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THE OPTIMALITY OF FIXED CHANNEL ASSIGNMENT POLICIES FOR CELLULAR RADIO SYSTEMS

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Abstract

In a cellular radio system a limited number of channels are available and neighbouring cells may not be allocated the same channel simultaneously because of the possibility of interference. Under heavy traffic, therefore, the average rate of losing calls may be reduced if certain calls are rejected as a matter of policy because of the potential inference they may cause. A fixed channel assignment policy is one in which each cell is allocated a fixed set of channels, this set typically being made smaller for cells likely to cause most interference. Sufficient conditions are found for the optimality of fixed channel assignment policies for a variety of layouts, and optimal and 'good' policies are found in a number of other cases.

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1. Cellular radio systems

In a cellular radio system, the geographical area covered is divided into cells. A user wishing to make a call from within a cell must first be allocated a channel; the call is then transmitted on this channel to the cell's radio base station. (The nature of the progress of the call beyond the base station does not concern us.) The same channel cannot be used for more than one call within a cell at any one time and, indeed, because of possible interference, the same channel cannot be used simultaneously in neighbouring cells. The system can thus be represented by a graph in which the nodes represent cells and linked nodes indicate 'neighbours', i.e. cells for which the same channel cannot be used simultaneously. In the layout shown in Figure 1, for example, cells 1 and 2 cannot share a channel but cells 1 and 3 can.

We assume that the objective in managing such a layout is to minimise the rate at which calls are lost. A call may be lost out of necessity (e.g. there are no channels available within a cell) or as a matter of policy (e.g. the potential interference with neighbouring cells does not make it worthwhile accepting the call).

One possible feature which adds flexibility in managing such a system is channel reallocation. If channel reallocation is allowed, we may change the channel allocated to a call during the call without affecting its transmission as far as the user is concerned. A shuffling of channels can often be useful in allowing a call to be accepted when initially there is no channel available in the appropriate cell.

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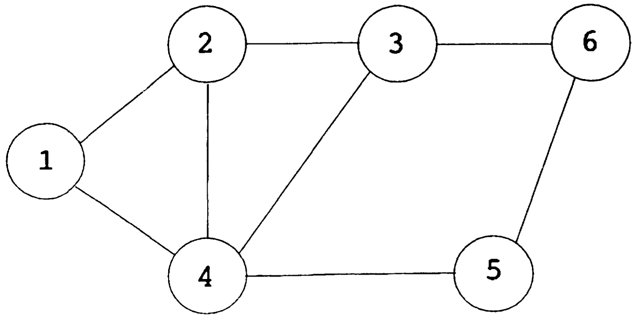

Figure 1

We shall assume that calls are initiated at different cells according to independent Poisson processes and that service times are independent and exponential. The arrival and service rates at the ith cell are denoted by λ_i and μ_i respectively.

The optimal policy for deciding when calls should be accepted is, in general, complicated. Two types of policy, which have the advantage of simplicity, do, however, present themselves: *maximum packing* and *fixed channel assignment.*

Under the maximum packing policy, a call is accepted whenever possible. This will often necessitate channel reallocation. Such a policy is optimal for a fully interconnected network (i.e. every cell is a neighbour of every other cell) with equal service rates at all cells. For such a layout there is nothing to be gained by rejecting a call if it can be accepted. Also, maximum packing is likely to be optimal if the arrival rates are all small and hence the chances of contention for channels between cells are small (Kelly (1985)).

Under a fixed channel assignment policy, each cell is allocated a fixed set of channels and a call is accepted at a cell only if at least one of the allocated channels is available. No two neighbours are allocated the same channel. Such a policy has the advantage of being simple to manage; it avoids channel reallocation and can be designed to try to minimise contention. The last factor is, of course, only likely to be of interest if arrival rates are large. It is the primary objective of this paper to find conditions where fixed channel assignment policies are optimal.

In Section 2, we show how the optimal policy can be found using the theory of Markov decision processes. Implementing this method typically involves solving a very large number of simultaneous linear equations. In Section 3, making use of a decomposition given in Zachary (1988), we find a relatively simple method for solving these equations for fixed channel assignment policies. Using this method, optimality conditions are derived in Section 4 for a particular type of layout (the 'star'). The case of an arbitrary layout with two channels and no channel reallocation is considered in Section 5, and simple sufficient conditions for the optimality of a certain type of fixed channel assignment policy are found. In Section 6, similar results are obtained for general, 2-colourable systems in which channel reallocation is allowed. In the last section, we show how optimal policies for more complex systems may be found by decomposing them into simpler systems.

2. Markov decision processes

The operation of a cellular radio network can be modelled by a Markov decision process and an optimal policy (i.e. one minimising the long-run average rate of losing calls) can then be found by using well-known results. This application has some features which simplify the theory: there is a finite number of states; for each state, there is a finite number of possible actions; for each policy the state 0, corresponding to no calls in the system, is accessible from all other states. For such a process, it is known that a stationary policy must be optimal, so we shall restrict attention to this type of policy. A stationary policy is one for which the choice of action depends only on the current state of the system. In the context of cellular radio, the only occasion at which a choice of action occurs is when a new call is being attempted for which there is a free channel (i.e. the call is 'admissible') and the possible actions are to accept it or reject it.

We denote the state space of the system by S. The states depend on the channels in use at the different cells and the cell at which any new admissible call is being attempted. We suppose that a cost of 1 is generated each time that a new call is not accepted, so that an optimal policy is one which minimises the average expected cost. Since state 0 is accessible from all other states, the average expected cost does not depend on the initial state. The behaviour of the system under a stationary policy can be modelled by a continuous-time Markov process, with instantaneous transitions between states occurring whenever an admissible call is rejected. (Inadmissible calls lead to a cost of 1 being generated, but do not change the state.)

With a given stationary policy, we can associate the transition matrix, P, the cost vector, c_P, and the set of instantaneous states, I_P. For $i \in S \backslash I_P$, row i of matrix P specifies the transition intensities from state i, and for $i \in I_P$ row i specifies the probability distribution of the next state. Similarly, if $i \in S \backslash I_P$, $c_P(i)$ is the rate at which inadmissible calls are attempted in state i, while if $i \in I_P$, $c_P(i)$ is equal to 1, the cost of rejecting the admissible call. The following results are well known. (See, for example, Howard (1960).)

Theorem 1. For stationary policy P, a unique potential function f exists satisfying

$$\begin{aligned} & f(i) = c_P(i) + \sum_j P_{ij} f(j) - g \qquad (i \in S \backslash I_P) \\ (2.1) \qquad & f(i) = c_P(i) + \sum_j P_{ij} f(j) \qquad (i \in I_P) \\ & f(0) = 0 \end{aligned}$$

where g is the average expected cost of P. If for any other stationary policy P',

$$(2.2)\qquad \begin{aligned} f(i) &\leqq c_{P'}(i) + \sum_j P'_{ij} f(j) - g \qquad && (i \in S \backslash I_{P'}) \\ f(i) &\leqq c_{P'}(i) + \sum_j P'_{ij} f(j) && (i \in I_{P'}) \end{aligned}$$

then the policy is optimal. On the other hand, if there exists a stationary policy P^* such that

$$(2.3)\qquad \begin{aligned} f(i) &\geqq c_{P^*}(i) + \sum_j P^*_{ij} f(j) - g \qquad && (i \in S \backslash I_{P^*}) \\ f(i) &\geqq c_{P^*}(i) + \sum_j P^*_{ij} f(j) && (i \in I_{P^*}) \end{aligned}$$

and at least one of the inequalities (2.3) is strict for a state i which is recurrent for policy P^*, then P^* has smaller average expected cost than P.

