Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

Ralf Borndörfer Andreas Eisenblätter
Martin Grötschel Alexander Martin

The Orientation Model for Frequency Assignment Problems

Technical Report TR 98-01 (April 1998)

The Orientation Model for Frequency Assignment Problems

Ralf Borndörfer Andreas Eisenblätter Martin Grötschel Alexander Martin*

April 3, 1998

Abstract. Mobile telecommunication systems establish a large number of communication links with a limited number of available frequencies; reuse of the same or adjacent frequencies on neighboring links causes interference. The task to find an assignment of frequencies to channels with minimal interference is the frequency assignment problem. The frequency assignment problem is usually treated as a graph coloring problem where the number of colors is minimized, but this approach does not model interference minimization correctly.

We give in this paper a new integer programming formulation of the frequency assignment problem, the orientation model, and develop a heuristic two-stage method to solve it. The algorithm iteratively solves an outer and an inner optimization problem. The outer problem decides for each pair of communication links which link gets the higher frequency and leads to an acyclic subdigraph problem with additional longest path restrictions. The inner problem to find an optimal assignment respecting an orientation leads to a min-cost flow problem.

Keywords. Minimum-Cost Flow Problems, Cellular Radio Telephone Systems, Frequency Assignment Problem, Integer Programming

Mathematics Subject Classification (1991). 90B10, 90B12, 90B80, 90C10

1 Introduction

Cellular radio telephone systems broadcast information on a limited number of available frequencies that serve as (communication) *channels.* The channels are spaced-out evenly along the electromagnetic spectrum such that adjacent channels have a constant inter-channel distance. Channels are the physical means to establish communication links between the mobile stations (shandies) and the antennae in a mobile telephone system. An abstract communication link is called a *carrier*; data on this link can be transmitted on any of the available channels.

A carrier works only inside a region around an antenna, because the signals broadcast by both antenna and mobile station get weaker and at some point useless as one moves away from the sender. The area of the whole mobile telephone system is thus subdivided into regions that can be serviced best by individual antennae, the *cells.* Each cell provides one or more carriers to service phone calls in its domain.

Now signals broadcast by an antenna or handy do not stop at cell boundaries but extend into neighboring cells. If a cell and its neighbor use the same channel, the two signals interfere with each other, resulting in erroneous data transmission or even failure of a carrier. In fact, interference can not only occur if two carriers in nearby cells use the same channel, but also if they use adjacent channels; this *adjacent channel interference*, however, is always less than the first mentioned *co-channel interference.*

Very large interference can not be accepted by the mobile telephone companies, resulting in a requirement that certain pairs of carriers must not use the same or adjacent channels. In other words, certain pairs of carriers have to obey a minimum *channel separation* of 1 (if co-channel interference is considered too large) or even 2 (in case of excessive adjacent channel interference). More separations, even larger than 2, are caused by technological restrictions: *co-cell separations* apply to channels broadcast on the same antenna, *co-site separations* result if several antennae are mounted on a common mast (servicing cells similar to sectors of a circle), *handover separations* protect the carrier switching process when a handy leaves one cell and enters another, and there are more.

*Konrad-Zuse-Zentrum Berlin, Takustr. 7, 14195 Berlin, Germany, Email surname@zib.de.

Other interferences may be acceptable, but they diminish the quality of communication in the network. Quality and costs, however, are the only two factors distinguishing different providers of mobile telephone systems. So the question is: how can one assign channels to the carriers such that all channel separations are obeyed and the resulting interference is minimal? This is the *frequency assignment problem.*

Clearly, this formulation is still vague; for example, what does 'minimal interference' mean? 'Minimal sum of all interferences' is a conservative suggestion (in fact we our going to use it), but not the only possible one. The point here is, however, that the problem in the above form calls for usage of the complete spectrum of available channels to reduce interference — a concept developed only lately.

The 'classic approach' treats the problem as a *graph coloring problem*, see Metzger [1970]. First, interferences are classified as unacceptable or negligible using thresholds; unacceptable interferences yield channel separations, negligible ones are ignored. Looking at the carriers as nodes of a graph, where two nodes are joined by an edge if they have a channel separation, and at the set of channels as a set of colors, the problem is to color the graph such that adjacent nodes use colors that are at least as far apart as the separation; usually, the *span* of the coloring, the largest minus the smallest color used, is minimized. The reason behind this objective is that 'shortage of the electromagnetic spectrum makes it necessary that each of these systems use its frequency channels as efficiently as possible', see Gamst [1986]. Important results for the coloring model are the development of fast heuristic algorithms, including adaptive and non-deterministic priority assignment algorithms, see Box [1978] and Gamst [1988], and simulated annealing approaches like Duque-Antón, Kunz, and Rüber [1993] or Quellmalz, Knälmann, and Krank [1993], the derivation and algorithmic use of local lower bounds, see Gamst [1986], and recently exact branch-and-cut approaches like Aardal and van Hoesel [1995].

Relatively little is known about the interference minimization problem: a first algorithmic result is the development of an adaptive and non-deterministic algorithm in Plehn [1994].

2 The Frequency Assignment Problem

Let us now state the frequency assignment problem in a mathematical way. Let

$$G = (V, E)$$

be a graph,

$$d \in \mathbb{N}_0^E, \quad \text{and} \quad \overline{p}, \overline{\overline{p}} \in [0,1]^E$$

be vectors with the property

$$\overline{\overline{p}} \leq \overline{p},$$

and

$$C = \{0, 1, \ldots, \zeta\} \subseteq \mathbb{N}_0$$

be the set of zero and the first ζ integers; each node $i \in V$ corresponds to a carrier, each edge $ij \in E$ has an associated minimum channel separation d_{ij}, an adjacent channel interference $\overline{\overline{p}}_{ij}$ and a co-channel interference $\overline{p}_{ij}$ (some of them possibly zero), and C is the set of available channels. Introducing variables

$$y_i \in C \quad \forall i \in V$$

for the channel assigned to carrier i, the frequency assignment problem is

$$\begin{array}{rrcll} \min & \sum\limits_{|y_i - y_j| = 0} \overline{p}_{ij} + \sum\limits_{|y_i - y_j| = 1} \overline{\overline{p}}_{ij} & & & \\ & |y_i - y_j| & \geq & d_{ij} & \forall ij \in E \\ & y_i & \leq & \zeta & \forall i \in C \\ & y_i & \geq & 0 & \forall i \in C \\ & y_i & \in & Z & \forall i \in C; \end{array} \qquad (\text{FAP}(G, d, p, \zeta))$$

This formulation is correct, but not an integer linear program because of the non-linear objective function and the non-linear separation constraints.

Note that the model FAP is *carrier oriented* and not *cell oriented* as usually in frequency assignment. The reasons for this are: first, channel separation data at our partner, the German mobile telephone system provider eplus, depends on different types of carriers, and second, it is conceptually easier to assign one channel to each carrier instead of sets of channels to a cell. A disadvantage of this concept is, on the other hand, that the data is 'blown up', because most relations are actually relations between two cells that have to be stored as relations between each pair of carriers of the two cells in a carrier oriented approach. Note also that the model is *symmetric*; for example, there is only one co-channel interference datum $\overline{p}_{ij}$ for each pair of carriers ij, although interference is in principle an asymmetric function. It is, however, clear, that either both i will co-interfere with j and j with i (if and only if both carriers use the same channel) or none of them; thus we can symmetrize by just adding interference values.

Frequency assignment problems often have additional constraints that do not show up in the model FAP or more general objective functions. For example, there is usually a set $A_i \subseteq C$ of eligible channels associated to each carrier such that $y_i \in A_i$ must hold. Such generalizations can, however, be handled using simple transformations; we refer the reader to Kaudewitz and Kürner [1995] for more details on frequency assignment at e-plus and to Borndörfer, Grötschel, and Martin [1995] for information about such modelling issues.

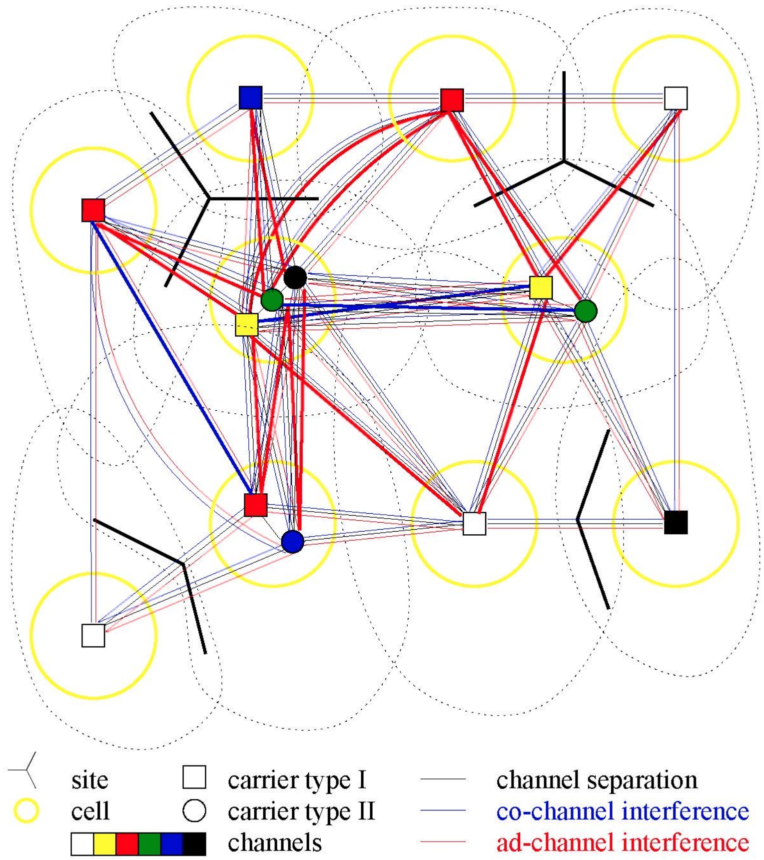

Figure 1: Frequency assignment problem.

Figure 1 shows a frequency assignment problem. There are four masts or sites; mounted on each mast are two or three antennae that service cells similar in shape to sectors of a circle. Each cell provides one to three carriers (of two different types) to service phone calls in its domain. However, the regions where signals of the individual antennae can be received overlap, causing co- and adjacent channel interference of carriers in neighboring cells. There are also channel separations arising from excessive interference and other technical constraints; the example assumes that co-cell separations are 2 and all other separations are 1. The problem is to assign channels from the available spectrum of size six to the carriers such that all separation relations are satisfied and the sum of the interferences is minimal. Figure 1 shows a frequency assignment. The channels in the spectrum are sorted with respect to 'darkness' (white is channel 0, yellow 1, red 2, green 3, blue 4 and black is 5); the resulting co- and adjacent channels interferences are highlighted.

