Frequency Assignment: Theory and Applications

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Abstract—**In this paper we introduce the minimum-order approach to frequency assignment and present a theory which relates this approach to the traditional one. This new approach is potentially more desirable than the traditional one. We model assignment problems as both frequency-distance constrained and frequency constrained optimization problems. The frequency constrained approach should be avoided if distance separation is employed to mitigate interference. A restricted class of graphs, called disk graphs, plays a central role in frequency-distance constrained problems. We introduce two generalizations of chromatic number and show that many frequency assignment problems are equivalent to generalized graph coloring problems. Using these equivalences and recent results concerning the complexity of graph coloring, we classify many frequency assignment problems according to the "execution time efficiency" of algorithms that may be devised for their solution. We discuss applications to important real world problems and identify areas for further work.**

I. Introduction

FREQUENCY assignment problems arise in a wide variety of real world situations. Many may be modeled as optimization problems having the following form: Given a collection of radio transmitters to be assigned operating frequencies, find an assignment that satisfies various constraints and that minimizes the value of a given objective function. Informal methods, which attempt to find such assignments, have been in use since the beginning of the twentieth century when maritime applications of Marconi's wireless telegraph first appeared [1].

The first frequency assignment problems arose from the discovery that transmitters, assigned to the same or to closely related frequencies, had the potential to interfere with one another. Thus the first approach to frequency assignment was to minimize or eliminate this potential interference (i.e., potential interference was the first objective function). In this approach, the major constraints were the operating bandwidth of the transmitters, the band of the electromagnetic spectrum which the transmitters were capable of using, and, combining these two, the total number of frequencies available for assignment to the transmitters (under the assumption that frequencies should be assigned to discrete, evenly spaced points in a dedicated portion of the spectrum). A simple way to minimize interference was to assign different transmitters to different noninterfering frequencies or to come as close to this as was possible within the constraints. Such an approach to frequency assignment tied up a lot of the spectrum but remained viable so long as the growth of the usable spectrum kept pace with the growth in demand placed upon it.

Recently (1950–1980), the growth of the usable spectrum has slowed while the demand placed upon it has grown exponentially [2]. This turn of events has induced spectrum managers to consider different approaches to frequency assignment. In one such approach, the amount of spectrum tied up by an assignment is the objective function to be minimized, and instead of eliminating unwanted interference, conditions which place acceptable upper bounds upon interference are included among the constraints which an assignment must satisfy. This approach also calls for an ongoing evaluation of the system (e.g., the constraints, conventions, regulations, policies, and procedures) that governs the way in which the spectrum is allocated, assigned, and used. In addition, the governing system may be modified if it can be demonstrated that such modifications lead to spectrum savings and that existing conditions (e.g., technologic, methodologic, and economic) make such actions feasible. This paper will provide tools for quantifying the effects on efficient spectrum use of such modifications to the governing system. For example, suppose that improvements in UHF-TV receivers allow for the relaxation of some of the UHF taboos. One can use the tools developed here to determine which taboo(s) to modify for the maximum gain in spectrum efficiency.

It is misleading to suggest that frequency assignment problems have always been formally modeled as optimization problems. In fact, investigations of formal mathematical models of assignment problems did not appear in the literature until the 1960's (e.g., [3], [4]). These early models seem to have enjoyed very limited application and together with other frequency assignment models in existence as of 1968, were not very well known, understood or accepted by the spectrum utilization experts of that day. As evidence for this conclusion, consider that the exhaustive report on spectrum engineering [2] mentions only two frequency assignment models and describes neither of these.

Since 1968, the interest in formal frequency assignment models has increased significantly as evidenced by the articles that have appeared in the literature [5]–[18]. In addition, as early as 1975, one of these approaches had been demonstrated to outperform older frequency assignment procedures on an important real world problem [13]. In spite of these developments, many policy makers, spectrum managers, and frequency assigners remain unconvinced that formal models are a viable approach to the wide range of assignment problems which arise in the real world. (For example, a 1977 encyclopedia volume [1], devoted to spectrum management techniques, does not mention a single formal frequency assignment model.) The reasons for this skepticism are clear. First of all, the existing formal models can handle only a limited range of the wide variety of real world problems. For example, the approach applied in [13] obtains significant spectrum savings over older methods when the only interference limiting constraints are cochannel constraints. However, if adjacent channel constraints are also considered, then these spectrum savings go to zero as the ratio of adjacent channel to cochannel constraints increases. There is an even more important reason for skepticism: *there exists no unifying theory which demonstrates that formal models are a viable approach to the wide range of prob-*

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lems which arise in the real world. The purpose of this paper is to provide such a unifying theory for a wide variety of real world problems.

Recent developments in the theory of computational complexity [19]–[22] allow for the classification of optimization problems according to the "execution time efficiency" of algorithms that may be devised for their solution. For example, the book [23] classifies well over 1000 combinatorial problems but not a single frequency assignment problem is included. An important feature of the theory developed here is that, for the first time, many frequency assignment problems are classified according to their complexity.

Graph coloring is perhaps the most famous optimization problem (e.g., the four-color theorem). That this problem also is one of the most intensively investigated and applied optimization problems is dramatically evidenced by the numerous books and articles that have appeared in the literature (e.g., [25]–[81]). A second important feature of the theory developed here is that a very close connection is established between each of the frequency assignment problems of this paper and graph coloring. Among the obvious benefits of this connection is the potential application of well-known graph coloring algorithms and/or heuristics to frequency assignment problems. Graph colorers will be interested to know that the theory of frequency assignment opens up new vistas in chromatic graph theory. Real world problems now make it important to find algorithms and/or heuristics for both classical and generalized graph coloring problems.

This paper is written primarily for spectrum planners, spectrum managers, and frequency assigners. We hope it will also be read by operations researchers, computer scientists and applied mathematicians. The mathematical (i.e., graph theory, optimization theory, complexity theory) and the spectrum engineering backgrounds of members of this audience are likely to range all over the scale. For this reason, we have attempted to provide motivation for formal definitions, describe the meanings of theorems, and to illustrate concepts with examples. We have proved theorems in their least general (but most understandable) form, while only stating or mentioning more general theorems which have the same proof.

In this paper, a frequency assignment is a *function* which assigns to each member of a set of transmitters an operating frequency from a set of available frequencies. Therefore, if A is an assignment for the set of transmitters V and if v is a transmitter belonging to V, then $A(v)$ denotes the frequency assigned to v by A. In a typical frequency assignment problem, one attempts to find a frequency assignment (i.e., a function from a given set of transmitters into a given set of frequencies) that satisfies certain constraints (e.g., a collection of interference limiting rules) and that minimizes the amount of spectrum tied up by the assignment.

It is sometimes convenient to differentiate between two types of frequency assignment problems. If the assignments are confined to discrete, but not necessarily evenly spaced frequencies and we wish to emphasize this fact then the problem is called a *channel assignment problem.* We sometimes conserve space and write assignment instead of channel (or frequency) assignment. It is important to differentiate between two types of interference limiting constraints. One type of constraint specifies that if the distance between two transmitters is less than a prescribed minimum number of miles then certain combinations of assignments to this pair of transmitters are taboo or forbidden. Such constraints employ both frequency and distance separation to mitigate interference and are called *frequency-distance* (F^*D) *constraints.* An assignment problem in which the interference limiting constraints are all F*D constraints is called a *frequency-distance constrained assignment problem.* The paper [82] discusses the origin and application of an elaborate set of F*D constraints called the UHF-TV taboos. A second type of interference limiting constraint specifies that certain combinations of assignments are forbidden for a given pair of transmitters. Superficially at least such constraints employ only frequency separation to mitigate interference and are called *frequency* (F) *constraints.* An assignment problem in which the interference limiting constraints are all frequency constraints is called a *frequency constrained assignment problem.* The papers [10], [18] investigate such problems.

We have mentioned that an assignment should not needlessly tie up spectrum. Traditionally, this has meant that the span of an assignment for a given set of transmitters must be minimized (where the span of an assignment is the largest frequency assigned to a transmitter in the set minus the smallest frequency assigned to a transmitter in the set). An assignment problem in which our objective is to minimize the span of an assignment is called *a minimum span assignment problem.* The papers [9], [13], [17], [18] investigate such problems.

Can a minimum span assignment waste spectrum? The answer is yes for channel assignment problems with interference limiting constraints other than cochannel constraints. That is, for such problems it is not uncommon for a minimum span assignment to assign transmitters to more frequencies than does a second assignment which may or may not be a minimum span assignment. In fact, for many common instances of assignment problems it is impossible to find a minimum span assignment which actually uses the minimum number of frequencies required. (See Examples One and Two in Section III for details). This potentially useful phenomenon makes it important to formalize a new type of assignment problem. The number of frequencies that an assignment actually uses is called its *order* and an assignment problem in which our objective is to minimize the span of an assignment subject to the additional constraint that its order is minimized is called *a minimum-order assignment problem.*

In Section II, we set down our conventions, notations, and other preliminary definitions. In Section III, we develop the elementary theory of frequency-distance constrained channel assignment problems (both minimum span and minimum order). Section IV presents a parallel development for the more general frequency constrained channel assignment problems. Section V presents other more complicated assignment problems and indicates how to develop a theory which parallels that of Section III for these problems. We also discuss other optimization problems some of which appear to be related to frequency assignment problems. In Section VI, guided by our efforts in Sections III, IV, and V, we formulate and investigate generalized graph coloring problems and, once again, indicate how to develop a theory for these problems that parallels that of Section III. In Section VII, we show that each of the assignment problems of Sections III, IV, and V is equivalent to a generalized graph coloring problem. Using these equivalences, we are able to classify many real world assignment problems according to their computational complexity. In Section VIII, we conclude with a summary and a discussion of real world applications of our findings. In addition, we suggest topics for further study.

II. Definitions and Notation

This paper contains the following notations and terminology. X is a subset of Y, $X \subseteq Y$; X is a proper subset of Y, $X \subset Y$; A is a function from X into Y (or A is an assignment of members of X to members of Y). $A: X \to Y$; the cardinal number of the set X, $|X|$; the empty set $\{\ \}$; the integers Z; the rationals Q; the positive integers, rationals and reals, respectively Z^+, Q^+, and R^+; the nonnegative integers, rationals and reals, respectively Z_0^+, Q_0^+, and R_0^+; the absolute value of the number a, $|a|$; the largest number in X, a finite nonempty subset of Z_0^+, $\max X$; the smallest number in X, a nonempty subset of Z_0, $\min X$; the greatest lower bound of X, a nonempty subset of Q_0^+, $\inf X$; the Euclidean distance between u and v, two points in the plane $D(u, v)$. If $A: X \to Y$ and x belongs to X, then $A(x)$ is the element of Y that A assigns to x and $A(X)$ equals $\{A(x) | x \text{ belongs to } X\}$. If a and b belong to Q then, $(a, b)_Q$ equals $\{c | c \text{ belongs to } Q \text{ and } a < c < b\}$ and $[a, b]_Q$ equals $\{c | c \text{ belongs to } Q \text{ and } a \leqslant c \leqslant b\}$.

If V is a finite set and E is a specified set of two element subsets of V, then $G = (V, E)$ is a *graph* with *vertex set* V and *edge set* E. To simplify notation, the two element subset $\{u, v\}$ belonging to E is denoted by uv. If $G = (V, E)$ and uv belongs to E then u and v are *adjacent vertices* in G. The graph $G = (V, E)$ is *complete* if uv belongs to E whenever $u \neq v$. The graph $G' = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If $G = (V, E)$, $V' \subseteq V$, and $E' = \{uv | uv \text{ belongs to } E, u \text{ and } v \text{ belong to } V'\}$, then the graph (V', E') is denoted $\langle V' \rangle$ and is called *the subgraph of G induced by* V'. If H is a complete subgraph of G and H is not properly contained in a complete subgraph of G, then H is a *clique* of G. The *clique number* of G is the number of vertices in the largest clique of G and is denoted by $W(G)$. The *chromatic number* of G is denoted by $X(G)$ and is the minimum number of colors necessary to color the vertices of G such that no two adjacent vertices receive the same color. A graph G is *perfect* if $X(H) = W(H)$ for every induced subgraph H of G.

A graph $G = (V, E)$ is called an *intersection graph* for F, a family of sets, if there exists a one-to-one correspondence, $f: V \to F$, such that uv is an element of E if and only if $f(u)$ and $f(v)$ have nonempty intersection. Conversely, F is called an *intersection model* for G if G is an intersection graph for F. If F is a finite collection of intervals on the real line then an intersection graph for F is called an *interval graph*. If F is a finite collection of arcs on a circle then an intersection graph for F is called *a circular-arc graph*. If, in addition, no arc in F contains another arc, G is called a *proper circular arc graph*.

III. Frequency-Distance Constrained Channel Assignment Problems

Complex frequency assignment problems are most easily discussed in terms of formal models. In this paper, all assignment problems are modeled as optimization problems. All but three of these are combinatorial optimization problems called search problems [23]. For our purposes, it is not necessary to give a rigorous definition of "search problem." The concept will be amply illustrated by many examples. In this section, we investigate cochannel, adjacent channel, and more complex frequency-distance constrained channel assignment problems. We assume that everything is uniform. That is, the terrain is uniform; the receivers are uniform; the transmitters are omnidirectional and all have the same power and operating bandwidth. Spectrum managers sometimes make these assumptions when taking a nationwide or regional approach to an assignment problem (e.g., UHF-TV as in [82]). We investigate the traditional minimum span approach to channel assignment and show that for some situations a new approach called the minimum-order approach may be more desirable. We show that the distinction between the minimum span approach and the minimum-order approach is lost on the cochannel assignment problem. In addition, we develop a theory which relates the two approaches and their common subproblem. Finally, for the reader who does not wish to work through the proofs of theorems, there is a summary at the end of this section in which we present an informal discussion of the theory.

