Diffraction of Matter Waves

Burak Kara
Boğaziçi University • Physics Department

***Abstract*—In 1924, Louid de Broglie postulated that matter exhibits wave properties with a wavelength depending on its momentum and mass. Such waves are called "Matter Waves". Despite the theoretical benefits of the matter waves, the wave properties of matter can be used for sensitive measurements. One of the techniques of sensitive measurements is the electron diffraction. In this experiment, we used electron diffraction to measure the crystal spacing of a graphite. We found 2 different spacings which are d_{10} = 0.22 $\pm$ 0.02 nm with 0.35σ away from the true value 0.213 $\pm$ 0.001 nm , and d_{11} = 0.11 $\pm$ 0.01 nm with 1.3σ away from the true value 0.123 $\pm$ 0.001 nm.**

I. Introduction

Christiaan Huygens proposed his well-known principle "Huygens Principle" In 1678. The principle proposes that the light acts as a wave[1]. However, Isaac Newton took another approach. He published his treatise "Optikcs" in 1704. In his treatise, he proposed that light is a particle with a small mass[2]. This opposition was solved in favor of Huygen by the experimental results of Thomas Young's interference experiment [3] and François Arago's detection of the Poisson spot experiment[4]. However, in 1905, Einstein proposed that light acts like a particle by photoelectric effect [5]. The photoelectric effect is also validated by experimental results. Therefore, these experimental and theoretical results lead to the Wave-Particle duality of light. Later, it was realized by Louis de Broglie that the wave-particle duality is not a special behavior of light. In 1924, he introduced his theory of electron waves[6]. The theory states that all matter exhibits wave-like properties depending on their momentum. The de Broglie's theory is also validated by experiments, which lead to the wave-particle duality.

This duality is one of the crucial parts of Quantum Mechanics. In this experiment, we will focus on the wave property of matter which is known as matter waves. Despite the theoretical benefits of the matter waves, it is also used for highly sensitive measurements. These measurement techniques include atom interferometers [7], neutron diffraction [8], and electron diffraction. In this experiment, we will conduct the electron diffraction method to measure the crystal structure of graphite. The graphite has two different crystal spacing, for this reason, there should be two diffraction patterns. We have successfully observed two different maxima and calculated the different spacings of the graphite [9]. The experiment aims to understand the matter waves and their benefits in measurements.

II. Theory

In this section, we will give the theoretical background of the experiment we have conducted.

de Broglie considered the following properties of photon:

Energy of a photon:

$$E = h\nu, E = pc$$

Speed of Light:

$$c = \lambda\nu$$

Putting Together:

$$E = \frac{hc}{\lambda} = pc$$

de Borglie Wavelength:

$$\lambda_B = \frac{h}{p} \tag{1}$$

where h is the Planck constant, ν is the frequency, λ is the wavelength, and p is the momentum.

As de Broglie states, the electrons exhibit wave-like properties depending on their momentum. Therefore, the electrons should exhibit a diffraction pattern which is originally a behavior of waves. We aim to obtain a diffraction pattern using a polycrystalline graphite foil.

The electron can gain speed with a potential difference by the following relation:

$$qV = \frac{p^2}{2m} \tag{3}$$

where q is the charge of the electron, V is the voltage, and m is the mass of the electron.

Also, we know from equation 1 that the momentum is related to the wavelength of the electron. Isolating the momentum from Eq.3 and inserting to the Eq.1 yields:

Wavelength:

$$\lambda_B = \frac{h}{\sqrt{2qmV}} \tag{4}$$

Notice that due to the relativistic effects, the formula above is not valid for high speeds. We can also derive the relativistic form of the formulation as follows:

Relativistic Momentum:

$$E = \sqrt{E_0^2 + c^2p^2} \Longrightarrow p = \frac{\sqrt{E^2 - E_0^2}}{c}$$

Putting to the Equation 1:

$$\lambda_B = \frac{hc}{\sqrt{E^2 - E_0^2}}$$

Relativistic Wavelength:

$$E = E_0 + E_k = m_e c^2 + qV$$
$$\Longrightarrow \lambda_B = \frac{hc}{\sqrt{2qVmc^2 + q^2V^2}} \quad (5)$$

where E is the total energy, E_0 is the rest energy, and E_k is the kinetic energy.

From equations 4 and 5, we have obtained the wavelength relation of the electron depending on the voltage we apply. We placed a polycrystalline graphite foil on the path of electrons. The spacing between latices of the graphite leads to a diffraction pattern. The relation obeys Bragg's Law [9].

