

1 Problem Setup

As input, we are given a graph $G = (V_G, E_G)$, where each vertex is a geographic position $s_i \in S^2$, and each edge (i, j) has an associated (Olivier-Ricci) curvature $R_{i,j} \in (-2, 1)$ and an associated latency $t_{i,j} \in \mathbb{R}_{\geq 0}$.

Intuitively, we want to return a surface in \mathbb{R}^3 that is the graph of a function $\rho : S^2 \rightarrow \mathbb{R}_{>0}$ whose geodesics $g_{i,j}$ between s_i and s_j (and their missing ρ -coordinates) have length $\phi_{i,j}$ that is in a linear relationship with the latency.

The strategy to realize this intuition is to create a mesh $M = (V_M, E_M)$ supported on a subset of S^2 that contains our input positions. We use a standard [half-edge](#) setup, so that E_M is a set of ordered pairs (edges are directed). Let P be the support. Then for each $s_i \in P$, we want to assign a $\rho_i \in \mathbb{R}_{>0}$, which in turn gives a point $v_i = (s_i, \rho_i) \in V$. This setup is made explicit in `mesh/sphere.py`.

A similar setup is found in `mesh/rectangle.py`, where we use $[0, 1]^2$ instead of S^2 . In general, this setup just requires that the position of any mesh vertex is controlled by a single scalar parameter.

2 Objective/Loss Functions

To enforce that the mesh approximates our desired surface, we define the objective functions

$$\begin{aligned}\mathcal{L}_{\text{geodesic}}(M) &\triangleq \sum_{e \in E_G} (\text{least squares residual of edge } e)^2, \\ \mathcal{L}_{\text{smooth}}(M) &\triangleq -\rho^\top L_C \rho, \\ \mathcal{L}_{\text{curvature}}(M) &\triangleq \sum_{\substack{v \in V_M \\ v \text{ close to } (i, j)}} (\kappa(v) - R_{i,j})^2, \\ \mathcal{L}(M) &\triangleq \lambda_{\text{geodesic}} \mathcal{L}_{\text{geodesic}}(M) + \lambda_{\text{curvature}} \mathcal{L}_{\text{curvature}}(M) + \lambda_{\text{smooth}} \mathcal{L}_{\text{smooth}}(M),\end{aligned}$$

where the λ 's are tunable hyperparameters. The other variables will be defined in the upcoming subsections. Our goal is then to minimize $\mathcal{L}(M)$.

Note that the loss functions (particularly the geodesic and total ones) also have a dependence on the measured latencies. We omit that as a written parameter because they are treated as fixed (we are really optimizing over the manifold, not over the measured latencies).

2.1 Laplacian

Some mesh notation first. If i and j are two indices vertices for which $(v_i, v_j) \in E_M$, let $c(i, j)$ be the index such that $v_i \rightarrow v_j \rightarrow v_{c(i,j)}$ traces a triangle counterclockwise. Note that this index exists and is unique assuming we have a mesh without boundary. On a mesh with boundary, if no $c(i, j)$ exists, then the half-edge (v_i, v_j) lies on the boundary.

We also write ∂M to represent the boundary of our mesh. Abusing notation, we can write things like $v_i \in \partial M$ or $(v_i, v_j) \in \partial M$.

We define the following variables:

$N_{i,j}$	Outward normal of triangle $v_i \rightarrow v_j \rightarrow v_{c(i,j)}$
$A_{i,j}$	Area of triangle $v_i \rightarrow v_j \rightarrow v_{c(i,j)}$
$D_{i,j}$	Vertex triangle areas; diagonal
$\theta_{i,j}$	Measure of $\angle v_i v_{c(i,j)} v_j$
L_C^N	Cotangent operator with zero-Neumann boundary condition
L_C^D	Cotangent operator with zero-Dirichlet boundary condition
L_C	Cotangent operator in the no-boundary case; sparse