A. The Frequency-Distance Constrained Cochannel Assignment Problem

The paper [13] discusses the following search problem called the *F*D constrained cochannel assignment problem* (*F*D-CCAP*). Given V a finite subset of the plane and d a positive rational number, the problem is to find an assignment $A: V \to Z^+$ which satisfies the conditions

$$\max A(V) \text{ is as small as possible and} \tag{1}$$

$$\text{if } u \text{ and } v \text{ are elements of } V, u \neq v \text{ and } D(u, v) \leqslant d \text{ then } A(u) \neq A(v). \tag{2}$$

The set V may be thought of as the locations of radio transmitters and $A: V \to Z^+$ as an assignment of channels to these transmitters. Thus the assignment A assigns the channel $A(v)$ to the transmitter located at v. Condition (2) requires that transmitters assigned to the same channel (i.e., cochannel transmitters) be separated by a distance greater than d. For this reason, condition (2) is called a *cochannel constraint*. An assignment $A: V \to Z^+$ which satisfies (2) is called a *feasible assignment for V and d*. The condition (1) is motivated by our desire to conserve spectrum; and if $A: V \to Z^+$ is a feasible assignment for V and d which satisfies (1) then A is called an *optimal assignment for V and d* and $\max A(V)$ is denoted $m(V, d)$. Thus $\{1, 2, \cdots, m(V, d)\}$ is the smallest set of channels which will accommodate an assignment of channels to the transmitters in V, which does not violate the cochannel constraint.

Throughout the rest of this paper we will use a standard format for specifying search problems. A restatement of F*D-CCAP illustrates this format.

*F*D-CCAP* (problem name)
INSTANCE: V a finite subset of the plane and $d > 0$ a rational number.
FIND: $A: V \to Z^+$ a feasible assignment for V and d such that $\max A(V)$ is as small as possible.

The standard format consists of three parts: the first part is the problem name, the second part specifies a generic instance of the problem, and the third part describes, in terms of the generic instance, the object(s) of the search. For each of the search problems of this paper, we establish that the search will not be fruitless; i.e., for each generic instance there exists at least one object of the search. Therefore, for our purposes, an algorithm (or computer program) is called *a solution of the search problem S* if it accepts as input any generic instance of the problem S and returns as output an object of the search. A search problem is *undecidable* if it is impossible to specify any algorithm which is a solution. An algorithm *runs in polynomial time* if it always terminates within a number of steps which is bounded above by some polynomial in the size of the input. A solution of a search problem is called an *efficient solution* if it runs in polynomial time. For example, given $V = \{v_1, v_2, \cdots, v_n\}$ and d an instance of F*D-CCAP, let $F(V, d)$ consist of all $A: V \to \{1, 2, \cdots, n\}$ which are feasible assignments for V and d. $F(V, d)$ is not empty since $A: V \to \{1, 2, \cdots, n\}$ defined $A(v_i) = i$ for $i = 1, \cdots, n$ is feasible. An exhaustive search of the finite set $F(V, d)$ will yield an optimal assignment for V and d. This exhaustive search can be formalized as an algorithm which solves F*D-CCAP. Therefore F*D-CCAP is decidable. For future reference we formalize these conclusions.

Theorem 1: If V is a finite subset of the plane and $d > 0$ is a rational number, then there exists $A: V \to \{1, \cdots, m(V, d)\}$ an optimal assignment for V and d.

Theorem 2: F*D-CCAP is decidable.

Exhaustive search algorithms are inefficient and, in practice, can only be applied to "small" problems. The likelihood of our finding an efficient solution for F*D-CCAP will be discussed in Sections VII and VIII. The reader who is interested in learning more about search problems and their computational complexity is referred to [23].

B. *The Frequency-Distance Constrained Adjacent Channel Assignment Problem (F*D-ACAP)*

The paper [13] discusses the following assignment problem. Let V be a finite subset of the plane and let $D = \{d(0), d(1)\}$ where $d(0) > d(1) > 0$ are rational numbers. If $A: V \to Z^+$ satisfies the condition, if u and v are elements of V,

$$u \neq v \quad \text{and} \quad D(u, v) \leqslant d(i)$$

then

$$|A(u) - A(v)| \neq i, \qquad \text{for } i = 0, 1, \tag{3}$$

then A is called *a feasible assignment for V and D*. When $i = 0$, (3) becomes a cochannel constraint and requires that cochannel transmitters be separated by a distance greater than $d(0)$. When $i = 1$, equation (3) becomes an adjacent channel constraint and requires that transmitters assigned to adjacent channels be separated by a distance larger than $d(1)$. An adjacent channel constraint is required, in practice, when a receiver tuned to a transmitter in V cannot tolerate the interference generated by adjacent channel transmitters which are "close" (in distance) to the receiver.

If A is a feasible assignment for V and D, then we say that $L = \max A(V)$ *accommodates V and D* and the smallest such L, denoted $m(V, D)$, is called *the minimum span of a feasible assignment for V and D*. If $A: V \to \{1, \cdots, m(V, D)\}$ is feasible for V and D, then A is called *a minimum span assignment for V and D*.

*F*D-ACAP*

INSTANCE: V a finite subset of the plane and $D = \{d(0), d(1)\}$ where $d(0)$ and $d(1)$ are positive rational numbers.

FIND: $A: V \to \{1, \cdots, m(V, D)\}$ a minimum span assignment for V and D.

Traditionally (see [9], [13], [17], and [18]), minimum span assignments have been regarded as mathematically optimal from the point of view of minimizing spectrum waste. Can a minimum span assignment waste spectrum?

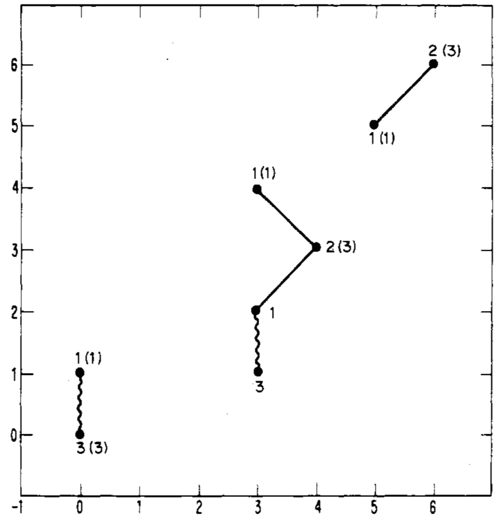

Fig. 1. Graphical depiction of the set of transmitter locations, the forbidden combinations of channel assignments, the minimum span assignment A, and the minimum-order assignment B of Example One.

C. *Example One: A Minimum Span Assignment That Wastes Spectrum*

Let $V = \{(0, 0), (0, 1), (3, 1), (3, 2), (3, 4), (4, 3), (5, 5), (6, 6)\}$ and $D = \{d(0), d(1)\}$ where $d(0) = 1.415$ and $d(1) = 1$. One can show by exhaustive search that $A: V \to \{1, 2, 3\}$ defined by $A(0, 1) = A(3, 2) = A(3, 4) = A(5, 5) = 1$, $A(0, 0) = A(3, 1) = 3$, and $A(4, 3) = A(6, 6) = 2$ is a minimum span assignment for V and D. However, $B: V \to \{1, 2, 3\}$ defined by $B(4, 3) = B(6, 6) = 3$ and $B(v) = A(v)$ otherwise is feasible for V and D. In addition, B uses only two channels whereas A uses three. Thus minimum span assignments may waste spectrum. Notice that B is also a minimum span assignment for V and D. One can show that no feasible assignment for V and D uses fewer than two channels. Therefore B is called a minimum order feasible assignment for V and D (where the order of an assignment is the number of channels actually used by the assignment). Fig. 1 depicts this example graphically. In this figure, transmitters separated by a distance equal to or less than the adjacent channel distance requirement ($d(1) = 1$) are connected by a wavy line and cannot be assigned the same or adjacent channels. The transmitters separated by a distance larger than the adjacent channel distance requirement but equal to or less than the cochannel distance requirement ($d(0) = 1.415$) are joined by a smooth line and cannot be assigned the same channel (but may be assigned adjacent channels). The numerals adjacent to the transmitter locations (but not in parentheses) constitute the minimum span assignment A. The numerals in parentheses constitute the minimum-order assignment B. Can it be that all of the minimum span assignments, for a particular problem, waste spectrum?

D. *Example Two: All Minimum Span Assignments Waste Spectrum*

Let $V = \{(2, -2), (2, 0), (2, 1), (2, 3), (3, 2), (4, 0)\}$ and $D = \{d(0), d(1)\}$ where $d(0) = 3$ and $d(1) = 2$. One can show by exhaustive search that six is the minimum span of a feasible assignment for V and D and that each of the minimum span assignments for V and D has order five or six. However, $A: V \to \{1, 2, \cdots, 7\}$ defined by $A(2, -2) = 3$, $A(2, 0) = 7$, $A(2, 1) = 5$, $A(2, 3) = 1$, $A(3, 2) = 3$ and $A(4, 0) = 1$ is a feasible assignment for V and D that uses only four channels. Therefore,

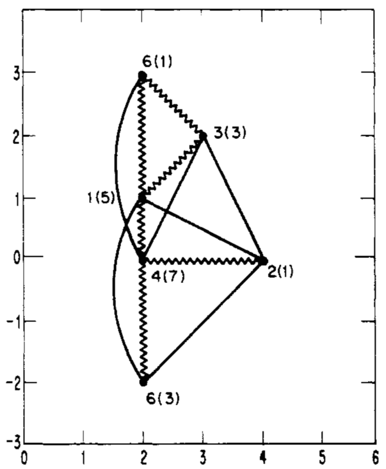

Fig. 2. Graphical depiction of the set of transmitter locations, the forbidden combinations of channel assignments, a minimum span assignment and the minimum order assignment A of Example Two.

in certain situations, A is more desirable than any minimum span assignment for V and D (since each of the minimum span assignments uses more channels). Fig. 2 depicts this example graphically.

The distinction between minimizing the span of an assignment versus minimizing the order is lost on F*D-CCAP since every minimum span assignment for an instance of F*D-CCAP is also a minimum-order assignment (see Theorems 15 and 26 below). However, the majority of real world assignment problems are more complex than F*D-CCAP. And for these more complex problems it is easy to find examples like the ones above. That is, minimum span assignments which fail to have minimum order abound in the real world. It is important that we investigate this potentially useful phenomenon. Before defining the search problems which capture the essence of minimum span and minimum order, we present another example to further motivate the following definitions and theory.

E. Example Three: UHF-TV

F*D constraints, other than cochannel and adjacent channel are often imposed in practice. For example, if V is a set of locations of UHF-TV transmitters in the Eastern U.S., then $A: V \to Z^+$ is a feasible assignment of channels for V if and only if the following condition is satisfied.

If u and v are elements of $V, u \neq v$, and $D(u, v) \leqslant M(i)$ then

$$|A(u) - A(v)| \neq i, \quad \text{for } i = 0, 1, 2, 3, 4, 5, 7, 8, 14, \text{ and } 15. \tag{4}$$

Where, $M(0) = 155$, $M(1) = 55$, $M(2) = M(3) = M(4) = M(5) = M(8) = 20$, $M(7) = M(14) = 60$ and $M(15) = 75$ are mileage separations required of transmitters assigned to channels separated by 0, 1, 2, 3, 4, 5, 8, 7, 14, and 15 channels, respectively. There is no mileage separation requirement for transmitters separated by $6, 9, 10, 11, 12, 13, 16, 17, \cdots$ channels. Let $R = \{(T(i), d(i)) | i = 0, 1, 2, 3, 4\}$ where $T(0) = \{0\}$, $T(1) = \{0, 15\}$, $T(2) = \{0, 7, 14, 15\}$, $T(3) = \{0, 1, 7, 14, 15\}$, $T(4) = \{0, 1, 2, 3, 4, 5, 7, 8, 14, 15\}$, $d(0) = 155$, $d(1) = 75$, $d(2) = 60$, $d(3) = 55$ and $d(4) = 20$. The following condition is a convenient way to express (4). If u and v are elements of V, $u \neq v$, and $D(u, v) \leqslant d(i)$, then

$$|A(u) - A(v)| \text{ is not an element of } T(i), \quad \text{for } i = 0, 1, 2, 3, 4. \tag{5}$$

The pair $(T(0), d(0))$ is called UHF-TV's cochannel constraint, since for $i = 0$, (5) requires that cochannel stations be separated by more than 155 mi. Similarly, the pair $(T(1), d(1))$ is called UHF-TV's 15th adjacent channel constraint since for $i = 1$, (5) requires that transmitters assigned to channels separated by exactly 15 channels be separated by more than 75 mi; the pair $(T(2), d(2))$ is called UHF-TV's 7th and 14th adjacent channel constraint since for $i = 2$ (5) requires that transmitters assigned to channels separated by exactly 7 or 14 channels be separated by more than 60 mi, etc. The pairs $(T(i), d(i))$ for $i = 1, 2, 3, 4$ are F*D constraints related to the UHF-TV receiver rejection characteristics. The paper [82] discusses this relationship and the possibility that improvements in receiver rejection characteristics may allow for the relaxation of some of these constraints.

The UHF-TV assignment problem is more involved than F*D-CCAP or F*D-ACAP and once again a minimum span assignment may fail to be a minimum-order assignment. To illustrate, let $V = \{v_1, v_2, v_3, v_4, v_5\}$ where $v_1 = (20, 10)$, $v_2 = (75, 100)$, $v_3 = (100, 80)$, $v_4 = (120, 80)$, and $v_5 = (100, 150)$. One can show by exhaustive search that $A: \to \{1, 2, 3, 4, 5, 6, 7\}$ defined by $A(v_1) = 4$, $A(v_2) = 3$, $A(v_3) = 1$, $A(v_4) = 7$, and $A(v_5) = 2$ is a minimum span assignment for V subject to the UHF taboos. Similarly $B: V \to \{1, 2, 3, 4, 5, 6, 7\}$ defined by $B(v_1) = 2$ and $B(v_i) = A(v_i)$ for $i = 2, 3, 4, 5$ is a minimum-order assignment for V which is more desirable than A. Fig. 3 depicts this example graphically. In this figure, transmitters separated by any distance d that is equal to or less than the cochannel distance requirement ($d(0) = 155$) are connected by a line and this line is labeled with $T(i)$ if $d(i + 1) < d \leqslant d(i)$. Thus, if two transmitters v_i and v_j are connected by a line that is labeled with $T(i)$ then any feasible assignment A must have the property $|A(v_i) - A(v_j)|$ is not an element of $T(i)$ (e.g., $|A(v_2) - A(v_5)|$ cannot belong to $\{0, 7, 14, 15\}$, $|A(v_2) - A(v_5)|$ cannot belong to $\{0, 1, 7, 14, 15\}$, etc.). The numerals adjacent to the transmitter locations constitute the minimum span assignment A. The numerals in parentheses constitute the minimum-order assignment B.

