Topic 2: Present Value

(a) Simple vs. Compound Interest

You have \$100. What can you count on getting if you invest this in a bank at a given interest rate r for 2 years? Assume $r = 7\%$ (annual).

2 possible assumptions:

1. *Simple Interest*:

$$\begin{aligned} \text{FV} &= \$100 + \$100r + \$100r \\ &= \$100(1 + 2r) \\ &= \$100(1 + 0.14) = \$114. \end{aligned}$$

   Under simple interest, you earn the interest rate twice. In general, you could go out for t periods, where P is the amount invested:

$$\text{FV} = P(1 + rt).$$

2. *Compound Interest*:

   Under compound interest, you earn interest on interest:

$$\begin{aligned} \text{FV} &= \$100 + \$100r + \$100r + (\$100r)r \\ &= \$100(1 + r) + \$100r(1 + r) \\ &= \$100(1 + r)^2. \end{aligned}$$

   In our example:

$$\text{FV} = \$100(1 + 0.07)^2 = \$114.49.$$

   The \$0.49 is the interest on interest. In general:

$$\text{FV} = P(1 + r)^t.$$

You may say: Who cares for a measly \$0.50? Let's make $t = 100$ years:

- *Simple Interest*: $\text{FV} = \$100(1 + 0.07(100)) = \$800.$
- *Compound Interest*: $\text{FV} = \$100(1 + 0.07)^{100} = \$86,771.$

Which is the correct formula? It depends on if you can reinvest the proceeds. If so, use the compound interest formula. Most often, you can, so that is our standing assumption.

We can also ask: What do I need to put aside today in order to have \$1000 in 2 years? Assuming that we can reinvest at r.

In other words, *what is the present value (PV) of \$1000?*

$$\$1000 = \mathrm{PV}(1+r)^t \Rightarrow$$

$$\mathrm{PV} = \frac{\$1000}{(1+r)^t}.$$

When $r = 7\%$, PV = \$873.

More generally, *how much do I need to put aside today to have C_t t years from now?*

$$\mathrm{PV} = \frac{C_t}{(1+r)^t}.$$

You are taking C_t and *discounting* back to the present. Discount means reduce.

$$\text{(Multiperiod) Discount Factor:} = \frac{1}{(1+r)^t}.$$

(b) Annuities and Perpetuities

Annuity: Equal payments that are known with certainty. This is one step more complicated than the single payment securities we were just considering.

Why are we interested?

- Many bonds on the market, such as those issued by the US government (Treasury bonds), are a stream of equal payments with a lump-sum payment at the end.
- You may want to find how much you have at the end, if starting now, you put a fixed sum into your retirement account every year for 40 years (future value of an annuity).
- We will also get a formula for what happens when the number of payments goes out to ∞. Some forms of equity resemble annuities that go out forever.

Example How much would you be willing to pay today for \$100 a year for 3 years, starting 1 year from now?

We can compute the present value by computing the present value of each of the components and summing up:

$$\text{PV} = \frac{\$100}{1+r} + \frac{\$100}{(1+r)^2} + \frac{\$100}{(1+r)^3}.$$

When $r = 5\%$:

$$\text{PV} = \$95.24 + \$90.70 + \$86.38 = \$272.32.$$

<u>In general</u>: *the present value of a cash flow C, for t years:*

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^t}.$$

We discount each payment appropriately.

We could just calculate each of these components separately, but it gets a bit tedious. It turns out that there is a formula:

$$\text{PV} = \begin{cases} C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] & r \neq 0 \\ Ct & r = 0. \end{cases}$$

Sometimes you will also see:

$$\text{PV} = C\frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

for $r \neq 0$. In the example:

$$\text{PV} = \$100\left[\frac{1}{0.05} - \frac{1}{0.05(1.05)^3}\right] = \$272.32.$$

Mathematical Digression on Series:

Consider a summation of a geometric series $x, x^2, x^3, \ldots$.

$$S_N = x + x^2 + \cdots + x^{N-1} + x^N.$$

Multiplying by x:

$$xS_N = x^2 + x^3 + \cdots + x^N + x^{N+1}.$$

Therefore:

$$S_N - xS_N = x - x^{N+1} = x(1 - x^N)$$

because the intermediate terms cancel!