The Bragg's Law:

$$2d\sin\theta = n\lambda \quad (6)$$

where d is the crystal spacing, n is the order.

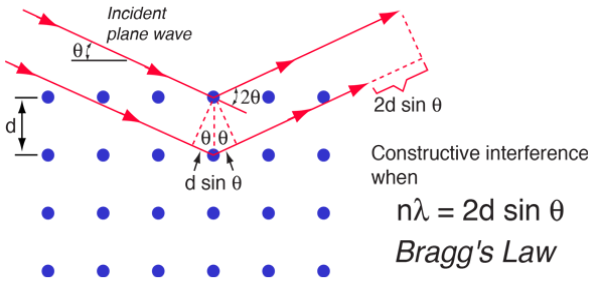

Fig. 1. Visualization of the Bragg's Law.
As can be seen from the figure, there is a $2d\sin\theta$ path difference.
The diffraction leads to a 2θ angle shift (from the original trajectory.)[10].

n = 1 for our configuration[9]. Combining equation 4 and 5 with 6 yields to:

Non Relativistic Relation:

$$\frac{1}{\sqrt{V}} = \frac{d\sqrt{8qm}\sin\theta}{h} \quad (7)$$

Relativistic Relation:

$$d\sin\theta = \frac{hc}{2\sqrt{2qVmc^2 + q^2V^2}} \quad (8)$$

For both cases, we can obtain the crystal spacing experimentally by obtaining the $\sin\theta$ value for various voltage differences. Fitting a line to the obtained results and calculating the slope will yield the spacing of the crystal scaled by a constant.

Even if we have a relativistic approach which will give a better result, the calculations are cumbersome. Now, we want to find out if the relativistic effect significantly affects the measurements or not.

Relative Error:

$$100 \cdot \frac{\left|\frac{1}{\sqrt{8qmV}} - \frac{c}{2\sqrt{2qVmc^2 + q^2V^2}}\right|}{\frac{c}{2\sqrt{2qVmc^2 + q^2V^2}}}$$

$$2400\text{V} \leq \text{Applied Volt} \leq 5000\text{V}$$
$$0.12\% \leq \text{Relative Error} \leq 0.24\%$$

$$h = 6.62607015 \times 10^{-34} J \cdot Hz^{-1}[11]$$
$$m = 9.1093837015 \times 10^{-31} kg[12]$$
$$q = 1.602176634 \times 10^{-19} C[13]$$
$$c = 299792458 m \cdot s^{-1}[14]$$

Since the relative error is not significant, we will calculate the crystal spacing using the non-relativistic approach.

III. The Experimental Setup & Method

For the experiment, we used the following apparatus:

- Electron Diffraction Tube.
- HV Power Supply.
- Ammeter.
- Plastic tape measure.

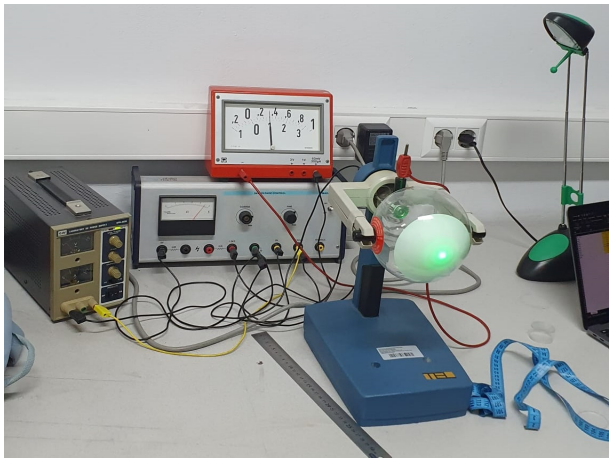

Fig. 2. Apparatus.

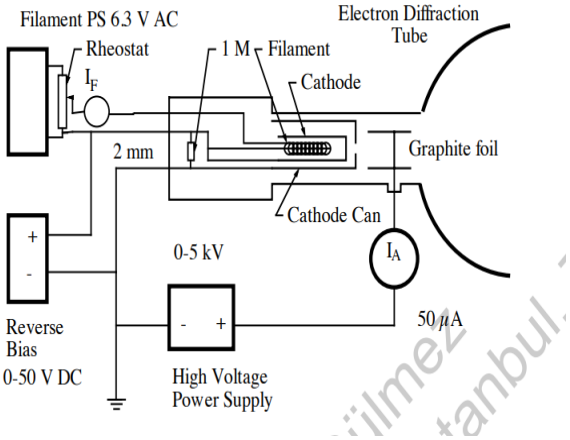

Fig. 3. Connection diagram [9].