*F. A General Minimum Span Channel Assignment Problem (F*D-CAP)*

If $d(0) > d(1) > \cdots > d(m) > 0$ are rational numbers and $\{0\} = T(0) \subset T(1) \subset \cdots \subset T(m)$ are finite subsets of Z_0^+ then $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$ is called *a set of F*D-constraints*. If $k \geqslant 0$ and k is an element of $T(j)$ but k is not an element of $T(j - 1)$, then the pair $(T(j), d(j))$ is called *R's kth-channel constraint*, and $d(j)$ is called *R's kth-channel distance constraint*. R's 0th-channel constraint is also called R's cochannel *constraint*; R's 1st-channel constraint is also called *R's adjacent channel constraint*; and for $k \geqslant 2$, R's kth channel constraint is also called *R's kth adjacent channel constraint*.

Let V be a finite subset of the plane and let $R = \{(T(i), d(i)) | i = 0, 1, \cdots, m\}$ be a set of F*D constraints. If $A: V \to Z^+$ satisfies: $|A(u) - A(v)|$ is not an element of

$$T(i) \text{ whenever } u \neq v \text{ and } D(u, v) \leqslant d(i), \quad \text{for } i = 0, 1, \cdots, m, \tag{6}$$

then A is called *a feasible assignment for V and R*. Thus, if the distance between two transmitters u and v is less than or equal to $d(i)$, then certain combinations of assignments to this pair of transmitters are taboo. In particular, any assignment in which $|A(u) - A(v)|$ is an element of $T(i)$ is forbidden by condition (6).

Let $F(V, R)$ denote the set of all feasible assignments for V and R. Let ℓ belong to Z^+ and let $F(V, R, \ell)$ denote $\{A | A$ is an element of $F(V, R)$ and $\max A(V) \leq 1\}$. If $|V| = n$, then let $M = 1 + \max\{\max T(i) | i = 0, \cdots, m\}$ and let $M(V, R) = 1 + (n - 1)M$.

Theorem 3: If $\ell \geq M(V, R)$, then $F(V, R, \ell)$ is not empty.

Proof: Let $v_1, v_2, \cdots, v_n$ be a list of V and define $A: V \to \{1, 2, \cdots, \ell\}$ by $A(v_i) = 1 + (i - 1)M$ for $i = 1, 2, \cdots, n$. It is easy to see that A is feasible for V and R and that $\max A(V) = M(V, R)$. Q.E.D.

Since $F(V, R, \ell) \subseteq F(V, R)$ for each ℓ, we have the following result as a corollary to Theorem 3.

Theorem 4: $F(V, R)$ is not empty.

If A is an element of $F(V, R)$ then we say that $\ell = \max A(V)$ *accommodates* V *and* R. The smallest such ℓ is denoted $m(V, R)$ and is called *the minimum span of a feasible assignment for* V *and* R. Thus $m(V, R) = \min\{\max A(V) | A$ is an element of $F(V, R)\}$ and the following results are immediate.

Theorem 5: $F(V, R, \ell)$ is not empty if and only if $\ell \geq m(V, R)$.

Theorem 6: $m(V, R) \leq M(V, R)$.

If A is an element of $F(V, R)$ and $\max A(V) = m(V, R)$ then A is called *a minimum span assignment for* V *and* R. We now formulate a general minimum span search problem called the *F*D constrained channel assignment problem.*

*F*D-CAP*

INSTANCE: V a finite subset of the plane, and R a set of F*D constraints.

FIND: $A: V \to \{1, \cdots, m(V, R)\}$ a minimum span assignment for V and R.

Recall that an algorithm solves F*D-CAP if, given V and R as input, it returns a minimum span assignment for V and R. Since $F(V, R, \ell)$ is finite and nonempty (where $\ell = M(V, R)$) an exhaustive search will yield a minimum span assignment for V and R.

Theorem 7: If V is a finite subset of the plane and R is a set of F*D constraints, then there exists $A: V \to \{1, \cdots, m(V, R)\}$ a minimum span assignment for V and R.

Theorem 8: F*D-CAP is decidable.

Example Four: An algorithm which solves F*D-CAP also solves any subproblem of F*D-CAP. Each of the minimum span search problems discussed earlier in the section are subproblems of F*D-CAP and may be obtained from F*D-CAP by restricting the form of R. For example, F*D-CCAP is obtained when R is restricted to have the form $\{(\{0\}, d(0))\}$; F*D-ACAP is obtained when R is restricted to have the form $\{(\{0\}, d(0)), (\{0, 1\}, d(1))\}$; and F*D-UHF is obtained when $R = \{(\{0\}, 155), (\{0, 15\}, 75), (\{0, 7, 14, 15\}, 60), (\{0, 1, 7, 14, 15\}, 55), (\{0, 1, 2, 3, 4, 5, 7, 8, 14, 15\}, 20)\}$.

G. A General Minimum-Order Assignment Problem with Limited Bandwidth (F*D-CAPOL)

If A is an element of $F(V, R)$, then $|A(V)|$ is called the order of A and is denoted $o(A)$. Thus $o(A)$ is the number of channels actually used by A. If $\ell \geq m(V, R)$ and $L = \{1, 2, \cdots, \ell\}$, then $\min\{o(A) | A$ is an element of $F(V, R, \ell)\}$ is called *the minimum order of a feasible assignment for* V *and* R *in* L and is denoted $o(V, R, \ell)$. Let $m_0(V, R, \ell)$ denote $\min\{\max A(V) | A$ is an element of $F(V, R, \ell)$ and $o(A) = o(V, R, \ell)\}$. If A belongs to $F(V, R, \ell)$, $o(A) = o(V, R, \ell)$, and $\max A(V) = m_0(V, R, \ell)$, then A is called *a minimum-order assignment for* V *and* R *in* L. We are now ready to formulate a minimum-order search problem called the *F*D-constrained minimum-order channel assignment problem with limited bandwidth.*

*F*D-CAPOL*

INSTANCE: V a finite subset of the plane, R a set of F*D-constraints and $\ell \geq m(V, R)$.

FIND: $A: V \to L = \{1, 2, \cdots, \ell\}$ a minimum-order assignment for V and R in L.

By Theorem 5, $F(V, R, \ell)$ is not empty when $\ell \geq m(V, R)$. Therefore, by exhaustive search there is a feasible assignment $A: V \to L$ which uses exactly $o(V, R, \ell)$ channels and no assignment A' which is an element of $F(V, R, \ell)$ uses fewer than $o(V, R, \ell)$ channels. Again by exhaustive search (this time on the nonempty finite set $\{A | A$ is an element of $F(V, R, \ell)$ and $o(A) = o(V, R, \ell)\}$) there is a feasible assignment $A: V \to \{1, 2, \cdots, m_0(V, R, \ell)\}$ which uses exactly $o(V, R, \ell)$ channels and $m_0(V, R, \ell) \leq \ell$ is the smallest number of channels that will accommodate such an assignment. We formalize these results as theorems.

Theorem 9: If $\ell \geq m(V, R)$, then there exists $A: V \to L = \{1, 2, \cdots, \ell\}$ a minimum-order assignment for V and R in L.

Theorem 10: F*D-CAPOL is decidable.

An algorithm which solves F*D-CAPOL also solves any subproblem of F*D-CAPOL including F*D-CCAPOL, F*D-ACAPOL, and F*D-UHFOL (where these subproblems are obtained by restricting the form of R exactly as was done in Example Four). In Example One above, A is a minimum span assignment for V and R but is not a minimum-order assignment for V and R in $L = \{1, \cdots, 6\}$. Therefore, F*D-ACAP is not equivalent to F*D-ACAPOL. We shall see (Theorem 15 below), however, that F*D-CCAPOL is equivalent to F*D-CCAP.

If $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$ is a set of F*D constraints, then let R_c denote $\{(T(0), d(0))\}$, R's cochannel constraint. Let $m_c(V, R)$ denote $\max A(V)$ where A is an optimal assignment for V and R_c. Notice that V and R_c is an instance of F*D-CCAP and that $\{1, \cdots, m_c(V, R)\}$ is the smallest set of channels which will accommodate a feasible assignment for V, when only R's cochannel constraint must be satisfied. We will show that for every $\ell \geq m(V, R)$, $m_c(V, R)$ is a lower bound on the minimum order of a feasible assignment for V and R in L.

Theorem 11: If A is an element of $F(V, R, \ell)$, then $o(A) \leq \max A(V)$.

Proof: $A(V) \subseteq \{1, \cdots, \max A(V)\}$ therefore $o(A) \leq \max A(V)$. Q.E.D.

Theorem 12: If A is an element of $F(V, R, \ell)$ and $\max A(V) = m_c(V, R)$, then A is an element of $F(V, R_c)$ and $o(A) = m_c(V, R)$.

Proof: If A is an element of $F(V, R, \ell)$, and u and v are elements of V, then $|A(u) - A(v)|$ is not an element of $T(i)$ whenever $u \neq v$ and $D(u, v) \leq d(i)$ for each $i = 0, 1, \cdots, m$. Therefore, $|A(u) - A(v)| \neq 0$ whenever $u \neq v$ and $D(u, v) \leq d(0)$ since 0 is an element of $T(i)$ for each $i = 0, \cdots, m$. In other words, A is an element of $F(V, R_c)$. By Theorem 11, $o(A) \leq \max A(V) = m_c(V, R)$. Assume that $o(A) \neq m_c(V, R)$ and therefore that $o(A) < m_c(V, R)$. It follows that $A(V)$ is a proper subset of $\{1, \cdots, m_c(V, R)\}$. Therefore, let M be the largest element of $\{1, \cdots, m_c(V, R)\}$ which is not in $A(V)$

and define $A: V \to \{1, \cdots, M\}$ by $A'(v) = M$ if $A(v) = m_c(V, R)$ and $A'(v) = A(v)$ otherwise. Now A' is an element of $F(V, R_c)$ since if u and v are elements of V and $u \neq v$:

Case 1: If $A(u) \neq m_c(V, R)$ and $A(v) = m_c(V, R)$, then $A'(u) = A(u)$ is an element of $A(V)$, $A'(v) = M$ is not an element of $A(V)$ and therefore $A'(u) \neq A'(v)$:

Case 2: If $A(u) = A(v) = m_c(V, R)$, then $D(u, v) > d(0)$;

Case 3: If $A(u) \neq m_c(V, R)$ and $A(v) \neq m_c(V, R)$, then $|A'(u) - A'(v)| = |A(u) - A(v)| \neq 0$ since A is an element of $F(V, R_c)$.

Therefore, we have A' is an element of $F(V, R_c)$ and by definition of A', $\max A'(v) = M < m_c(V, R)$. This is impossible since by definition $m_c(V, R) = \min \{\max A(V) | A$ is an element of $F(V, R_c)\}$. Therefore $o(A) = m_c(V, R)$. Q.E.D.

Theorem 13: If A is an element of $F(V, R, \ell)$ and $\max A(V) = m_c(V, R)$, then $A(V) = \{1, 2, \cdots, m_c(V, R)\}$.

Proof: $A(V) \subseteq \{1, 2, \cdots, m_c(V, R)\}$ by Theorem 11, and inclusion cannot be proper since otherwise $o(A) < m_c(V, R)$ which is impossible by Theorem 12. Q.E.D.

Theorem 14: If A is an element of $F(V, R, \ell)$ and $\max A(V) = m_c(V, R)$, then $o(A) = m_c(V, R) = o(V, R, \ell) = m_0(V, R, \ell)$.

Proof: By Theorem 12, $o(A) = m_c(V, R)$ and by definition $o(V, R, \ell) \leqslant o(A) = m_c(V, R)$. Suppose that, $o(V, R, \ell) < o(A)$ and let A' be an element of $F(V, R, \ell)$ such that $o(A') = o(V, R, \ell)$. Let C_i, $i = 1, \cdots, o(A')$ be an indexing of $A'(V)$ and define $A'': V \to \{1, 2, \cdots, o(A')\}$ by $A''(v) = i$ if $A'(v) = C_i$. Now A'' is an element of $F(V, R_c)$ and $\max A''(V) = o(A') < m_c(V, R)$ which is impossible. Therefore $o(A) = o(A') = o(V, R, \ell)$. We now have A is an element of $F(V, R, \ell)$ and $o(A) = o(V, R, \ell)$, therefore, by definition of $m_0(V, R, \ell)$, $\max A(V) \geqslant m_0(V, R, \ell)$. But $m_c(V, R) = \max A(V) > m_0(V, R, \ell)$ is impossible since if A^* is a minimum-order assignment for V and R in L, then A^* is an element of $F(V, R_c)$ (by Theorem 12) and $\max A^*(V) = m_0(V, R, \ell) \geqslant m_c(V, R)$ by definition of $m_c(V, R)$. Q.E.D.

The following result is an immediate corollary.

Theorem 15: F*D-CCAP is equivalent to F*D-CCAPOL.

Theorem 16: If A is an element of $F(V, R)$, then $m_c(V, R) \leqslant o(A)$.

Proof: Suppose $o(A) < m_c(V, R)$ and let C_i, $i = 1, \cdots, o(A)$, be an indexing of $A(V)$. Define $A': V \to \{1, 2, \cdots, o(A)\}$ by $A'(v) = i$ if $A(v) = C_i$. Clearly A is an element of $F(V, R_c)$ and $\max A'(V) = o(A) < m_c(V, R)$ which is impossible. Q.E.D.

As promised, we have the following result as a corollary.

Theorem 17: If $\ell \geqslant m(V, R)$, then $m_c(V, R) \leqslant o(V, R, \ell)$.

By Example One, $m_c(V, R)$ is the best possible lower bound. Taking into account Theorem 15, we say that A an element of $F(V, R)$ is *a minimum-order feasible assignment for V and R* if $o(A) = m_c(V, R)$. Let $F_0(V, R)$ denote the set of all such assignments. Let ℓ be an element of Z^+ and let $F_0(V, R, \ell)$ denote $\{A | A$ is an element of $F_0(V, R)$ and $\max A(V) \leqslant \ell\}$. Let $M = 1 + \max \{\max T(i) | i = 0, \cdots, m\}$ and let $M_0(V, R) = 1 + (m_c(V, R) - 1) M$.