2.1.1 Forward Computation

We have the following (standard) definition of the Laplace-Beltrami operator on a mesh:

$$\begin{aligned}N_{i,j} &= \begin{pmatrix} v_i - v_{c(i,j)} \\ v_j - v_{c(i,j)} \end{pmatrix} \times \begin{pmatrix} v_i - v_{c(i,j)} \\ v_j - v_{c(i,j)} \end{pmatrix}, \\ A_{i,j} &= \frac{1}{2} \|N_{i,j}\|_2,\end{aligned}$$

$$\begin{aligned}
D_{i,j} &= \begin{cases} \frac{1}{3} \sum_{\substack{k \\ (v_i, v_k) \in E_M}} A_{i,k} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \\
\cot(\theta_{i,j}) &= \frac{\left(v_i - v_{c(i,j)} \right) \cdot \left(v_j - v_{c(i,j)} \right)}{2A_{i,j}}, \\
\left(L_C^N \right)_{i,j} &= \begin{cases} \frac{1}{2} \cot(\theta_{i,j}) & \text{if } (v_i, v_j) \in \partial M, \\ \frac{1}{2} \cot(\theta_{j,i}) & \text{if } (v_j, v_i) \in \partial M, \\ \frac{1}{2} \left(\cot(\theta_{i,j}) + \cot(\theta_{j,i}) \right) & \text{if } (v_i, v_j) \in E_M \text{ and } (v_j, v_i) \in E_M, \\ -\frac{1}{2} \left(\sum_{\substack{k \\ (v_i, v_k) \in E_M}} \cot(\theta_{i,k}) + \sum_{\substack{k \\ (v_k, v_i) \in E_M}} \cot(\theta_{k,i}) \right) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \\
\left(L_C^D \right)_{i,j} &= \begin{cases} \frac{1}{2} \left(\cot(\theta_{i,j}) + \cot(\theta_{j,i}) \right) & \text{if } (v_i, v_j) \in E_M, v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2} \sum_{\substack{k \notin \partial M \\ (v_i, v_k) \in E_M \\ (v_k, v_i) \in E_M}} \left(\cot(\theta_{i,k}) + \cot(\theta_{k,i}) \right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Flipping our attention back to meshes without boundary, the two definitions above coincide, so we can write

$$L_C = L_C^{\text{Neumann}} = L_C^{\text{Dirichlet}}.$$

2.1.2 Reverse Computation

For the ease of notation, assume that we are using the spherical setup, so $v_\ell = \rho_\ell s_\ell$.

We compute

$$\begin{aligned}
\frac{\partial v_i}{\partial \rho_\ell} &= \begin{cases} s_i & \text{if } \ell = i, \\ 0 & \text{otherwise,} \end{cases} \\
\frac{\partial N_{i,j}}{\partial \rho_\ell} &= \begin{cases} \left(v_{c(i,j)} - v_j \right) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = i, \\ \left(v_i - v_{c(i,j)} \right) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = j, \\ (v_j - v_i) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = c(i,j), \\ 0 & \text{otherwise,} \end{cases} \\
\frac{\partial A_{i,j}}{\partial \rho_\ell} &= \frac{1}{4A_{i,j}} N_{i,j} \cdot \frac{\partial N_{i,j}}{\partial \rho_\ell}, \\
\left(\frac{\partial D}{\partial \rho_\ell} \right)_{i,j} &= \begin{cases} \frac{1}{3} \sum_{\substack{k \\ (v_i, v_k) \in E_M}} \frac{\partial A_{i,k}}{\partial \rho_\ell} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) &= \begin{cases} \frac{\left(v_j - v_{c(i,j)}\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2 \cot(\theta_{i,j}) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = i, \\ \frac{\left(v_i - v_{c(i,j)}\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2 \cot(\theta_{i,j}) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = j, \\ \frac{\left(2v_{c(i,j)} - v_i - v_j\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2 \cot(\theta_{i,j}) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = c(i,j), \\ 0 & \text{otherwise,} \end{cases} \\
\left(\frac{\partial L_C^N}{\partial \rho_\ell}\right)_{i,j} &= \begin{cases} \frac{1}{2} \frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) & \text{if } (v_i, v_j) \in \partial M, \\ \frac{1}{2} \frac{\partial}{\partial \rho_\ell} \cot(\theta_{j,i}) & \text{if } (v_j, v_i) \in \partial M, \\ \frac{1}{2} \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) + \frac{\partial}{\partial \rho_\ell} \cot(\theta_{j,i}) \right) & \text{if } (v_i, v_j) \in E_M \text{ and } (v_j, v_i) \in E_M, \\ -\frac{1}{2} \left(\sum_{\substack{k \\ (v_i, v_k) \in E_M}} \frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,k}) + \sum_{\substack{k \\ (v_k, v_i) \in E_M}} \frac{\partial}{\partial \rho_\ell} \cot(\theta_{k,i}) \right) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \\
\left(\frac{\partial L_C^D}{\partial \rho_\ell}\right)_{i,j} &= \begin{cases} \frac{1}{2} \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) + \frac{\partial}{\partial \rho_\ell} \cot(\theta_{j,i}) \right) & \text{if } (v_i, v_j) \in E_M, v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2} \sum_{\substack{k \notin \partial M \\ (v_i, v_k) \in E_M \\ (v_k, v_i) \in E_M}} \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,k}) + \frac{\partial}{\partial \rho_\ell} \cot(\theta_{k,i}) \right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