Now, when $x \neq 1$, we can divide both sides by $1 - x$:

$$S_N = \frac{x(1 - x^N)}{1 - x}.$$

What happens when $x = 1$?

$$S_N = N.$$

Application to Annuities:

We have:

$$\begin{aligned} \text{PV} &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^t} \\ &= C\left[\frac{1}{1+r} + \cdots + \frac{1}{(1+r)^t}\right]. \end{aligned}$$

This is a geometric progression, with $x = 1/(1+r)$!

So, (for $r \neq 0$):

$$\text{PV} = C\frac{\frac{1}{1+r}(1 - (\frac{1}{1+r})^t)}{1 - \frac{1}{1+r}} \quad \frac{\times(1+r)}{\times(1+r)} = \frac{1 - \frac{1}{(1+r)^t}}{1 + r - 1}.$$

Fortunately, it simplifies (multiply and divide by $1 + r$) to the equation we here.

Annuity factor notation:

$$\text{Annuity factor} = \text{AF}_r^t = \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right].$$

Annuities in Practice:

Example You have taken out a 15-year mortgage for \$0.5M. Suppose $r = 4\%$. What must your annual payment be?

Here, we have the PV and r. We need to calculate C. [Note: the owner of this asset is the bank. The cash flows on the asset are your payments to the bank. So in solving this problem, we set C so that the value to the bank is correct.]

Solution: Find C such that

$$0.5 = C\left(\text{AF}_{0.04}^{15}\right).$$

Because

$$\text{AF}_{0.04}^{15} = \frac{1}{r}\left(1 - \frac{1}{(1+r)^{15}}\right) = 11.12,$$

$$C = \frac{0.5}{11.12} = 0.045M = \$45,000.$$

Sometimes we want to calculate the future value of an annuity. An important application is how much you will have if you invest a fixed amount each year into a retirement plan.

How do we find the future value of an annuity? You already know the answer to this. Yon't even need a new formula. How do we find the future value of anything? Let's say we've got some cash flow $\square$ in t years:

$$\mathrm{FV} = \square(1+r)^t.$$

So, FV of annuity in t years:

$$\mathrm{FV} = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right](1+r)^t.$$

Example Invest \$2000 at 6% interest for 50 years:

$$\mathrm{FV} = 2000\left[\frac{(1.06)^{50}-1}{.06}\right] = \$580,670.$$

Perpetuities: now we want to know what happens when the number of payments goes out to ∞. A perpetuity is a fixed payment that occurs every year, forever.

Because we can't sum to ∞, we really need a formula now. Recall the earlier formula:

$$\mathrm{PV} = C\left[\frac{1-\frac{1}{(1+r)^t}}{r}\right].$$

The second term disappears, so we have:

$$\mathrm{PV} = \frac{C}{r}.$$

That's nice. Note what's happening here: the payments go on forever:

$$\mathrm{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

However, each term is getting smaller fast enough so that the overall sum is finite.

What assumption do we really need here?

Assume: $r > 0$.

Otherwise, the terms aren't getting smaller, they are getting bigger.

Example Assume $r = 10\%$ and $C = \$100$:

$$\text{PV} = \frac{C}{r} = \frac{\$100}{0.10} = \$1000.$$

Suppose r went from 10% to 20%. What would you pay?

$$\text{PV} = \frac{\$100}{0.2} = \$500.$$

So, as interest rate doubles ⇒ price halves.

This is only literally true for consols, but it is almost true for more interesting things.

Some stocks behave like perpetuities. Some high-yield stocks tend to offer the same dividend payment over time (utilities used to, but not anymore).

Definition *Preferred Stock* (as opposed to common stock): **dividends are guaranteed** unless the company defaults on its debt.