Theorem 18: If $\ell \geqslant M_0(V, R)$, then $F_0(V, R, \ell)$ is not empty.

Proof: By Theorem 1, there exists A' an element of $F(V, R_c)$ such that $A': V \to \{1, 2, \cdots, m_c(V, R)\}$. For each $i = 1, \cdots, m_c(V, R)$ denote $\{v | A'(v) = i\}$ by V_i. Define $A: V \to \{1, 2, \cdots, \ell\}$ by $A(v) = 1 + (i - 1) M$ for each v an element of V_i (where $M = 1 + \max \{\max T(i) | i = 0, \cdots, m\}$). It is easy to show that A is an element of $F_0(V, R)$ and that $\max A(V) = M_0(V, R) \leqslant \ell$. Q.E.D.

Theorem 19: $F_0(V, R)$ is not empty.

*H. A General Minimum-Order Assignment Problem With Unlimited Bandwidth (F*D-CAPO)*

The integer $\min \{\max A(V) | A$ is an element of $F_0(V, R)\}$ is called *the minimum span of a minimum-order feasible assignment for V and R* and is denoted $m_0(V, R)$. If A is an element of $F_0(V, R)$ and $\max A(V) = m_0(V, R)$, then A is called *a minimum-order assignment for V and R*. The following results are immediate.

Theorem 20: $F_0(V, R, \ell)$ is not empty if and only if $\ell \geqslant m_0(V, R)$.

Theorem 21: $m_0(V, R) \leqslant M_0(V, R)$.

We are now ready to formulate a second minimum-order search problem called *the F*D-constrained minimum-order assignment problem (with unlimited bandwidth)*.

*F*D-CAPO*

INSTANCE: V a finite subset of the plane, and R a set of F*D-constraints.

FIND: $A: V \to \{1, 2, \cdots, m_0(V, R)\}$ a minimum-order assignment for V and R.

Theorem 22: If V is a finite subset of the plane, and R is a set of F*D rules, then there exists a minimum-order assignment for V and R.

Proof: By Theorem 18, if $\ell = M_0(V, R)$, then $F_0(V, R, \ell)$ is finite and not empty. Thus an exhaustive search will yield a minimum-order assignment for V and R. Q.E.D.

Theorem 23: F*D-CAPO is decidable.

Any algorithm which solves F*D-CAPO also solves F*D-CCAPO, F*D-ACAPO, and F*D-UHFO (where these subproblems of F*D-CAPO are obtained by restricting the form of R exactly as was done in Example Four).

Theorem 24: If A is an element of $F(V, R, \ell)$ and $\max A(V) = m_c(V, R)$, then $o(A) = m_c(V, R) = o(V, R, \ell) = m(V, R) = m_0(V, R, \ell) = m_0(V, R)$.

Proof: See Theorem 14. Q.E.D.

We have the following two results as corollaries.

Theorem 25: If $m_c(V, R) = m(V, R)$ and $\ell \geqslant m(V, R)$, then $m_c(V, R) = o(V, R, \ell) = m(V, R) = m_0(V, R, \ell) = m_0(V, R)$.

Theorem 26: F*D-CCAPO is equivalent to F*D-CCAP.

F*D-ACAPO is not equivalent to F*D-ACAP by Example One.

Theorem 27: If $\ell \geqslant m(V, R)$, then $m_c(V, R) \leqslant m(V, R) \leqslant m_0(V, R, \ell)$.

Proof: By the definitions of $m_c(V, R)$, $m(V, R)$, $m_0(V, R, \ell)$ and the fact that $F(V, R, \ell) \subseteq F(V, R) \subseteq F(V, R_c)$. Q.E.D.

Theorem 28: If $\ell \geqslant m(V, R)$, then $m_c(V, R) \leqslant o(V, R, \ell) \leqslant m(V, R)$.

Proof: By definition $o(V, R, \ell) = \min \{o(A) | A$ is an element of $F(V, R, \ell)\}$. Therefore, $m_c(V, R) \leqslant o(V, R, \ell)$ by Theorem 16, and $o(V, R, \ell) \leqslant m(V, R)$ by Theorem 11. Q.E.D.

Theorem 29: If $\ell \geqslant m(V, R)$, then $m_c(V, R) \leqslant o(V, R, \ell) \leqslant m(V, R) \leqslant m_0(V, R, \ell) \leqslant m_0(V, R)$.

Proof: It remains to show that $m_0(V, R, \ell) \leqslant m_0(V, R)$. Suppose to the contrary that $m_0(V, R, \ell) > m_0(V, R)$. Let A

be an element of $F(V, R, \ell)$ such that $o(A) = o(V, R, \ell)$ and $\max A(V) = m_0(V, R, \ell)$. Now $m_0(V, R, \ell) > m_0(V, R)$ implies that if A' is an element of $F(V, R)$ such that $o(A') = m_c(V, R)$, then $\max A'(V) < m_0(V, R, \ell) = \max A(V)$. Thus it follows that $A'(V) \subseteq \{1, \cdots, \max A(V)\} \subseteq \{1, \cdots, \ell\}$ and A' is an element of $F(V, R, \ell)$. By definition of $o(V, R, \ell)$, we have, $o(V, R, \ell) \leqslant o(A') = m_c(V, R)$ and by Theorem 16, $o(V, R, \ell) = m_c(V, R)$. Now, by definition, $m_0(V, R, \ell) \leqslant \max A'(V)$ which contradicts the above result that $\max A'(V) < m_0(V, R, \ell)$. Q.E.D.

Example Two shows that each of the inequalities in Theorem 29 may be strict. That is, for $\ell = 6$, $m_c(V, R) < o(V, R, \ell)$ and $m_0(V, R, \ell) < m_0(V, R)$; and for $\ell = 7$, $o(V, R, \ell) < m(V, R)$ and $m(V, R) < m_0(V, R, \ell)$.

Theorem 30: If $\ell' \geqslant \ell \geqslant m(V, R)$ then $o(V, R, \ell') \leqslant o(V, R, \ell)$ and $m_0(V, R, \ell) \leqslant m_0(V, R, \ell')$.

Proof: $o(V, R, \ell') \leqslant o(V, R, \ell)$ by definition of $o(V, R, \ell)$ since $F(V, R, \ell) \subseteq F(V, R, \ell')$. $m_0(V, R, \ell) > m_0(V, R, \ell')$ leads to a contradiction by an argument identical to the one in the Proof of Theorem 29. Q.E.D.

Theorem 31: If, $\ell \geqslant m_0(V, R)$, then $m_c(V, R) = o(V, R, \ell)$ and $m_0(V, R, \ell) = m_0(V, R)$.

Proof: By Theorem 29, $m_c(V, R) \leqslant o(V, R, \ell)$. Suppose $m_c(V, R) < o(V, R, \ell)$, and let A be an element of $F(V, R, \ell)$ such that $o(A) = o(V, R, \ell)$ and $\max A(V) = M_0(V, R, \ell)$. Now, $o(V, R, \ell) > m_c(V, R)$ implies that if A' is an element of $F(V, R)$ such that $o(A') = m_c(V, R)$, then $m_c(V, R) = o(A') \leqslant \max A'(V) < m_0(V, R, \ell) = \max A(V)$. Thus A' is an element of $F(V, R, \ell)$ and $o(V, R, \ell) \leqslant o(A') = m_c(V, R)$, a contradiction. By Theorem 29, $m_0(V, R, \ell) \leqslant m_0(V, R)$. Now, let A be an element of $F(V, R, \ell)$ such that $o(A) = o(V, R, \ell)$. By what has already been shown, we have $o(V, R, \ell) = m_c(V, R)$. Therefore A is an element of $F(V, R, \ell) \subseteq F(V, R)$ and $o(A) = m_c(V, R)$. By definition, $m_0(V, R) \leqslant \max A(V) = m_0(V, R, \ell)$. Q.E.D.

I. A Summary of the Theory

At the beginning of this section, we promised to investigate the cochannel, minimum span, and minimum-order assignment problems, and to develop a theory which would illuminate the relationships among these problems. We have shown that the cochannel assignment problem plays a central role and serves to tie the other two problems together. In particular, we have demonstrated that: all of the search problems considered in this section (i.e., F*D-CAP, F*D-CAPOL, F*D-CAPO, and all subproblems of these) have algorithmic solutions (Theorems 2, 8, 10, and 23); if one is interested in merely minimizing the span of an assignment, then $m(V, R)$ channels will suffice and if any more are allocated they are wasted (Theorem 7); a minimum span assignment may tie up more channels than necessary (Examples One and Two); a minimum-order assignment (bandwidth limited or not) never ties up more channels than does a minimum span assignment and may tie up fewer channels (Theorem 28, Examples One and Two); a minimum-order assignment ties up at least $m_c(V, R)$ (the number of channels tied up by an optimal assignment for the cochannel subproblem) channels and possibly more (Theorem 16 and Example Two); as the number of channels available for assignment increases from $m(V, R)$ to $m_0(V, R)$ the number of channels tied up by a minimum-order assignment decreases from $m(V, R)$ to $m_c(V, R)$ (Theorem 30); if one is interested in minimizing the order of an assignment, then $m_0(V, R)$ channels will suffice and if any more are allocated they are wasted (Theorems 22 and 31); the minimum span and minimum-order assignment problems intersect in the cochannel assignment problem (Theorems 15 and 26).

More concretely, if V is a set of locations of UHF-TV stations in the Eastern U.S., and R is the collection of UHF taboos (see Example Three) for this region, then a minimum span assignment for V and R requires exactly $m(V, R)$ channels. If the number of contiguous channels available for assignment is smaller than $m(V, R)$ then there is no assignment of channels to V that satisfies each of the UHF taboos. If R_c denotes R's cochannel constraint (i.e., $R_c = \{(\{0\}, 155)\}$) then $m_c(V, R) = m(V, R_c)$ the span (and the order) of an optimal assignment for V and R_c is equal to or less than both the span and the order of any feasible assignment for V and R. If there are $m_0(V, R)$ contiguous channels available for assignment, then there is a feasible assignment for V and R which uses exactly $m_c(V, R)$ of the $m_0(V, R)$ contiguous channels. The span of such an assignment is exactly $m_0(V, R)$. If there are C contiguous channels available for assignment and $m(V, R) \leqslant C < m_0(V, R)$, then each feasible assignment for V and R in C has order greater than $m_c(V, R)$.

IV. Frequency Constrained Channel Assignment Problems

The search problems discussed in Section III model existing and potential real world problems (e.g., the present UHF-TV problem and potential future variations upon this problem). These search problems are important but limited in scope. Important existing and potential problems do not use distance separation to mitigate interference. For example, fixed distance separation plays no role if the transmitters are mobile, if the transmitters are colocated, or if the distance between transmitters is insignificant. In this section, we develop an approach to channel assignment problems for this more complex situation. In particular, we model frequency constrained channel assignment problems (both minimum span and minimum order) as search problems; we show that these search problems extend the search problems of Section III to this more complex situation; and for these more general problems we indicate how to develop a theory that parallels the theory of Section III.

A. The Problems: F-CCAP, F-CAP, F-CAPOL, and F-CAPO

In each of the papers [8]–[10], [17] a frequency separation matrix serves as a set of interference limiting constraints. To illustrate, if $V = \{1, 2, \cdots, n\}$ denotes a set of transmitters and $t(i, j)$ is the set of forbidden channel separations for transmitters i and j, then the nxn matrix $[t(i, j)]$ is a convenient way to display and/or store these forbidden channel separations (see Example Five below for concrete illustrations). A matrix such as $[t(i, j)]$ is a natural way to model interference limiting constraints which employ only frequency separation to mitigate potential intereference (e.g., if the transmitters to which channels must be assigned are colocated, the distance separations are small enough to be insignificant or the transmitters are mobile). We now formalize such a matrix approach to frequency constrained channel assignment problems.

Let $P^*(Z_0^+) = \{S \subset Z_0^+ | S$ is empty or S is finite and 0 is an element of $S\}$, and let $V = \{1, 2, \cdots, n\}$. If $t: V \times V \rightarrow P^*(Z_0^+)$ satisfies $t(i, j) = t(j, i)$ and $t(i, i) = \{\ \}$ for all i and j in V, then t is called *a channel separation matrix* for V. If $A: V \rightarrow Z^+$ satisfies

$$|A(i) - A(j)| \text{ does not belong to } t(i, j), \quad \text{for all } i \text{ and } j \text{ in } V \tag{1}$$

then A is called *a feasible assignment for V and t*. The elements of V may be thought of as names for a set of n transmitters. Thus (1) requires that transmitters i and j not be assigned to channels with forbidden channel separations. If t is a channel separation matrix for V, then define $t_c : V \times V \to P^*(Z_0^+)$ by

$$t_c(i,j) = \begin{cases} \text{the empty set if } t(i,j) \text{ is empty} \\ \{0\} \text{ if } t(i,j) \text{ is not empty.} \end{cases}$$

Clearly, t_c is a channel separation matrix for V, and if t' is any channel separation matrix for V such that $t'(i,j) \subseteq t(i,j)$ for each i and j in V, then $t_c(i,j) \subseteq t'(i,j)$ for each i and j in V. Also, if A is a feasible assignment for V and t_c, then (1) requires that $A(i) \neq A(j)$ whenever $t_c(i,j) = \{0\}$. Therefore, t_c is called *t's cochannel submatrix*.

If t is a channel separation matrix for V, then let $F(V, t)$ denote the set of all feasible assignments for V and t. Let $m(V, t)$ denote min $\{\max A(V) | A$ is an element of $F(V, t)\}$. If A is an element of $F(V, t_c)$ and max $A(V) = m(V, t_c)$, then A is called *an optimal assignment for V and t_c*. We now formulate the *frequency constrained cochannel assignment problem*.

F-CCAP
INSTANCE: t a channel separation matrix for $V = \{1, \cdots, n\}$.
FIND: An optimal assignment for V and t_c.

If A is an element of $F(V, t)$ and max $A(V) = m(V, t)$ then A is called *a minimum span assignment for V and t*. We now formulate the *frequency constrained minimum span assignment problem*.

F-CAP
INSTANCE: t a channel separation matrix for V.
FIND: A minimum span assignment for V and t.

