# 1 Problem Setup

As input, we are given a graph $G = (V_G, E_G)$, where each vertex is a geographic position $s_i \in S^2$, and each edge $(i, j)$ has an associated (Olivier-Ricci) curvature $R_{i,j} \in (-2, 1)$ and an associated latency $t_{i,j} \in \mathbb{R}_{\geq 0}$.

Intuitively, we want to return a surface in $\mathbb{R}^3$ that is the graph of a function $\rho : S^2 \to \mathbb{R}_{>0}$ whose geodesics $g_{i,j}$ between $s_i$ and $s_j$ (and their missing $\rho$-coordinates) have length $\phi_{i,j}$ that is in a linear relationship with the latency.

The strategy to realize this intuition is to create a mesh $M = (V_M, E_M)$ supported on a subset of $S^2$ that contains our input positions. We use a standard half-edge setup, so that $E_M$ is a set of ordered pairs (edges are directed). Let $P$ be the support. Then for each $s_i \in P$, we want to assign a $\rho_i \in \mathbb{R}_{>0}$, which in turn gives a point $v_i = (s_i, \rho_i) \in V$. This setup is made explicit in `mesh/sphere.py`.

A similar setup is found in `mesh/rectangle.py`, where we use $[0,1]^2$ instead of $S^2$. In general, this setup just requires that the position of any mesh vertex is controlled by a single scalar parameter.

# 2 Objective/Loss Functions

To enforce that the mesh approximates our desired surface, we define the objective functions

$$\begin{aligned}
\mathcal{L}_{\text{geodesic}}(M) &\triangleq \sum_{e \in E_G} (\text{least squares residual of edge } e)^2, \\
\mathcal{L}_{\text{smooth}}(M) &\triangleq -\rho^\intercal L_C \rho, \\
\mathcal{L}_{\text{curvature}}(M) &\triangleq \sum_{\substack{v \in V_M \\ v \text{ close to } (i,j)}} \big(\kappa(v) - R_{i,j}\big)^2, \\
\mathcal{L}(M) &\triangleq \lambda_{\text{geodesic}} \mathcal{L}_{\text{geodesic}}(M) + \lambda_{\text{curvature}} \mathcal{L}_{\text{curvature}}(M) + \lambda_{\text{smooth}} \mathcal{L}_{\text{smooth}}(M),
\end{aligned}$$

where the $\lambda$'s are tunable hyperparameters. The other variables will be defined in the upcoming subsections. Our goal is then to minimize $\mathcal{L}(M)$.

Note that the loss functions (particularly the geodesic and total ones) also have a dependence on the measured latencies. We omit that as a written parameter because they are treated as fixed (we are really optimizing over the manifold, not over the measured latencies).

## 2.1 Laplacian

Some mesh notation first. If $i$ and $j$ are two indices vertices for which $\big(v_i, v_j\big) \in E_M$, let $c(i, j)$ be the index such that $v_i \to v_j \to v_{c(i,j)}$ traces a triangle counterclockwise. Note that this index exists and is unique assuming we have a mesh without boundary. On a mesh with boundary, if no $c(i, j)$ exists, then the half-edge $\big(v_i, v_j\big)$ lies on the boundary.

We also write $\partial M$ to represent the boundary of our mesh. Abusing notation, we can write things like $v_i \in \partial M$ or $\big(v_i, v_j\big) \in \partial M$.

We define the following variables:

| | |
|---|---|
| $N_{i,j}$ | Outward normal of triangle $v_i \to v_j \to v_{c(i,j)}$ |
| $A_{i,j}$ | Area of triangle $v_i \to v_j \to v_{c(i,j)}$ |
| $D_{i,j}$ | Vertex triangle areas; diagonal |
| $\theta_{i,j}$ | Measure of $\angle v_i v_{c(i,j)} v_j$ |
| $L_C^{\text{N}}$ | Cotangent operator with zero-Neumann boundary condition |
| $L_C^{\text{D}}$ | Cotangent operator with zero-Dirichlet boundary condition |
| $L_C$ | Cotangent operator in the no-boundary case; sparse |