Our aim in this paper is to discuss a new approach to the frequency assignment problem based on a two-stage integer programming model. The outer stage consists of an acyclic subdigraph problem with additional longest path restrictions, the inner stage of a minimum-cost flow problem. These two stages are alternately optimized. The algorithm is used in a project with the German mobile telephone system provider eplus. The frequency assignment problems appearing there currently have up to 4000 carriers, more than 25.000 separations of 1 to 3 and about 50.000 interference relations (note that the corresponding graph has a density of about 40%!); the available spectrum contains 50 to 75 channels.

3 The Orientation Model

In this section we want to give an integer linear programming formulation for the frequency assignment problem. Our starting point is the model FAP.

This model makes a very natural choice to represent channel assignments by introducing integer variables y_i (for a different 'stable set approach', see Borndörfer, Grötschel, and Martin [1995]). Note however, that deciding to use these variables is a major design step since the y_i are genuinely integer and not binary variables.

A first property of any model using y-variables is illustrated in Figure 2. Consider for all edges with non-zero separation the sets

$$Y_{ij} := \left\{ (y_i, y_j) \in \mathbb{Z}^2 : \begin{array}{rcl} |y_j - y_i| & \geq & d_{ij} \\ y_i, y_j & \leq & \zeta \\ y_i, y_j & \geq & 0 \end{array} \right\} \qquad \forall ij \in E : d_{ij} > 0$$

of channels assignments to two carriers i and j that are feasible with respect to the separation d_{ij}. Figure 2 shows the case $d_{ij} = 1$ and $\zeta = 5$. The convex hull

$$P_{ij} := \operatorname{conv} Y_{ij} \qquad \forall ij \in E : d_{ij} > 0$$

of Y_{ij} contains integer points not in Y_{ij}; in fact, all of these points violate the separation condition. In Figure 2 we have $Y_{ij} = \{0,1,2,3,4,5\}^2 \setminus \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$ and P_{ij} is the region inside the thick black line. We see that the integer points in $P_{ij} \setminus Y_{ij}$ are $\{(1,1),(2,2),(3,3),(4,4)\}$; they form a 'gap'.

This simple observation has far-reaching consequences, as we will point out now. Let FAP(G, d, p, ζ) be a frequency assignment problem and Y be the set of its integer solutions. We would like to investigate this set by means of integer programming techniques, that is, we look at the convex hull

$$P := \operatorname{conv} Y$$

of Y. Since every extreme point of P is a member of Y we can optimize a linear objective over Y by solving a linear program over P — if we know a description of P in terms of linear inequalities. If we can not obtain such a complete description, we might, less ambitious, at least try to find a description of a (relaxed) polytope $P' \supseteq P$ such that

$$\mathbb{Z}(P') = \mathbb{Z}(P)$$

(where $\mathbb{Z}(Q)$ denotes the set of integral points $Q \cap \mathbb{Z}^V$ for any polytope $Q \subseteq R^V$). Then, $\mathbb{Z}(P') = \mathbb{Z}(P) \supseteq Y$ is an integer linear description of a set of integer points that has only members of Y as extreme points and we can optimize over Y by applying a branch-and-bound or branch-and-cut algorithm to P'. In Figure 2, $\mathbb{Z}(P_{ij}) = \{0,1,2,3,4,5\}^2 \setminus \{(0,0),(5,5)\}$. Remember that our current model FAP does not provide an integer linear formulation because of the absolute values in the separation conditions.

A natural approach to get such a description is to try to write P as the intersection of the polytopes P_{ij}. It seems to be irrelevant that we get infeasible points in the interior of the polytopes P_{ij} as long as the vertices are integral — and it is easy to describe each of the individual polytopes P_{ij} by means of linear inequalities.

Following this idea, however, we get into trouble. Clearly,

$$\mathbb{Z}(P) \subseteq \mathbb{Z}\left(\bigcap_{d_{ij}>0} P_{ij} \cap \{0 \leq y \leq \zeta \mathbb{1}\} \right) =: \mathbb{Z}(Q)$$

holds where $\mathbb{1}$ is the vector having a one in each coordinate. But unfortunately, equality does not hold. Assuming $\lfloor \zeta/2 \rfloor \geq \max_{ij \in E} d_{ij}$, the point

$$\lfloor \zeta/2 \rfloor \mathbb{1}$$

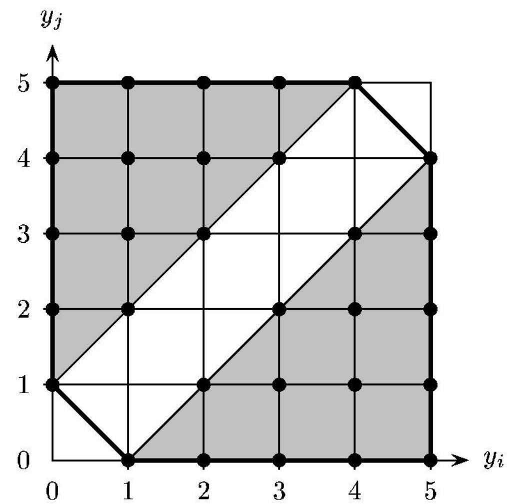

Figure 2: Dichotomy in y-Solution Space.

which lies in the gaps of the sets Y_{ij} is in $\mathbb{Z}(Q)$, but not in Y because it violates every single separation (given that there is a single non-zero separation). So what? — if, as in the two-dimensional case, $\mathbb{Z}(Q) = \mathbb{Z}(P)$, we have reached our (less ambitious) goal and found an integer linear description. Unfortunately this is not so: $Z(Q)$ always contains the point $\lfloor \zeta/2 \rfloor \mathbb{1}$ while it is $\mathcal{NP}$-complete to decide whether $Z(P)$ is empty or not. This is because

$$\mathbb{Z}(P) = \emptyset \iff Y = \emptyset \iff \text{FAP}(G, d, p, \zeta) \text{ has no feasible solution;}$$

but $\text{FAP}(G, d, p, \zeta)$ becomes a $\mathcal{NP}$-complete graph coloring problem by ignoring the interferences.

Our conclusion is that it seems to be difficult to describe $Z(P)$ by means of linear inequalities plus integrality stipulations and we do not know how to write down such an integer linear formulation of the feasible region of $\text{FAP}(G, d, p, \zeta)$, much less of the objective function. So was all this effort for nothing? Of course not! We can give an integer linear formulation by introducing additional variables.

We observe that the situation depicted in Figure 2 is a dichotomy. Both below and above the 'gap' exist sets of integral points whose convex hulls (the shaded triangles) contain no other integral points. If we always knew whether we are 'above the gap', or, equivalently, whether $y_j - y_i \geq 0$ holds, or 'below the gap', that is, $y_j - y_i \leq 0$ holds, we probably could give an integer linear description! We will show now how this can be done.

We introduce binary decision variables

$$\Delta_{ij} \in \{0, 1\} \quad \forall ij \in E$$

with the meaning

$$\Delta_{ij} = \begin{cases} 1 & \text{if } y_j - y_i \geq 0 \\ 0 & \text{if } y_j - y_i \leq 0. \end{cases}$$

(Let us assume here and in what follows $i < j$ for an edge $ij \in E$ to get a convenient notation although the edges of a graph —strictly speaking— are unordered pairs of vertices.) In other words, Δ_{ij} is 1 if the edge ij is oriented 'upward' from the lower channel y_i to the higher (or equal) channel y_j and 0 if the edge ij is oriented 'downward', that is, channel y_i is larger than or equal to channel y_j; if y_i and y_i are equal, Δ_{ij} can be both zero or one.

To express the objective function, we introduce binary *co- and adjacent channel interference variables*

$$\overline{z}_{ij}, \overline{\overline{z}}_{ij} \in \{0, 1\} \quad \forall ij \in E$$

that are

$$\overline{z}_{ij} = \begin{cases} 1, & \text{if } y_j - y_i = 0 \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad \overline{\overline{z}}_{ij} = \begin{cases} 1, & \text{if } |y_j - y_i| = 1 \\ 0, & \text{else} \end{cases}$$

(this is slightly incorrect but let us leave it like this for the moment). The variables $\overline{z}_{ij}$ are 1 in case of a co-channel interference and the $\overline{\overline{z}}_{ij}$s are 1 in case of an adjacent channel interference.

We can now state the frequency assignment problem as an integer linear program.

$$\begin{array}{lrcllll}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & & & \\
& y_j - y_i & \geq & d_{ij}\Delta_{ij} & -\zeta(1-\Delta_{ij}) & \forall ij \in E & (1) \\
& -y_j + y_i & \geq & d_{ij}(1-\Delta_{ij}) & -\zeta\Delta_{ij} & \forall ij \in E & (2) \\
& y_j - y_i + \overline{z}_{ij} & \geq & \Delta_{ij} & -\zeta(1-\Delta_{ij}) & \forall ij \in E & (3) \\
& -y_j + y_i + \overline{z}_{ij} & \geq & (1-\Delta_{ij}) & -\zeta\Delta_{ij} & \forall ij \in E & (4) \\
& y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2\Delta_{ij} & -\zeta(1-\Delta_{ij}) & \forall ij \in E & (5) \\
& -y_j + y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2(1-\Delta_{ij}) & -\zeta\Delta_{ij} & \forall ij \in E & (6) \\
& y_i & \leq & \zeta & & \forall i \in V & (7) \\
& \Delta_{ij} & \leq & 1 & & \forall ij \in E & (8) \\
& \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \leq & 1 & & \forall ij \in E & (9) \\
& y_i & \geq & 0 & & \forall i \in V & (10) \\
& \Delta_{ij}, \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \geq & 0 & & \forall ij \in E & (11) \\
& y_i & \in & \mathbb{Z} & & \forall i \in V & () \\
& \Delta_{ij}, \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \in & \mathbb{Z} & & \forall ij \in E & ()
\end{array}$$

$$(\mathrm{TIP}(G, d, p, \zeta))$$

Conditions (7), (10), and (12) state that the channel assignment variables have to be chosen from the spectrum, while conditions (8), (9), (11), and (13) characterize the Δ *orientation variables* and the z-interference variables as binary decision variables. The objective sums up total interference; note that the coefficients are all non-negative.

Suppose now $\Delta_{ij} = 1$, that is, ij is oriented upward from i to j, and consider inequality (1): it simplifies to $y_j - y_i \geq d_{ij}$. Thus, y_j and y_i have to obey the channel separation; also, y_j must in fact be larger than or equal to y_i. Now consider inequality (2): it simplifies to $-y_j + y_i \geq -\zeta$ which is redundant because of the upper bounds on y. The downward case $\Delta_{ij} = 0$ is symmetric. This time, inequality (1) is redundant, and inequality (2) reads $-y_j + y_i \geq d_{ij}$. Again y_j and y_i obey the separation constraint but now y_j is less than or equal to y_i. Thus, inequalities (1) and (2) model the channel separations correctly!