2.2 Geodesic Distance via the Heat Method

Here are the variables used for this part of the computation:

γ	Set of points in V_M
h	Mean half-edge length
δ^γ	Heat source (indicator on γ)
$u^{\gamma, N}$	Heat flow with zero-Neumann boundary condition
$u^{\gamma, D}$	Heat flow with zero-Dirichlet boundary condition
u^γ	Heat flow
$q_{i,j}^\gamma$	Intermediate value for computation
$m_{i,j}^\gamma$	Intermediate value for computation
$X_{i,j}^\gamma$	Unit vector in same direction as $\nabla u_{i,j}^\gamma$
$p_{i,j}^\gamma$	Intermediate value for computation
ϕ^γ	Vector of geodesic distances

2.2.1 Forward Computation

Say we want to find the geodesic distances to a set of points $\gamma \subseteq V_M$. Following the [Crane et al's Heat Method](#), we use the (approximate) heat flow u^γ , where

$$\begin{aligned}
h &= \text{TODO}, \\
\delta^\gamma &= \begin{cases} 1 & \text{if } v_i \in \gamma, \\ 0 & \text{if } v_i \notin \gamma, \end{cases} \\
u^{\gamma, N} &= \left(D - h^2 L_C^N \right)^{-1} \delta^\gamma, \\
u^{\gamma, D} &= \left(D - h^2 L_C^D \right)^{-1} \delta^\gamma,
\end{aligned}$$

$$\begin{aligned}
u^\gamma &= \frac{1}{2} \left(u^{\gamma, \text{N}} + u^{\gamma, \text{D}} \right), \\
q_{i,j}^\gamma &= u_i^\gamma \left(v_{c(i,j)} - v_j \right), \\
m_{i,j}^\gamma &= q_{i,j}^\gamma + q_{j,c(i,j)}^\gamma + q_{c(i,j),i}^\gamma, \\
(\nabla u^\gamma)_{i,j} &= N_{i,j} \times m_{i,j}^\gamma, \\
X_{i,j}^\gamma &= - \frac{(\nabla u^\gamma)_{i,j}}{\|(\nabla u^\gamma)_{i,j}\|_2}, \\
p_{i,j} &= \cot(\theta_{i,j}) (v_j - v_i), \\
(\nabla \cdot X^\gamma)_i &= \frac{1}{2} \sum_{\substack{k \\ (v_i, v_k) \in E_M}} \left(p_{i,k} - p_{c(i,k),i} \right) \cdot X_{i,k}^\gamma, \\
\phi^\gamma &= \left(L_C^{\text{N}} \right)^+ \cdot (\nabla \cdot X^\gamma).
\end{aligned}$$

Here, $\left(L_C^{\text{N}} \right)^+$ is the [pseudoinverse](#) of L_C^{N} (this is necessary as it is singular).

Note that we're being careful about which pieces have a dependence on γ , as we can reuse certain computations if we want to compute distances from multiple sources. We can get the pairwise distance matrix (that is, get rid of the γ dependence) from

$$\phi_{i,j} = \left(\phi^{\{v_j\}} \right)_i.$$

2.2.2 Reverse Computation

Note that $c(i, c(j, i)) = j$. This is helpful for reindexing some sums (in particular, the one for $\nabla \cdot X$).