The potential function has the following interpretation: $f(i)$ is the expected cost until state 0 is first reached less the product of the expected time until state 0 is first reached and g, given that the initial state is i, i.e. $f(i)$ is the expected cost, in excess of that generated by the average expected cost, until state 0 is first reached.

Although the optimality conditions (2.2) have been phrased in terms of alternative stationary policies, it is sufficient to check that the appropriate inequality holds for each state where there is a choice of action and for each possible action. If the optimality condition does not hold, then a policy P^* satisfying (2.3) can be constructed which chooses the same action as policy P except in states i for which condition (2.2) does not hold (i.e. for such states, if policy P specifies 'accept call', policy P^* will specify 'reject call', and vice versa). Policy P^* is the first iterate in Howard's policy improvement procedure. Continuing this procedure, the potential vector of P^* is found and the optimality of P^* ascertained. If P^* is optimal the procedure finishes, otherwise the iterative step is repeated. Eventually, an optimal policy will be found.

3. Evaluating the potential function for a fixed channel assignment policy

The first step in establishing the optimality of a fixed channel assignment policy is to find its potential vector, f, satisfying Equation (2.1). Recall that under a fixed channel assignment policy each cell is allocated a fixed set of channels and a call is accepted at a cell only if at least one of the allocated channels is available. For such a policy, we can define the epochs to occur at 'arrivals' of accepted calls and when calls are completed. The state of the system can most generally be specified by $\mathbf{s} = (s_1, s_2, \cdots, s_N)$, where N is the number of cells and s_i is the set of channels currently in use at the ith cell ($i = 1, 2, \cdots, N$). However, for the recurrent states of the fixed channel assignment policy, a simpler state description may be employed.

Suppose that under this policy, m_i channels are allocated to cell i, then for recurrent states the state can be represented by $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ where n_i is the number of channels in use at cell i $(i = 1, 2, \cdots, N)$. The recurrent states of a fixed channel assignment policy are those for which only the channels assigned to each cell by the policy are in use. In particular, for recurrent states $0 \leqq n_i \leqq m_i$ for $i = 1, 2, \cdots, N$. Once a recurrent state has been reached, the system becomes a decoupled Erlang loss system. The following result can then be easily confirmed.

Theorem 2. The potential vector $f(n_1, n_2, \cdots, n_N)$ of the fixed channel assignment policy is given by

$$f(n_1, n_2, \cdots, n_N) = \sum_{i=1}^{n} f_i(n_i) \qquad (n_i = 0, 1, \cdots, m_i; i = 1, 2, \cdots, N) \tag{3.1}$$

where, for $i = 1, 2, \cdots, N$, f_i is the potential vector for cell i alone, uniquely determined by

$$\begin{aligned} &f_i(0) = 0 \\ &0 = \delta_i \nu_i - ((1-\delta_i)\nu_i + n_i\mu_i) f_i(n_i) + (1-\delta_i)\nu_i f_i(n_i+1) \\ &\qquad + n_i\mu_i f_i(n_i - 1) - g_i \qquad (n_i = 0, 1, \cdots, m_i) \end{aligned} \tag{3.2}$$

where $\delta_i = \delta_i(\boldsymbol{n})$ is the indicator variable taking the value 1 if $n_i = m_i$ and 0 if $n_i < m_i$ and $n_i\mu_i f_i(n_i - 1)$ is understood to equal 0 when $n_i = 0$.

The average expected cost of the fixed channel assignment policy is $g = \sum_{i=1}^{N} g_i$.

Equations (3.2) may be solved by making the substitution $h_i(n_i) = f_i(n_i) - f_i(n_i - 1)$ to yield

$$h_i(n_i + 1) = g_i n_i!\, \mu_i^{n_i} \nu_i^{-(n_i+1)} \sum_{j=0}^{n_i} (\nu_i \mu_i^{-1})^j / j! \qquad (n_i = 0, 1, \cdots, m_i - 1) \tag{3.3}$$

$$g_i = \frac{\nu_i (\nu_i \mu_i^{-1})^{m_i}}{m_i! \sum_{j=0}^{m_i} (\nu_i \mu_i^{-1})^j / j!}. \tag{3.4}$$

The right-hand side of (3.4) is just the Erlang loss rate for a queue with m_i servers and capacity m_i. Since this formula occurs several times below, we shall take the opportunity to make the definition

$$g_i(k) = \frac{\nu_i (\nu_i \mu_i^{-1})^{k}}{k! \sum_{j=0}^{k} (\nu_i \mu_i^{-1})^j / j!} \qquad (k = 0, 1, 2, \cdots). \tag{3.5}$$

Summing (3.3) and using the general result

$$\sum_{j=j_0}^{j_1} \frac{\Gamma(a+j)}{\Gamma(b+j)} = \frac{1}{a-b+1} \left(\frac{\Gamma(a + j_1 + 1)}{\Gamma(j_1 + b)} - \frac{\Gamma(a + j_0)}{\Gamma(b + j_0 - 1)} \right) \tag{3.6}$$

for $a \neq b-1$ and $j_0+b>1$, it may be confirmed that

$$f_i(k)=g_i\nu_i^{-1}k!\sum_{l=0}^{k-1}(\mu_i\nu_i^{-1})^l/((l+1)(k-l-1)!) \qquad (k=0,1,\cdots,m_i). \tag{3.7}$$

We have thus found the value of the potential vector at the recurrent states. Unfortunately, in order to check for optimality, it is also necessary to evaluate the potential vector at the transient states, i.e. where some cells are using unallocated channels. For such states, we have to revert to the more general form of the state description. The components of Equation (2.1) corresponding to the transient states yield a very large number of equations which appear very difficult to solve, especially if channel reallocation is permitted. In order to make progress, we shall temporarily disallow channel reallocation; i.e. once a call has been allocated a channel, it must stay on that channel until it is completed.

We define an 'interfering neighbour' of cell i to be any neighbouring cell of cell i which is using a channel allocated to cell i under the fixed channel allocation policy. We shall also extend the definition of the indicator variable δ_i, so that $\delta_i(s)$ takes the value 1 if cell i cannot accept calls (i.e. either cell i or its interfering neighbours are using all the channels allocated to cell i) and takes the value 0 if cell i can accept calls. Equation (2.1) can be written

$$0=f(\varnothing,\varnothing,\cdots,\varnothing)$$

$$\begin{aligned}0=&\sum_{i=1}^N \nu_i\delta_i-\Big(\sum_{i=1}^N \nu_i(1-\delta_i)+\sum_{i=1}^N n_i\mu_i\Big)f(s)\\&+\sum_{i=1}^N \nu_i(1-\delta_i)f(\cdots,s_i\cup\{x_i\},\cdots)+\sum_{i=1}^N \mu_i\sum_{k\in s_i}f(\cdots,s_i\backslash\{k\},\cdots)-g\end{aligned} \tag{3.8}$$

for all admissible states s. In (3.8), x_i indicates the lowest numbered channel allocated to cell i which is not already in use. Thus, if a call arrives at cell i, and there is an available channel, we suppose that it is allocated to channel x_i. The expression $s_i\backslash\{k\}$ indicates that the call on channel k in cell i has been completed, so that this channel is to be omitted from the set s_i. If $s_i=\varnothing$, the empty set, then $\sum_{k\in s_i}f(\cdots,s_i\backslash\{k\},\cdots)$ is to be interpreted as taking the value 0. Solving (3.8) directly is, in general, extremely difficult, but we shall see that this can be greatly simplified by expressing the solution as the sum of solutions relating to individual cells.