### 2.1.1 Forward Computation

We have the following (standard) definition of the Laplace-Beltrami operator on a mesh:

$$\begin{aligned}
N_{i,j} &= \Big(v_i - v_{c(i,j)}\Big) \times \Big(v_j - v_{c(i,j)}\Big), \\
A_{i,j} &= \frac{1}{2} \big\| N_{i,j} \big\|_2,
\end{aligned}$$

$$
D_{i,j} = \begin{cases} \frac{1}{3} \sum\limits_{\substack{k \\ (v_i, v_k) \in E_M}} A_{i,k} & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\cot(\theta_{i,j}) = \frac{\left(v_i - v_{c(i,j)}\right) \cdot \left(v_j - v_{c(i,j)}\right)}{2A_{i,j}},
$$

$$
\left(L_C^{\mathrm{N}}\right)_{i,j} = \begin{cases} \frac{1}{2}\cot(\theta_{i,j}) & \text{if } (v_i, v_j) \in \partial M, \\ \frac{1}{2}\cot(\theta_{j,i}) & \text{if } (v_j, v_i) \in \partial M, \\ \frac{1}{2}\left(\cot(\theta_{i,j}) + \cot(\theta_{j,i})\right) & \text{if } (v_i, v_j) \in E_M \text{ and } (v_j, v_i) \in E_M, \\ -\frac{1}{2}\left( \sum\limits_{\substack{k \\ (v_i, v_k) \in E_M}} \cot(\theta_{i,k}) + \sum\limits_{\substack{k \\ (v_k, v_i) \in E_M}} \cot(\theta_{k,i}) \right) & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\left(L_C^{\mathrm{D}}\right)_{i,j} = \begin{cases} \frac{1}{2}\left(\cot(\theta_{i,j}) + \cot(\theta_{j,i})\right) & \text{if } (v_i, v_j) \in E_M,\ v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2} \sum\limits_{\substack{k \notin \partial M \\ (v_i, v_k) \in E_M \\ (v_k, v_i) \in E_M}} \left(\cot(\theta_{i,k}) + \cot(\theta_{k,i})\right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise}. \end{cases}
$$

Flipping our attention back to meshes without boundary, the two definitions above coincide, so we can write

$$
L_C = L_C^{\text{Neumann}} = L_C^{\text{Dirichlet}}.
$$

#### 2.1.2 Reverse Computation

For the ease of notation, assume that we are using the spherical setup, so $v_\ell = \rho_\ell s_\ell$.

We compute

$$
\frac{\partial v_i}{\partial \rho_\ell} = \begin{cases} s_i & \text{if } \ell = i, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\frac{\partial N_{i,j}}{\partial \rho_\ell} = \begin{cases} \left(v_{c(i,j)} - v_j\right) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = i, \\ \left(v_i - v_{c(i,j)}\right) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = j, \\ (v_j - v_i) \times \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = c(i,j), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\frac{\partial A_{i,j}}{\partial \rho_\ell} = \frac{1}{4A_{i,j}} N_{i,j} \cdot \frac{\partial N_{i,j}}{\partial \rho_\ell},
$$

$$
\left(\frac{\partial D}{\partial \rho_\ell}\right)_{i,j} = \begin{cases} \frac{1}{3} \sum\limits_{\substack{k \\ (v_i, v_k) \in E_M}} \frac{\partial A_{i,k}}{\partial \rho_\ell} & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}
$$