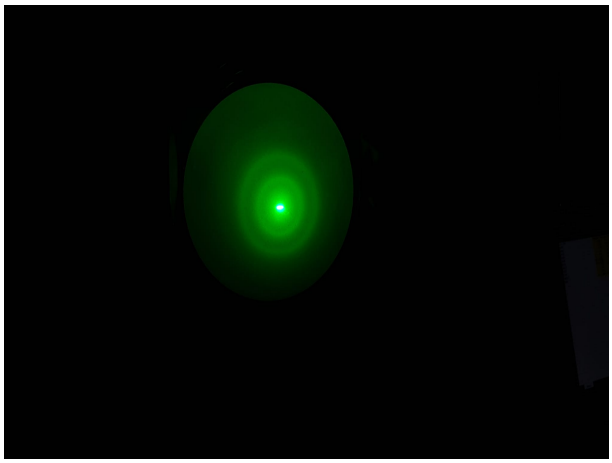

Fig. 5. Observed Diffraction Pattern on the Fluorescent Screen. The dot at the center is the central maxima. The inner and outer circles are due to the diffraction from two different spacings.

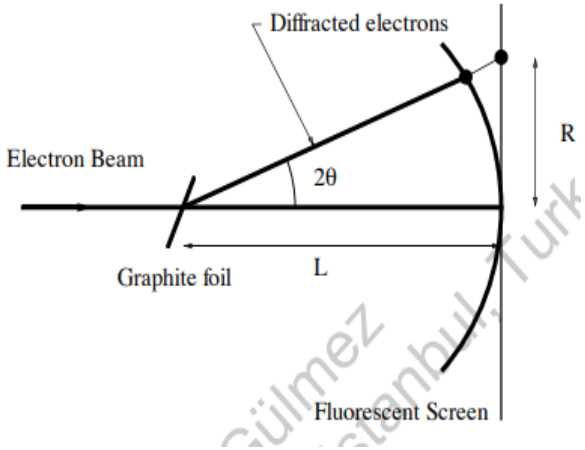

Fig. 4. Geometry of the electron diffraction tube. $R = 4.3 \pm 0.1cm$, $L = 14.0 \pm 0.3cm$. [9]

1) The connections are satisfied according to Fig3.
2) The electrons are produced via heating the filament.
3) Electrons are accelerated with HV.
4) The scattered electrons formed a ring-shaped pattern. (See Fig5.
5) The arc of the circle is measured with a plastic tape measure. The measurements are done from the X, and Y axes and for both inner radius and outer radius. (See. TableI and TableII.)

IV. The Data

The measurement we have done is listed in this section. The measurements of the inner(first) ring are listed in TableI, and the outer(second) ring are listed in TableII.

TABLE I
First Ring Measurements

Voltage(kV)	X_{in}(cm)	X_{out}(cm)	Y_{in}(cm)	Y_{out}(cm)
5.0 ± 0.1	2.1 ± 0.1	2.7 ± 0.1	2.0± 0.1	2.8± 0.1
4.8 ± 0.1	2.1 ± 0.1	2.7 ± 0.1	2.2± 0.1	2.5± 0.1
4.5 ± 0.1	2.2 ± 0.1	2.6 ± 0.1	2.1± 0.1	2.8± 0.1
4.2 ± 0.1	2.2 ± 0.1	2.7 ± 0.1	2.3± 0.1	2.7± 0.1
3.9 ± 0.1	2.2 ± 0.1	3.0 ± 0.1	2.4± 0.1	3.1± 0.1
3.6 ± 0.1	2.4 ± 0.1	3.2 ± 0.1	2.5± 0.1	3.1± 0.1
3.4 ± 0.1	2.6 ± 0.1	3.5 ± 0.1	2.5± 0.1	3.1± 0.1
2.9 ± 0.1	2.5 ± 0.1	3.4 ± 0.1	2.4± 0.1	3.5± 0.1
2.7 ± 0.1	2.8 ± 0.1	3.4 ± 0.1	2.8± 0.1	3.5± 0.1
2.4 ± 0.1	2.9 ± 0.1	3.7 ± 0.1	2.8± 0.1	3.6± 0.1