Define the indicator variable $\varepsilon_{ijk}(s)$, or ε_{ijk} for short, to take the value 1 if cell j interferes with cell i on channel k and to take the value 0 otherwise. (Thus, $\varepsilon_{ijk}=1$ indicates that cell j is a neighbour of cell i and is using channel k, a channel allocated by the fixed channel assignment policy to cell i.) Consider now the function

$f_i(n_i, \varepsilon_{i11}, \varepsilon_{i12}, \cdots)$—the 'local potential'—satisfying

$$f_i(0, 0, \cdots, 0) = 0 \tag{3.9}$$

$$\begin{aligned} 0 = \nu_i\delta_i - \Big(\nu_i(1-\delta_i) + n_i\mu_i + \sum_{j=1}^{N} \mu_j \sum_{k=1}^{c} \varepsilon_{ijk}\Big) f_i(n_i, \varepsilon_{i11}, \cdots) \\ + \nu_i(1-\delta_i) f_i(n_i+1, \varepsilon_{i11}, \cdots) + n_i\mu_i f_i(n_i - 1, \varepsilon_{i11}, \cdots) \\ + \sum_{j=1}^{N} \mu_j \sum_{k=1}^{c} \varepsilon_{ijk} f_i(n_i, \cdots, \varepsilon_{ijk} - 1, \cdots) - g_i \end{aligned} \tag{3.10}$$

for $n_i = 0, 1, \cdots, c$, $\varepsilon_{ijk} = 0, 1$ $(i = 1, 2, \cdots, N; j = 1, 2, \cdots, N; k = 1, 2, \cdots, c)$. As usual, terms of the form $0f_i(\cdots, -1, \cdots)$ are understood to take the value 0. By comparing (3.10) with $\varepsilon_{ijk} = 0$ $(i = 1, \cdots, N; j = 1, \cdots, N; k = 1, \cdots, c)$ with (3.2) it is easy to verify that

$$f_i(n_i, 0, \cdots, 0) = f_i(n_i) \quad (n_i = 0, 1, \cdots, m_i; i = 1, 2, \cdots, N)$$

and that the g_i's are the same as before. Thus for the recurrent states, f_i is the potential at cell i, ignoring the rest of the system. For the transient states, f_i includes the cost to cell i (in additional lost calls) imposed by interfering neighbours, in as much as they may prevent cell i taking its full quota of calls. This is why, for instance, completion of service at neighbouring cells of calls on non-interfering channels does not appear in (3.10), while the completion of interfering calls does appear. Even in transient states, interfering calls are not allowed to arrive, under the fixed channel assignment policy. Also note that Equations (3.9) and (3.10) constitute the vector equation and constraint for a potential (of the form (2.1)) and therefore have a unique solution.

Theorem 3. The solution of (3.8) is

$$f(\mathbf{s}) = \sum_{l=1}^{N} f_l(n_l, \varepsilon_{l11}, \cdots), \qquad g = \sum_{l=1}^{N} g_l.$$

Proof. We substitute $\sum f_l(n_l, \varepsilon_{l11}, \cdots)$ for $f(\mathbf{s})$ and $\sum g_l$ for g in the right-hand side of (3.8). Note that a call which is allowed to arrive at cell i by the fixed channel assignment policy must be given an allocated channel and cannot, therefore, interfere with any neighbours; i.e. the value of ε_{ljk} is unaffected for all values of l, j, k. On the other hand, a call which is completed may change ε_{ljk} for some values of l, j, k. Specifically, if a call is completed at cell i on channel k, then ε_{lik} will have its value changed from 1 to 0 for all cells l for which $\varepsilon_{lik} = 1$ (i.e. cell i is a neighbour of cell l and channel k is allocated to cell l). Thus, the right-hand side of (3.8) becomes

$$\sum_{i=1}^{N} \nu_i\delta_i - \Big(\sum_{i=1}^{N} \nu_i(1-\delta_i) + \sum_{i=1}^{N} n_i\mu_i\Big)\sum_{l=1}^{N} f_l(n_l, \varepsilon_{l11}, \cdots)$$
$$+ \sum_{i=1}^{N} \nu_i(1-\delta_i)\Big(f_i(n_i+1, \varepsilon_{i11}, \cdots) + \sum_{l\neq i}^{N} f_l(n_l, \varepsilon_{l11}, \cdots)\Big)$$
$$+ \sum_{i=1}^{N} \mu_i \sum_{k\in s_i} \Big[f_i(n_i-1, \varepsilon_{i11}, \cdots) + \sum_{l=1}^{N} \varepsilon_{lik} f_l(n_l, \cdots, \varepsilon_{lik}-1, \cdots)$$
$$+ \sum_{l\neq i}^{N} (1-\varepsilon_{lik}) f_l(n_l, \varepsilon_{l11}, \cdots)\Big] - \sum_{l=1}^{N} g_l.$$

Cancelling terms and noting that $\varepsilon_{lik} = 0$ if $k \notin s_i$, this becomes

$$\begin{aligned}
&\sum_{i=1}^{N} \nu_i\delta_i - \Big(\sum_{i=1}^{N} \nu_i(1-\delta_i) + \sum_{i=1}^{N} n_i\mu_i\Big) f_i(n_i, \varepsilon_{i11}, \cdots) \\
&\quad - \sum_{i=1}^{N} \mu_i \sum_{k=1}^{c} \sum_{l=1}^{N} \varepsilon_{lik} f_l(n_l, \varepsilon_{l11}, \cdots) \\
&\quad + \sum_{i=1}^{N} \nu_i(1-\delta_i) f_i(n_i+1, \varepsilon_{i11}, \cdots) + \sum_{i=1}^{N} \mu_i n_i f_i(n_i-1, \varepsilon_{i11}, \cdots) \\
&\quad + \sum_{i=1}^{N} \mu_i \sum_{k=1}^{c} \sum_{l=1}^{N} \varepsilon_{lik} f_l(n_l, \cdots, \varepsilon_{lik}-1, \cdots) - \sum_{l=1}^{N} g_l.
\end{aligned} \tag{3.11}$$

Now

$$\sum_{i=1}^{N} \mu_i \sum_{k=1}^{c} \sum_{l=1}^{N} \varepsilon_{lik} f_l(n_l, \varepsilon_{l11}, \cdots) = \sum_{i=1}^{N} \sum_{l=1}^{N} \mu_l \sum_{k=1}^{c} \varepsilon_{ilk} f_i(n_i, \varepsilon_{i11}, \cdots)$$

and

$$\sum_{i=1}^{N} \mu_i \sum_{k=1}^{c} \sum_{l=1}^{N} \varepsilon_{lik} f_l(n_l, \cdots, \varepsilon_{lik}-1, \cdots) = \sum_{i=1}^{N} \sum_{l=1}^{N} \mu_l \sum_{k=1}^{c} \varepsilon_{ilk} f_i(n_i, \cdots, \varepsilon_{ilk}-1, \cdots).$$

Substituting into (3.11) and rearranging, we obtain

$$\sum_{i=1}^{N} \Big\{\nu_i\delta_i - \Big(\nu_i(1-\delta_i) + n_i\mu_i + \sum_{l=1}^{N} \mu_l \sum_{k=1}^{c} \varepsilon_{ilk}\Big) f_i(n_i, \varepsilon_{i11} \cdots)$$
$$+ \nu_i(1-\delta_i) f_i(n_i+1, \varepsilon_{i11}, \cdots) + n_i\mu_i f_i(n_i-1, \varepsilon_{i11}, \cdots)$$
$$+ \sum_{l=1}^{N} \mu_l \sum_{k=1}^{c} \varepsilon_{ilk} f_i(n_i, \cdots, \varepsilon_{ilk}-1, \cdots) - g_i\Big\}.$$

But this is equal to 0, by (3.10). Thus, the given solution satisfies (3.8). We know from the general theory that the solution of (3.8) is unique.