Inequalities (3) and (4) deal with co-channel interference in a very similar way. Again, (4) will be trivially satisfied if $\Delta_{ij} = 1$ and (3) if $\Delta_{ij} = 0$; in the first case, (3) becomes $y_j - y_i + \overline{z}_{ij} \geq 1$. If y_j and y_i are more than 1 channel apart, this inequality is always satisfied no matter what value $\overline{z}_{ij}$ takes; since we are minimizing, $\overline{z}_{ij}$ will be zero if $\overline{p}_{ij} > 0$ and doesn't matter otherwise. If, on the other hand, $y_j - y_i = 0$, $\overline{z}_{ij}$ has to be one, indicating a co-channel interference. The case $\Delta_{ij} = 0$ is similar. So conditions (3) and (4) set $\overline{z}$ correctly.

The remaining inequalities (5) and (6) are for adjacent channel interference. Let us consider the case $\Delta_{ij} = 1$ and the non-redundant condition (5): it reads $y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} \geq 2$. If $y_j - y_i > 2$ we can do without setting any of the interference variables, if $y_j - y_i = 1$ we could set $\overline{z}_{ij}$, or $\overline{\overline{z}}_{ij}$, or both to one but choosing $\overline{\overline{z}}_{ij}$ is best with respect to the objective, and finally, if $y_j - y_i = 0$, we have to set $\overline{z}_{ij} = 1$ because of (3) and this already fulfills (5).

Thus the interpretation of the interference variables given above was not entirely correct; however, any optimal solution of TIP can be easily modified to be of this form: if (y, z) is a solution of $\mathrm{TIP}(G, d, p, \zeta)$, then (y, z') is also a solution with the same or smaller objective value where

$$\overline{z}'_{ij} := \begin{cases} 1, & \text{if } |y_j = y_i| = 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \overline{\overline{z}}'_{ij} := \begin{cases} 1, & \text{if } |y_j - y_i| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have arrived at a correct integer linear programming formulation of the frequency assignment problem. Clearly, the *projection* of this formulation on the space of the y-variables yields an implicit integer linear description of the set $\mathbb{Z}(P)$ that we investigated some paragraphs ago and naturally it is also $\mathcal{NP}$-complete to decide whether $\mathrm{TIP}(G, d, p, \zeta)$ has a solution or not. But the point is that this difficulty is now *only* due to the integrality stipulations (12) and (13) while all of the inequalities in the system are quite simple.

In contrast, we were not able to give a similar formulation for $\mathbb{Z}(P)$ because here already the inequalities seem to be complex.

Before we proceed, let us do a couple of easy modifications to the model TIP that will be important later. A first observation is that there are a lot of superfluous variables in the model that can be fixed to zero: if we have a distance of 2 or more on some edge ij, there will never be any interference and we can fix $\overline{z}_{ij} = \overline{\overline{z}}_{ij} = 0$. We can also fix $\overline{z}_{ij} = 0$ on edges with $d_{ij} > 1$, $\overline{\overline{z}}_{ij} = 0$ if $\overline{\overline{p}}_{ij} = 0$, and finally $\overline{z}_{ij} = 0$ if $\overline{p}_{ij} = 0$. Second, the upper bounds (9) on the interference variables are irrelevant because with any solution (y, z, Δ) of TIP(G, d, p, ζ) without constraint (9) the point $(y, \min\{z, \mathbb{1}\}, \Delta)$ (with the minimum taken for each component) is also a solution of TIP(G, d, p, ζ) with the same or smaller cost. So let us remove all variables that we just fixed to zero and the inequalities (9), too. More precisely, we introduce edge sets

$$\overline{E} = \{ij \in E : d_{ij} = 0 \text{ and } \overline{p}_{ij} > 0\} \quad \text{and} \quad \overline{\overline{E}} = \{ij \in E : d_{ij} \leq 1 \text{ and } \overline{\overline{p}}_{ij} > 0\}.$$

$\overline{E}$ is the set of edges with possible non-zero co-channel interference, $\overline{\overline{E}}$ is the set of edges with possible non-zero adjacent channel interference. After removing all fixed variables and the constraints (9) our model looks as follows.

$$\begin{array}{llcllr}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & & \\
& y_j - y_i & \geq & d_{ij}\Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in E & (1) \\
& -y_j + y_i & \geq & d_{ij}(1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in E & (2) \\
& y_j - y_i + \overline{z}_{ij} & \geq & \Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in \overline{E} & (3) \\
& -y_j + y_i + \overline{z}_{ij} & \geq & (1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in \overline{E} & (4) \\
& y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2\Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in \overline{\overline{E}} & (5) \\
& -y_j + y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2(1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in \overline{\overline{E}} & (6) \\
& -y & \geq & -\zeta\mathbb{1} & & (7) \\
& \Delta & \leq & \mathbb{1} & & (8) \\
& y, \Delta, z & \geq & 0 & & (9) \\
& y, \Delta, z & \in & \mathbb{Z}^{V \times E \times \overline{E} \times \overline{\overline{E}}} & & (10)
\end{array} \qquad (\mathrm{TIP}(G, d, p, \zeta))$$

(We assumed here $\overline{p}, \overline{z} \in \mathbb{R}^{\overline{E}}$ and $\overline{\overline{p}}, \overline{\overline{z}} \in \mathbb{R}^{\overline{\overline{E}}}$ for a simpler notation.)

We can write the model in a more compact form using matrix notation.

$$\begin{array}{rrcl}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & \\
& Ay + Bz & \geq & d - D\Delta \\
& -y & \geq & -\zeta\mathbb{1} \\
& \Delta & \leq & \mathbb{1} \\
& y, \Delta, z & \geq & 0 \\
& y, \Delta, z & \in & \mathbb{Z}^{V \times E \times \overline{E} \times \overline{\overline{E}}}
\end{array} \qquad (\mathrm{TIP}(G, d, p, \zeta))$$

We already know that y is the channel assignment and z indicates interference; what about the orientation variable Δ? It turns out that Δ corresponds to an 'acyclic subdigraph of restricted diameter'.

1 Definition (Diameter of a Digraph) *The diameter of a digraph with respect to a vector w of arc weights is the length of a longest directed path in the digraph with respect to w (in a path, node repetitions are allowed but not arc repetitions).*

2 Definition (Orientation of a Graph) *Let $G = (V, E)$ be a graph with $V \subseteq \mathbb{N}_0$ and $\Delta \in \{0, 1\}^E$. The* Δ-orientation of G *is the digraph*

$$\overrightarrow{G(\Delta)} := \big(V, \{(i, j) : \Delta_{ij} = 1\}\big) \cup \big(V, \{(j, i) : \Delta_{ij} = 0\}\big).$$

(We assume $i < j$ when writing Δ_{ij}.)

In the orientation of a graph, edges with $\Delta_{ij} = 1$ will be oriented upward from i to j, edges with $\Delta_{ij} = 0$ will be oriented downward from j to i. With this definition, we can characterize the set of feasible vectors Δ for TIP(G, d, p, ζ).

3 Observation (Feasible Orientations) *Let TIP(G,d,p,ζ) be a frequency assignment problem. The vector $\Delta \in \{0,1\}^E$ is feasible for TIP(G,d,p,ζ) if and only if $\overrightarrow{G(\Delta)}$ does not contain a directed cycle of positive length and is of diameter at most ζ (both with respect to d).*

Proof.

$\Longrightarrow$: Let Δ be feasible for TIP(G,d,p,ζ). Then there exists a solution (y,z,Δ) of TIP(G,d,p,ζ). The definition of $\overrightarrow{G(\Delta)}$ implies

$$(i,j) \in A(\overrightarrow{G(\Delta)}) \iff y_i \leq y_j \text{ and } ij \in E.$$

Suppose $\overrightarrow{G(\Delta)}$ contains a directed cycle $\{(i_1,i_2),(i_2,i_3),\ldots,(i_k,i_1)\}$ of positive length. Without loss of generality we can assume $y_{i_1} < y_{i_2}$; but then we get the contradiction

$$y_{i_1} < y_{i_2} \leq \cdots \leq y_{i_k} \leq y_{i_1}.$$

Suppose $\overrightarrow{G(\Delta)}$ contains a directed path $\{(i_1,i_2),(i_2,i_3),\ldots,(i_{k-1},i_k)\}$ of length more that ζ with respect to d; but then we get the contradiction

$$\zeta \geq y_{i_k} \geq y_{i_{k-1}} + d_{i_{k-1}i_k} \geq \cdots \geq y_{i_1} + \sum_{j=1}^{k-1} d_{i_j i_{j+1}} > y_{i_1} + \zeta \geq \zeta.$$

$\Longleftarrow$: We will iteratively construct a solution (y,z,Δ) of TIP(G,d,p,ζ). Let us assume first that $\overrightarrow{G(\Delta)}$ does not contain any directed cycle, neither of zero nor of positive length. Then initialize (y,z) as 'undefined'. Take a node j all of whose predecessors $V(\delta^-(j))$ have defined y-values (in the first step there will be a node whose set of predecessors is empty). Set

$$y_j := \min\left\{0, \min_{i \in V(\delta^-(j))} y_i + d_{ij}\right\}$$

and iterate until y is completely assigned. Now set

$$\begin{aligned}
\overline{z}_{ij} &:= \max\{1-(y_j-y_i),0\} && \forall (i,j) \in A(\overrightarrow{G(\Delta)}) : ij \in \overline{E} \\
\overline{\overline{z}}_{ij} &:= \max\{2-2\overline{z}_{ij}-(y_j-y_i),0\} && \forall (i,j) \in A(\overrightarrow{G(\Delta)}) : ij \in \overline{\overline{E}}.
\end{aligned}$$

Then (y,z,Δ) satisfies the separation constraints due to construction of y and the upper bound constraint because channel 0 is assigned to the first node of each longest path of $\overrightarrow{G(\Delta)}$ and the length of such a path is at most ζ; it fulfills the interference constraints due to construction of z. Thus (y,z,Δ) is a feasible solution and Δ feasible for TIP(G,d,p,ζ).

If Δ contains directed cycles of length zero we note that all nodes on such a cycle must get the same frequency. The case can be reduced to the cycle-free case by contracting all directed cycles of length zero; the details are left to the reader. □

The occurrence of directed cycles of length zero results from the freedom in the choice of Δ_{ij} if $y_j = y_i$; such edges ij can always be reoriented in such way that one gets an acyclic orientation.