TABLE II
Second Ring Measurements

Voltage(kV)	X_{in}(cm)	X_{out}(cm)	Y_{in}(cm)	Y_{out}(cm)
5.0 ± 0.1	3.9 ± 0.1	4.5 ± 0.1	3.8 ± 0.1	4.5 ± 0.1
4.8 ± 0.1	3.8 ± 0.1	4.3 ± 0.1	3.8 ± 0.1	4.3 ± 0.1
4.5 ± 0.1	3.8 ± 0.1	4.4 ± 0.1	3.8 ± 0.1	4.4 ± 0.1
4.2 ± 0.1	4.0 ± 0.1	4.6 ± 0.1	4.2 ± 0.1	4.7 ± 0.1
3.9 ± 0.1	4.2 ± 0.1	5.1 ± 0.1	4.3 ± 0.1	5.1 ± 0.1
3.6 ± 0.1	4.5 ± 0.1	5.2 ± 0.1	4.4 ± 0.1	5.3 ± 0.1
3.4 ± 0.1	4.6 ± 0.1	5.5 ± 0.1	4.5 ± 0.1	5.4 ± 0.1
2.9 ± 0.1	5.0 ± 0.1	6.0 ± 0.1	5.0 ± 0.1	6.0 ± 0.1
2.7 ± 0.1	5.2 ± 0.1	6.0 ± 0.1	5.4 ± 0.1	6.2 ± 0.1
2.4 ± 0.1	5.5 ± 0.1	6.3 ± 0.1	5.4 ± 0.1	6.5 ± 0.1

V. The Analysis and Results

As Eq.4 suggests, we can obtain the de Broglie wavelength of the electrons directly using the voltage difference. Using TableI or TableII, we have found the wavelengths as:

TABLE III
DE BROGLIE WAVELENGTS

Voltage(V)	λ (m)
5000 ± 100	$1.72 \times 10^{-11} \pm 0.02 \times 10^{-11}$
4800 ± 100	$1.76 \times 10^{-11} \pm 0.02 \times 10^{-11}$
4500 ± 100	$1.82 \times 10^{-11} \pm 0.02 \times 10^{-11}$
4200 ± 100	$1.90 \times 10^{-11} \pm 0.02 \times 10^{-11}$
3900 ± 100	$1.96 \times 10^{-11} \pm 0.02 \times 10^{-11}$
3600 ± 100	$2.04 \times 10^{-11} \pm 0.03 \times 10^{-11}$
3400 ± 100	$2.10 \times 10^{-11} \pm 0.03 \times 10^{-11}$
2900 ± 100	$2.28 \times 10^{-11} \pm 0.04 \times 10^{-11}$
2700 ± 100	$2.36 \times 10^{-11} \pm 0.04 \times 10^{-11}$
2400 ± 100	$2.50 \times 10^{-11} \pm 0.05 \times 10^{-11}$

As can be seen from TableIII, the wavelengths are smaller than a nanometer. In addition to the wavelengths, we can calculate the θ values (See. Fig4). We have used Table I and Table II to obtain the θ value. The θ value can be calculated with the following equations:

Half of the arc:

$$arc = \frac{X_{in} + X_{out} + Y_{in} + Y_{out}}{8} \tag{9}$$

Theta Value:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{R \cdot \sin(arc/R)}{L - R + R \cdot \cos(arc/R)}\right) \tag{10}$$

As Eq.7 suggests, we can extract the crystal spacing with a line fit. Therefore, we need to obtain the error formulas for both $V^{-1/2}$ and $\sin\theta$. We considered the following equation for error propagation, assuming there is not any correlation between variables.

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2 + \cdots} \tag{11}$$

For the exact formula of $\sigma_{V^{-1/2}}$ and $\sigma_{\sin\theta}$ see VIIAppendix .

We have used the ROOT Framework for line fit (See. VII.Appendix). Using the data we measured, Eq.7, Eq. 8, Eq.9, and the error propagation formula; we obtained the following plots:

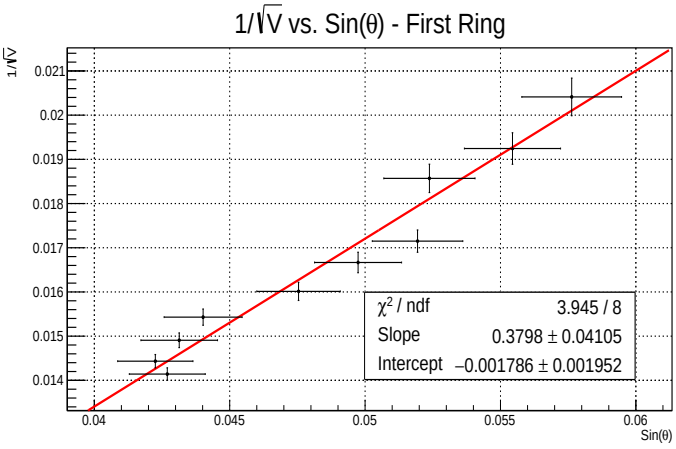

Fig. 6. First Ring $V^{-1/2}$ vs $\sin\theta$

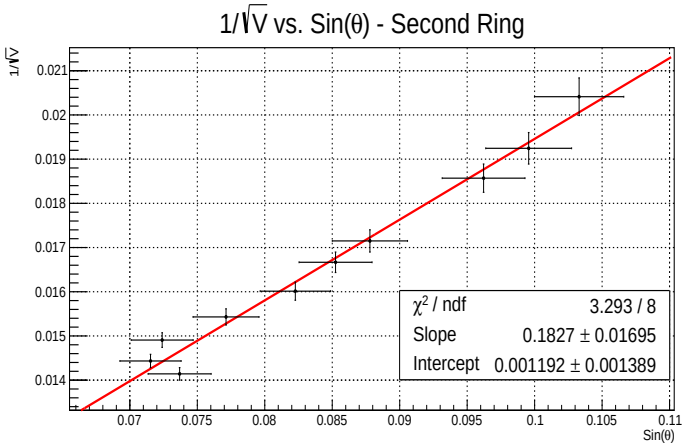

Fig. 7. Second Ring $V^{-1/2}$ vs $\sin\theta$

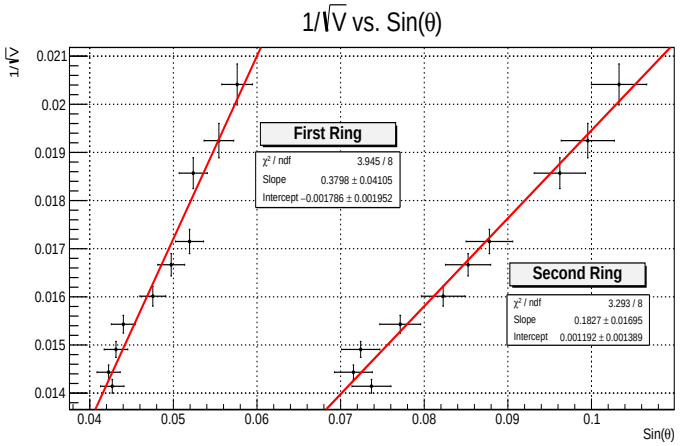

Fig. 8. Both Rings $V^{-1/2}$ vs $\sin\theta$

We see that the χ^2 values are reasonable for both fits. The slope of the line is not exactly what we wanted to obtain. As Eq.7 suggests, to extract the crystal spacing, we need to divide the slope by $\frac{\sqrt{8qm}}{h}$. Which yields to:

d_{10} = 0.22 ± 0.02 nm, 0.35σ away from the true value 0.213 ± 0.001 nm with 3.3% relative error.

d_{11} = 0.11 ± 0.01 nm , 1.3σ away from the true value 0.123 ± 0.001 nm with 11% relative error.

VI. THE CONCLUSION

From the experiment that we conducted, we have found the crystal spacing of the graphite as:

d_{10} = 0.22 ± 0.02 nm, 0.35σ away from the true value 0.213 ± 0.001 nm with 3.3% relative error.

d_{11} = 0.11 ± 0.01 nm , 1.3σ away from the true value 0.123 ± 0.001 nm with 11% relative error. Therefore, we concluded that the experiment was partly successful.

We are 0.35 and 1.3 σ away from the true value for the first and second rings respectively. The main difference between these two rings is that the measurements of the outer ring have higher uncertainty (which we did not add to the calculations) because it is more faded out. Other than this issue we have observed that measuring the rings with a plastic tape measure in a dark room is unsuitable for this experiment. More sophisticated measurement devices designed for this experiment could provide a better result.

Observing a ring or a sphere is 'natural' for physicists; however, this symmetry needs further explanations. Since we shoot the electrons from a single source with a straight line, we cannot explain this symmetry with them. Therefore, the effect of the circular symmetry should come from the crystal. The polycrystal that we used is not perfectly ordered. Since the crystal has more than one layer, the electrons pass through a series of layers, producing a superposition of single diffraction patterns with various orientations. Therefore, we see a ring instead of other shapes.