In the case $\mu_1 = \mu_2 = \cdots = 1$, say, the notation can be simplified somewhat. Let $f_i(n_i, a_1, a_2, \cdots)$ denote the local potential at cell i when there are n_i calls at cell i and a_j of cell i's neighbours are using the jth channel allocated to cell i, for $j = 1, 2, \cdots, m_i$. Then (3.10) can be written

$$\begin{aligned} 0 = {} & v_i\delta_i - \Big(v_i(1-\delta_i) + n_i + \sum_j a_j\Big) f_i(n_i, a_1, \cdots) + v_i(1-\delta_i) f_i(n_i + 1, a_1, \cdots) \\ & + n_i f_i(n_i - 1, a_1, \cdots) + \sum_j a_j f_i(n_i, \cdots, a_j - 1, \cdots) - g_i. \end{aligned} \tag{3.12}$$

Equations (3.10) or (3.12) are, of course, very much easier to solve than Equation (3.8). We shall see below how the local potentials may be derived for quite complicated layouts and how these can be used to check for the optimality of the fixed channel assignment policy.

4. Star system

In this section, we consider layouts in which one cell (the 'sun') is linked to all other cells (the 'planets') in the manner shown in Figure 2. We suppose that there are c channels available and N planets, labelled $1, 2, \cdots, N$. The 'sun' is labelled 0. (Note: in the previous section, N denoted the number of cells.) We assume that the service rates at all cells are equal, choosing the time scaling so that the common service rate is 1. We denote the number of calls at cell i by n_i. We show that the only fixed channel assignment policy which may be optimal is the '$0/c$' policy, under which the sun is allocated no channels and the planets are each allocated all channels. We shall further show that the '$0/c$' policy is optimal only if the arrival rates of calls at the planets are sufficiently large.

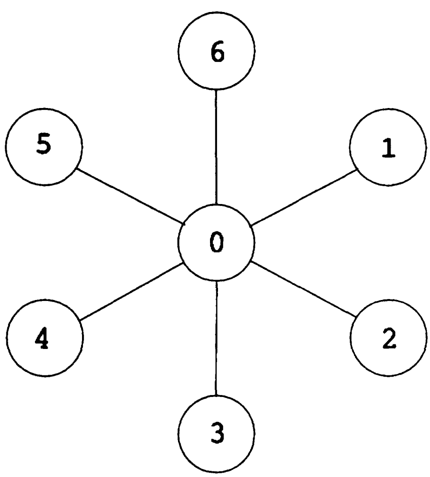

Figure 2

4.1. '0/c' *fixed channel assignment.*

Theorem 4. The '0/c' policy is optimal for the star system if and only if

$$\sum_{i=1}^{N} \frac{v_i^c}{c! \sum_{j=0}^{c} v_i^j/j!} \geqq 1.$$

Proof. We assume initially that channel reallocation is not allowed. We must first evaluate the local potentials. Let us take cell 0, the sun, first. Since this cell has no allocated channels, its local potential is denoted by $f_0(n_0)$. Equations (3.9) and (3.10) can be written

$$f_0(0) = 0$$

$$0 = v_0 - n_0 f_0(n_0) + n_0 f_0(n_0 - 1) - g_0 \qquad (n_0 = 0, 1, \cdots, c).$$

The solution is

$$g_0 = v_0$$

$$f_0(n_0) = 0 \qquad (n_0 = 0, 1, \cdots, c).$$

The local potential is always 0 since the cost rate is the same for all states.

Now we consider the local potential at a planet cell, i. Such a cell has only one neighbour—the sun—and interference from this neighbour can be summarised by the number of calls at the sun, n_0. Writing $f_i(n_i, \varepsilon_{i11}, \cdots)$ as $f_i(n_i, n_0)$, Equations (3.9) and (3.10) become

$$f_i(0, 0) = 0$$

$$\begin{aligned} 0 = v_i\delta_i - (v_i(1 - \delta_i) + n_i + n_0)f_i(n_i, n_0) + v_i(1 - \delta_i)f_i(n_i + 1, n_0) \\ + n_i f_i(n_i - 1, n_0) + n_0 f_i(n_i, n_0 - 1) - g_i \\ (n_i = 0, 1, \cdots, c; n_0 = 0, \cdots, c - n_i - 1) \end{aligned}$$

where $\delta_i = 1$ when $n_i + n_0 = c$. By (3.7), (3.4) and (3.5)

$$f_i(n_i, 0) = g_i v_i^{-1} n_i! \sum_{l=0}^{n_i - 1} v_i^{-l}/((l + 1)(n_i - l - 1)!) \qquad (n_i = 0, 1, \cdots, c) \tag{4.1}$$

and $g_i = g_i(c)$, defined in (3.5). Recalling that $f_i(n_i, n_0)$ is the expected cost till state $(0, 0)$ is reached less the product of the expected time and g_i, it is clear that

$$f_i(n_i, n_0) = f_i(n_i + n_0, 0) \qquad (n_i = 0, 1, \cdots, c; n_0 = 0, 1, \cdots, c - n_i). \tag{4.2}$$

Having found the potentials of the '0/c' policy, we can check for the optimality of the policy, i.e. we must check that the inequalities (2.2) hold for all alternative actions. First, suppose that a call arrives at cell i (>0) and $n_i + n_0 < c$. Under the '0/c' policy the call is accepted and the number of calls at cell i increases to $n_i + 1$.

Under the alternative action of rejecting the call, the system remains in the same state and a cost of 1 is generated. Optimality condition (2.2) for this state is then

$$f_i(n_i+1,\, n_0) \leqq 1 + f_i(n_i,\, n_0). \tag{4.3}$$

Note that accepting the call at cell i might also affect the local potential at neighbouring cells, but in this case the only neighbouring cell is cell 0 whose potential does not depend on its neighbours. Using (4.2) and noting that $f_i(n_i+1, 0) - f_i(n_i, 0) = h_i(n_i+1)$, given by (3.3), it can be verified that condition (4.3) must hold.

The other decision we must check is whether an arriving call at cell 0 should be rejected, as is specified by the '$0/c$' policy. The effect of rejecting the call is to incur a 'cost' of 1, but there is no change of state. If instead the call is accepted, the number of calls at cell 0 is increased from n_0 to n_0+1 and the resulting increase in potential at cell 0 and its neighbours is

$$f_0(n_0+1) - f_0(n_0) + \sum_{i=1}^{N} (f_i(n_i,\, n_0+1) - f_i(n_i,\, n_0)).$$

The first two terms make no contribution, so that the optimality condition is

$$\sum_{i=1}^{N} (f_i(n_i,\, n_0+1) - f_i(n_i,\, n_0)) \geqq 1 \qquad (n_i = 0, 1, \cdots, c;\, n_0 = 0, 1, \cdots, c - n_i - 1). \tag{4.4}$$

Again using (4.2) and (3.3), this condition simplifies to $\sum_i g_i(c)/g_i(n_i+n_0) \geqq 1$. Since $g_i(k)$ is a decreasing function in k, for each i, condition (4.4) will be satisfied if

$$\sum_{i=1}^{N} g_i(c)/g_i(0) \geqq 1,$$

i.e.

$$\sum_{i=1}^{N} \frac{v_i^c}{c! \sum_{j=0}^{c} v_i^j/j!} \geqq 1. \tag{4.5}$$

Condition (4.5) is sufficient for the '$0/c$' policy to be optimal. It is also a necessary condition; this follows by a similar argument to the one used in Section 4.2 to show that no other fixed channel allocation policy is optimal. Finally we note that in the above argument it is only the number of calls at each cell that affects the value of the potential, not the channels which are being used. (The key factor is that ε_{0jk} is 0 for all states j and channels k.) It follows that the potentials are unaffected by channel reallocation and that condition (4.5) is also the optimality condition for the '$0/c$' policy if channel reallocation is allowed.