The point is that preferred stock offers a perpetual payment that is almost fixed. Even if the perpetuity formula isn't exact, we can still use the general principle of a perpetuity.

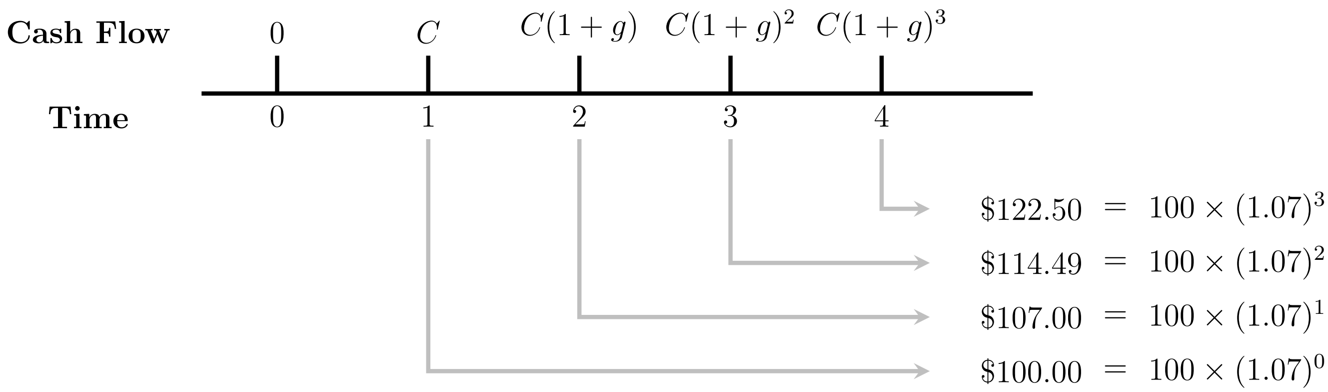
What we will see is that the same principle holds for all stocks. This explains why interest rate matters for stock prices. The stock price is like the present value of future dividends. When the interest rate goes up, the present value, and therefore the stock price, goes down.

(c) Growing Annuities and Perpetuities; Delayed Annuities and Perpetuities

Growing Annuity: an asset where the payments grow at rate g for t years.

Cash Flow	0	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^{t-1}$
Time	0	1	2	3	t

This is a bit confusing, because the last payment we have $(1+g)^{t-1}$. But that is the way its supposed to be. Suppose $t = 4$ and $g = .07$. Then:



So we contribute \$100, \$107, \$114.49, and \$122.50 in years 1-4, respectively. Then:

$$\text{PV} = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{t-1}}{(1+r)^t}.$$

Unfortunately, this asymmetry follows us into the formula. However, if we look hard, we will be able to see the geometric progression:

$$\text{PV} = \frac{C}{1+g}\left[\frac{1+g}{1+r} + \cdots + \left(\frac{1+g}{1+r}\right)^t\right].$$

We use the same idea as before, with $x = (1+g)/(1+r)$:

$$\text{PV} = \frac{C}{1+g}\frac{1+g}{1+r}\left[\frac{1-\left(\frac{1+g}{1+r}\right)^t}{1-\frac{1+g}{1+r}}\right].$$

This leads us to the formula:

$$\text{PV} = C\left[\frac{1}{r-g} - \frac{1}{r-g}\left(\frac{1+g}{1+r}\right)^t\right].$$

However, we can't use this formula when $r = g$. So, when $r = g$:

$$\text{PV} = t\frac{C}{1+g}$$

For a reality check, we can substitute $g = 0$ into our earlier formulas..

Growing Perpetuities: the present value is:

$$\text{PV} = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

We want to have an expression for this formula. But first, can we even find an expression? We have an infinite number of terms here. How do we even know that there is a finite price?

What we need (at least) is for the terms in the sum to get smaller each year. For that to happen, we need $r > g$. When $r > g$, we can apply the annuity formula, "substituting in" ∞, just like last time. The second term becomes 0 since $1 + g < 1 + r$, so :

$$\text{PV} = \frac{C}{r-g}$$

For $r > g$. By setting $g = 0$, we can recover our earlier formulas.