If $\ell \geq m(V, t)$ then, let $F(V, t, \ell)$ denote $\{A | A$ is an element of $F(V, t)$ and max $A(V) \leq \ell\}$; let $o(V, t, \ell)$ denote min $\{o(A) | A$ is an element of $F(V, t, \ell)\}$; and let $m_0(V, t, \ell)$ denote min $\{\max A(V) | A$ is an element of $F(V, t, \ell)$ and $o(A) = o(V, t, \ell)\}$. If A is an element of $F(V, t, \ell)$, $o(A) = o(V, t, \ell)$, max $A(V) = m_0(V, t, \ell)$, and $L = \{1, 2, \cdots, \ell\}$, then A is called *a minimum-order assignment for V and t and in L*. We now formulate the *frequency constrained minimum-order assignment problem* (with limited bandwidth).

F-CAPOL
INSTANCE: t a channel separation matrix for V and $\ell \geq m(V, t)$.
FIND: A minimum-order assignment for V and t in $L = \{1, \cdots, \ell\}$.

Let $m_c(V, t)$ denote $m(V, t_c)$. If A is an element of $F(V, t)$ and $o(A) = m_c(V, t)$ then A is called *a minimum-order assignment for V and t*. Let $F_0(V, t)$ denote the set of all such assignments. The integer min $\{\max A(V) | A$ belongs to $F_0(V, t)\}$ is called *the minimum span of a minimum-order feasible assignment for V and t* and is denoted $m_0(V, t)$. If A is an element of $F_0(V, t)$ and max $A(V) = m_0(V, t)$, then A is called *a minimum-order assignment for V and t*. We now formulate the *frequency constrained minimum-order assignment problem*.

F-CAPO
INSTANCE: t a channel separation matrix for V.
FIND: A minimum-order assignment for V and t.

*B. Frequency Constrained Problems Generalize F*D Constrained Problems*

The problems F-CCAP, F-CAP, F-CAPOL, and F-CAPO are called frequency constrained problems since superficially, at least, they employ only frequency separation to mitigate interference. We now show that any set of F*D constraints may be replaced by an equivalent channel separation matrix, and therefore the problems of this section contain the corresponding problems of Section III as subproblems.

Let $V = \{v_1, \cdots, v_n\}$ be a subset of the plane and let $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$ be a set of F*D constraints. Let $V' = \{1, \cdots, n\}$ and define $t' : V' \times V' \to P^*(Z_0^+)$ as follows: if $i \neq j$ and k is the smallest integer for which $D(v_i, v_j) \leq d(k)$, then $t'(i,j) = T(k)$ and otherwise $t'(i,j)$ equals the empty set. If $A : V \to Z^+$, then define $A' : V' \to Z^+$ by $A'(i) = A(v_i)$ for $i = 1, \cdots, n$.

Theorem 32: A is feasible for V and R if and only if A' is feasible for V' and t'.

Proof: Suppose A is feasible for V and R. Case 1: if $v_i \neq v_j$ and k is the smallest integer for which $D(v_i, v_j) \leq d(k)$ then $|A'(i) - A'(j)| = |A(v_i) - A(v_j)|$ is not an element of $T(k) = t'(i,j)$. Case 2: otherwise $|A'(i) - A'(j)|$ is not an element of $\{\ \} = t(i,j)$. Therefore A' is feasible for V' and t'. Conversely, suppose $|A'(i) - A'(j)|$ is not an element of $t'(i,j)$ for all i and j in V'. Also, suppose $v_i \neq v_j$ and $D(v_i, v_j) \leq d(\ell)$. Now $i \neq j$ and if k is the smallest integer for which $D(v_i, v_j) \leq d(k)$, then $t'(i,j) = T(k)$ and $|A(v_i) - A(v_j)| = |A'(i) - A'(j)|$ is not an element of $t'(i,j) = T(k) \subseteq T(\ell)$. Therefore, A is feasible for V and R. Q.E.D.

Example Five: F-ACAP, F-ACAPOL, and F-ACAPO and the subproblems of F-CAP, F-CAPOL and F-CAPO, respectively obtained by restricting $t(i,j)$ to be an element of $\{\{\ \}, \{0\}, \{0, 1\}\}$. F-UHF, F-UHFOL, and F-UHFO are the subproblems of F-CAP, F-CAPOL, and F-CAPO, respectively obtained by restricting $t(i,j)$ to be an element of $\{\{\ \}, \{0\}, \{0, 15\}, \{0, 7, 14, 15\}, \{0, 1, 7, 14, 15\}, \{0, 1, 2, 3, 4, 5, 7, 8, 14, 15\}\}$.

We have the following results as immediate corollaries of Theorem 32.

Theorem 33: F*D-CCAP, F*D-CAP, F*D-CAPOL, and F*D-CAPO are subproblems of F-CCAP, F-CAP, F-CAPOL, and F-CAPO, respectively.

Theorem 34: F*D-ACAP, F*D-ACAPOL, and F*D-ACAPO are subproblems of F-ACAP, F-ACAPOL and F-ACAPO, respectively.

Theorem 35: F*D-UHF, F*D-UHFOL, and F*D-UHFO are subproblems of F-UHF, F-UHFOL, and F-UHFO, respectively.

In Section VII, we will see that the converses of Theorems 33, 34, and 35 are not valid. Thus the frequency constrained matrix approach of this section is more general than the F*D approach of Section III. In addition, the matrix approach obscures the role played by distance separation and in this respect may hide (or encode) potentially useful information. In particular, efficient solutions for some important subproblems of F-CAP require that this encoded distance separation information be decoded. From this point of view F*D formulations of problems are more desirable. (This important point will be discussed at greater length in Section VII.)

The reader may easily verify that Theorems 1–31 of Section

III remain valid if F*D and R are replaced in every occurrence by F and t, respectively. It may be beneficial to reread the summary at the end of Section III with this transformation in mind.

V. Other Problems

The main purpose of this paper is to provide a unifying theory which demonstrates that our formal modeling approach to assignment problems is a viable one that can handle the wide range of problems which arise (or may arise) in the real world. Up until now, we have restricted our attention to situations in which assignments are confined to discrete evenly spaced frequencies. As we shall see, our formal modeling approach is not limited to such problems.

Suppose there are two or more classes of transmitters C_i where all the transmitters in C_i have the same operating power P_i and the same operating bandwidth b_i but that $P_i \neq P_j$ and/or $b_i \neq b_j$ for $i \neq j$. In addition, suppose that ideally all the transmitters should be assigned to operating frequencies in the same region of the spectrum. Results of Section III (e.g., Examples One and Two, and Theorems 29, 30, and 31) suggest that spectrum may be conserved if these different classes were to share the same band in an interwoven fashion. In this section, we investigate F*D and frequency constrained assignment problems that model this potentially useful interwoven approach to spectrum sharing. Two of these problems are not channel assignment problems, i.e., assignments are not restricted to discrete frequencies. We conclude this section with a discussion of other well-known assignment problems that are not channel or frequency assignment problems and note that some of these problems appear to be closely related to frequency assignment problems.

A. *Interwoven Mixed Service with Variable Power Transmitters*

For $i = 1, \cdots, p$, let V_i denote the set of locations of all transmitters having power P_i and let d_i denote the cochannel separation distance for a pair of transmitters in V_i. The following search problem is a natural extension of F*D-CCAP to this more complex variable power situation.

*F*D-CCAP(*)*
INSTANCE: V a finite subset of the plane, $p \leqslant |V|$, $\{V_1, V_2, \cdots, V_p\}$ a partition of V, and d_i an element of Q^+ for $i = 1, \cdots, p$.
FIND: $A : V \to Z^+$ which satisfies the conditions:

$$\max A(V) \text{ is as small as possible} \tag{7}$$

and

$$|A(u) - A(v)| > 0 \text{ whenever } u \neq v,\ u \text{ is an element of } V_i,\ v \text{ is an element of } V_j \text{ and } D(u, v) \leqslant (d_i + d_j)/2. \tag{8}$$

More generally, if V_i and P_i are as above, let $R_j = \{(T_j(i), d_j(i)) | i = 0, \cdots, m\}$ be a set of F*D constraints for transmitters in V_j. Let $m = \max\{m_j | j = 1, \cdots, p\}$ and for j, $k = 1, \cdots, p$ defined T_{jk} and d_{jk} as follows:

$$T_{kk}(i) = \begin{cases} T_k(i), & \text{for } i = 0, \cdots, m_k \\ T_k(m_k), & \text{for } i = m_k + 1, \cdots, m \end{cases}$$

$$T_{jk}(i) = T_{jj}(i) \text{ union } T_{kk}(i), \quad \text{for } j, k = 1, \cdots, p \text{ and } i = 0, \cdots, m$$

$$d_{kk}(i) = \begin{cases} d_k(i), & \text{for } i = 0, \cdots, m_k \\ 0, & \text{for } i = m_k + 1, \cdots, m \end{cases}$$

$$d_{jk}(i) = (d_{jj}(i) + d_{kk}(i))/2, \quad \text{for } j, k = 1, \cdots, p \text{ and } i = 0, \cdots, m.$$

Clearly, $\{0\} = T_{jk}(0) \subseteq T_{jk}(1) \subseteq \cdots \subseteq T_{jk}(m)$ and

$$d_{jk}(0) > d_{jk}(1) > \cdots > d_{jk}(m) > 0.$$

Therefore, let $R = \{(T_{jk}(i), d_{jk}(i)) | j, k = 1, \cdots, p \text{ and } i = 0, \cdots, m\}$ be called a set of F*D constraints for the mixed service $V = \{V_1, V_2, \cdots, V_p\}$. The following search problem is a natural extension of F*D-CAP to this more complex situation.

*F*D-CAP(*)*
INSTANCE: V is a finite subset of the plane, $p \leqslant |V|$, and R a set of F*D constraints for the mixed service $\{V_1, \cdots, V_p\}$.
FIND: $A : V \to Z^+$ which satisfies (7) and $|A(u) - A(v)|$ is not an element of $T_{jk}(i)$ whenever $u \neq v$, u is an element of V_j, v is an element of V_k, and

$$D(u, v) \leqslant d_{jk}(i), \quad \text{for } j, k = 1, \cdots, p \text{ and } i = 0, \cdots, m. \tag{9}$$

An assignment $A : V \to Z^+$ which satisfies (7) and (9) is called *a minimum span assignment for V and R*. We must remark that the search problems F*D-CAPOL(*) and F*D-CAPO(*) in which we search for minimum-order assignments for V and R in L and minimum-order assignments for V and R (with obvious definitions), respectively, have the expected interrelationships with F*D-CCAP(*) and F*D-CAP(*). That is, Theorems 1–31, and analogs to Theorems 32, 33, and 34, and 35 remain valid for these problems. (Hint: if $i \neq j$, v_i is an element of V_k, v_j is an element of V_ℓ and h is the smallest integer for which $D(v_i, v_j) \leqslant d_{k\ell}(h)$ then define $t'(i, j) = T_{k\ell}(h)$ and otherwise $t'(i, j) = \{\ \}$). In Section VII, we will see that the converses of Theorems 33, 34, and 35 are not valid for these problems.

B. *Interwoven Mixed Service With Unevenly Spaced Discrete Frequencies*

For $j = 1, \cdots, p$, let V_j, P_j and R be as in the last paragraph, and let all the transmitters in V_j have operating bandwidth b_j belonging to Q^+ with $b_i \neq b_j$ when $i \neq j$. Let $B = \{b_1, \cdots, b_p\}$. Now, in addition to allowing these different classes of transmitters to share the same band in an interwoven fashion also allow frequencies to be assigned to any element of $C'(B) = \{kb_i/2 | i = 1, \cdots, p; k = 1, 3, 5, \cdots\}$.

As motivation for this approach, consider that, in practice, many F*D constraints result from the fact that assigning transmitters to discrete evenly spaced frequencies increases the potential for intolerable interference [82]. Intuition leads one to believe, therefore, that spectrum may be conserved by allowing transmitters to be assigned to unevenly spaced frequencies. In order, not to violate our convention that search problem assignments have the form $A : V \to Z^+$, let us rename the elements of $C'(B)$. That is, if $b_i = r_i/s_i$, then let 1cm be the least common multiple of $2, s_1, \cdots, s_p$, and for $i = 1, \cdots, p$

$$\text{let } r_i' = \begin{cases} 1\text{cm} \cdot r_i/s_i, & \text{if } s_1 \cdot s_2 \cdots s_p \text{ is even} \\ 1\text{cm} \cdot r_i/2 \cdot s_i, & \text{otherwise.} \end{cases}$$

Now let $C(B) = \{1 + kr_i' | i = 1, \cdots, p; k = 0, 1, 2, \cdots\} \subseteq Z^+$. The following search problem is a natural extension of F*D-CAP(*) to this situation.

*F*D-CAP(*; *)*

INSTANCE: V a finite subset of the plane, $p \leqslant |V|$, R a set of F*D constraints for the mixed service $\{V_1, \cdots, V_p\}$, and $B = \{b_1, \cdots, b_p\} \subset Q^+$.

FIND: $A: V \to C(B)$ which satisfies (7) and $1/r_\ell' \cdot |A(u) - A(v)|$ is not an element of $T_{jk}(i)$ whenever $u \neq v$, u is an element of V_j, v is an element of V_k, and $D(u, v) \leqslant d_{jk}(i)$ for $j, k, \ell = 1, \cdots, p$ and $i = 0, \cdots, m$. (10)

If $A: V \to C(B)$ satisfies (7) and (10) then A is called *a minimum span assignment for V, R, and B.* Condition (10) requires that u and v not be assigned to certain frequencies (namely those whose differences divided by r_ℓ' belongs to $T_{jk}(i)$) when u and v are separated by less than the minimum separation distance required by R. In particular, if R is restricted to have the form $\{(T_{jk}(0), d_{jk}(0)) | j, k = 1, \cdots, p\}$, then the resulting subproblem is denoted F*D-CCAP(*; *), since in this case only cochannel assignments are constrained. The reader may define F*D-CAPOL(*; *) and F*D-CAPO(*; *), and verify that propositions analogous to Theorems 1-35 remain valid for these problems.