Condition (4.5) affords an intriguing interpretation. By a well-known result from queueing theory, the summand in the left-hand side of (4.5) is the long-run probability of all c channels allocated to cell i being busy, so that the left-hand side of (4.5) is equal to the long-run mean number of 'full' planet cells under the '$0/c$' policy. The result that this mean must equal or exceed 1 in order that the '$0/c$' policy is optimal does not seem at all intuitive. Another surprising feature about the optimality condition is that it does not involve ν_0, the arrival rate at the sun. Again this seems counterintuitive; we might expect that it would be less wise to 'switch off the sun' if the arrival rate at the sun greatly exceeded the arrival rate at the planets.

4.2. *'$k/c-k$' fixed channel assignment*

Theorem 5. The '$k/l-k$' fixed channel assignment, under which k (>0) channels are allocated to the sun and $c-k$ channels are allocated to the planets, is always suboptimal.

Proof. We assume initially that channel reallocation is not allowed. It suffices to find a policy for which condition (2.3) holds and where the inequality is strict for a recurrent state. The case we shall consider is when the system is empty except for $c-k$ calls at cell 1, all on allocated channels. Suppose now that a call arrives at cell 1 and we must decide whether to accept it. If the call is accepted, there will be an increase in the local potential at cell 0 since it will have an interfering neighbour. We can denote the increase in potential by $f_0(0,1)-f_0(0,0)$. (The first zero in the argument refers to the number of calls at cell 0.) Now $f_0(n_0, n_1)$, for $n_0=0, 1, \cdots, k$ and $n_1=0, 1$, satisfy the same equations as $f_i(n_i, n_0)$ in Section 4.1 with i replaced by 0, n_0 replaced by n_1 and c replaced by k. In particular $f_0(0,1)=f_0(1,0)=g_0(k)\nu_0^{-1}$ and, of course, $f_0(0,0)=0$. The value of the local potential at cell 1 is not changed by accepting the call since this call is on an unallocated channel and therefore does not affect the future performance of cell 1 under the '$k/c-k$' policy. The increase in system potential if the call is accepted is therefore equal to $g_0(k)\nu_0^{-1}$. Under the '$k/c-k$' policy, the call at cell 1 is rejected, resulting in a 'cost' of 1, but no change in state. The optimality condition is, therefore, $1 \leqq g_0(k)\nu_0^{-1}$. It is easily confirmed that this condition never holds.

Consider now the policy which behaves just as the '$k/c-k$' policy except that whenever all the allocated channels at cell 1 are in use and the rest of the system is empty, a further call arriving at cell 1 is accepted. Such an event is recurrent under the modified policy, so that condition (2.3) holds with a component of δ positive for a recurrent state. It follows that the '$k/c-k$' policy has average cost strictly greater than the modified policy.

Since the '$k/c-k$' policy has average cost greater than some other policy not making use of channel reallocation, it follows that the '$k/c-k$' policy is also suboptimal when channel reallocation is allowed.

4.3. *Policy improvement.* We have shown that for the star layout, when arrival rates at the planets are large (condition (4.5), to be precise), it is optimal to allocate all c channels to the planets and no channels to the sun. What policy should be adopted when condition (4.5) does not hold? Certainly not a '$k/c-k$' policy, it would seem. Instead we can use policy improvement ideas, taking as the initial policy the '$0/c$' policy with potential function as found in Section 4.1.

Condition (4.3) was found always to hold, suggesting that calls at planets should never be rejected. Condition (4.4) will not always hold, if condition (4.5) is not satisfied. It may hold, however, for the particular state when a call arrives, which would suggest that the call should be rejected. In brief, policy improvement suggests the following policy: accept all calls at the planets, and calls at the sun unless

$$\sum_{i=1}^{N} (f_i(n_i, n_0+1) - f_i(n_i, n_0)) \geqq 1$$

i.e.

$$\sum_{i=1}^{N} g_i(c)/g_i(n_0+n_i) \geqq 1 \tag{4.6}$$

where $g_i(k)$ is given by (3.5). By Theorem 1 this policy will have average cost less than the '$0/c$' policy. The following example shows that for the case $N=2$ at least, the resulting policy will often be optimal.

Example. A two-channel system has the layout shown in Figure 3. The arrival rate of calls at each cell is v and channel reallocation is allowed. We shall firstly find the policies suggested by the policy improvement method. It is easily confirmed that $g_i(0)=v$, $g_i(1)=v^2/(1+v)$ and $g_i(2)=v^3/(2+2v+v^2)$. Optimality condition (4.5) can be confirmed to hold if $v \geqq 1+\sqrt{3}$. Suppose that $v<1+\sqrt{3}$. If $n_0=1$, $n_1=n_2=0$ or $n_0=0$, $n_1=n_2=1$, condition (4.6) becomes $2\, g_i(2)/g_i(1) \geqq 1$, which is satisfied if $v \geqq \sqrt{2}$. Thus, policy improvement suggests that a call arriving at the sun when $n_0=1$, $n_1=n_2=0$ or $n_0=0$, $n_1=n_2=1$ should be rejected if $v \geqq \sqrt{2}$. Similarly if $n_1=1$, $n_0=n_2=0$ or $n_2=1$, $n_0=n_1=0$, inequality (4.6) suggests that a call at the sun should be rejected if $v \geqq 2$.

Since there are not too many states in this case, it is possible, with the aid of a computer algebra package, to use Howard's policy improvement procedure to find optimal policies for all values of v. In Table 1 we summarise the actions specified by the policy improvement policy and by the optimal policy when a call arrives at the sun and the state is as indicated.

5. General layout with two channels; channel reallocation not allowed

We now consider an arbitrary layout with two channels, with channel reallocation not allowed, and seek conditions for the optimality of a '0/2' fixed channel

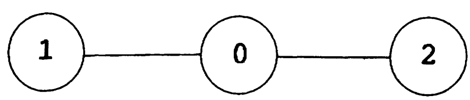

Figure 3

TABLE 1

state	policy improvement policy	optimal policy
$n_0=1,\ n_1=n_2=0$	Reject if $v \geqq \sqrt{2} \approx 1{\cdot}414$	Reject if $v \geqq 1{\cdot}307$
$n_0=0,\ n_1=n_2=1$	Reject if $v \geqq \sqrt{2} \approx 1{\cdot}414$	Reject if $v \geqq 1{\cdot}341$
$\left.\begin{array}{l} n_1=1,\ n_0=n_2=0 \\ n_2=1,\ n_0=n_1=0 \end{array}\right\}$	Reject if $v \geqq 2$	Reject if $v \geqq 1{\cdot}967$
$n_0=n_1=n_2=0$	Reject if $v \geqq 1+\sqrt{3}$	Reject if $v \geqq 1+\sqrt{3}$

assignment policy. For this type of policy, cells are divided into '0-cells', which are allocated no channels, and '2-cells', which are allocated both channels. In order for the policy to be admissible, we suppose that no two 2-cells are neighbours. We assume that the service rate is 1 at all cells. Only brief details will be given; further details and some discussion of layouts with more than two channels are given in Robinson (1990).