$$\frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,j}\right) = \begin{cases} \dfrac{\left(v_j - v_{c(i,j)}\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2\cot\left(\theta_{i,j}\right) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = i, \\ \dfrac{\left(v_i - v_{c(i,j)}\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2\cot\left(\theta_{i,j}\right) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = j, \\ \dfrac{\left(2v_{c(i,j)} - v_i - v_j\right) \cdot \frac{\partial v_\ell}{\partial \rho_\ell} - 2\cot\left(\theta_{i,j}\right) \frac{\partial A_{i,j}}{\partial \rho_\ell}}{2A_{i,j}} & \text{if } \ell = c(i,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\frac{\partial L_C^{\mathrm{N}}}{\partial \rho_\ell}\right)_{i,j} = \begin{cases} \frac{1}{2} \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,j}\right) & \text{if } \left(v_i, v_j\right) \in \partial M, \\ \frac{1}{2} \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{j,i}\right) & \text{if } \left(v_j, v_i\right) \in \partial M, \\ \frac{1}{2} \left(\frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,j}\right) + \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right) \in E_M \text{ and } \left(v_j, v_i\right) \in E_M, \\ -\frac{1}{2} \left( \displaystyle\sum_{\substack{k \\ \left(v_i, v_k\right) \in E_M}} \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,k}\right) + \sum_{\substack{k \\ \left(v_k, v_i\right) \in E_M}} \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{k,i}\right) \right) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\frac{\partial L_C^{\mathrm{D}}}{\partial \rho_\ell}\right)_{i,j} = \begin{cases} \frac{1}{2} \left(\frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,j}\right) + \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right) \in E_M,\ v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2} \displaystyle\sum_{\substack{k \notin \partial M \\ \left(v_i, v_k\right) \in E_M \\ \left(v_k, v_i\right) \in E_M}} \left(\frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{i,k}\right) + \frac{\partial}{\partial \rho_\ell} \cot\left(\theta_{k,i}\right)\right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise.} \end{cases}$$

### 2.2 Geodesic Distance via the Heat Method

Here are the variables used for this part of the computation:

| | |
|---|---|
| $\gamma$ | Set of points in $V_M$ |
| $h$ | Mean half-edge length |
| $\delta^\gamma$ | Heat source (indicator on $\gamma$) |
| $u^{\gamma,\mathrm{N}}$ | Heat flow with zero-Neumann boundary condition |
| $u^{\gamma,\mathrm{D}}$ | Heat flow with zero-Dirichlet boundary condition |
| $u^\gamma$ | Heat flow |
| $q_{i,j}^\gamma$ | Intermediate value for computation |
| $m_{i,j}^\gamma$ | Intermediate value for computation |
| $X_{i,j}^\gamma$ | Unit vector in same direction as $\nabla u_{i,j}^\gamma$ |
| $p_{i,j}^\gamma$ | Intermediate value for computation |
| $\tilde{\phi}^\gamma$ | Vector of offset geodesic distances |
| $\phi^\gamma$ | Vector of offset geodesic distances |

#### 2.2.1 Forward Computation

Say we want to find the geodesic distances to a set of points $\gamma \subseteq V_M$. Following the Crane et al's Heat Method, we use the (approximate) heat flow $u^\gamma$, where

$$h = \frac{1}{|E_M|} \sum_{\substack{i,j \\ \left(v_i, v_j\right) \in E_M}} \left\|v_i - v_j\right\|_2,$$

$$\delta^\gamma = \begin{cases} 1 & \text{if } v_i \in \gamma, \\ 0 & \text{if } v_i \notin \gamma, \end{cases}$$

$$u^{\gamma,\mathrm{N}} = \left(D - h^2 L_C^{\mathrm{N}}\right)^{-1} \delta^\gamma,$$