Delayed Annuities and Perpetuities: one feature of many investments is the cash flows grow very rapidly at the beginning, then more slowly, or not at all, later. We can value these by breaking them into building blocks: an annuity plus a delayed perpetuity.

Example

What is the PV of an asset that pays cash flows C in perpetuity beginning 4 years from now? Discount rate is r. (Assume $g = 0$). Answer:

$$\text{PV} = \frac{1}{(1+r)^3}\frac{C}{r}$$

Whenever we value something that is complicated, we need to write out the cash flows:

Cash Flow	0	0	0	0	C	C	$\ldots$
Time	0	1	2	3	4	5	$\ldots$

At $t = 3$, PV $= \frac{C}{r}$, since payments start at $t = 4$. So more generally, this cash flow is worth exactly the same as:

Cash Flow	0	0	0	C/r
Time	0	1	2	3

That brings us to the more general formula: what is the PV of an asset that pays C in perpetuity beginning s years from now?

$$\text{PV} = \frac{1}{(1+r)^{s-1}} \frac{C}{r}$$

What is the PV of an asset that pays C for t years beginning s years from now?

$$\text{PV} = \frac{1}{(1+r)^{s-1}} C \left(\text{AF}_r^t\right)$$

We've seen how to calculate PV of annuities and perpetuities, as well as of growing annuities, perpetuities, and delayed perpetuities. You will see these concepts will be very important when we value bonds and stocks and apply NPV to more realistic situations.

So far, we have made several simplifying assumptions. One is that interest is compounded once a year. Once we relax this, allowing for compounding within the year, we need to look at Effective Annual Rates (EAR).

(d) Compounding within the Year & EAR

When computing PV, we made a specific assumption about compounding, namely that interest is paid and reinvested once a year. Now, suppose that interest is paid and reinvested more frequently, say, 2 times a year.

Example

Bank offers a stated annual interest rate of 8%, compounded semi-annually. [Note: this means 4% is paid twice a year]. If we invest \$100:

- After 6 months, have $\$100 + \$100(0.04) = \$100(1.04)$
- After 1 year, have $\$100(1.04) + \$100(1.04)(0.04) = \$100(1.04)^2 = \$108.16$.

Notation: r_a = stated annual interest rate (SAIR). This is also called an Annual Percentage Rate (APR).

We will call r_a/m the *period rate*. In this example, the period rate is $0.08/2 = 0.04$. Our investment earns 4% for two six-month periods.

Generally, we say r_a is the SAIR, compounded m times a year. Then, our initial investment of \$100 at a given period in the future becomes:

$$\text{FV} = \$100 \left(1 + \frac{r_a}{m}\right)^m$$

The period rate here is r_a/m. That's the rate we earn per period. In general, if we invest \$100 for t years, we have:

$$\text{FV} = \$100\left(1+\frac{r_a}{m}\right)^{mt}$$

We can also use this formula for discounting. How much do we need to invest to have \$100 in t years at SAIR r_a, compounded m times a year?

$$\text{PV} = \$100\left(1+\frac{r_a}{m}\right)^{-mt}$$

Where m could be:

- $m = 1$, annually: \$108
- $m = 2$, semiannually: \$108.16
- $m = 4$, quarterly: \$108.24
- $m = 12$, monthly: \$108.30
- $m = 365$, daily: \$108.3278
- $m = 8760$, hourly: \$108.32867
- $m = 31,536,000$, secondly: \$108.32871
- $m = \infty$, continuously

There is no actual such thing as compounding interest continuously – it is a mathematical abstraction that's a limit of compounding more and more frequently. What is this limit?

$$\lim_{m\to\infty}\left(1+\frac{r_a}{m}\right)^{m}$$

We can't put in ∞ here because then we have 1, and 1 raised to any power is 1. Fortunately, Leibniz considered this very problem 350 years ago. He showed that:

$$\lim_{m\to\infty}\left(1+\frac{r_a}{m}\right)^{m} = e^{r_a}$$

where $e = 2.718282...$ an irrational number.