Suppose then that a '0/2' fixed channel assignment policy is being used. Let $f_0(n_0)$ be the local potential for a 0-cell. By the argument at the beginning of Section 4.1

$$f_0(n_0)=0 \qquad (n_0=0,1,2,\cdots). \tag{5.1}$$

Interest is therefore confined to the local potentials of 2-cells which we can find by solving (3.12). Thus, denoting by $f_i(n_i, a_1, a_2)$ the local potential at a 2-cell, i, when it has n_i calls, a_1 neighbouring cells using channel 1 and a_2 neighbouring cells using channel 2:

$$f_i(0,0,0)=0 \tag{5.2}$$

$$0=-(v_i+n_i)f_i(n_i,0,0)+v_if_i(n_i+1,0,0)+n_if_i(n_i-1,0,0)-g_i \qquad (n_i=0,1) \tag{5.3}$$

$$0=v_i-2f_i(2,0,0)+2f_i(1,0,0)-g_i \tag{5.4}$$

$$0=-(v_i+a_1)f_i(0,a_1,0)+v_if_i(1,a_1,0)+a_1f_i(0,a_1-1,0)-g_i \qquad (a_1=1,2,\cdots) \tag{5.5}$$

$$0=v_i-(1+a_1)f_i(1,a_1,0)+f_i(0,a_1,0)+a_1f_i(1,a_1-1,0)-g_i \qquad (a_1=1,2,\cdots) \tag{5.6}$$

$$0=v_i-(a_1+a_2)f_i(0,a_1,a_2)+a_1f_i(0,a_1-1,a_2)+a_2f_i(0,a_1,a_2-1)-g_i \qquad (a_1=1,2,\cdots;a_2=1,2,\cdots). \tag{5.7}$$

By symmetry,

$$f_i(n_i,a_1,a_2)=f_i(n_i,a_2,a_1) \qquad (n_i=0,1,2;a_1=0,1,\cdots;a_2=0,1,\cdots). \tag{5.8}$$

The solution to (5.2), (5.3), (5.4) is given by (3.4), (3.5) and (3.7): $g_i=g_i(2)$ and

$$f_i(n_i,0,0)=g_iv_i^{-1}n_i!\sum_{l=0}^{n_i-1} v_i^{-l}/((l+1)(n_i-l-1)!) \qquad (n_i=0,1,2). \tag{5.9}$$

After some algebra, (5.5) and (5.6) can be solved to obtain

$$f_i(0, a_1, 0) = \frac{v_i^2(2+v_i)}{(1+v_i)(2+2v_i+v_i^2)}\left(\sum_{j=1}^{a_1}\frac{1}{j}+\frac{\Gamma(1+v_i)a_1!}{\Gamma(2+v_i+a_1)}-\frac{1}{1+v_i}\right) \tag{5.10}$$
$$(a_1 = 0, 1, \cdots).$$

Values of $f(0, a_1, a_2)$ for $a_1 = 1, 2, \cdots, a_2 = 1, 2, \cdots$, can now be found numerically using (5.7).

We can now consider the optimality conditions. First suppose that a call arrives at 2-cell i, and that it is admissible to accept it. Optimality condition (2.2) amounts to checking that the increase in potential associated with accepting the call is less than or equal to 1, the 'cost' of rejecting it. Suppose that there are no calls currently at cell i, a_1 of cell i's neighbours are using channel 1 and none of cell i's neighbours is using channel 2. The optimality condition is

$$f_i(1, a_1, 0) - f_i(0, a_1, 0) \leqq 1. \tag{5.11}$$

If the call arrives when there is already one call at cell i, the optimality condition is

$$f_i(2, 0, 0) - f_i(1, 0, 0) \leqq 1. \tag{5.12}$$

(Note that a_1 and a_2 must be 0, otherwise it would be inadmissible to accept the call.) It can be confirmed that conditions (5.11) and (5.12) will always be satisfied.

The other decision we must check is whether an arriving call at a 0-cell should be rejected. Suppose first that there are no calls at the 0-cell. The increase in local potential if the call is accepted, on channel 1, say, at the 0-cell, or at any of its 0-cell neighbours is zero. The increase in local potential at a 2-cell neighbour, l, which currently has 0 calls, a_{1l} neighbours on channel 1 and a_{2l} neighbours on channel 2 is

$$f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l}) \qquad (a_{1l} = 0, 1, \cdots, N_l - 1; a_{2l} = 0, 1, \cdots, N_l - 1)$$

where N_l is the number of neighbours of cell l. (Because of interconnections between cells, not all combinations of a_{1l} and a_{2l} may be possible. We shall discuss the effects of this below.) The increase in local potential at a 2-cell neighbour, l, which currently has 1 call, on channel 2, and a_{1l} neighbours using channel 1 is

$$f_l(1, a_{1l}+1, 0) - f_l(1, a_{1l}, 0) \qquad (a_{1l} = 0, 1, \cdots, N_l - 1).$$

For optimality, we must show that the increased potential associated with accepting the call must exceed 1, the 'cost' of rejecting it, i.e. the optimality condition is:

$$\sum_{l\in A_0}(f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) + \sum_{l\in A_1}(f_l(1, a_{1l}+1, 0) - f_l(1, a_{1l}, 0)) \geqq 1 \tag{5.13}$$
$$(a_{1l} = 0, 1, \cdots, N_l - 1; a_{2l} = 0, 1, \cdots, N_l - 1)$$

where A_0 is the set of 2-cell neighbours of the 0-cell which have no calls and A_1 is the set of 2-cell neighbours which have one call. It can be shown that

$$f_l(1, a_{1l}+1, 0) - f_l(1, a_{1l}, 0) > f_l(0, a_{1l}+1, 0) - f(0, a_{1l}, 0) \qquad (a_{1l}=0, 1, \cdots)$$

so that it is enough to verify that

$$\sum_{l\in A} (f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) \geqq 1$$

$$(a_{1l}=0, 1, \cdots, N_l-1; a_{2l}=0, 1, \cdots, N_l-1)$$

where A is the set of 2-cell neighbours of the 0-cell. The same condition is obtained if the call is accepted on channel 2.

Finally, we must evaluate the increase in potential if a call is accepted at the 0-cell when there is already one call at the cell. It is easily confirmed that the increase is the same as in the case above, though in this case a_{2l} can take the additional value N_l. Combining the two cases the condition for optimality is

$$(5.14) \quad \sum_{l\in A} (f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) \geqq 1$$

$$(a_{1l}=0, 1, \cdots, N_l-1; a_{2l}=0, 1, \cdots, N_l).$$

Define $\psi(N_l)$ by

$$(5.15) \quad \psi(N_l) = \min_{\substack{a_{1l}=0,1,\cdots,N_l-1\\ a_{2l}=0,1,\cdots,N_l}} (f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) \qquad (N_l=1, 2, \cdots).$$

We have now established the following result.

Theorem 6. For a general cellular radio layout with two channels and no channel reallocation, a '0/2' policy is optimal if for each 0-cell

$$(5.16) \qquad \sum_{l\in A} \psi(N_l) \geqq 1$$

where A is the set of 2-cell neighbours of the 0-cell, N_l is the number of neighbours of cell l ($l=1, 2, \cdots, N$) and $\psi(\,)$ is defined by (5.15).