4 Observation (Feasible Acyclic Digraphs) *If (y,z,Δ) is a solution of TIP(G,d,p,ζ) there is a solution (y,z,Δ') with the same objective value such that $\overrightarrow{G(\Delta')}$ is acyclic.*

Proof. Let (y,z,Δ) be a feasible solution of TIP(G,d,p,ζ). Then (y,z,Δ') is a feasible solution of TIP(G,d,p,ζ) with the same objective value where

$$\Delta'_{ij} := \begin{cases} 1, & \text{if } y_j - y_i > 0 \\ 1, & \text{if } y_j - y_i = 0 \text{ and } j > i \\ 0, & \text{otherwise.} \end{cases}$$

□

Observation 4 allows us to restrict the set of feasible orientations of TIP(G,d,p,ζ) to acyclic ones. We can do this using *transitivity inequalities*. Let $P = \{i_1i_2, i_2i_3, \ldots, i_{k-1}i_k\}$ be a path in G and $e_P = i_1i_k \in E$ be the *transitive edge* for this path; without loss of generality we may assume $i_1 < i_k$. We can orient this path in two ways upward from i_1 to i_k and downward from i_k to i_1 obtaining directed paths $\overrightarrow{P}$ and $\overleftarrow{P}$ with corresponding settings of orientation variables $\Delta^{\overrightarrow{P}}$ and $\Delta^{\overleftarrow{P}}$ (with $\Delta^{\overrightarrow{P}} + \Delta^{\overleftarrow{P}} = \mathbb{1}$). Then the inequalities

$$\begin{array}{rrcl} & \sum_{ij\in P:\Delta^{\overrightarrow{P}}_{ij}=1} \Delta_{ij} + \sum_{ij\in P:\Delta^{\overrightarrow{P}}_{ij}=0}(1-\Delta_{ij}) - \Delta_{i_1 i_k} & \leq & |P|-1 \\ :\Longleftrightarrow & \overrightarrow{P}(\Delta) - \Delta_{e_P} & \leq & |P|-1 \end{array}$$

$$\begin{array}{rrcl} & \sum_{ij\in P:\Delta^{\overleftarrow{P}}_{ij}=1}(1-\Delta_{ij}) + \sum_{ij\in P:\Delta^{\overleftarrow{P}}_{ij}=0} \Delta_{ij} - (1-\Delta_{i_1 i_k}) & \leq & |P|-1 \\ :\Longleftrightarrow & \overleftarrow{P}(\Delta) - (1-\Delta_{e_P}) & \leq & |P|-1 \end{array}$$

make sure that if P if oriented upward so is e_P and that e_P will be oriented downward if all edges on P are oriented downward. Adding the transitivity inequalities to the model TIP(G,d,p,ζ), we obtain an equivalent model for the frequency assignment problem where all orientations are acyclic.

$$\begin{array}{lrcll} \min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & \\ & Ay + Bz & \geq & d - D\Delta & \\ & -y & \geq & -\zeta\mathbb{1} & \\ & \Delta & \leq & \mathbb{1} & \\ & y,\Delta,z & \geq & 0 & \\ & y,\Delta,z & \in & \mathbb{Z}^{V\times E\times\overline{E}\times\overline{\overline{E}}} & \\ \overrightarrow{P}(\Delta) - \Delta_{e_P} & & \leq & |P|-1 & \forall e_P, P \in G \\ \overleftarrow{P}(\Delta) - (1-\Delta_{e_P}) & & \leq & |P|-1 & \forall e_P, P \in G \end{array} \qquad (\mathrm{ATIP}(G,d,p,\zeta))$$

We call the model TIP the *orientation model* and the model ATIP(G,d,p,ζ) the *acyclic orientation model* of the frequency assignment problem.

4 Basic Properties of the Orientation Model

4.1 The Acyclic Orientation Polytope

We want to solve the frequency assignment problem using integer programming techniques. Given an instance TIP(G,d,p,ζ) or ATIP(G,d,p,ζ), let us denote by

$$P(G,d,\zeta) \qquad \text{or} \qquad \overrightarrow{P(G,d,\zeta)}$$

the convex hull of the set of feasible solutions of TIP(G,d,p,ζ) or ATIP(G,d,p,ζ); we call both $P(G,d,\zeta)$ and $\overrightarrow{P(G,d,\zeta)}$ the *frequency assignment polyhedron*. The frequency assignment can now be stated as a linear program

$$\begin{array}{l} \min p^{\mathrm{T}}x \\ \qquad x \in P(G,d,\zeta) \quad \text{or} \quad \overrightarrow{P(G,d,\zeta)} \end{array}$$

— if we knew a description of the frequency assignment polyhedron in terms of inequalities. A first approximation to this description is the LP-relaxation of the formulation TIP(G,d,p,ζ) or ATIP(G,d,p,ζ). How good is the LP-relaxation? Unfortunately, it is miserable.

5 Observation (LP-relaxation) *Let TIP(G,d,p,ζ) or ATIP(G,d,p,ζ) be a frequency assignment problem with*

$$\zeta \geq \max d_{ij} \geq 2.$$

Then the point

$$(y_0, z_0, \Delta_0) := (0, 0, 1/2\mathbb{1})$$

is feasible and its objective value is 0.

Proof. If we set $\Delta_{ij} = 1/2$ for all $ij \in E$, the right hand side of inequality (1) and (2) becomes $d_{ij}/2 - \zeta/2$, of inequality (3) and (4) $1/2-\zeta/2$, and (5) and (6) get a right hand side of $1-\zeta/2$. Assuming $\zeta \geq \max d_{ij} \geq 2$, these right hand sides are negative and make constraints (1)–(6) redundant. The transitivity constraints in the model ATIP(G, d, p, ζ) are also fulfilled. Setting y and z to zero, the remaining constraints (7)–(9) are also satisfied. The objective value of the solution is $p^T z = p^T 0 = 0$. □

Even worse, if ζ is much larger than $\max d_{ij}$ like in the frequency assignment problems at eplus, the right hand sides of inequalities (1)–(6) are very negative at (y_0, z_0, Δ_0) and the constraints start to get tight only short before Δ gets integral. In other words, the Big-M link between the Δ and the other variables in the model is weak. If we want to solve the model TIP using a cutting plane approach, we would thus have to tighten the LP-relaxation substantially. A first class of inequalities is inherited from the *acyclic subdigraph polytope*, see Grötschel and Jünger [1982]. More precisely, let us denote by

$$\overrightarrow{P_\Delta(G, d, \zeta)}$$

the convex hull of all acyclic orientations (of diameter at most ζ), that is, $\overrightarrow{P_\Delta}$ is the projection of $\overrightarrow{P}$ on the space of the orientation variables. $\overrightarrow{P_\Delta}$ is the *acyclic orientation polytope* associated to the problem ATIP(G, d, p, ζ).

6 Observation (Acyclic Subdigraph Inequalities) *Each valid inequality $a^T x \leq b$ for the acyclic subdigraph polytope of a graph G is also valid for the acyclic orientation polytope $\overrightarrow{P(G, d, \zeta)}$ for any distance function d and any maximum diameter ζ; it is also valid for the frequency assignment problem ATIP(G, d, p, ζ).*

Proof. Each acyclic orientation is an acyclic subdigraph. □

Acyclic subdigraph inequalities do not take into account that only acyclic subdigraphs of diameter at most ζ are feasible for the frequency assignment problem and the acyclic subdigraph polytope is thus probably not a good approximation to the acyclic orientation polytope. However, additional inequalities could provide a better description and analogously to Observation 6 they are valid for the frequency assignment problem.

7 Observation (Tournament Inequalities) *Any inequality that is valid for the acyclic orientation polytope $\overrightarrow{P_\Delta(G, d, \zeta)}$ is valid for the frequency assignment polyhedron $\overrightarrow{P(G, d, \zeta)}$.*

It makes thus sense to study the acyclic orientation polytope. Interesting questions are:

- What is an integer linear formulation for the set of acyclic orientations (of diameter at most ζ)?
- We know that it is in general $\mathcal{NP}$-complete to decide whether the acyclic orientation polytope is empty or not. But are there reasonable conditions such that we nevertheless can make statements about the dimension of the acyclic orientation polytope?
- When is a facet of the acyclic subdigraph polytope facet-defining for the acyclic orientation polytope?
- What are additional facets? Can we separate them in polynomial time?

4.2 The Frequency Assignment Problem for Fixed Orientation

Clearly, the inequalities for the acyclic orientation polytope deal only with the orientation variables and we will need further inequalities for the frequency and the interference variables and inequalities linking the Δ and the (y, z) variables to get a good approximation of the frequency assignment polyhedron. Inequalities involving only the frequency and the interference variables, however, may be not so hard to find; in fact, for a large class of frequency assignment problems the inequalities in the IP-formulation do already suffice. Let us denote for each fixed orientation $\Delta \in P_\Delta$ the convex hull of the matching assignments (y, z) by

$$P^{\Delta}_{(y,x)}(G, d, \zeta) := \operatorname{conv}\{(y, x) : (y, x, \Delta) \in P(G, d, \zeta)\}.$$

By definition, we have

$$\mathbb{Z}\left(P^{\Delta}_{(y,x)}(G, d, \zeta)\right) = \left\{ \begin{array}{rcl} Ay + Bx & \leq & d - D\Delta \\ -y & \geq & -\zeta \mathbb{1} \\ y, x & \geq & 0 \\ (y, x) & \in & \mathbb{Z}^{V \times E} \end{array} \right\}.$$

We saw in the last section that for fixed orientation half of the system $Ay + Bx \le d - D\Delta$ is redundant; let us eliminate all inequalities with negative right hand sides and denote the remaining system by

$$A_\Delta y + B_\Delta x \le d_\Delta - D_\Delta \Delta,$$

such that

$$\left\{ \begin{array}{rcl} Ay + Bx & \le & d - D\Delta \\ -y & \ge & -\zeta \mathbb{1} \\ y, x & \ge & 0 \\ (y,x) & \in & \mathbb{Z}^{V\times E} \end{array} \right\} = \left\{ \begin{array}{rcl} A_\Delta y + B_\Delta x & \le & d_\Delta - D_\Delta \Delta \\ -y & \ge & -\zeta \mathbb{1} \\ y, x & \ge & 0 \\ (y,x) & \in & \mathbb{Z}^{V\times E} \end{array} \right\}.$$

Then it is easy to see that one can drop the integrality requirements if $\overline{E} \cap \overline{\overline{E}} = \emptyset$.

8 Theorem *Let TIP(G, d, p, ζ) be a frequency assignment problem with $\overline{E} \cap \overline{\overline{E}} = \emptyset$. Then the matrix (A_Δ, B_Δ) is totally unimodular for any $\Delta \in P_\Delta(G, d, \zeta)$ and*

$$P^\Delta_{(y,x)} = \left\{ \begin{array}{rcl} A_\Delta y + B_\Delta x & \le & d_\Delta - D_\Delta \Delta \\ -y & \ge & -\zeta \mathbb{1} \\ y, x & \ge & 0 \end{array} \right\}.$$

Proof. A —as the arc/node-incidence matrix of a digraph— is a totally unimodular network matrix and so is its submatrix A_Δ. We will now show that B_Δ contains only a single 1 per column and is thus a submatrix of the identity matrix. Given this, the Theorem follows.