We then have the following partial derivatives:

$$\begin{aligned}
\frac{\partial h}{\partial \rho_\ell} &= \text{TODO}, \\
\frac{\partial u^{\gamma, \text{N}}}{\partial \rho_\ell} &= - \left(D - h^2 L_C^{\text{N}} \right)^{-1} \left(\frac{\partial D}{\partial \rho_\ell} - 2h \frac{\partial h}{\partial \rho_\ell} L_C^{\text{N}} - h^2 \frac{\partial L_C}{\partial \rho_\ell} \right) u^{\gamma, \text{N}}, \\
\frac{\partial u^{\gamma, \text{D}}}{\partial \rho_\ell} &= - \left(D - h^2 L_C^{\text{D}} \right)^{-1} \left(\frac{\partial D}{\partial \rho_\ell} - 2h \frac{\partial h}{\partial \rho_\ell} L_C^{\text{D}} - h^2 \frac{\partial L_C}{\partial \rho_\ell} \right) u^{\gamma, \text{D}}, \\
\frac{\partial u^\gamma}{\partial \rho_\ell} &= \frac{1}{2} \left(\frac{\partial u^{\gamma, \text{N}}}{\partial \rho_\ell} + \frac{\partial u^{\gamma, \text{D}}}{\partial \rho_\ell} \right), \\
\frac{\partial q_{i,j}^\gamma}{\partial \rho_\ell} &= \begin{cases} \frac{\partial u_i^\gamma}{\partial \rho_\ell} \left(v_{c(i,j)} - v_j \right) - u_i^\gamma \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = j, \\ \frac{\partial u_i^\gamma}{\partial \rho_\ell} \left(v_{c(i,j)} - v_j \right) + u_i^\gamma \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = c(i, j), \\ \frac{\partial u_i^\gamma}{\partial \rho_\ell} \left(v_{c(i,j)} - v_j \right) & \text{otherwise,} \end{cases} \\
\frac{\partial m_{i,j}^\gamma}{\partial \rho_\ell} &= \frac{\partial q_{i,j}^\gamma}{\partial \rho_\ell} + \frac{\partial q_{j,c(i,j)}^\gamma}{\partial \rho_\ell} + \frac{\partial q_{c(i,j),i}^\gamma}{\partial \rho_\ell}, \\
\frac{\partial (\nabla u^\gamma)_{i,j}}{\partial \rho_\ell} &= \frac{\partial N_{i,j}}{\partial \rho_\ell} \times m_{i,j}^\gamma + N_{i,j} \times \frac{\partial m_{i,j}^\gamma}{\partial \rho_\ell}, \\
\frac{\partial X_{i,j}^\gamma}{\partial \rho_\ell} &= - \frac{1}{\|(\nabla u^\gamma)_{i,j}\|_2} \left(I - X_{i,j}^\gamma \left(X_{i,j}^\gamma \right)^\top \right) \frac{\partial (\nabla u^\gamma)_{i,j}}{\partial \rho_\ell},
\end{aligned}$$

$$\frac{\partial p_{i,j}}{\partial \rho} = \begin{cases} \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) \right) (v_j - v_i) - \cot(\theta_{i,j}) \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = i, \\ \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) \right) (v_j - v_i) + \cot(\theta_{i,j}) \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = j, \\ \left(\frac{\partial}{\partial \rho_\ell} \cot(\theta_{i,j}) \right) (v_j - v_i) & \text{if } \ell = c(i,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial(\nabla \cdot X^\gamma)_i}{\partial \rho_\ell} = \frac{1}{2} \sum_{\substack{k \\ (v_i, v_k) \in E_M}} \left(\left(\frac{\partial p_{i,k}}{\partial \rho_\ell} - \frac{\partial p_{c(i,k),i}}{\partial \rho_\ell} \right) \cdot X_{i,k}^\gamma + \left(p_{i,k} - p_{c(i,k),i} \right) \cdot \frac{\partial X_{i,k}^\gamma}{\partial \rho_\ell} \right),$$

$$\frac{\partial \phi^\gamma}{\partial \rho_\ell} = \left(L_C^N \right)^+ \left(\frac{\partial(\nabla \cdot X^\gamma)}{\partial \rho_\ell} - \frac{\partial L_C^N}{\partial \rho_\ell} \phi^\gamma \right).$$