Since N_l will typically not be large for any cell, $\psi(N_l)$ is straightforward to compute. In fact, numerical studies indicate that the minimum is achieved when $a_{2l}=0$, $a_{1l}=N_l-1$. It follows that

$$(5.17) \quad \psi(N_l) = \frac{\nu_l^2(2+\nu_l)}{(2+2\nu_l+\nu_l^2)(1+\nu_l)N_l}\left(1 - \frac{\Gamma(2+\nu_l)N_l!}{\Gamma(2+\nu_l+N_l)}\right) \qquad (N_l=1, 2, \cdots).$$

Finally, we note that the aforementioned possibility of interconnection between neighbours of 2-cells means that the condition (5.16) is sufficient but not necessary for particular systems.

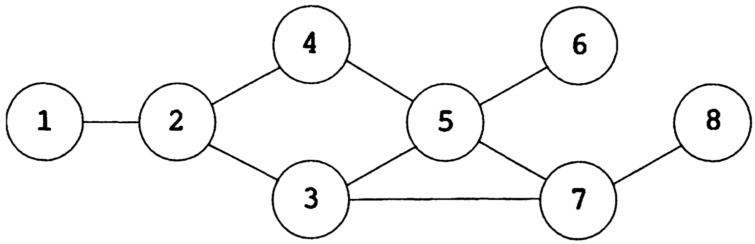

Figure 4

Example. For the two-channel layout shown in Figure 4, the arrival rate of calls at each cell is 5. We can show that the optimal policy is to allocate no channels to cells 2, 5 and 7 and to allocate both channels to cells 1, 3, 4, 6 and 8.

For $v = 5$, $\psi(1) \approx 0{\cdot}676$, $\psi(2) \approx 0{\cdot}380$, $\psi(3) \approx 0{\cdot}260$ (by (5.17)). Therefore, the ψ-values of 2-cells are:

2-cell	1, 8	4, 6	3
ψ-value	0·676	0·380	0·260

The sums of the ψ-values of the neighbouring 2-cells for each 0-cell are:

0-cell	2	5	7
Sum of ψ-values	1·316	1·020	1·316

Since the sum of ψ-values for all three 0-cells is at least 1·0, we can conclude that the given policy is optimal.

6. Channel reallocation allowed

6.1. *2-colourable layouts.* The method introduced in Section 3 for evaluating potentials for transient states relies on transient effects being 'local' for fixed channel assignment policies. For instance, if a call is accepted, its effect on the potential of the system is felt only through its effect on the local potential at the cell where it occurs and the immediate neighbours of that cell. Unfortunately, once channel reallocation is allowed, this will not happen in general. Consider for example, a two-channel system with the layout shown in Figure 5.

Suppose there is one call at cells 2, 3 and 5 and no calls at cells 1 and 4. It is clear that, by reallocating channels if necessary, a call could be accepted at either cell 1 or cell 4 (or, indeed, cell 5). Suppose now that a call is accepted at cell 1. It can easily be verified that, however the channels are reallocated, it is no longer possible to accept a call at cell 4, i.e. the effect of accepting a call at cell 1 is felt beyond its immediate neighbours. We can describe this as 'action at a distance'.

Although the methods of this paper do not apply to general layouts when channel reallocation is allowed, it turns out that they do apply to 2-colourable layouts.

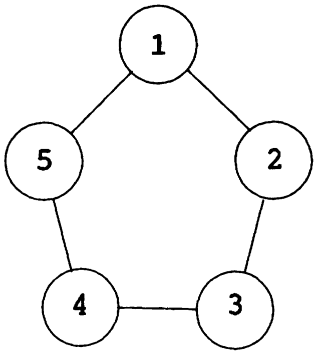

Figure 5

(Some layouts which are not 2-colourable are considered in Section 7.) A 2-colourable layout is one where the cells can be labelled A or B in such a way that no two neighbours have the same letter. The following result is required.

Theorem 7. For a 2-colourable layout with c channels in an admissible state, the channels can be allocated so that all A-labelled cells with a calls use channels $\{1, 2, \cdots, a\}$, for $a = 0, 1, \cdots, c$, and all B-labelled cells with b calls use channels $\{c-b+1, c-b+2, \cdots, c\}$, for $b = 0, 1, \cdots, c$.

Proof. Fix attention on some A cell with a calls. By relabelling throughout the system, if necessary, ensure that the channels allocated to this cell are $1, 2, \cdots, a$. Now move to neighbouring B cells, noting that since the state is admissible the number of calls at each one cannot exceed $c - a$. Thus we can label the channels at their B cells according to the above prescription without the B cells sharing any channels with the A cell. (Notice that because the layout is 2-coloured, the B cells cannot be neighbours of each other.) Now move to the A-cell neighbours of these B cells. Once again, the condition that the state is admissible allows us to label the channels at the A cells in the above manner. We can continue in this way until the whole layout is covered. At no point is there any danger of a channel appearing twice in neighbouring cells, since state admissibility means that the total number of calls at two neighbouring cells cannot exceed c.

We shall call this allocation of channels to A and B cells the canonical allocation. Notice that under the canonical allocation, in order to check whether it is admissible to accept a new call at a cell, it is only necessary to consider the channels currently being used at the cell and at its neighbours. Thus, there is no 'action at a distance'. We shall assume in the rest of this section that the layout is 2-colourable and that the canonical allocation is always used.

The assumption that the canonical allocation is being used allows us to specify the local potential in an economical way. Consider, for instance, an A cell, i, which under a fixed channel assignment policy is allocated m_i channels, $\{1, 2, \cdots, m_i\}$.

The local potential at cell i is $f_i(n_i, a_1, a_2, \cdots, a_{m_i})$, where n_i is the number of calls at cell i and a_j is the number of neighbours of cell i which are using $c - m_i + j$ channels ($j = 1, 2, \cdots, m_i$). Thus, a_j neighbours are preventing cell i using j of its allocated channels. The recurrent states correspond to $a_1 = a_2 = \cdots = a_{m_i} = 0$ and $n_i \leqq m_i$.

6.2. *Two-channel networks.* As in Section 5, we consider two-channel networks and find conditions for the optimality of a '0/2' fixed channel assignment policy, under which cells are divided into 0-cells, which are allocated no channels, and 2-cells, which are allocated both channels. We assume that the layout is 2-colourable, and further assume that the 0-cells are labelled A and the 2-cells are labelled B.

As we have seen on previous occasions, the local potential at 0-cells is 0, regardless of the number of calls at the cell (or its neighbours). We concentrate, therefore, on $f_i(n_i, a_1, a_2)$, the local potential at a 2-cell, i, with n_i calls, a_1 neighbours with one call and a_2 neighbours with two calls. (Recall that under the canonical allocation, the cells with one call are all using channel 1, since all neighbours of 2-cells are 0-cells, which are labelled A.)

It is straightforward to check that $f_i(n_i, a_1, 0)$ satisfies the same equations as the corresponding potential in Section 5.1. Therefore, (5.9) and (5.10) apply equally to this case. The components of (2.1) corresponding to the cases $a_2 = 1, 2, \cdots$, are

$$\begin{aligned} 0 = v_i - (a_1 + 2a_2)f_i(0, a_1, a_2) + a_1 f_i(0, a_1 - 1, a_2) \\ + 2a_2 f_i(0, a_1 + 1, a_2 - 1) - g_i \qquad (a_1 = 0, 1, \cdots; a_2 = 1, 2, \cdots) \end{aligned} \tag{6.1}$$

where, as usual, '$0f(0, -1, a_2)$' is to be understood to equal 0. Given that $f_i(0, a_1, 0)$ is known for $a_1 = 0, 1, \cdots, f_i(0, a_1, a_2)$ can be found numerically for $a_1 = 0, 1, 2, \cdots, a_2 = 1, 2, \cdots$.