Let us look at the interference variables in the system $A_\Delta y + B_\Delta x \le d_\Delta - D_\Delta \Delta$. For each edge ij, the co- and adjacent channel interference variables $\overline{z}_{ij}$ and $\overline{\overline{z}}_{ij}$ appear in only two rows, namely

$$\begin{array}{rclll} y_j - y_i + \overline{z}_{ij} & \ge & 1 & \forall ij \in \overline{E} & (3) \\ y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \ge & 2 & \forall ij \in \overline{\overline{E}} & (5) \end{array}$$

if $\Delta_{ij} = 1$ and

$$\begin{array}{rclll} -y_j + y_i + \overline{z}_{ij} & \ge & 1 & \forall ij \in \overline{E} & (4) \\ -y_j + y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \ge & 2 & \forall ij \in \overline{\overline{E}} & (6) \end{array}$$

if $\Delta_{ij} = 0$. If $\overline{E} \cap \overline{\overline{E}} = \emptyset$ these simplify to

$$\begin{array}{rclll} y_j - y_i + \overline{z}_{ij} & \ge & 1 & \forall ij \in \overline{E} & (3) \\ y_j - y_i \quad + \overline{\overline{z}}_{ij} & \ge & 2 & \forall ij \in \overline{\overline{E}} & (5) \end{array}$$

if $\Delta_{ij} = 1$ and simplify to

$$\begin{array}{rclll} -y_j + y_i + \overline{z}_{ij} & \ge & 1 & \forall ij \in \overline{E} & (4) \\ -y_j + y_i \quad + \overline{\overline{z}}_{ij} & \ge & 2 & \forall ij \in \overline{\overline{E}} & (6) \end{array}$$

if $\Delta_{ij} = 0$: each interference variable appears only in a single column with a coefficient of 1. □

In the case $\overline{E} \cap \overline{\overline{E}} = \emptyset$ where we can have either co- or adjacent channel interference on an edge but not both, we know thus a complete description of the polytope $P^\Delta_{(y,z)}(G, d, \zeta)$. Let us call this case the *simple case* in contrast to the *mixed case*. In the mixed case, (A_Δ, B_Δ) contains 2-entries and is thus no longer totally unimodular; in fact it is easy to construct examples where fractional solutions appear.

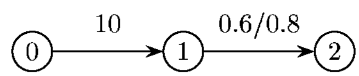

Figure 3: Mixed Interference Case.

9 Example (Fractional Solution for the Mixed Case) *In the example in Figure 3, we have to assign frequencies to three carriers such that* $y_0 \leq y_1 \leq y_2$. *There is a large co-channel interference* $\overline{p}_{01}$ *and small* $\overline{p}_{12}$ *and* $\overline{\overline{p}}_{12}$. *If we have 2 frequencies available, the best assignment will be* $y = (0, 1, 1)$ *with an interference of* 0.8. *However, the optimal solution of the LP*

$$\begin{array}{lcccccccccccccl}
\min & & & & & & & 10\overline{z}_{01} & + & 0.8\overline{z}_{12} & + & 0.6\overline{\overline{z}}_{12} & & \\
-y_0 & + & y_1 & & & & & & & & & & \geq & 0 \\
-y_0 & + & y_1 & & & + & \overline{z}_{01} & & & & & & \geq & 1 \\
 & - & y_1 & + & y_2 & & & & & & & & \geq & 0 \\
 & - & y_1 & + & y_2 & & & + & \overline{z}_{12} & & & & \geq & 1 \\
 & - & y_1 & + & y_2 & & & + & 2\overline{z}_{12} & + & \overline{\overline{z}}_{12} & & \geq & 2 \\
 & & & & & & & & & 0 \leq & y_0, y_1, y_2 & & \leq & 2
\end{array}$$

is $(y, z) = (0, 0, 1, 0, 1/2, 0)$ *with an objective value of* 0.4.

But, as we will show now, co-channel interference variables with values of $1/2$ are the worst that can happen.

10 Theorem (Halfintegrality of Interference Variables) *Let* $TIP(G, d, p, \zeta)$ *be a frequency assignment problem and* $\Delta \in P_\Delta(G, d, \zeta)$ *be a feasible orientation. Then all vertices* (y_0, z_0) *of* $P^\Delta_{(y,z)}$ *have integral components except for the* $\overline{z}$*-component which may be halfintegral, that is,*

$$(y_0, \overline{z}_0, \overline{\overline{z}}_0) \in \mathbb{Z}^V \times 1/2 \cdot \mathbb{Z}^{\overline{E}} \times \mathbb{Z}^{\overline{\overline{E}}}.$$

Proof. (y_0, z_0) is a vertex of $P^\Delta_{(y,z)}$ if and only if it is a basic feasible solution of the system

$$\begin{array}{rcl}
A_\Delta y + B_\Delta z & \geq & d_\Delta - D_\Delta \Delta \\
-y & \geq & -\zeta \mathbb{1} \\
y, z & \geq & 0.
\end{array} \qquad (I_\Delta(G, d, \zeta))$$

Let M be a basis for (y_0, z_0), that is, a set of $|V| + |\overline{E}| + |\overline{\overline{E}}|$ row indices such that (y_0, z_0) is the unique solution of the regular linear equation system

$$\begin{pmatrix} A_\Delta & B_\Delta \\ -I & \\ I & \\ & I \end{pmatrix}_{M\cdot} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} d_\Delta - D_\Delta \Delta \\ -\zeta \\ 0 \\ 0 \end{pmatrix}.$$

Consider an adjacent channel interference variable $\overline{\overline{z}}_{ij}$ where $ij \in \overline{\overline{E}} \setminus \overline{E}$ (there is no co-channel interference on the edge ij): there are only two inequalities that contain this variable, namely

$$\overline{\overline{z}}_{ij} \geq 0 \quad \text{and} \quad y_j - y_i + \overline{\overline{z}}_{ij} \geq 2$$

(assuming $\Delta_{ij} = 1$, the other case is analogous). If the non-negativity condition is constraint k in the basis and the adjacent channel constraint is also in the basis, we can eliminate $\overline{\overline{z}}_{ij}$ from the adjacent channel interference constraint doing a pivot operation; after the pivot, $\overline{\overline{z}}_{ij}$ occurs only in constraint k. If $\overline{\overline{z}}_{ij} = 0$ is not in the basis, the adjacent channel constraint must be in; denote this inequality by k in this case. In both cases, after probably doing a pivot, we obtain a system

$$\begin{pmatrix} A'_\Delta & B'_\Delta \\ -I & \\ I & \\ & I \end{pmatrix}_{M\cdot} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} d'_\Delta - D'_\Delta \Delta \\ -\zeta \\ 0 \\ 0 \end{pmatrix}$$

that contains only a single 1 in the $\overline{\overline{z}}_{ij}$-column; clearly, the solution of the system is the same as before. By deleting row k we get a system that is still regular but does not contain $\overline{\overline{z}}_{ij}$ any more. We can solve this system and then compute $\overline{\overline{z}}_{ij}$ using constraint k. Clearly, if the reduced system has an integral solution, $\overline{\overline{z}}_{ij}$ will also turn out to be integral. Applying this technique for all adjacent channel interference variables, we can assume that our system contains no simple adjacent channel interference variables.

The argument that we just used works exactly the same way for co-channel interference variables from $\overline{E} \setminus \overline{\overline{E}}$, so we can assume that we have only mixed interference left.

Let us thus consider the mixed case. There are co- and adjacent channel interference variables $\overline{z}_{ij}$ and $\overline{\overline{z}}_{ij}$; they appear in the four inequalities

$$\begin{array}{rcll} y_j - y_i + \ \overline{z}_{ij} & \geq & 1 & (1) \\ y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2 & (2) \\ \overline{z}_{ij} & \geq & 0 & (3) \\ \overline{\overline{z}}_{ij} & \geq & 0 & (4) \end{array}$$

(assuming $\Delta_{ij} = 1$). Suppose $\overline{z}_{ij} = 0$. Then we may assume without loss of generality that inequality (3) is in the basis. Doing pivots, we eliminate $\overline{z}_{ij}$ from all other inequalities in the basis (at most two). This will not change the solution of the system; dropping (3) from the modified system, we again get a smaller system, solve it, and use (3) to determine $\overline{z}_{ij}$ (which will be zero). The pivots also produced a simple adjacent channel interference that we can eliminate. So we may assume that all mixed co-channel interferences are non-zero. Now suppose $\overline{\overline{z}}_{ij} = 0$. Analogously, we can assume (4) to be basic, and eliminate $\overline{\overline{z}}_{ij}$ and (4) from the system. Now $\overline{z}_{ij} > 0$. If (1) is basic, (2) is not, we eliminate $\overline{z}_{ij}$ and (1), and determine from an integral solution of the reduced system $\overline{z}_{ij} = 1$ from (1) and $\overline{\overline{z}}_{ij} = 0$ from (4). However, if (1) is not basic, (2) is; (2) will then be the only basic constraint containing $\overline{z}_{ij}$. Suppose the reduced system with deleted (2) yields an integral y; then (2) determines $\overline{z}_{ij} = 1$ if $y_j - y_i = 0$ and $\overline{z}_{ij} = 1/2$ if $y_j - y_i = 1$. Since we have analyzed the cases $\overline{z}_{ij} = 0$ and $\overline{\overline{z}}_{ij} = 0$, we are left with the case $\overline{\overline{z}}_{ij} > 0$ and $\overline{z}_{ij} > 0$. Here, (1) and (2) must be in the basis while (3) and (4) are not. Deleting rows (1) and (2) from the system, we obtain again a reduced system. Having solved this, (1) and (2) determine $\overline{z}_{ij}$ and $\overline{\overline{z}}_{ij}$ — we will analyze their values in a minute.

Applying the technique just outlined, we reduce the system to the point where there are no more interference variables. All reductions, however, had no effect on A and maintained integrality of the right hand side; thus, y will be integral. All simple interference variables are then set to integral values as well as all mixed interference variables where $\overline{z}_{ij} = 0$; in the case $\overline{\overline{z}}_{ij} = 0$ either an integral or halfintegral $\overline{z}_{ij}$ may result. Thus we are left with the case of both non-zero mixed interference variables. Here, subtraction of the tight inequalities (1) and (2) yields $\overline{z}_{ij} + \overline{\overline{z}}_{ij} = 1$. If $y_j - y_i = 0$, $\overline{z}_{ij}$ has to be one and $\overline{\overline{z}}_{ij} = 0$, a contradiction. In case $y_j - y_i = 1$, $\overline{z}_{ij}$ is zero, a contradiction again. If $y_j - y_1 \geq 2$, (1) is not tight, which is also impossible. □

The proof of Theorem 10 gives us also a complete characterization of the feasible bases of the system $I_\Delta(G, d, \zeta)$. A little informally speaking, the bases arise from collections of trees plus additional linear independent inequalities to fix the interference variables. More precisely, but still informal, all feasible bases will be obtained using the following scheme.