The value of \$100 under continuous compounding in t years:

$$\text{FV} = \$100e^{r_a t}.$$

The amount you need to set aside today to have \$100 in t years under continuous compounding:

$$\text{PV} = \$100e^{-r_a t}.$$

Note: $e^{-r_a t}$ is the continuously compounded discount factor.

Example What is \$100 one year from now worth today, assuming SAIR an $r_a = 6\%$?

- Annual compounding: $P = \$100/1.06 = 94.34$
- Quarterly compounding: $P = \frac{\$100}{\left(1+\frac{0.06}{4}\right)^4} = 94.22$
- Monthly compounding: $P = \frac{\$100}{\left(1+\frac{0.06}{12}\right)^{12}} = 94.19$
- Continuous compounding: $P = \$100e^{-0.06} = 94.18$

This example shows that the greater the compounding frequency, the smaller the PV. Consider the meaning of PV: this is the amount you must put away today to have \$100 in one year. Under a greater compounding frequency, you can put away fewer dollars today because the money you invest will grow at a faster rate.

Now we have a problem, however. We have all these different rates floating around. How do we compare them? We need to put them on some equal footing.

What we ask: what rate of return will you actually earn if you invest for one year? The answer to this question is the *effective annual interest rate (EAR)*:

Definition *Effective Annual Interest Rate (EAR)*: The rate that, when compounded annually, produces the same return as r_a, when r_a is compounded m times a year.

$$1 + \text{EAR} = \left(1 + \frac{r_a}{m}\right)^m,$$

Rearranging the equation allows us to find EAR for compounding m times a year:

$$\text{EAR} = \left(1 + \frac{r_a}{m}\right)^m - 1$$

And also for continuous compounding:

$$\text{EAR} = e^{r_a} - 1$$

Using the EAR, we are able to translate everything into annual rates. We can translate monthly, quarterly, and even continuous compounding into compounding at an annual rate.

Example Assume SAIR = 6%. Solve for the EAR (r) when:

- Semi-annual: $m = 2$: $r = 6.09\%$
- Quarterly: $m = 4$: $r = 6.14\%$
- Continuous: $m = \infty$: $r = 6.18\%$

Note: as EAR increases, so does compounding frequency. This is unsurprising: the more often interest is compounded for a set SAIR, the more you have at the end of the year.

There is another name for the SAIR r: the Annual Periodic Rate, or APR. This represents simple interest and thus is the *incorrect* way to measure annual returns. To help explain why, recall that our premise for discussion is that interest was compounded within the year.

Now suppose you earn simple interest during the year. SAIR = 6%, paid semiannually:

$$\$100 + \$100(0.06/2) + \$100(0.06/2) = \$100(1.06)$$

Under simple interest, the SAIR is exactly what you earn. I emphasize this because you need to know when it is correct to annualize using the EAR versus the SAIR. For example, your credit card statement probably says something like:

$$\text{Period rate } = 1.5\%/\text{month}$$

$$\text{APR} = 1.5\% \times 12 = 18\%$$

18% is the annual rate you would pay if company charged you simple interest over the year.

However, the credit card company compounds interest monthly. Suppose you spend \$100 on your credit card, don't buy anything else, and never pay your bill. What will your statement be at the end of each month?

- After 1 month $\Rightarrow$ owe $100(1.015) = 101.50$
- After 2 months $\Rightarrow$ owe $100(1.015)^2$ ...
- After 12 months $\Rightarrow$ owe $100(1.015)^{12} = 119.56$

You pay \$19.56 in interest. Thus: $\text{EAR} = (1.015)^{12} - 1 = 19.56\%$

Notice the connection to the EAR definition above:

$$\text{EAR} = \left(1 + \frac{.18}{12}\right)^{12} - 1 = 19.56\%$$

$\Rightarrow$Thus the EAR is 19.56%. That's what the credit card company *should* be reporting.