We start checking optimality by considering the case of a call arriving at a 2-cell. We must show that the increment in the potential induced by accepting the call, the action of the '0/2' policy, is less than or equal to 1, the 'cost' of rejecting the call. It turns out that the condition is always satisfied; the details are exactly the same as for the case considered in Section 5.1.

Suppose now that a call arrives at a 0-cell which currently has no calls. By an argument similar to the one leading to inequality (5.13), the optimality condition is

$$\sum_{l \in A_0} (f_l(0, a_{1l} + 1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) + \sum_{l \in A_1} (f_l(1, a_{1l} + 1, 0) - f_l(1, a_{1l}, 0)) \geqq 1 \tag{6.2}$$

$$(a_{1l} = 0, 1, \cdots, N_l - 1; a_{2l} = 0, 1, \cdots, N_l - 1 - a_{1l})$$

for each 0-cell, where A_0 is the set of neighbours of the 0-cell which currently have no calls, A_1 is the set of neighbours which currently have one call and N_l is the number of neighbours of neighbour l. As in Section 5.1, this condition may be simplified to

$$\sum_{l \in A} (f_l(0, a_{1l}+1, a_{2l}) - f_l(0, a_{1l}, a_{2l})) \geqq 1 \qquad (6.3)$$
$$(a_{1l} = 0, 1, \cdots, N_l - 1; a_{2l} = 0, 1, \cdots, N_l - 1 - a_{1l})$$

where A is the set of all neighbours of the 0-cell.

The remaining case to consider is when a call arrives at a 0-cell, i, which already has one call in progress. Proceeding as above, the optimality condition can be shown to be

$$\sum_{l \in A} (f_l(0, a_{1l}-1, a_{2l}+1) - f_l(0, a_{1l}-1, a_{2l}+1)) \geqq 1 \qquad (6.4)$$
$$(a_{1l} = 1, 2, \cdots, N_l; a_{2l} = 0, 1, \cdots, N_l - a_{1l}).$$

We can summarise our results as follows.

Theorem 8. For a general cellular radio layout with two channels and channel reallocation, a '0/2' policy is optimal if for each 0-cell conditions (6.3) and (6.4) hold.

When $f_l(0, a_1, a_2)$ is evaluated numerically, it turns out that the critical optimality condition is (6.3), with $a_{1l} = 0$, $a_{2l} = N_l - 1$. Thus, checking for the optimality of the '0/2' policy is very similar to the method described in Section 5.1, except that the function ψ defined by (5.17) is replaced by ψ^* defined by

$$\psi^*(N_l) = f_l(0, 1, N_l - 1) - f_l(0, 0, N_l - 1) \qquad (N_l = 1, 2, \cdots). \qquad (6.5)$$

It is found numerically that $\psi^*(N_l) \leqq \psi(N_l)$ for $N_l = l, 2, \cdots$ and all $v_i > 0$. For example, if $v_i = 10$, the values of the two functions are:

N_l	1	2	3	4	5
$\psi(N_l)$	0·820	0·441	0·297	0·273	0·179
$\psi^*(N_l)$	0·820	0·294	0·178	0·127	0·099

We see, therefore, that it is 'harder' for the '0/2' policy to be optimal when channel reallocation is allowed. Roughly speaking, the flexibility allowed by channel reallocation tends to count against the 'rigid' fixed channel assignment policy.

7. Connecting sub-layouts

We saw in Section 6 that the local potential method can be used only for systems for which channel reallocation is allowed when the layout is 2-colourable. In this section we shall briefly indicate how optimal policies can sometimes be found for layouts whose chromatic number exceeds 2.

Suppose that a layout can be split into a number of sub-layouts and that, ignoring links between sub-layouts, optimal 'sub-policies' can be found for each sub-layout. In general, it will not be possible to combine these sub-policies into a single optimal

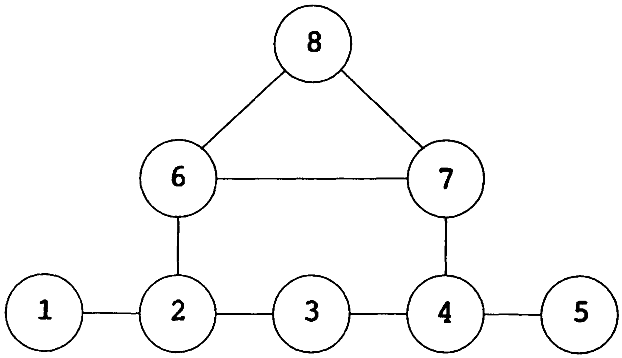

Figure 6

policy for the complete layout because of contention associated with the links between the sub-layouts. If however, at least one cell at the end of each such link is to receive no calls under the appropriate sub-policy, then no such contention can arise and the combined policy will be optimal.

Example. The arrival rate at the two-channel system shown in Figure 6 is 10 and the service rate is 1. We show that the optimal policy is to accept no calls at cells 2 and 4 and all admissible calls at the remaining cells. Split the layout into sub-layouts L_1, consisting of cells 1–5, and L_2, consisting of cells 6–8. Using the methods of Section 6 it is easy to show that for $\nu = 10$, the optimal policy within L_1 is to make cells 2 and 4 0-cells and cells 1, 3 and 5 2-cells. Sub-layout L_2 is fully interconnected, so that the maximal packing policy is optimal. Since the two sub-layouts are joined via 0-cells, the overall optimal policy is as described.

When a sub-layout is fully interconnected and all members of the sub-layout have the same neighbours, it is sometimes possible to consider the sub-layout as a single cell with arrival and service rates equal to the sums of the arrival and service rates within the sub-layout.

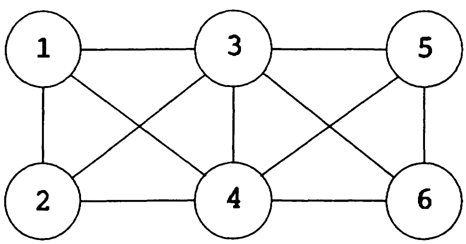

Figure 7

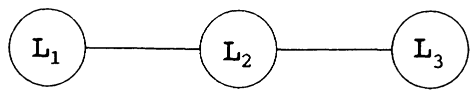

Figure 8

Example. We show that the optimal policy for the three-channel layout shown in Figure 7, where the common service rate is 0·5 and the common arrival rate is 2·5, is to accept all admissible calls at cells 1, 2, 5 and 6 and no calls at cells 3 and 4.

Let cells 1 and 2 constitute sub-layout L_1, cells 3 and 4 constitute sub-layout L_2 and cells 5 and 6 constitute sub-layout L_3. Note that L_1, L_2 and L_3 are each fully interconnected and that cells within each sub-layout have the same neighbours. Assuming that we are going to adopt a policy depending only on the total numbers of calls in sub-layouts L_1, L_2 and L_3, we can consider reducing the system to the one shown in Figure 8, where the arrival rate at each 'cell' is $2 \times 2{\cdot}5 = 5{\cdot}0$ and the service rate is $2 \times 0{\cdot}5 = 1{\cdot}0$. It may be confirmed that condition (4.5) holds for the reduced system, so that the optimal policy is to accept all admissible calls at cells L_1 and L_3 and no calls at cell L_2. The optimality of the equivalent policy for the original layout can be confirmed after noting that its potentials are given by

$$f(n_1, n_2, n_3, n_4, n_5, n_6) = f_1(n_1 + n_2, n_3 + n_4) + f_1(n_5 + n_6, n_3 + n_4)$$
$$(n_i = 0, 1, 2, 3; i = 1, 2, \cdots, 6)$$

where n_i is the number of calls at cell i ($i = 1, 2, \cdots, 6$) and f_1 is given by (4.1), with $v_1 = 5$, $c = 3$.

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