- Select a forest $F \subseteq G$ covering all nodes in G.
- Let $T_1, \ldots, T_k$ be the trees of the forest. Select a vertex $r(T_i) \in T_l$ for every tree T_l.
- Choose inequalities from $I_\Delta(G, d, \zeta)$ for the each edge in each tree and the vertices $r(T_l)$ as follows:
  - For $r(T_l)$ choose either $y_{r(T_l)} \geq 0$ or $y_{r(T_l)} \leq \zeta$.
  - If $ij \in T_l$ is an edge with no interference, choose the distance inequality for edge ij.
  - If $ij \in T_l$ is an edge with simple interference, choose either the distance or the interference inequality; in the first case, choose either the interference or the non-negativity constraint to fix the interference variable (to zero), in the second case, choose the non-negativity constraint for the interference variable (which will also be zero).
  - If $ij \in T_l$ is an edge with mixed interference, 'decide' whether $y_j - y_i$ (assuming $\Delta_{ij} = 1$) should be 0, 1, or 2. In the first case, choose the distance and two additional linearly independent inequalities from (1)–(4) to fix the interference variables to $\overline{z}_{ij} = 1$ and $\overline{\overline{z}}_{ij} = 0$. In the second case, choose both interference and the non-negativity constraint $\overline{\overline{z}}_{ij} \geq 0$ to fix $\overline{\overline{z}}_{ij}$ to 0 and $\overline{z}_{ij}$ to 1. In the third case choose the adjacent channel interference and the two non-negativity constraints to fix the interference variables to 0.
- Compute y and z_F from the resulting system; (y, z_F) will be feasible because the paths in the trees are of length at most ζ, where edges oriented away from $r(T_l)$ are counted with weight d_{ij} and edges oriented toward the root count with weight d_{ij}; mixed interference edges count with 0, ± 1, or ± 2, according to the 'distance decision' and the orientation.

- For all other edges with simple interferences, select —depending on the value of $y_j - y_i$— either the interference or the non-negativity constraint to fix the interference variable. For mixed interferences, do the same; in the case $y_j - y_i = 1$, however, the additional choice to select $\overline{\overline{z}}_{ij} = 0$ and the adjacent channel constraint to fix $\overline{z}_{ij}$ to $1/2$ is also available.

Theorem 8 allows us to solve the *frequency assignment problem for fixed orientation* Δ

$$\begin{array}{lrcl} \min & p^{\mathrm{T}}x & & \\ & Ay + Bx & \leq & d - D\Delta \\ & -y & \geq & -\zeta\mathbb{1} \\ & y, x & \geq & 0 \end{array}$$

in polynomial time as a linear program — in the simple case. Theorem 10 tells us that this does not work in the mixed case, but that we are not far from integrality. An interesting question is:

- Can we solve the mixed case also in polynomial time or is this perhaps $\mathcal{NP}$-hard?

4.3 A Two-Stage Heuristic

We can summarize the results of the last two subsections as follows. The frequency assignment problem TIP(G, d, p, ζ) seems to be hard to solve, its LP-relaxation is bad and has to be tightened substantially before we will be able to derive significant lower bounds or good solutions from it. The acyclic orientation problem alone as well as the frequency assignment problem for fixed orientation seem to be 'easier' to solve. We can exploit this observation to invent a heuristic using a two step approach: we try to determine a good Δ and solve the frequency assignment problem for fixed Δ. From this solution, we try to gain information how to improve Δ and iterate until a good solution is found.

Our starting point to make this idea more precise is as follows.

$$\begin{array}{lrcl} \min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & \\ & Ay + Bz & \geq & d - D\Delta \\ & -y & \geq & -\zeta\mathbb{1} \\ & \Delta & \leq & \mathbb{1} \\ & y, \Delta, z & \geq & 0 \\ & y, \Delta, z & \in & \mathbb{Z}^{V\times E\times\overline{E}\times\overline{\overline{E}}} \end{array} = \min_{\Delta \in P_\Delta(G,d,\zeta)} \begin{array}{lrcl} \min & p^{\mathrm{T}}z & & \\ & Ay + Bz & \geq & d - D\Delta \\ & -y & \geq & -\zeta\mathbb{1} \\ & y, z & \geq & 0 \\ & (y, z) & \in & \mathbb{Z}^{V\times E} \end{array} =: \min_{\Delta \in P_\Delta(G,d,\zeta)} f(\Delta)$$

where f is the solution of the inner minimization problem. But Δ appears in this inner problem only on the right hand side and f is all but a linear function. Since a value is put on Δ in a very indirect way, it is hard to compare two values of Δ or to improve Δ. A possibility to get a linear objective function with respect to Δ is as follows.

$$\begin{aligned} \min_{\Delta \in P_\Delta(G,d,\zeta)} f(\Delta) &= \min_{\Delta \in P_\Delta(G,d,\zeta)} \begin{array}{lrcl} \min & p^{\mathrm{T}}z & & \\ & Ay + Bz & \geq & d - D\Delta \\ & -y & \geq & -\zeta\mathbb{1} \\ & y, z & \geq & 0 \\ & (y, z) & \in & \mathbb{Z}^{V\times E} \end{array} \\ &\geq \min_{\Delta \in P_\Delta(G,d,\zeta)} \begin{array}{lrcll} \min & p^{\mathrm{T}}z & & & \\ & Ay + Bz & \geq & d - D\Delta & (x) \\ & -y & \geq & -\zeta\mathbb{1} & (x_0) \\ & y, z & \geq & 0 & \end{array} \\ &= \min_{\Delta \in P_\Delta(G,d,\zeta)} \begin{array}{lrcl} \max & (d - D\Delta)^{\mathrm{T}}x - \zeta\mathbb{1}^{\mathrm{T}}x_t & & \\ & x^{\mathrm{T}}A - x_t^{\mathrm{T}} & \leq & 0 \\ & x^{\mathrm{T}}B & \leq & p^{\mathrm{T}} \\ & x, x_t & \geq & 0 \end{array} \end{aligned}$$

$$= \min_{\Delta \in P_\Delta(G,d,\zeta)} \quad \max \quad \begin{array}{rcl} d^{\mathrm{T}}x - \Delta^{\mathrm{T}}D^{\mathrm{T}}x - \zeta \mathbb{1}^{\mathrm{T}}x_t & & \\ x^{\mathrm{T}}A - x_t^{\mathrm{T}} & \leq & 0 \\ x^{\mathrm{T}}B & \leq & p^{\mathrm{T}} \\ x, x_t & \geq & 0 \end{array}$$

We first go to the LP-relaxation of the inner optimization problem; we loose something here, but Theorems 8 and 10 justify this step as acceptable. Dualization then brings Δ up into a linear objective function

$$x^{\mathrm{T}}D\Delta + \text{terms not depending on } \Delta$$

and our idea is to use $D^{\mathrm{T}}x$ as objective function in the outer optimization.

11 Algorithm (Two-Stage Heuristic for the Frequency Assignment Problem)
Input: *Frequency Assignment Problem TIP*(G, d, p, ζ)
Output: *Solution* (Δ, y, z) *for TIP*(G, d, p, ζ)
begin

$i := 0;$

$c_i := 0;$

do

/* *outer optimization* */

compute

$$\Delta_i := \operatorname*{argmin}_{\Delta \in P_\Delta(G,d,\zeta)} c_i^T\Delta \quad \text{or a 'good' } \Delta;$$

/* *inner optimization* */

compute

$$(x_i, x_{t_i}) := \operatorname{argmax} \quad \begin{array}{rcl} (d - D\Delta)^{\mathrm{T}}x - \zeta \mathbb{1}^{\mathrm{T}}x_t & & \\ x^{\mathrm{T}}A - x_t^{\mathrm{T}} & \leq & 0 \\ x^{\mathrm{T}}B & \leq & p^{\mathrm{T}} \\ x, x_t & \geq & 0 \end{array}$$

and the corresponding dual solution

$(y_i, z_i);$

$i{+}{+};$

/* *update of objective* */

$c_i := D^{\mathrm{T}}x_i;$

while *want to continue;*

output $(\Delta_i, y_i, z_i);$

end

Let us state here that we do not know at present whether Algorithm 11 or a similar approach works; in fact, the computation of Δ in the outer minimization is not even completely specified and first numerical experiments suggest that the procedure in its current form does not converge. Also, it is true that Δ appears now linearly in the objective — but not in a linear program, in a min / max problem with quadratic objective. So questions to consider are:

- Suppose we could compute the optimum Δ in the outer minimization. Does the algorithm converge? To the optimum?
- Can we still say something if we compute Δ with a heuristic?
- What is a good heuristic? This question is subject of the forthcoming Master's Thesis Haberland [1995].
- The inner minimization is correct in the simple case. What do we do in the mixed case?

Note also that the inner maximization requires the solution of an LP of large scale: there will be a column for any interference variable and a row for each distance or interference relation. In the case of eplus with

50,000 interference relations this results in a $50,000 \times 50,000$ constraint matrix. In first tests, such LPs could not be solved within hours of computation time — and after all, we want to iterate this step to get a heuristic. However, this difficulty can be overcome as we will show in the next section.

5 A Minimum-Cost Flow Approach

We have seen in the last section that the frequency assignment problem for fixed orientation can be solved *in the simple case* as a linear program

$$\begin{array}{rrcl} \min & p^{\mathrm{T}} z & & \\ & A_\Delta y + B_\Delta z & \geq & d_\Delta - D_\Delta \Delta \\ & -y & \geq & -\zeta \mathbb{1} \\ & y, z & \geq & 0. \end{array} \qquad (\mathrm{TIP}_\Delta(G,d,\zeta))$$

However, the LP may be quite large. But perhaps we can exploit the special structure of the problem by, for example, making use of the large network matrix A_Δ? And in fact, as we will show now, the dual of $\mathrm{TIP}_\Delta(G,d,\zeta)$ is a minimum cost flow problem — only in the simple case, of course.