C. Interwoven Mixed Service With No Restriction to Discrete Frequencies

Let V be a set of locations of transmitters. Let $|V| = n$, $p \leqslant n$, and $P = \{V_1, \cdots, V_p\}$ be a partition of V. For $i = 1, \cdots, p$ let P_i, and b_i denote respectively the operating power and bandwidth of transmitters in V_i where $P_i \neq P_j$ and/or $b_i \neq b_j$ whenever $i \neq j$. If u is an element of V_j, and v is an element of V_k, then let $S_{jk}(x)$ denote the minimum frequency separation required for assignments to u and v when $x = D(u, v)$; let $T_{jk}(x)$ denote the forbidden combinations of frequency assignments for u and v when $x = D(u, v)$; and let d_{jk} denote the cofrequency distance separation required of u and v. In practice, S_{jk} and T_{jk} may be functions of P_i, P_k, b_j, b_k, the rejection characteristics of receivers that tune to transmitters in V_j and V_k, etc. If m_{jk} and m_{jk} denote respectively the min and max of $\{D(u, v) |\ u$ is in V_j and v is in $V_k\}$, then S_{jk}: $[m_{jk},\ m_{jk}]_Q \to [0,\ S_{jk}(m_{ik})]_Q$ and T_{jk}: $[m_{jk},\ M_{ik}]_Q \to P^*(Z^+)$ where we, also, require that $D(u,\ v) = 0$ iff $u = v$; $S_{jk}(x) = 0$ and $T_{jk}(x) = 0$ iff $d_{jk} < x \leqslant m_{jk}$ or $j = k$ and $x = 0$; and if $x > y$ then $S_{jk}(x) \leqslant S_{jk}(y)$ and $T_{jk}(x) \subseteq T_{jk}(y)$. Let $B = \{b_1, \cdots, b_p\}$, $b = \min \{b_i/2 | i = 1, \cdots, p\}$, $S = \{S_{jk} | j, k = 1, \cdots, p\}$, $T = \{T_{jk} | j, k = 1, \cdots, p\}$ and $I = (V, P, B, T, S)$.

If $A: V \to [b, \infty)$ satisfies: $A(v)$ is an element of Q for all v with $A(v) < \max A(V)$, $|A(u) - A(v)| \geqslant S_{jk}(D(u,\ v))$, and $\max \{A(u)/A(v),\ A(v)/A(u)\}$ is not an element of $T_{jk}(D(u, v))$ whenever $u \neq v$, u is in V_j and v is in V_k, then A is called *a feasible assignment for I.* Let $F(I)$ denote the set of all such assignments and for q an element of Q^+ let $F(I,\ q)$ denote $\{A | A$ is an element of $F(I)$ and $\max A(V) \leqslant q\}$. Let $v_1, \cdots, v_n$ be a list of V such that v_1 is an element of V_i and $b = b_i/2$. If $A: V \to [b, \infty)$ is defined by $A(v_1) = b$ and for $i = 2, \cdots, n$, $A(v_i) = (q_i + 1/n_i)A(v_{i-1})$ (where $q_i = A(v_{i-1}) + S_{jk}(D(v_{i-1}, v_i))$, and n_i is the smallest element of Z^+ such that $(q_i + 1/n_i)/A(v_{i-1})$ is not an element of $T_{jk}(D(v_{i-1}, v_i))$, v_{i-1} is an element of V_j, and v_i is an element of V_k), then A is an element of $F(I,\ q)$ when $q \geqslant A(v_n)$. Therefore, let $m(I)$ denote $\inf \{\max A(V) | A$ is an element of $F(I)\}$. If A is an element of $F(I)$ and $\max A(V) = m(I)$ then A is called *a minimum span assignment for I.* The following optimization problem is called *the frequency-distance constrained frequency assignment problem.*

*F*D-FAP*

INSTANCE: $I = (V, P, B, T, s)$

FIND: A minimum span assignment for I.

We now formulate a frequency constrained generalization of F*D-FAP. Let $V = \{1, \cdots, n\}$ be a set of transmitters, $[s(i,j)]$ be a symmetric nxn matrix where $s(i, j)$ is a nonnegative rational number that denotes the minimum frequency separation required of transmitters i and j, and let $[t(i,j)]$ be a symmetric nxn matrix where $t(i,j)$ (an element of $P^*(Z^+)$) represents the set of forbidden combinations of frequency assignments for i and j. In practice $s(i,j)$ and $t(i,j)$ may be functions of $D(i,j)$, the terrain surrounding i and j, the powers P_i and P_j of i and j, the bandwidths b_i and b_j of i and j, the rejection characteristics of receivers that tune to i and j, etc. We require that $s(i, i) = 0$ and that $t(i,j) = \{\ \}$ iff $s(i,j) = 0$ for all i and j elements of V. Let $B = \{b_1, \cdots, b_n\}$, $b = \min \{b_i/2 | i = 1, \cdots, n\}$ and $I = (V, B, t, s)$.

If $A: V \to [b, \infty)$ satisfies: $A(v)$ is in Q for all v in V with $A(v) < \max A(V)$, $|A(i) - A(j)| \geqslant s(i,\ j)$ and $\max \{A(i)/A(j),\ A(j)/A(i)\}$ is not an element of $t(i, j)$ for all i and j in V, then A is called *a feasible assignment for I.* Let $F(I)$ denote the class of all such assignments and for q in Q^+ and let $F(I, q)$ denote $\{A | A$ is an element of $F(I)$ and $\max A(V) \leqslant q\}$. Now $F(I, q) \subset F(I)$ and if $A: V \to [b, \infty)$ is defined by $A(1) = b$ and for $i = 2, \cdots, n$, $A(i) = (q_i + 1/n_i)/A(i-1)$ (where $q_i = A(i-1) + s(i,\ i-1)$, and n_i is the smallest element of Z^+ such that $(q_i + 1/n_i)/A(i-1)$ is not an element of $t(i-1, i)$), then A is an element of $F(I,\ q)$ when $q \geqslant \max A(V)$. Therefore, let $m(I)$ denote $\inf \{\max A(V) | A$ is an element of $F(I)\}$. If A is an element of $F(I)$ and $\max A(V) = m(I)$, then A is called *a minimum span assignment for I.* The following optimization problem is called *the frequency constrained frequency assignment problem.*

F-FAP

INSTANCE: $I = (V, B, t, s)$

FIND: A minimum span assignment for I.

The reader may define $m_c(I)$, $o(I, q)$, $m_0(I, q)$, $m_0(I)$, F*D-CFAP, F*D-FAPOL, F*D-FAPO, F-CFAP, F-FAPOL, F-FAPO and verify that Theorems 1–35 (excepting 2, 8, 10, and 23) remain valid. The decidability of F*D-CFAP, F-CFAP, etc., are left as exercises for complexity theorists.

D. Other Combinatorial Optimization Problems

The assignment problems of this paper are special cases of a more general assignment problem. Given a collection of consumers who place demands upon a set of resources, find an assignment of consumers to resources that satisfies various constraints and that minimizes (or in some cases maximizes) a given objective function. The approach of this paper (i.e., the modeling of frequency assignment problems as search problems) has been effectively applied to other problems of the assignment type. To illustrate, in network routing problems, calls or packets are assigned to paths or links of the network in such a way that the number of simultaneous calls through the network is maximized or the average packet delay is minimized; in school timetabling problems, instructional units are assigned to time periods, teachers, rooms, instructional equipment, or other resources in such a way that the consumed resource(s) is minimized; in computer job scheduling problems, jobs to be executed are assigned to starting times and processors in such a way that the earliest time at which all jobs are completed is minimized; and in the graph coloring problem, vertices are assigned colors in such a way that adjacent vertices are assigned different colors and the number of colors used is minimized.

We have not investigated frequency assignment problems in which the spectrum is time shared, but must remark that these problems appear to be closely related to the extensively studied computer job scheduling problems [83]-[88].

It is well known that the graph coloring problem is closely related to the cochannel assignment problem [5], [11], [12], [13] and to school timetabling problems [50], [51], [53]. In the next three sections, we extend and exploit the former relationship.

VI. Generalized Coloring Problems

The papers [10], [12], [13] discuss the relationship of F*D-CCAP to the following search problem called *the coloring problem (CP)*.

CP
INSTANCE: $G = (V, E)$ a graph.
FIND: $A : V \to Z^+$ such that $\max A(V)$ is as small as possible and $A(u) \neq A(v)$ whenever uv is an element of E (such an assignment is called an *optimal coloring for* G, $\max A(V)$ is called *the chromatic number of* G and is denoted $X(G)$).

If $G = (V, E)$ is a graph and $t : E \to P(Z_0^+)$ then t is called *an edge constraint for* G and $G_t = (G, t)$ is called *an edge constrained graph*. If $A : V \to Z^+$ satisfies $|A(u) - A(v)|$ is not an element of $t(uv)$ whenever uv is an element of E, then A is called *a feasible coloring for* G *and* t. Let $F(G, t)$ denote the class of all such colorings, and let $X_1(G, t)$ denote min $\{\max A(V) | A$ is an element of $F(G, t)\}$. If A is an element of $F(G, t)$ and $\max A(V) = X_1(G, t)$, then A is called *a minimum span coloring for* G *and* t. The integer $X_1(G, t)$ is called *the minimum span chromatic number for* G *and* t.

The following search problem extends CP to edge constrained graphs and is called the *generalized coloring problem*.

GCP
INSTANCE: $G = (V, E)$, and an edge constraint for G.
FIND: A minimum span coloring for G and t.

If A is an element of $F(G, t)$, then $|A(V)|$ is called *the order of* A and is denoted $o(A)$. If ℓ is an element of Z^+, and $\ell \geqslant X_1(G, t)$, then let $F(G, t, \ell)$ denote $\{A | A$ is an element of $F(G, t)$ and $\max A(V) \leqslant \ell\}$, let $o(G, t, \ell)$ denote min $\{o(A) | A$ is an element of $(G, t, \ell)\}$, and let $X_2(G, t, \ell)$ denote min $\{\max A(V) | A$ is in $F(G, t, \ell)$ and $o(A) = o(G, t, \ell)\}$. If $L = \{1, \cdots, \ell\}$, then $X_2(G, t, \ell)$ is called *the minimum-order chromatic number for* G *and* t *in* L. If A is an element of $F(G, t, \ell)$, $o(A) = o(G, t, \ell)$ and $\max A(V) = X_2(G, t, \ell)$, then A is called *a minimum-order coloring for* G *and* t *in* L. The following search problem extends the notion of bandwidth limited minimum-order assignment to graph coloring.

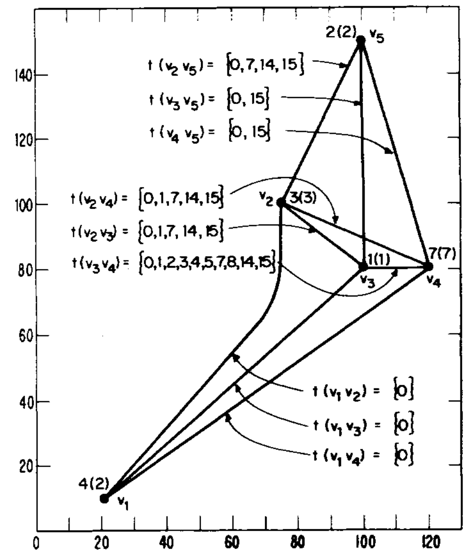

Fig. 3. Graphical depiction of the set of transmitter locations, the taboo combinations of channel assignments, the minimum span assignment A and the minimum-order assignment B of Example Three.

GCPOL
INSTANCE: $G = (V, E)$, t an edge constraint for G, and $L = \{1, \cdots, \ell\}$ where $X_1(G, t) \leqslant \ell$.
FIND: A minimum-order coloring for G and t in L.

Let $G = (V, E)$ and define $c : E \to P(Z_0^+)$ by $c(uv) = \{0\}$ for all uv in E. Clearly, c is an edge constraint for G and if t is any edge constraint for G then $c(uv) \subseteq t(uv)$ for all uv in E. Therefore, c is called *the minimal edge constraint for* G, and by definition $x_1(G, c) = X(G)$. Also, as in Section III Theorem 16, $X(G) \leqslant o(A)$ for any A in $F(G, t)$. Thus min $\{\max A(V) | A$ is an element of $F(G, t)$ and $o(A) = X(G)\}$ is denoted $X_2(G, t)$ and is called *the minimum order chromatic number for* G *and* t. If A is an element of $F(G, t)$, $o(A) = X(G)$ and $\max A(V) = X_2(G, t)$ then A is called *a minimum-order coloring for* G *and* t. The following search problem extends the notion of minimum order assignment (with unlimited bandwidth) to graph coloring.

GCPO
INSTANCE: $G = (V, E)$ and t an edge constraint for G.
FIND: A minimum-order coloring for G and t.

Example Six: Fig. 3 of Section III depicts an edge constrained graph generated by the UHF taboos and the set of transmitter locations V of Example Three. The $t(v_iv_j)$'s in Fig. 3 are the values of the edge constraint imposed by the UHF taboos. Similarly, Figs. 1 and 2 depict edge constrained graphs generated by instances of F*D-ACAP. In addition, the minimum span (minimum-order) assignments illustrated in these figures are minimum span (minimum-order) colorings for the corresponding edge constrained graphs.

By the definitions, if $\ell \geqslant X(G)$, then $X(G) = o(G, c, \ell) = X_1(G, c) = X_2(G, c, \ell) = X_2(G, c)$ and the reader may verify that Theorems 1-31 and their proofs remain valid if V, R, m_c, m, m_0, F*D-CCAP, F*D-CAP, F*D-CAPOL, and F*D-CAPOL are replaced in every instance by G, t, X, X_1, X_2, CP, GCP, GCPOL, and GCPO, respectively.

Let G and t be as above and let $C \subseteq Z^+$. If $A: V \to C$ satisfies $|A(u) - A(v)|$ is not an element $t(uv)$ for all uv in E then A is called *a feasible coloring for G, t, and C*. Let $F(G, t, C)$ denote the class of all such colorings, and let $X_1(G, t, C)$ denote $\min \{\max A(V) | A$ is an element $F(G, t, C)\}$. If A is an element of $F(G, t, C)$ and $\max A(V) = X_1(G, t, C)$, then A is called *a minimum span coloring for G, T and C*. The following search problem extends GCP to the situation of unevenly spaced colors.

GCP(*)
INSTANCE: $G = (V, E)$, t an edge constraint for G, and $C \subseteq Z^+$.
FIND: A minimum span coloring for G, t, and C.