According to Bragg's Law, there should exist higher-order rings. However, in our experiment, we could not detect them. The reason is that the higher-order diffractions can occur on a larger screen. One can show this by using Eq.10 and Eq.6, to solve for R. Since the most contracted ring is the first ring with the highest voltage, we calculated R with $\lambda = 1.72^{-11}m$, and $d = 0.213nm$ (See. TableIII). For this configuration $R = 29cm$ which exceeds the radius of the diffraction tube. The same argument can be used to show that the second ring is not a second-order diffraction, but it is a different crystal spacing.

REFERENCES

[1] *Huygens' Principle*. URL: https://www.mathpages.com/home/kmath242/kmath242.htm (visited on 11/24/2023).

[2] Sir Isaac Newton. *Opticks: or, A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light*. 1st. 1704.

[3] *ThomasYoung and the wave theory of light*. URL: https://web.archive.org/web/20160915001601/https://www.bibnum.education.fr/sites/default/files/71-young-analysis.pdf (visited on 11/24/2023).

[4] *Poisson-Arago dot.* URL: https://www.animations.physics.unsw.edu.au/jw/light/Poisson-Arago-dot.html (visited on 11/24/2023).

[5] *The Nobel Prize in Physics 1921*. URL: https://www.nobelprize.org/prizes/physics/1921/summary/ (visited on 11/24/2023).

[6] *On the Theory of Quanta*. URL: https://fondationlouisdebroglie.org/LDB-oeuvres/De_Broglie_Kracklauer.pdf (visited on 11/24/2023).

[7] *Gravitational Wave Detection with Atom Interferometry*. URL: https://arxiv.org/abs/0712.1250 (visited on 11/24/2023).

[8] *Measurement of residual stress in materials using neutrons*. URL: https://www-pub.iaea.org/MTCD/publications/PDF/te_1457_web.pdf (visited on 11/24/2023).

[9] E. Gülmez. *Advanced Physics Experiments*. 1st. Boğaziçi University Publications, 1999.

[10] *Bragg's Law*. URL: http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/bragg.html (visited on 11/24/2023).

[11] *CODATA Value: Planck constant*. URL: https://physics.nist.gov/cgi-bin/cuu/Value?h (visited on 11/24/2023).

[12] *CODATA Value: electron mass*. URL: https://physics.nist.gov/cgi-bin/cuu/Value?me%7Csearch_for=electron+mass (visited on 11/24/2023).

[13] *CODATA Value: elementary charge*. URL: https://physics.nist.gov/cgi-bin/cuu/Value?e (visited on 11/24/2023).

[14] *CODATA Value: speed of light in vacuum*. URL: https://physics.nist.gov/cgi-bin/cuu/Value?c (visited on 11/24/2023).

VII. Appendix

Error Propagation Formula:

$$\frac{\partial\theta}{\partial R} = \frac{R\sin\left(\frac{arc}{R}\right)\left(2L - arc\sin\left(\frac{arc}{R}\right)\right)}{R\left(R\cos\left(\frac{arc}{R}\right) + 2L - 2R\right)^2}$$

$$+\frac{2\cdot arc\cdot(R-L)\cos\left(\frac{arc}{R}\right) - arc\cdot R\cos^2\left(\frac{arc}{R}\right)}{R\left(R\cos\left(\frac{arc}{R}\right) + 2L - 2R\right)^2}$$

$$\frac{\partial\theta}{\partial L} = \frac{-2\cdot R\cdot\sin\left(\frac{arc}{R}\right)}{\left(R\cdot\cos\left(\frac{arc}{R}\right) + 2\cdot L - 2\cdot R\right)^2}$$

$$\frac{\partial\theta}{\partial arc} = \frac{2\cdot(L-R)\cdot\cos\left(\frac{arc}{R}\right) + R\cdot\sin^2\left(\frac{arc}{R}\right)}{\left(R\cdot\cos\left(\frac{arc}{R}\right) + 2\cdot L - 2\cdot R\right)^2}$$

$$+\frac{R\cdot\cos^2\left(\frac{arc}{R}\right)}{\left(R\cdot\cos\left(\frac{arc}{R}\right) + 2\cdot L - 2\cdot R\right)^2}$$

$$\sigma_\theta = \sqrt{\left(\frac{\partial\theta}{\partial R}\cdot\sigma_R\right)^2 + \left(\frac{\partial\theta}{\partial L}\cdot\sigma_L\right)^2 + \left(\frac{\partial\theta}{\partial arc}\cdot\sigma_{arc}\right)^2}$$

