Documentation of Facility Location Game with Best Response Dynamics process environment

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Abstract. This document briefly reflects the theoretical basis of a Facility Location Game and Best Response Dynamics process and then describes the associated implementation in python.

Keywords: Multi-agent systems · Facility Location Games · Game Theory · Best Response Dynamics

1 Theoretical Framework

1.1 Game Theory Foundations

As stated by [6], "Game Theory aims to model situations in which multiple participants interact or affect each other's outcomes". These situations are often considered as strategic games, and, according to [1], involve:

- A set of players (the participants) $N = 1, 2, ..., n$,
- Strategy profiles $A = A_1 \times A_2 \times ... \times A_n$, which are the combination of actions chosen by all players in the game, where A_i is the set of actions available to player i
- Utility (or payoff) functions $u_i : A \to \Re$

A *Nash Equilibrium* (NE) is a strategy profile x^* satisfying:

$$x^* \in \mathrm{BR}(x^*)\,,$$

where BR denotes the best-response mapping. In other words, no player can increase their payoff by unilaterally deviating from x^*.

According to [5], *potential game* are the ones where a function $\Phi : S \to \mathbb{R}$ exists such that for every player i, any strategy profile $s = (s_i, s_{-i})$, and any alternative strategy s'_i, the change in player i's utility satisfies:

$$u_i(s'_i, s_{-i}) - u_i(s_i, s_{-i}) = \Phi(s'_i, s_{-i}) - \Phi(s_i, s_{-i}). \tag{1}$$

where u_i denotes the utility function of player i.

1.2 Best Response Dynamics (BRD)

Best Response Dynamics (BRD) models rational decision-making in strategic games by assuming agents iteratively update their strategies to maximize individual utilities based on others' actions. This process can be represented as a directed graph where nodes correspond to action profiles, and edges denote transitions via unilateral best-response deviations. At each step, a player switches to a strategy that maximizes their payoff given the current actions of others, driving the system toward equilibrium states. [4]

While BRD converges to Nash equilibria in potential games [7], traditional models face critical limitations:

- They cannot incorporate external interventions,
- They fail to account for learning processes or collaborative behaviors among agents,
- Many existing regulatory frameworks are static and unable to adapt to dynamic systems.

1.3 Facility Location Games

The Facility Location Game is an optimization problem where the goal is to determine which facilities to open and how to assign customers cost-effectively. Given a set of facilities F and a set of customers U, each facility $i \in F$ has a fixed, non-negative opening cost f_i. Additionally, serving a customer $j \in U$ from a facility $i \in F$ incurs a non-negative service cost c_{ij}, which depends on the specific facility–customer pair. The objective is to minimize the total cost, which consists of the sum of the opening costs of the selected facilities and the service costs of assigning each customer to an open facility. This requires making two key decisions: selecting the facilities to open and determining the optimal assignment of customers to these facilities while ensuring every customer is served [2, 3].

Facility Location Games, as potential games, share the property of guaranteed convergence to a Nash equilibrium, although the upper bound is considered exponential (i.e., $O(n^m)$). Another important characteristic of potential games is that they have a central function, called the potential function, which is optimized by the actions of all the players [8].

2 Description of the simulated environment

This simulated environment is a graph-based representation of a facility location game. The base consists of a weighted, undirected, connected tree graph $G = (V, D, E, W)$, where:

- V (nodes):
  - Each with an associated client demand.
  - Potential facility locations $F \subseteq V$.

– D (node weights): node weights represent the demand at each node, such that $\forall d \in D,\ d \in \mathbb{N}$.
– E (edges): connections between nodes (e.g., roads, transit links).
– W (edge weights): edge weights represent serving costs (e.g., distance, congestion, transportation fees), such that $\forall w \in W,\ w \in \mathbb{N}$.

The decision to use a tree was made to ensure convergence via BRD. There are three main reasons: the potential is bounded ($\phi \geq 0$); each best response reduces ϕ or leaves it unchanged if equilibrium is reached; and the absence of cycles prevents infinite loops.

The group of players acting under Best Response Dynamics can be defined as a set $N = \{player_1, player_2, \ldots, player_n\}$, where each player i chooses a location $f_i \in F$ to build its unique uncapacitated facility with no associated building cost, and this location becomes exclusive to i while i decides to keep it. We also define a distance function $d_G(x, y)$ giving the shortest-path distance between any two vertices $x, y \in V$ and a profit function $U_i(f_i, f_{-i})$ depending on the locations of all players, defined in equation (4):

$$U_i(f_i, f_{-i}) = \sum_{\substack{c \in V \\ f_i = \text{nearest}(c)}} D(c) \cdot \big(1 - d_G(c, f_i)\big) \tag{2}$$

Since this FLG is considered under the potential games framework, we use a global potential function ϕ that reflects system-wide efficiency:

$$\phi(f) = \sum_{c \in V} D(c) \cdot d_G\big(c, \text{nearest}(f)\big) \tag{3}$$

Here, ϕ represents the total weighted distance from all clients to their nearest facility. Thus, players' strategies directly impact ϕ.

The best-response update for player i at time step $t + 1$ is described in equation (6):

$$f_i^{(t+1)} = \arg \max_{f \in F \setminus f_{-i}^{(t)}} U_i\big(f, f_{-i}^{(t)}\big) \tag{4}$$

where $f_{-i}^{(t)}$ are the locations of the other players at time t. When the distance calculation between customers and facilities results in a tie, it is broken by random assignment.

Given these definitions, the BRD process proceeds as follows:

1. Start with an initial random configuration $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ with no overlap.
2. At each time step t, select a player at random to update their position to their best response given the current positions of others.
3. Halt when no player can improve their utility (PNE).

Under this design, a Nash equilibrium arises when no player can unilaterally move to capture more clients (i.e., $\nexists f_i' : U_i(f_i', f_{-i}) > U_i(f_i, f_{-i})$).

After success with this simple simulation, certain changes could improve the model's real-life applicability. For example, we could simulate a more dynamic graph where road conditions change over time, include traffic congestion, model evolving client demand with stochastic variations, etc.

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