have found acceptable values; however, the experiment could be done in much better environment. Shielding the gravitational force effectively is impossible with our current knowledge [6]. Therefore, we could experiment in more mass-symmetric environment which would not effect the apparatus significantly.

References

[1] *Notices of the Proceedings*. URL: https://books.google.com.tr/books?id=ZrloHemOmUEC&pg=PA355&redir_esc=y#v=onepage&q&f=false (visited on 04/17/2023).

[2] Onur Efe. *Cavendish Experiment Guide*. 1st. Boğaziçi University, 2022.

[3] *Cavendish Experiment: Weighing the World*. URL: http://ffden-2.phys.uaf.edu/211_fall2010.web.dir/Smith_Elliot/Cav_World.html (visited on 03/07/2022).

[4] E. Gülmez. *Advanced Physics Experiments*. 1st. Boğaziçi University Publications, 1999.

[5] *CODATA Value: Newtonian constant of gravitation*. URL: https://physics.nist.gov/cgi-bin/cuu/Value?bg (visited on 04/17/2023).

[6] *General Theory of Relativity: Will it survive the next decade?* URL: https://arxiv.org/pdf/gr-qc/0602016.pdf (visited on 04/17/2023).

V. Appendix

The fit has been done with Root's built-in function (See. Fig3-5). We have used root release 6.28/00 for Ubuntu22

```
{
  TTree *tree = new TTree("tree", "tree");
  TString file("cavendish.csv");
  tree->ReadFile(file);
  float t,v;
  tree->SetBranchAddress("t", &t);
  tree->SetBranchAddress("v", &v);
  float * time, * volt;
  int n = tree->GetEntries();
  time = new float[n];
  volt = new float[n];

  //Filling Arrays
  for (int i = 0; n > i; i++){
  tree->GetEntry(i);
   time[i] = t;
   volt[i] = v;
  }
  gStyle->SetOptFit(1111);

  // Sinusoidal Fit
  TCanvas *c1 = new TCanvas();
  TGraph *graph = new TGraph(n,time,volt);
  graph->Draw("A*");
  graph->GetXaxis()->SetLimits(1850, 3300);
   graph->SetTitle("Resonance
       Fit;Time(t);Volt(V)");
  TF1 *sin_fit = new TF1("sin_fit","[0] +
      [1]*sin([2]*x + [3])",2150,2900);
  sin_fit->SetParameters(-0.35, 0.15, 0.04, 0);
  sin_fit->SetParNames("A", "B", "w_0",
      "Phase");
```

```
graph->Fit(sin_fit,"R");

//Damping Fit
TCanvas *c2 = new TCanvas();
TGraph *graph2 = new TGraph(n,time,volt);
graph2->Draw("A*");
graph2->GetXaxis()->SetLimits(3000, 4150);
graph2->SetTitle("Damping
    Fit;Time(t);Volt(V)");
TF1 *damp_fit = new TF1("damp_fit","[0] +
    [1]*sin([2]*x +
    [3])*exp(-[4]*x)",3150,3950);
damp_fit->SetParameters(-0.35, 0.15, 0.04,
    0, 0.001);
damp_fit->SetParNames("A'","B'","w_d",
    "Phase'", "Damp Constant");
graph2->Fit(damp_fit,"R");

//All Data
 TCanvas *c3 = new TCanvas();
TGraph *graph3 = new TGraph(n,time,volt);
graph3->Draw("A*");
graph3->GetXaxis()->SetLimits(0, 4200);
graph3->SetTitle("All Data;Time(t);Volt(V)");

 //Calculatin G
float inertia = 1.43e-4;
float b = 0.0461025;
float r = 0.066653;
float M = 1.0385;
float m = 0.014573;
float w_0 = sin_fit->GetParameter(2);
float amplitude = sin_fit->GetParameter(1);
float A = sin_fit->GetParameter(0);
float A_p = damp_fit->GetParameter(0);
float length = 1.64 - 0.09;
float del_S = 0.248 - 0.223;

float k = inertia * (w_0 * w_0);
float del_theta = del_S * abs(A -
    A_p)/(4*amplitude*length);
float G = (b * b * k * del_theta)/(2*m*M*r);

//Calculating Errors
float sd_inertia = inertia*6.17e-3;
float sd_b = b*3.42e-3;
float sd_r = r*5.57e-4;
float sd_M = M*9.63e-4;
float sd_m = m*6.86e-5;
float sd_w_0 = sin_fit->GetParError(2);
float sd_amplitude = sin_fit->GetParError(1);
float sd_A = sin_fit->GetParError(0);
float sd_A_p = damp_fit->GetParError(0);
float sd_length = 0.01;
float sd_del_S = 0.003;


float sd_k = sqrt(
pow(w_0*w_0,2)*pow(sd_inertia,2) +
pow(2*inertia*w_0,2)*pow(sd_w_0,2)
);

float sd_del_theta = sqrt(
pow(abs(A-A_p)/(4*length*amplitude),2)*pow(sd_del_S,2)
    +
pow(del_S*abs(A-A_p)/(4*length*length*amplitude),2)*pow(sd_length,2)
    +
pow(del_S*abs(A-A_p)/(4*length*amplitude*amplitude),2)*pow(sd_amplitude,2)
    +
pow(del_S/(4*length*amplitude),2)*pow(sd_A,2)
    +
pow(del_S/(4*length*amplitude),2)*pow(sd_A_p,2)
);

float sd_G = sqrt(
pow((b*k*del_theta)/(m*M*r),2)*pow(sd_b,2) +
 pow((b*b*del_theta)/(2*m*M*r),2)*pow(sd_k,2)
    +
 pow((b*b*k)/(2*m*M*r),2)*pow(sd_del_theta,2)
    +
 pow((b*b*k*del_theta)/(2*m*m*M*r),2)*pow(sd_m,2)
    +
 pow((b*b*k*del_theta)/(2*m*M*M*r),2)*pow(sd_M,2)
    +
 pow((b*b*k*del_theta)/(2*m*M*r*r),2)*pow(sd_r,2)
);

//Calculating Mass of Earth
float Earth_R = 6.3781e06;
float g = 9.81;
float Earth_M = g*Earth_R*Earth_R/G;

//Calculating Error of Mass of Earth
    (Assuming R and g are exact)
sd_Earth_M = sqrt(
pow((g*Earth_R*Earth_R)/(G*G),2)*pow(sd_G,2)
);

cout << "Kappa: "<< k << " +- "<< sd_k <<
    endl;
cout << "Delta Theta: " << del_theta << " +-
    " << sd_del_theta << endl;
cout << "Gravitational Constant: " << G << "
    +- " << sd_G << endl;
cout << "Mass of Earth: " << Earth_M << " +-
    " << sd_Earth_M << endl;

 //Creating Files
 c1->Print("resonance.pdf");
 c2->Print("damp.pdf");
 c3->Print("all.pdf");
}
```