$$arc = \frac{X_{in} + X_{out} + Y_{in} + Y_{out}}{8}$$

$$R = 4.3\times10^{-2}m$$

$$L = 14.0\times10^{-2}m$$

$$\sigma_V = 100V$$

$$\sigma_R = 0.1\times10^{-2}m$$

$$\sigma_L = 0.3\times10^{-2}m$$

$$\sigma_{measure} = 0.1\times10^{-2}m$$

$$\sigma_{arc} = \frac{\sigma_{measure}}{8} = 0.01\times10^{-2}m$$

With the error calculations above,

we obtained the following error propagations:

$$\sigma_{\sin\theta} = \cos\theta\cdot\sigma_\theta$$

$$\sigma_{V^{-1/2}} = V^{\frac{-3}{2}}\cdot\sigma_V$$

We have used ROOT Version: 6.28/04 on Ubuntu 22.04.2 LTS machine for the following macros.

```
#include <iostream>
#include <cmath>
#include <vector>
using namespace std;

// Constants:
double R = 4.3 * 1e-2;
double L = 14.0 * 1e-2;
double h = 6.62607015 * 1e-34; // exact
double q = 1.602176634 * 1e-19; // exact
double m = 9.1093837015 * 1e-31;

double V_error = 100;
double R_error = 0.1 * 1e-2;
double L_error = 0.3 * 1e-2;
double measure_error = 0.1 * 1e-2;

double significant_round(double value, int
    digits)
{
   if (value == 0.0)
      return 0.0;
   double factor = pow(10.0, digits -
       ceil(log10(fabs(value))));
   return round(value * factor) / factor;
}

double calculateTheta(double a)
{
   double numerator = R * sin(a / R);
   double denominator = L - R + R * cos(a / R);
   double result = 0.5 * atan2(numerator,
       denominator);
   return result;
}

double calculateThetaError(double a)
{
   double numerator_R = R * sin(a / R) * (2 *
       L - a * sin(a / R)) + 2 * a * (R - L) *
       cos(a / R) - a * R * pow(cos(a / R), 2);
   double denominator_R = R * pow(R * cos(a /
       R) + 2 * L - 2 * R, 2);
   double partial_R = numerator_R /
       denominator_R;

   double numerator_L = -2 * R * sin(a / R);
   double denominator_L = pow(R * cos(a / R) +
       2 * L - 2 * R, 2);
   double partial_L = numerator_L /
       denominator_L;

   double numerator_a = 2 * (L - R) * cos(a /
       R) + R * pow(sin(a / R), 2) + R *
       pow(cos(a / R), 2);
   double denominator_a = pow(R * cos(a / R) +
       2 * L - 2 * R, 2);
   double partial_a = numerator_a /
       denominator_a;

   double a_error = measure_error / 8;
   double error = sqrt(pow(partial_R *
       R_error, 2) + pow(partial_L * L_error,
       2) + pow(partial_a * a_error, 2));

   return error;
}

double calculateWavelength(double V)
{
   return h / sqrt(2 * q * m * V);
}

double calculateWavelengthError(double V)
{
   double constant = h / sqrt(2 * q * m);
```

```
    return sqrt(pow(constant * -0.5 * pow(V,
        -1.5) * V_error, 2));
}

void print_results(double true_value, double
    value, double error)
{
    double relative_error = abs(true_value -
        value) * 100 / true_value;
    double sigma = abs(true_value - value) /
        error;
    cout << value << " +- " << error << " nm ";
    cout << ", True Value = " << true_value <<
        " nm ";
    cout << ", Sigma = " <<
        significant_round(sigma, 2);
    cout << ", Relative Error = " <<
        significant_round(relative_error, 2) <<
        "%" << endl;
}

void matterwave()
{
    vector<double> V_vec = {5000, 4800, 4500,
        4200, 3900, 3600, 3400, 2900, 2700,
        2400};
    vector<vector<double>> Arc_in_matrix = {
        {0.021, 0.027, 0.020, 0.028},
        {0.021, 0.027, 0.022, 0.025},
        {0.022, 0.026, 0.021, 0.028},
        {0.022, 0.027, 0.023, 0.027},
        {0.022, 0.030, 0.024, 0.031},
        {0.024, 0.032, 0.025, 0.031},
        {0.026, 0.035, 0.025, 0.031},
        {0.025, 0.034, 0.024, 0.035},
        {0.028, 0.034, 0.028, 0.035},
        {0.029, 0.037, 0.028, 0.036}};