If $t: E \to P(Z_0^+)$ is an edge constraint for $G = (V, E)$, and $s: E \to Q^+$, then (V, E, t, s) is denoted G_{ts} and is called *a doubly edge constrained graph*. If $A: V \to R^+$ satisfies: $A(v)$ is an element of Q^+ for all v in of V such that $A(v) < \max A(V)$, $|A(u) - A(v)| \geqslant s(uv)$ and $\max \{A(u)/A(v), A(v)/A(u)\}$ is not an element of $t(uv)$ for all uv in E, then A is called *a feasible coloring for G, t and s*. Let $F(G, t, s)$ denote the class of all such colorings and $X_1(G, t, s)$ denote $\inf \{\max A(V) | A$ is an element of $F(G, t, s)\}$. If A is an element of $F(G, t, s)$ and $\max A(V) = X_1(G, t, s)$, then A is called *a minimum span coloring for G, t and s*. The following optimization problem extends GCP to the situation in which there is a nondiscrete set of allowable colors.

GCP(; *)*
INSTANCE: $G = (V, E)$, $t: E \to P(Z_0^+)$, and $s: E \to Q^+$
FIND: A minimum span coloring for G, t and s.

The reader may define CP(*), GCPOL(*) and verify that theorems analogous to Theorems 1–31 remain valid for these problems. Similarly, analogs to Theorems 1–31 (excepting the proofs for 2, 8, 10, and 23) remain valid for the problems C(*; *), GCP(*; *), GCPOL(*; *) and GCPO(*; *).

VII. Assignment Problems as Generalized Coloring Problems

An important element of the scientific approach to problem solving consists of attempting to show that the problem under study is closely related to a known, well-studied problem. In this section, we show that each of the frequency constrained problems of Sections IV and V is equivalent to a generalized graph coloring problem and that each of the F*D constrained problems of Sections III and V is equivalent to a generalized coloring problem restricted to a narrow subclass of the class of all graphs. From these results, it follows that the frequency constrained approach is more general than the frequency-distance constrained approach to assignment problems.

Recent developments in the theory of computational complexity allow for the classification of optimization problems according to the "execution time efficiency" of algorithms that may be devised for their solution. Informally, if an optimization problem is *NP-hard*, then it is very unlikely that a polynomial time solution will ever be devised. (The reader who is unfamiliar with the concept of NP-hardness and who would like more information on what is meant by "very unlikely" is referred to [23] for both a low-level introductory and a rigorous treatment of these matters. In addition, the paper [24] presents an expository discussion of these matters.) Using recent results on the computational complexity of graph coloring together with the equivalences between frequency assignment problems and generalized graph coloring problems, we show that each of the assignment problems of Sections III, IV, and V is NP-hard, but that some important subproblems of these problems have efficient solutions.

A. Frequency Constrained Problems and Their Complexity

It is well known that CP is NP-hard [20] and that F-CCAP is related to CP [10]. In this paragraph, we show that F-CCAP, F-CAP, F-CAPOL, F-CAPO, F-CCAP(*; *), F-CAP(*; *), F-CAPOL(*; *), F-CAPO(*; *), F-CFAP, F-FAP, F-FAPOL and F-FAPO are equivalent to CP, GCP, GCPOL, GCPO, CP(*), GCP(*), GCPOL(*), GCPO(*), CP(*; *), GCP(*, *), GCPOL(*; *), and GCPO(*; *), respectively. From these equivalences and the NP-hardness of CP it follows that each of the twelve frequency constrained problems listed above is NP-hard.

Theorem 36: F-CCAP is equivalent to CP.

Proof: If $V = \{1, \cdots, n\}$ and t_c is an instance of F-CCAP then let $E = \{ij | t_c(i, j) \neq \{\ \}\}$. Now $G = (V, E)$ is an instance of CP and $A: V \to \{1, \cdots, X(G)\}$ an optimal coloring for G is also an optimal assignment for V and t_c with $m(V, t_c) = X(G)$. Conversely, if $G = (V, E)$ is an instance of CP where $V = \{v_1, \cdots, v_n\}$ then let $V' = \{1, \cdots, n\}$ and define

$$t_c(i, j) = \begin{cases} \{0\} & \text{if } v_i v_j \text{ is in } E \\ \{\ \} & \text{otherwise.} \end{cases}$$

Now V' and t_c is an instance of F-CCAP and if $A': V' \to \{1, \cdots, m(V', t_c)\}$ is an optimal assignment for V' and t_c, then $A: V \to Z^+$ defined $A(v_i) = A(i)$ for all $i = 1, \cdots, n$ is an optimal coloring for G and $X(G) = m(V, t_c)$. Q.E.D.

Theorem 37: F-CAP, F-CAPOL, and F-CAPO are equivalent to GCP, GCPOL, and GCPO, respectively.

Proof: If $V = \{1, \cdots, n\}$, t and 1 is an instance of F-CAPOL, then let $E = \{ij | t(i, j) \neq$ the empty set$\}$ and continue as in the proof of Theorem 36. Q.E.D.

Theorem 38: F-CCAP(*; *) is equivalent to CP(*).

Proof: If $V = \{1, \cdots, n\}$, t_c, and $C \subseteq Z^+$ is an instance of F-CCAP(*; *), then let $E = \{ij | t_c(i, j) \neq$ the empty set$\}$. Now $G = (V, E)$ and C is an instance of CP(*). Continue as in the proof of Theorem 36. Q.E.D.

Theorem 39: F-CAP(*; *), F-CAPOL(*; *), and FCAPO(*; *) are equivalent to GCP(*), GCPOL(*) and GCPO(*), respectively.

Proof: If $V = \{1, \cdots, n\}$, t and $C \subseteq Z^+$ is an instance of F-CAP(*; *), then let $E = \{ij | t(i, j) \neq$ the empty set$\}$ and define $t': E \to P(Z_0^+)$ by $t'(ij) = t(i, j)$ for all ij in E. Now $G = (V, E)$, t' and C is an instance of GCP(*), etc. Q.E.D.

Theorem 40: F-CFAP is equivalent to CP(*; *).

Proof: If $V = \{1, \cdots, n\}$, t_c, and S_c is an instance of F-CFAP, then let $E = \{ij | t_c(i, j) \neq$ the empty set$\}$ and define $t': E \to P(Z_0^+)$, $s': E \to Q^+$ by $t'(ij) = t_c(i, j) = \{1\}$, $s'(ij) = S_c(i, j)$ for ij in E. Now $G = (V, E)$, t' and s' is an instance of CP(*; *), etc. Q.E.D.

Theorem 41: F-FAP, F-FAPOL, and F-FAPO are equivalent to GCP(*; *), GCPOL(*; *) and GCPO(*; *), respectively.

Proof: If $V = \{1, \cdots, n\}$, t, and s is an instance of F-FAP, then let $E = \{ij | t(i, j) \neq$ the empty set$\}$ and define $t': E \to$

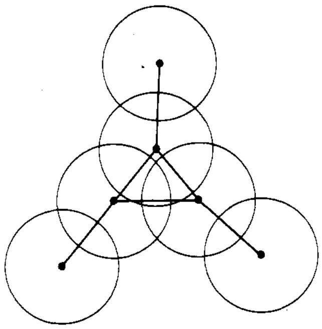

Fig. 4. A unit disk graph together with one of its intersection models.

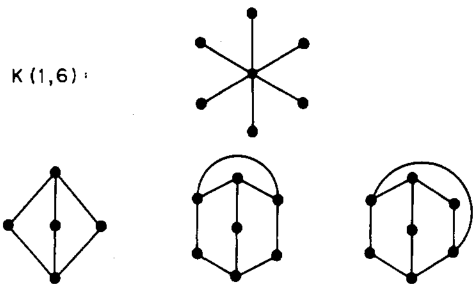

Fig. 5. The six pointed star $K(1, 6)$ and other graphs that are not unit disk graphs.

$P(Z_0^+)$, $s':E \to Q^+$ by $t'(ij) = t(i, j)$, $s'(ij) = s(i, j)$ for all ij in E. Now $G = (V, E)$, t', s' is an instance of GCP(*; *), etc. Q.E.D.

Theorem 42: F-CCAP, F-CAP, F-CAPOL, F-CAPO, F-CCAP(*; *), F-CAP(*; *), F-CAPOL(*; *), F-CAPO(*; *), F-CFAP, F-FAP, F-FAPOL, and F-FAPO are NP-hard.

Proof: It is well known that CP is NP-hard and the theorem follows since F-CCAP is a subproblem of each of the others. Q.E.D.

B. Frequency-Distance Constrained Problems and Their Complexity

It is known that F*D-CCAP is related to CP [5], [12], [13], and that other F*D constrained problems are equivalent to generalized coloring problems [89]. In this paragraph, we show that F*D-CCAP is equivalent to CP restricted to a narrow class of graphs called unit disk graphs. Similarly, we show that each of the F*D constrained problems of Sections III and IV is equivalent to a generalized coloring problem restricted to a narrow class of edge constrained graphs called disk graphs. From these equivalences and the recent discovery that CP restricted to unit disk graphs is NP-hard it follows that the F*D constrained problems of Sections III and IV are each NP-hard. Thus open questions concerning the complexity of F*D constrained problems [89] are resolved. Finally, the frequency-constrained approach is more general than the F*D approach since $K(1, 6)$ the six-pointed star (and many other graphs) is not a unit disk graph [90] (see Figs. 4 and 5).

Theorem 43: F*D-CAP(*; *) is equivalent to a subproblem of GCP(*).

Proof: If V, $P = \{V_1, \cdots, V_p\}$, $R = \{(T_{jk}(i), d_{jk}(i)) | i = 0, \cdots, m; j, k = 1, \cdots, p\}$ and $B = \{b_1, \cdots, b_p\} \subset Q^+$ is an instance of F*D-CAP(*; *) and r'_ℓ, $\ell = 1, \cdots, p$ and $C(B)$ are as in Section V-B, then let $E = \{uv | u \neq v$, u is an element of V_j, v is an element of V_k and $D(u, v) \leqslant d_{jk}(0), j, k = 1, \cdots, p\}$ and define $t:E \to P(Z_0^+)$ by $t(uv) = \{hr'_\ell | \ell = 1, \cdots, p$; h is in $T_{jk}(i)$ where u is in V_j, v is in V_k and i is the largest integer for which $D(u, v) \leqslant d_{jk}(i)\}$. Now $G = (V, E)$, t and $C(B)$ is an instance of GCP and $A: V \to C(B)$ a minimum span coloring for G, t and $C(B)$ is also a minimum span assignment for V, P, R, and B. Q.E.D.

For future reference, the edge constrained graph $G_t = (V, E, t)$ in the proof above is denoted $G_t[I]$ and is called *the edge constrained graph generated by* $I = (V, P, R, B)$. Let $G_t[*; *]$ denote the class of all such graphs, let $G_t[*]$ denote the subclass that results when $b_i = b$ for $i = i, \cdots, p$. (Note: in this case, $r'_\ell = 1$ for $\ell = 1, \cdots, p$ and $C(B) = Z^+$ so $G_t[*]$ is the class of edge constrained graphs generated by instances of F*D-CAP(*)), let $G_t[p; p]$ and $G_t[p]$ denote the subclasses that result when p is fixed. (Note: $G_t[1]$ is the class of edge constrained graphs generated by instances of F*D-CAP.)

Theorem 44: F*D-FAP is equivalent to a subproblem of GCP(*; *).

Proof: If $I = (V, P, B, T, s, d)$ is an instance of F*D-FAP, then let $E = \{uv | u \neq v$, u is in V_j, v is in V_k, $D(u, v) \leqslant d_{jk}$ $j, k = 1, \cdots, p\}$. Define $t:E \to P(Z_0^+)$ and $s':E \to Q^+$ by $t(uv) = T_{jk}(D(u, v))$ and $s'(uv) = s_{jk}(D(u, v))$ when u is in V_j, v is in V_k, and $j, k = 1, \cdots, p$. Now $G = (V, E)$, t and s' is an instance of GCP(*; *) and if $A: V \to R^+$ is a minimum span coloring for G, t and s', $\min A(V) = q'$ (an element of Q^+) and $q = b/q'$ then $A: V \to [b, \infty)$ defined $A'(v) = qA(v)$ for all v in V is a minimum span assignment for I. Suppose, to the contrary, that B is an element of $F(I)$ and $\max B(V) < \max A(V)$. Let $e = \max A'(V) - \max B(V)$ and let q'' belong to $(0, e)_Q$ such that $q'' < 1/q$. Now $A'': V \to R^+$ defined $A''(v) = q''A'(v)$ is a feasible coloring for G, t and s' such that $\max A''(V) < \max A(V)$ which is impossible. Q.E.D.

For future reference, the doubly edge constrained graph (V, E, t, s') in the proof above is denoted $G_{ts}[I]$ and is called *the doubly edge constrained graph generate by I.* Let $G_{ts}[*; *]$ denote the class of all such graphs. (Note: $G_t[*; *]$ is a proper subclass of $G_{ts}[*; *]$). In order to classify the F*D constrained problems as to their complexity, it is convenient to first characterize the edge constrained graphs generated by these problems. A closed disk in the plane is called a *unit disk* if it has diameter one, and $G = (V, E)$ is called a *unit disk graph* if it has an intersection model $\{D_v | v$ is in $V\}$ consisting of unit disks in the plane (e.g., see Fig. 4). Let $B_1(2)$ denote the class of all unit disk graphs. If $G = (V, E)$ belongs to $B_1(2)$, $\{D_v | v$ is in $V\}$ is an intersection model for G, and $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$ is a set of F*D constraints, then let $d'(i) = d(i)/d(0)$ for $i = 0, \cdots, m$ and let v' denote the center of D_v for each v in V. Define $t:E \to P(Z_0^+)$ by $t(uv) = T(i)$ where i is the largest integer for which $D(u', v') \leqslant d'(i)$. Now $G_t = (V, E, t)$ is an edge constrained $B_1(2)$ graph generated by R. Let $B_1^*(2)$ denote the class of all edge constrained $B_1(2)$ graphs obtained in this way.

Theorem 45: $G_t[1] = B_1^*(2)$.