Let us start by looking at a simple $\mathrm{TIP}_\Delta(G,d,\zeta)$ in more detail.

$$\begin{array}{lllcll} \min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & & \\ (x_{ij}) & y_j - y_i & & \geq & d_{ij} & \forall (i,j) \in A(\overrightarrow{G(\Delta)}) \\ (\overline{x}_{ij}) & y_j - y_i + \overline{z}_{ij} & & \geq & 1 & \forall (i,j) \in \overline{A}(\overrightarrow{G(\Delta)}) \\ (\overline{\overline{x}}_{ij}) & y_j - y_i & + \overline{\overline{z}}_{ij} & \geq & 2 & \forall (i,j) \in \overline{\overline{A}}(\overrightarrow{G(\Delta)}) \\ (x_{ij}) & -y_j + y_i & & \geq & d_{ij} & \forall (j,i) \in A(GD) \\ (\overline{x}_{ij}) & -y_j + y_i + \overline{z}_{ij} & & \geq & 1 & \forall (j,i) \in \overline{A}(\overrightarrow{G(\Delta)}) \\ (\overline{\overline{x}}_{ij}) & -y_j + y_i & + \overline{\overline{z}}_{ij} & \geq & 2 & \forall (j,i) \in \overline{\overline{A}}(\overrightarrow{G(\Delta)}) \\ (x_{tj}) & -y_j & & \geq & -\zeta & \forall j \in V \\ & & y, z & \geq & 0 & \end{array}$$

Introducing the dual variables on the left of the inequalities, the dual looks as follows.

$$\begin{array}{lrcll} \max & d^{\mathrm{T}}x + \mathbb{1}^{\mathrm{T}}\overline{x} + 2\mathbb{1}^{\mathrm{T}}\overline{\overline{x}} - \zeta\mathbb{1}^{\mathrm{T}}x_t & & & \\ & (x + \overline{x} + \overline{\overline{x}})\left(\delta^-(j)\right) - (x + \overline{x} + \overline{\overline{x}})\left(\delta^+(j)\right) - x_{tj} & \leq & 0 & \forall j \in V \quad (1) \\ & \overline{x}_{ij} & \leq & \overline{p}_{ij} & \forall ij \in \overline{E} \quad (2) \\ & \overline{\overline{x}}_{ij} & \leq & \overline{\overline{p}}_{ij} & \forall ij \in \overline{\overline{E}} \quad (3) \\ & x, x_t & \geq & 0 & \quad (4) \end{array}$$

$$(\mathrm{MCF}_\Delta(G,d,\zeta))$$

($\delta^-(j)$ denotes the arcs entering node j, $\delta^+(j)$ the arcs leaving j). If we interpret x and x_t as flow variables, $\mathrm{MCF}_\Delta(G,d,\zeta)$ becomes a min-cost flow problem. Let us consider the following example that is illustrated in Figure 4.

$$\begin{array}{lrrrrrrrcl} \min & \multicolumn{7}{c}{0.2\overline{z}_{01} + 0.3\overline{\overline{z}}_{14} + \overline{z} + 0.4\overline{z} + 0.8\overline{z}} & & \\ (x_{01}) & -y_0 & +y_1 & & & & & \geq & 0 \\ (\overline{x}_{01}) & -y_0 & +y_1 & & & & +\overline{z}_{01} & \geq & 1 \\ (x_{12}) & & -y_1 & +y_2 & & & & \geq & 1 \\ (x_{03}) & -y_0 & & & +y_3 & & & \geq & 0 \\ (\overline{x}_{03}) & -y_0 & & & +y_3 & & +\overline{z} & \geq & 1 \\ (x_{14}) & & -y_1 & & & +y_4 & & \geq & 1 \\ (\overline{\overline{x}}_{14}) & & -y_1 & & & +y_4 & +\overline{\overline{z}}_{14} & \geq & 2 \\ (x_{24}) & & & -y_2 & & +y_4 & & \geq & 0 \\ (\overline{x}_{24}) & & & -y_2 & & +y_4 & +\overline{z}_{24} & \geq & 1 \\ (x_{34}) & & & & -y_3 & +y_4 & & \geq & 0 \\ (\overline{x}_{34}) & & & & -y_3 & +y_4 & +\overline{z}_{34} & \geq & 1 \\ & & & & & \multicolumn{2}{r}{y_0, y_1, y_2, y_3, y_4} & \leq & 1 \\ & & & & & & y, z & \geq & 0. \end{array} \qquad (\mathrm{P})$$

Its dual is

$$
\begin{array}{rl}
\max & \overline{x}_{01} + \overline{x}_{03} + x_{12} + x_{14} + 2\overline{\overline{x}}_{14} + \overline{x}_{24} + \overline{x}_{34} - x_{0t} - \cdots - x_{4t} \\
 & - x_{01} - \overline{x}_{01} - x_{03} - \overline{x}_{03} - x_{0t} \leq 0 \\
 & x_{01} + \overline{x}_{01} - x_{12} - x_{14} - \overline{\overline{x}}_{14} - x_{1t} \leq 0 \\
 & x_{12} - x_{24} - \overline{x}_{24} - x_{2t} \leq 0 \\
 & x_{03} + \overline{x}_{03} - x_{34} - \overline{x}_{34} - x_{3t} \leq 0 \\
 & x_{14} + \overline{\overline{x}}_{14} + x_{24} + \overline{x}_{24} + x_{34} + \overline{x}_{34} - x_{4t} \leq 0 \\
 & \overline{x}_{01} \leq 0.2 \\
 & \overline{\overline{x}}_{14} \leq 0.3 \\
 & \overline{x}_{24} \leq 0.1 \\
 & \overline{x}_{03} \leq 0.4 \\
 & \overline{x}_{34} \leq 0.4 \\
 & x, \overline{x}, \overline{\overline{x}} \geq 0.
\end{array}
\qquad \text{(D)}
$$

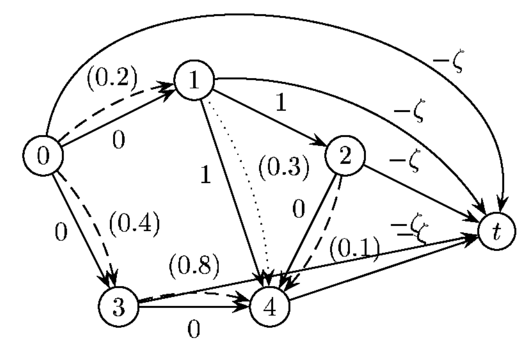

Figure 4: Min-Cost Flow Problem.

We have flow along the distance and the interference arcs in $\overrightarrow{G(\Delta)}$. The first five conditions in (D) state that there is at least as much flow out of each node in V as into the node; each node in V can thus act as a source. In addition to the network $\overrightarrow{G(\Delta)}$, there is an additional node t and an arc (i, t) for each node $i \in V$. t has no flow conservation constraint and no outgoing arcs. Since all nodes in V pass the flow they receive and possibly create new flow, the whole flow will end up at t which is thus the sink of the network. All arcs have unlimited capacity except for the interference arcs which have capacity equal to the interference. Distance arcs (i, j) increase the objective by the distance d_{ij}, co-channel interference arcs by 1 and adjacent channel interference arcs by 2. ζ is 1, that is, we have two frequencies.

The best solution for Example 4 is to assign frequency 0 to nodes 0, 1, and 3 and frequency 1 to nodes 2 and 4; this solution gives rise to a total interference of $0.2 + 0.1 + 0.3 + 0.4 = 1.0$. Let us now examine how this solution comes up in the min-cost flow model.

To maximize the objective function, we are looking for a path in the network from some node in V to the sink t where we can send flow. To maximize the objective, we want the path to contain as much distance and interference arcs as possible because these contribute to a positive objective function; at least, these values have to outweigh the $-\zeta$-weight on the obligatory final arc to the sink. A good path consists of the arcs

$$\{\overline{(0,1)}, (1,2), \overline{(2,4)}, (4,t)\}.$$

We can send 0.1 units of flow on it before $\overline{x}_{24}$ reaches its upper bound and get an increase in the objective from 0 to

$$0.1(1 + 1 + 1 - 1) = 0.2.$$

Replacing the saturated arc $\overline{(2,4)}$ in this path by the arc $(2,4)$ with unlimited capacity, we can send another 0.1 units until $\overline{(0,1)}$ reaches its capacity limit of 0.2. Since $d_{24} = 0$, this will cost us, however, one unit in the objective: the increase this time is only

$$0.1(1 + 1 + 0 - 1) = 0.1.$$

We could now replace $\overline{(0,1)}$ with $(0,1)$ and send an unlimited amount of flow on this path, however, this will not affect the objective function. Note that we have now accounted for the interferences $\overline{p}_{01}$ and $\overline{p}_{24}$. Another path, however, is

$$\{\overline{\overline{(1,4)}}, (4,t)\}.$$

We can send 0.3 units along this path and the objective goes up by

$$0.3(2-1) = 0.3,$$

which is exactly $\overline{\overline{p}}_{14}$. A last path is

$$\{\overline{(0,3)}, \overline{(3,4)}, (4,t)\}.$$

Sending 0.4 units along this path, we account for $\overline{p}_{03}$:

$$0.4(1+1-1) = 0.4.$$

All other paths do not increase the objective.

Figure 5 shows the primal and dual solutions of Example 4.

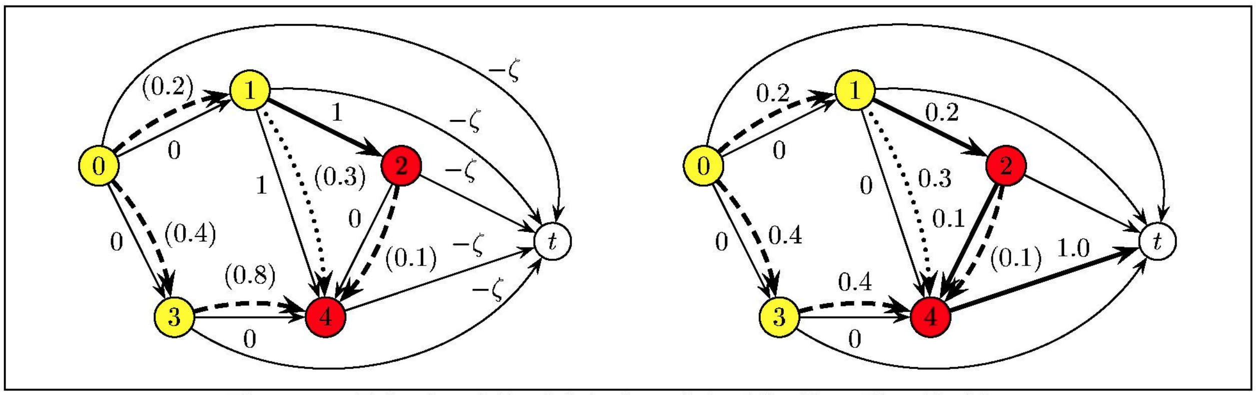

Figure 5: Primal and Dual Solution of the Min-Cost Flow Problem.