    vector<vector<double>> Arc_out_matrix = {
        {0.039, 0.045, 0.038, 0.045},
        {0.038, 0.043, 0.038, 0.043},
        {0.038, 0.044, 0.038, 0.044},
        {0.040, 0.046, 0.042, 0.047},
        {0.042, 0.051, 0.043, 0.051},
        {0.045, 0.052, 0.044, 0.053},
        {0.046, 0.055, 0.045, 0.054},
        {0.050, 0.060, 0.050, 0.060},
        {0.052, 0.060, 0.054, 0.062},
        {0.055, 0.063, 0.054, 0.065}};

    gStyle->SetOptFit(1);
    gErrorIgnoreLevel = kFatal;
    TF1 *fitFcn = new TF1("fitFcn", "[0]*x +
        [1]");
    fitFcn->SetParName(0, "Slope");
    fitFcn->SetParName(1, "Intercept");
    auto *graph1 = new TGraphErrors();
    graph1->GetXaxis()->SetTitle("Sin(#theta)");
    graph1->GetYaxis()->SetTitle("1/#sqrt{V}");

    auto *graph2 = new TGraphErrors();
    graph2->GetXaxis()->SetTitle("Sin(#theta)");
    graph2->GetYaxis()->SetTitle("1/#sqrt{V}");

    int row_num = Arc_in_matrix.size();
    for (int i = 0; i < row_num; i++)
    {
        int ncol = Arc_out_matrix[i].size();
        double sum1 = 0;
        double sum2 = 0;
        for (int j = 0; j < ncol; j++)
        {
            sum1 += Arc_in_matrix[i][j] / ncol;
            sum2 += Arc_out_matrix[i][j] / ncol;
        }
        double theta1 = calculateTheta(sum1 / 2);
        double theta2 = calculateTheta(sum2 / 2);
        double theta1_error =
            calculateThetaError(sum1 / 2);
        double theta2_error =
            calculateThetaError(sum2 / 2);
        double wavelength =
            calculateWavelength(V_vec[i]);
        double wavelength_error =
            calculateWavelengthError(V_vec[i]);
        cout << "V = " << V_vec[i] << " +- " <<
            V_error << ", Lambda = " <<
            significant_round(wavelength, 2) <<
            " +- " <<
            significant_round(wavelength_error,
            2) << endl;
        double y_axis = 1 / (sqrt(V_vec[i]));
        double y_axis_error = wavelength_error /
            (h / sqrt(2 * q * m));
        double x1_axis_error = cos(theta1) *
            theta1_error;
        double x2_axis_error = cos(theta2) *
            theta2_error;
        graph1->SetPoint(i, sin(theta1), y_axis);
        graph1->SetPointError(i, x1_axis_error,
            y_axis_error);
        graph2->SetPoint(i, sin(theta2), y_axis);
        graph2->SetPointError(i, x2_axis_error,
            y_axis_error);
    }

    // First Ring
    TCanvas *c1 = new TCanvas("c1", "c1", 200,
        10, 600, 400);
    c1->SetGrid();
    c1->Draw();
    graph1->Fit("fitFcn", "QR");
    string title = "1/#sqrt{V} vs. Sin(#theta)
        - First Ring";
    graph1->SetTitle(title.c_str());
    graph1->Draw("A*");
    double slope = fitFcn->GetParameter(0);
    double slope_error = fitFcn->GetParError(0);
    double d = slope * h / sqrt(8 * q * m);
    double d_error = slope_error * h / sqrt(8 *
        q * m);
    cout << "d10 = ";
    print_results(0.213, significant_round(d *
        1e9, 2), significant_round(d_error *
        1e9, 2));

    // Second Ring
    TCanvas *c2 = new TCanvas("c2", "c2", 200,
        10, 600, 400);
    c2->SetGrid();
    c2->Draw();
    graph2->Fit("fitFcn", "QR");
    title = "1/#sqrt{V} vs. Sin(#theta) -
        Second Ring";
    graph2->SetTitle(title.c_str());
    graph2->Draw("A*");
    slope = fitFcn->GetParameter(0);
    slope_error = fitFcn->GetParError(0);
```

```
    d = slope * h / sqrt(8 * q * m);
    d_error = slope_error * h / sqrt(8 * q * m);
    cout << "d11 = ";
    print_results(0.123, significant_round(d *
        1e9, 2), significant_round(d_error *
        1e9, 2));
}
```