Proof: $G_t[1] \subseteq B_1^*(2)$: Let $G_t = (V, E, t) = G_t[V, R]$ belong to $G_t[1]$ where $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$. For each v in V, let D_v be the unit disk centered at $v' = v/d(0)$. Now uv is an element of E if and only if $u \neq v$ and $D(u, v) \leqslant d(0)$ if and only if $u \neq v$ and $D(u', v') \leqslant 1$ if and only if $u \neq v$ and D_u and D_v have nonempty intersection. Therefore, $\{D_v | v$ is in $V\}$ is an intersection model for $G = (V, E)$ and G belongs to $B_1(2)$. By definition, t is an edge constraint generated by R and therefore G_t belongs to $B_1^*(2)$. $B_1^*(2) \subset G_t[1]$: Let $G_t = (V, E, t)$ belong to $B_1^*(2)$. By definition $G = (V, E)$ has an intersection model $\{D_v | v \text{ is in } V\}$ and if v' is the center of D_v, then $V' = \{v' | v \text{ is in } V\}$ is a subset of the plane. Also by definition of $B_1^*(2)$, t is generated by $R = \{(T(i), d(i)) | i = 0, \cdots, m\}$ a set of F*D constraints and G_t is an edge constrained graph generated by V' and $R' = \{(T(i), d'(i)) | i = \ , \cdots, m\}$ where $d'(i) = d(i)/d(0)$ for $i = 0, \cdots, m$. Q.E.D.

We have the following results as corollaries to Theorem 45.

Theorem 46: F*D-CCAP is equivalent to CP restricted to $B_1(2)$.

Theorem 47: F*D-CAP is equivalent to GCP restricted to $B_1^*(2)$.

Theorem 48: F*D-CAPOL is equivalent to GCPOL restricted to $B_1^*(2)$.

Theorem 49: F*D-CAPO is equivalent to GCPO restricted to $B_1^*(2)$.

A graph $G = (V, E)$ is called a *disk graph* if it has an intersection model $\{D_v | v \text{ is in } V\}$ consisting of closed disks in the plane each of which has rational diameter $\leqslant 1$ (e.g., see Fig. 4). Let $B(2)$ denote the class of all disk graphs. If $G = (V, E)$ belongs to $B(2)$ and $\{D_v | v \text{ is in } V\}$ is an intersection model for G, then let $P = \{q | \text{the diameter of } D_v = q \text{ for some } v \text{ in } V\} \subset Q^+$, let $|P| = p$, let $q_1, \cdots, q_p$ be a list of P, let $V_i = \{v | v \text{ is in } V \text{ and the diameter of } D_v = q_i\}$ for $i = 1, \cdots, p$, let $R = \{(T_{ik}(i), d_{ik}(k)) | i = 0, \cdots, m; j, k = 1, \cdots, p\}$ be a set of F*D constraints for the mixed service $\{V_1, \cdots, V_p\}$ and let $B = \{b_1, \cdots, b_p\} \subset Q^+$. Let $d'_{jk}(i) = d_{ij}(i)/d_{ik}(0)$ for $i = 0, \cdots, m; j, k = 1, \cdots, p$, let v' denote the center of D_v for each v in V, and let r'_ℓ, $\ell = 1, \cdots, p$ be as in Section V-B. Define $t: E \to P(Z_0^+)$ by $t(uv) = \{hr'_\ell | \ell = 1, \cdots, p, \ h \text{ is in } T_{jk}(i)$ where u is in V_j, v is in V_k and i is the largest integer for which $D(u', v') \leqslant d_{ij}(k)\}$. Now $G_t = (V, E, t)$ is an edge constrained $B(2)$ graph generated by R and B. Let $B^*(2)$ denote the class of all edge constrained $B(2)$ graphs obtained in this way, and let $B_b^*(2)$ denote the subclass of $B^*(2)$ that results when $b_i = b$ for $i = 1, \cdots, p$. Note that $B_1^*(2)$ is the subclass that results when $p = 1$.

Theorem 50: $B_b^*(2) = G_t[*]$ and $B^*(2) = G_t[*; *]$

Proof: The proof is very similar to the proof of Theorem 45. Q.E.D.

We have the following results as corollaries.

Theorem 51: F*D-CCAP(*) and F*D-CCAP(*; *) are equivalent to CP restricted to $B(2)$ and CP(*) restricted to $B(2)$, respectively.

Theorem 52: F*D-CAP(*), F*D-CAPOL(*), and F*D-CAPO(*) are equivalent, respectively, to GCP, GCPOL, and GCPO restricted to $B_b^*(2)$.

Theorem 53: F*D-CAP(*; *), F*D-CAPOL(*; *), and F*D-CAPO(*; *) are equivalent, respectively, to GCP(*), GCPOL(*), and GCPO(*) restricted to $B^*(2)$.

Let $G = (V, E)$ belong to $B(2)$, let $\{D_v | v \text{ is in } V\}$, $P = \{q_1, \cdots, q_p\}$, $\{V_1, \cdots, V_p\}$, and B be as above. Let $s = \{s_{jk} | j, k = 1, \cdots, p\}$, $d = \{d_{jk} | j, \ k = 1, \cdots, p\}$, and $T = \{T_{jk} | j, k = 1, \cdots, p\}$ be as in Section V-C. Define $t: E \to P(Z_0^+)$ and $s: E \to Q^+$ by $t(uv) = T_{jk}(d_{jk}D(u', v'))$ and $s'(uv) = s'_{jk}(d_{jk} D(u', v'))$ when u is in V_j, v is in V_k, and $j, k = 1, \cdots, p$. Now, $G_t = (V, E, t, s')$ is a doubly edge constrained $B(2)$ graph generated by T, s, and d. Let $B^{**}(2)$ denote the class of all doubly edge constrained $B(2)$ graphs obtained in this way.

Theorem 54: $G_t[*; *] = B^{**}(2)$.

Proof: The proof is very similar to the proof of Theorem 45.

The following results are immediate corollaries. Q.E.D.

Theorem 55: F*D-CFAP is equivalent to CP(*; *) restricted to $B(2)$.

Theorem 56: F*D-FAP, F*D-FAPOL and F*D-FAPO are equivalent, respectively, to GCP(*; *), GCPOL(*; *), and GCPO(*; *) restricted to $B^{**}(2)$.

Theorem 57: F*D-CCAP, F*D-CAP, F*D-CAPOL, F*D-CAPO, F*D-CCAP(*), F*D-CAP(*), F*D-CAPOL(*; *), F*D-CAPO(*), F*D-CCAP(*; *), F*D-CAP(*; *), F*D-CAPOL(*; *), F*D-CAPO(*; *), F*D-CFAP, F*D-FAP, F*D-FAPOL, and F*D-FAPO are NP-hard.

Proof: In April of 1980, J. B. Orlin demonstrated that CP restricted to $B_1(2)$ is NP-hard. The result follows from the fact that F*D-CCAP is a subproblem of each of the others. Q.E.D.

C. Polynomial Time Subproblems

We now exploit the equivalence between frequency assignment problems and graph coloring to obtain efficient solutions for some important real world problems. For instance, if the transmitters and receivers that tune to them are restricted to lie on a straight line (as along a highway, a pipe line, etc.) then the subproblem of F*D-CCAP(*) corresponding to this situation is equivalent to a subproblem of CP restricted to interval graphs and the algorithm [30] is an efficient solution for this problem. Also, an obvious modification of this algorithm yields an efficient solution for the analogous subproblem of F*D-CCAP(*;*). Indeed the transmitters need not be omnidirectional and the highway need not be straight; it is sufficient to require that the highway intersect the coverage area of each transmitter in a simple arc. More generally, if the locations of transmitters are restricted in such a way that the resulting generated subclass of B(2) is made up of perfect graphs then L. Lovasz has found, using L. G. Khachian's ellipsoid method, an efficient solution for this subproblem of F*D-CCAP(*) (soon to be published).

If the transmitters are restricted to lie on the circumference of a circle (as in a ring network) then the subproblem of F*D-CCAP corresponding to this situation is equivalent to a subproblem of CP restricted to proper circular-arc graphs and the algorithm [49] is an efficient solution for this problem. More generally, if the transmitters are restricted to lie on the circumferences of concentric circles, then the corresponding subproblem of F*D-CCAP(*) (where if v is an element of v_i, then v lies upon the circle centered at (0, 0) with radius $\sqrt{1 + (d_i/2)^2}$) is equivalent to CP restricted to circular-arc graphs in which arcs are restricted to be on the unit circle and to have arc length no longer than pi. (To see this, if D_i is a disk with radius r_i centered at polar coordinates (R_i, a_i) where $R_i = \sqrt{1 + r_i^2}$, then let a'_i be the circular arc with midpoint at $(1, a_i)$ and radian measure $2 \cdot \arctan r_i$. Clearly, a'_i and a'_j are disjoint iff D_i and D_j are disjoint.) Therefore, if the number of channels (or colors) available is fixed at $k \leq |V|$ (as is usually the case in practical problems), then there is a polynomial time algorithm which produces a minimum span assignment using k or fewer channels, if such an assignment exists [48].

In Section III, we remarked that potentially useful information may be encoded when F*D constrained problems are modeled as frequency constrained problems, and we indicated that the frequency constrained approach should only be used for problems in which distance separation plays no role. In support of this view, if the preceeding problem is modeled as a frequency constrained problem, then the circular-arc nature of the problem may be encoded. Although there is a polynomial time algorithm which decodes this information [91] this algorithm, its proof, and its efficient implementation are all nontrivial. A quick reading of [91] should convince the reader that one should model any problem in which distance separation plays a role as an F*D constrained problem.

VIII. Concluding Remarks

In this paper, we have introduced the minimum-order approach to frequency assignment and have developed a theory which relates this potentially useful approach to the traditional minimum span approach. We have modeled existing (e.g., cochannel, adjacent channel, UHF-TV) and potential (e.g., mixed service with interwoven spectrum sharing) real world assignment problems as both F*D constrained and frequency constrained optimization problems. We have demonstrated that the frequency constrained approach is more general than the frequency-distance approach and should be avoided if distance separation is employed to mitigate interference. We have shown that a restricted class of graphs, called disc graphs, plays a central role in F*D constrained problems. We have introduced two generalized chromatic numbers and have shown that each of the frequency assignment problems studied in this paper is equivalent to a generalized graph coloring problem. Using these equivalences and recent results concerning the complexity of graph coloring, we have shown that each of the general assignment problems studied in this paper is NP-hard, but that several important subproblems have polynomial time solutions. We have noted that the theory relating the minimum span and the minimum order approaches, as developed for the F*D constrained problems, remains valid for the frequency constrained and the generalized graph coloring problems.

What is the significance of all of this? First of all, there are the standard benefits of knowing the complexity classifications of problems. That is, employers may choose not to spend money for the development of polynomial time solutions for the assignment problems now known to be NP-hard. They may instead invest in the development of polynomial time solutions for subproblems of NP-hard problems, nonpolynomial time solutions for the NP-hard problems, which are almost always fast for practical problems, and/or heuristics for the NP-hard problems, which almost always perform well for practical problems, etc. In addition, there are several ways to exploit the close connection between frequency assignment and graph coloring. The many existing solutions [26]-[37], [39],[40], [42], [45]-[47] (heuristics [50]-[68]) for the graph coloring problem may be applied directly (without modification) to cochannel assignment problems. If it can be demonstrated that one of these almost always runs sufficiently fast (produces sufficiently good approximate solutions), then it may be modified to handle more general assignment problems. If none of the existing solutions (heuristics) shows promise even for cochannel problems, then a solution (heuristic) which exploits the special structure of disk graphs may be devised for the F*D constrained cochannel assignment problem and subsequently extended to more general problems.

More generally, using the approach of this paper, one may evaluate various proposed conventions, policies, and procedures which are to govern a new or existing communications service. For example, suppose that improvements in UHF-TV receivers allow for the relaxations of some of the distance separation requirements. One can use our approach to determine which taboo(s) to modify for the maximum gain in spectrum efficiency [92]. As another example, consider the FM-broadcast service. One may use our approach to evaluate the effectiveness of assigning frequencies to the different classes of FM transmitters in an interwoven fashion or of allowing frequencies to be assigned to unevenly spaced discrete frequencies, etc. Finally, one may use our approach to accurately determine the amount of spectrum to allocate for a proposed new service (given a projected saturated environment).

What remains to be done? As indicated in our discussion of exploiting the graph coloring connection: We need to devise good algorithms and/or heuristics for assignment problems. The smallest last [58], largest first[51] and saturation degree [47] graph coloring heuristics have been generalized to handle minimum span assignment problems [10], [92] and the minimum residual difficulty heuristic has recently appeared [18], but very little else has been done.

What about performance guarantees [63]-[65] for these heuristics restricted to F*D constrained problems (i.e., to disk graphs)? None of the exact graph coloring algorithms has been applied to the cochannel assignment problem. How fast do these algorithms run when restricted to disk (or to unit disk) graphs? There is no graph coloring algorithm or heuristic which exploits the special structure of unit disk or disk graphs. There is no known intrinsic characterization of unit disk graphs (a reasonable forbidden subgraph characterization seems out of the question, as a large list of infinite families of forbidden subgraphs continues to grow). Such a characterization would be helpful in other applications involving unit disk graphs [93]-[98]. What is the complexity of the clique problem [23] for unit disk graphs? Exhaustive search algorithms are the only known exact solutions to nontrivial (i.e., problems involving constraints other than cochannel) minimum span and minimum-order assignment problems; and although minimum span algorithms may be readily obtained from graph coloring algorithms, it is not obvious that trivial modifications of these or other known algorithms will work for minimum order problems. There is no known polynomial time heuristic for nontrivial minimum order assignment problems. Except for what we have presented in this paper, there is no chromatic graph theory for edge constrained graphs. For example, there are no nontrivial upper or lower bounds on $X_1(G,T)$, $X_2(G,T,\mathfrak{L})$, and $X_2(G,t)$ either for edge constrained graphs restricted to $B^*(2)$ or for the general case (one expects many of the results found in [69]-[81] to have analogs here).

In a 1964 U.S. Government report (see [2]), the annual economic value of the electromagnetic spectrum was estimated to be $17 billion. Every facet of personal, commercial, and governmental life in the developed world relies heavily upon successful use of the spectrum. It is widely acknowledged that successful communication contributes to world health, safety, understanding, and peace. Clearly, it is important that we address the unsolved problems discussed above.

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