$$
\begin{aligned}
u^{\gamma,\mathrm{D}} &= \left(D - h^2 L_C^{\mathrm{D}}\right)^{-1} \delta^\gamma, \\
u^\gamma &= \frac{1}{2}\left(u^{\gamma,\mathrm{N}} + u^{\gamma,\mathrm{D}}\right), \\
q_{i,j}^\gamma &= u_i^\gamma \left(v_{c(i,j)} - v_j\right), \\
m_{i,j}^\gamma &= q_{i,j}^\gamma + q_{j,c(i,j)}^\gamma + q_{c(i,j),i}^\gamma, \\
\left(\nabla u^\gamma\right)_{i,j} &= N_{i,j} \times m_{i,j}^\gamma, \\
X_{i,j}^\gamma &= -\frac{\left(\nabla u^\gamma\right)_{i,j}}{\left\|\left(\nabla u^\gamma\right)_{i,j}\right\|_2}, \\
p_{i,j} &= \cot\left(\theta_{i,j}\right)\left(v_j - v_i\right), \\
\left(\nabla \cdot X^\gamma\right)_i &= \frac{1}{2} \sum_{\substack{k \\ \left(v_i, v_k\right) \in E_M}} \left(p_{i,k} - p_{c(i,k),i}\right) \cdot X_{i,k}^\gamma, \\
\widetilde{\phi}^\gamma &= \left(L_C^{\mathrm{N}}\right)^+ \cdot \left(\nabla \cdot X^\gamma\right), \\
\phi^\gamma &= \widetilde{\phi}^\gamma - \min\left(\widetilde{\phi}^\gamma\right).
\end{aligned}
$$

Here, $\left(L_C^{\mathrm{N}}\right)^+$ is the pseudoinverse of $L_C^{\mathrm{N}}$ (this is necessary as it is singular).

Note that we're being careful about which pieces have a dependence on $\gamma$, as we can reuse certain computations if we want to compute distances from multiple sources. We can get the pairwise distance matrix (that is, get rid of the $\gamma$ dependence) from

$$
\phi_{i,j} = \left(\phi^{\{v_j\}}\right)_i .
$$

### 2.2.2 Reverse Computation

Note that $c\big(i, c(j,i)\big) = j$. This is helpful for reindexing some sums (in particular, the one for $\nabla \cdot X$).

We then have the following partial derivatives:

$$
\begin{aligned}
\frac{\partial h}{\partial \rho_\ell} &= \frac{1}{|E_M|}\left(\sum_{\substack{k \\ \left(v_\ell, v_k\right) \in E_M}} \frac{\left(v_\ell - v_k\right)}{\left\|v_\ell - v_k\right\|_2} \cdot \frac{\partial v_\ell}{\partial \rho_\ell} + \sum_{\substack{k \\ \left(v_k, v_\ell\right) \in E_M}} \frac{\left(v_\ell - v_k\right)}{\left\|v_\ell - v_k\right\|_2} \cdot \frac{\partial v_\ell}{\partial \rho_\ell}\right), \\
\frac{\partial u^{\gamma,\mathrm{N}}}{\partial \rho_\ell} &= -\left(D - h^2 L_C^{\mathrm{N}}\right)^{-1}\left(\frac{\partial D}{\partial \rho_\ell} - 2h\frac{\partial h}{\partial \rho_\ell} L_C^{\mathrm{N}} - h^2 \frac{\partial L_C^{\mathrm{N}}}{\partial \rho_\ell}\right) u^{\gamma,\mathrm{N}}, \\
\frac{\partial u^{\gamma,\mathrm{D}}}{\partial \rho_\ell} &= -\left(D - h^2 L_C^{\mathrm{D}}\right)^{-1}\left(\frac{\partial D}{\partial \rho_\ell} - 2h\frac{\partial h}{\partial \rho_\ell} L_C^{\mathrm{D}} - h^2 \frac{\partial L_C^{\mathrm{D}}}{\partial \rho_\ell}\right) u^{\gamma,\mathrm{D}}, \\
\frac{\partial u^\gamma}{\partial \rho_\ell} &= \frac{1}{2}\left(\frac{\partial u^{\gamma,\mathrm{N}}}{\partial \rho_\ell} + \frac{\partial u^{\gamma,\mathrm{D}}}{\partial \rho_\ell}\right), \\
\frac{\partial q_{i,j}^\gamma}{\partial \rho_\ell} &= \begin{cases}
\frac{\partial u_i^\gamma}{\rho_\ell}\left(v_{c(i,j)} - v_j\right) - u_i^\gamma \frac{\partial v_\ell}{\rho_\ell} & \text{if } \ell = j, \\
\frac{\partial u_i^\gamma}{\rho_\ell}\left(v_{c(i,j)} - v_j\right) + u_i^\gamma \frac{\partial v_\ell}{\partial \rho_\ell} & \text{if } \ell = c(i,j), \\
\frac{\partial u_i^\gamma}{\rho_\ell}\left(v_{c(i,j)} - v_j\right) & \text{otherwise,}
\end{cases} \\
\frac{\partial m_{i,j}^\gamma}{\partial \rho_\ell} &= \frac{\partial q_{i,j}^\gamma}{\partial \rho_\ell} + \frac{\partial q_{j,c(i,j)}^\gamma}{\partial \rho_\ell} + \frac{\partial q_{c(i,j),i}^\gamma}{\partial \rho_\ell}, \\
\frac{\partial \left(\nabla u^\gamma\right)_{i,j}}{\partial \rho_\ell} &= \frac{\partial N_{i,j}}{\partial \rho_\ell} \times m_{i,j}^\gamma + N_{i,j} \times \frac{\partial m_{i,j}^\gamma}{\partial \rho_\ell},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial X^{\gamma}_{i,j}}{\partial \rho_\ell} &= -\frac{1}{\left\|\left(\nabla u^{\gamma}\right)_{i,j}\right\|_2}\left(I - X^{\gamma}_{i,j}\left(X^{\gamma}_{i,j}\right)^{\mathsf{T}}\right)\frac{\partial\left(\nabla u^{\gamma}\right)_{i,j}}{\partial \rho_\ell},\\
\frac{\partial p_{i,j}}{\partial \rho} &= \begin{cases}
\left(\frac{\partial}{\partial \rho_\ell}\cot\left(\theta_{i,j}\right)\right)\left(v_j - v_i\right) - \cot\left(\theta_{i,j}\right)\frac{\partial v_\ell}{\rho_\ell} & \text{if } \ell = i,\\
\left(\frac{\partial}{\partial \rho_\ell}\cot\left(\theta_{i,j}\right)\right)\left(v_j - v_i\right) + \cot\left(\theta_{i,j}\right)\frac{\partial v_\ell}{\rho_\ell} & \text{if } \ell = j,\\
\left(\frac{\partial}{\partial \rho_\ell}\cot\left(\theta_{i,j}\right)\right)\left(v_j - v_i\right) & \text{if } \ell = c(i,j),\\
0 & \text{otherwise,}
\end{cases}\\
\frac{\partial\left(\nabla \cdot X^{\gamma}\right)_i}{\partial \rho_\ell} &= \frac{1}{2}\sum_{\substack{k\\ \left(v_i, v_k\right)\in E_M}}\left(\left(\frac{\partial p_{i,k}}{\partial \rho_\ell} - \frac{\partial p_{c\left(i,k\right),i}}{\partial \rho_\ell}\right)\cdot X^{\gamma}_{i,k} + \left(p_{i,k} - p_{c\left(i,k\right),i}\right)\cdot \frac{\partial X^{\gamma}_{i,k}}{\partial \rho_\ell}\right),\\
\frac{\partial \widetilde{\phi}^{\gamma}}{\partial \rho_\ell} &= \left(L^{\mathrm{N}}_C\right)^{+}\left(\frac{\partial\left(\nabla \cdot X^{\gamma}\right)}{\partial \rho_\ell} - \frac{\partial L^{\mathrm{N}}_C}{\partial \rho_\ell}\phi^{\gamma}\right),\\
\frac{\partial \phi^{\gamma}}{\partial \rho_\ell} &= \frac{\partial \widetilde{\phi}^{\gamma}}{\partial \rho_\ell} - \left(\frac{\partial \widetilde{\phi}^{\gamma}}{\partial \rho_\ell}\right)_{\gamma}.
\end{aligned}
$$

Note that $\gamma = \arg\min(\phi)$, which is where the final subtraction comes from.