We can interpret these findings such that interference is 'caused' by long paths in $\overrightarrow{G(\Delta)}$. In our example, we sent 0.1 units of flow along the path

$$\{\overline{(0,1)}, (1,2), \overline{(2,4)}, (4,t)\}.$$

This path has one distance arc $(1,2)$ contributing 1 to the length, and two co-channel interference arcs, contributing 1 each. If we assign the same frequency to all nodes on the path, this would cause a violation of 1 of separation constraints and two times a co-channel interference, that is a total of 3. But we have $\zeta + 1 = 2$ frequencies available and can thus increase the frequency on the path $\zeta = 1$ times getting rid of $\zeta = 1$ in the sum of the violation and interferences.

The same holds for the path

$$\{\overline{\overline{(1,4)}}, (4,t)\}.$$

Using frequency 0 on all nodes on this path causes a distance violation of 1 plus 1 adjacent channel interference, a total of 2 and 2 is exactly the coefficient of $\overline{\overline{z}}_{14}$ in the objective function; 2 minus $\zeta = 1$ increases of the frequency on the path leaves 1 unit of adjacent channel interference.

An interesting case is also when (P) has no feasible solution. Then (D) will be unbounded because a path consisting of distance arcs with unlimited capacity will be found whose length exceeds ζ.

Although the original motivation for the min-cost flow model was to solve Algorithm 11's inner minimization problem in a way more efficient than using linear programming, the interpretation of problem (P)'s dual variables as a min-cost flow gives rise to a couple of further ideas.

- The min-cost flow model can be combined with any primal heuristic because each frequency assignment gives rise to an orientation via $\Delta_{ij} = 1 :\iff y_j > y_i$ or $y_j = y_i$ and $j > i$. Given this orientation, the assignment can be reoptimized with respect to Δ.

- After solving the min-cost flow problem, we can decompose the flow into its interference causing paths. A natural idea is to try to break long paths by reorienting arcs in some way. An idea is to try to identify an arc (i, j) occurring in 'many paths' or with high flow value and take this as an indication that the orientation from i upward to j is false and should be reversed. (The idea to reorient arcs is already mentioned in Borndörfer, Grötschel, and Martin [1995], however, the question how to find good candidates for reorientations remained open.)

- Another point is that the min-cost flow algorithm can potentially be used as a separation algorithm in a cutting plane approach to the frequency assignment problem $\mathrm{TIP}(G, d, p, \zeta)$. Suppose we could solve acyclic subdigraph problems (plus additional linear constraints) to integral optimality. To be an orientation, the acyclic digraph has to be of diameter at most ζ. Normally, it is difficult to check a longest path condition, but here, the min-cost flow problem can identify paths longer than ζ for us. Adding an inequality forbidding such a path P like, for example, $\sum_{a \in P} \Delta_a \leq |P| - 1$, we iterate.

Let us close this section with two remarks. The first is that the min-cost flow problem can be easily transformed to standard textbook form introducing a supersink s supplying all nodes in V and a return arc (t, s). In the linear program (D) this amounts to changing the flow conservation constraints to equations by introducing slack variables. Figure 6 shows the resulting network, D$'$ shows the transformed LP.

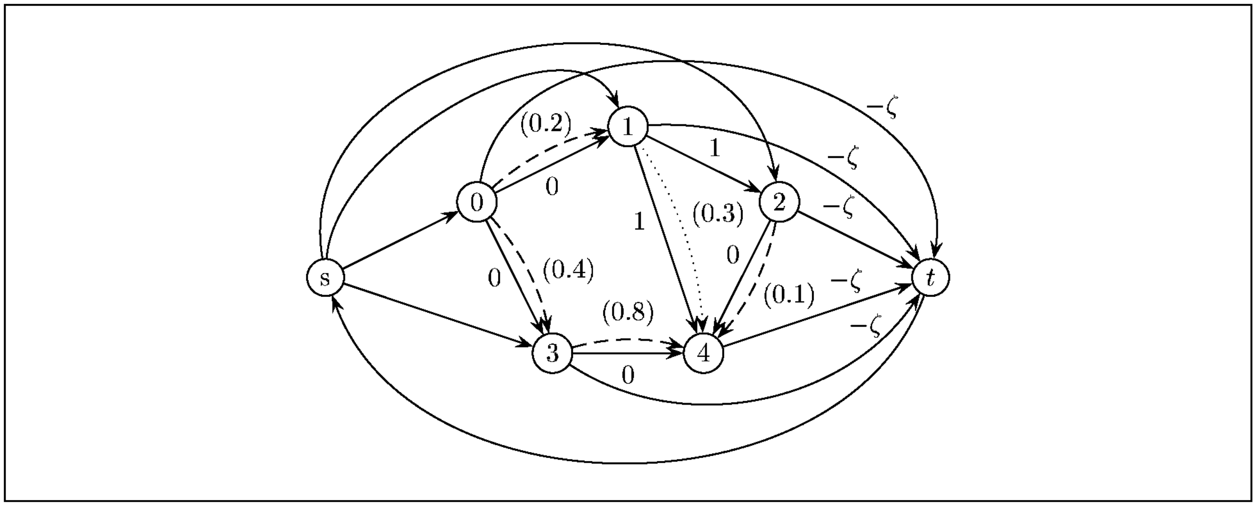

Figure 6: Textbook Min-Cost Flow Problem.

$$
\begin{aligned}
\max \quad & \overline{x}_{01} + \overline{x}_{03} + x_{12} + x_{14} + 2\overline{\overline{x}}_{14} + \overline{x}_{24} + \overline{x}_{34} - x_{0t} - \cdots - x_{4t} \\
& - x_{01} - \overline{x}_{01} - x_{03} - \overline{x}_{03} - x_{0t} + x_{s0} = 0 \\
& x_{01} + \overline{x}_{01} - x_{12} - x_{14} - \overline{\overline{x}}_{14} - x_{1t} + x_{s1} = 0 \\
& x_{12} - x_{24} - \overline{x}_{24} - x_{2t} + x_{s2} = 0 \\
& x_{03} + \overline{x}_{03} - x_{34} - \overline{x}_{34} - x_{3t} + x_{s3} = 0 \\
& x_{14} + \overline{\overline{x}}_{14} + x_{24} + \overline{x}_{24} + x_{34} + \overline{x}_{34} - x_{4t} + x_{a4} = 0 \\
& - x_{s0} - x_{s1} - x_{s2} - x_{s3} - x_{s4} + x_{ts} = 0 \\
& x_{0t} + x_{1t} + x_{2t} + x_{3t} + x_{4t} - x_{ts} = 0 \\
& \overline{x}_{01} \leq 0.2 \\
& \overline{\overline{x}}_{14} \leq 0.3 \\
& \overline{x}_{26} \leq 0.1 \\
& \overline{x}_{03} \leq 0.4 \\
& \overline{x}_{34} \leq 0.4 \\
& x, \overline{x}, \overline{\overline{x}} \geq 0.
\end{aligned}
\tag{D$'$}
$$

The second remark is about a simple transformation that allows to reduce some mixed interferences to the simple co-channel case. Consider a frequency assignment problem with fixed orientation $\mathrm{TIP}_{\Delta}(G, d, \zeta)$

with a mixed interference arc (i,j) that satisfies

$$2\overline{\overline{p}}_{ij} \le \overline{p}_{ij}.$$

We call (i,j) a *mixed arc with small adjacent channel interference.* The transformation consists in replacing (i,j) by two arcs with co-channel interference only. We delete (i,j) from $\overrightarrow{G(\Delta)}$, introduce an auxiliary node v_{ij} and two arcs

$$(i, v_{ij}) \quad \text{and} \quad (v_{ij}, j)$$

with

$$d_{iv_{ij}} := d_{v_{ij}j} := d_{ij} = 0, \quad \overline{p}_{iv_{ij}} = \overline{\overline{p}}_{ij}, \quad \text{and} \quad \overline{p}_{v_{ij}j} = \overline{p}_{ij} - \overline{\overline{p}}_{ij}.$$

Note that $\overline{p}_{v_{ij}j} \le \overline{p}_{ij} - \overline{\overline{p}}_{ij}$.

Lets call the frequency assignment problem resulting from this transformation $\mathrm{TIP}_\Delta(G,d,\zeta)'$. An example of this transformation is shown in Figure 7.

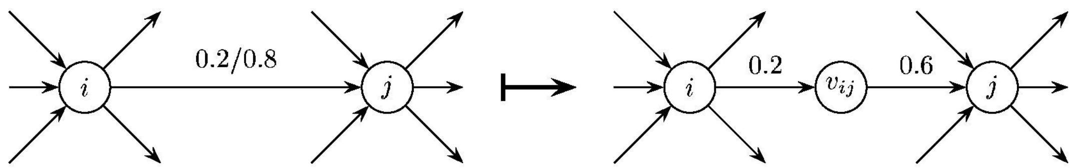

Figure 7: Mixed Interference with Small Adjacent Channel Interference.

It is easy to see that there is a one to one correspondence between *optimal* frequency assignments for $\mathrm{TIP}_\Delta(G,d,\zeta)$ and $\mathrm{TIP}_\Delta(G,d,\zeta)'$. Let y be an optimal frequency assignment for P. Then we can construct an assignment y' for $\mathrm{TIP}_\Delta(G,d,\zeta)'$ with the same objective value as follow. We set $y'_k := y_k$ for all $k \in V$. Now if $y'_i = y'_j$, we set $y_{v_{ij}}$ to y_i resulting in a co-channel interference of $\overline{\overline{p}}_{ij}$ on the arcs (i, v_{ij}) and a co-channel interference of $\overline{p}_{ij} - \overline{\overline{p}}_{ij}$ on arc (v_{ij}, j). If $y'_j - y'_i = 1$, the best we can do is to set $y'_{v_{ij}} := y'_i$ resulting in an interference of $\overline{\overline{p}}_{ij}$. If $y'_j - y'_i \ge 2$ we can do without any interference. The reverse transformation is even simpler; if y' is an optimal assignment for (P$'$), we just set $y_k := y'_k$ for all $k \in V$ and get an assignment with the same objective value.

We can thus deal with mixed interference where the adjacent channel interference is small. Although in the problems at eplus these form the bulk of mixed interferences, there remains a small rest of about 100 or so mixed interferences with large adjacent channel interference and the question arises whether it is possible to invent a similar but more complicated transformation to deal with this case. This, as we will point out now, is not possible using a 'local' transformation. Suppose we could construct a frequency assignment problem with fixed orientation $P_{(i,j)}$ involving some digraph $D_{(i,j)}$ (containing nodes i and j) with the following (local) property:

if y is an optimal solution of $P_{(i,j)}$ with objective value $f(y)$, then

$$f(y) = \begin{cases} \overline{p}_{ij} + M & \text{if } y_j - y_i = 0 \\ \overline{\overline{p}}_{ij} + M & \text{if } y_j - y_i = 1 \\ 0 + M & \text{if } y_j - y_i \ge 2. \end{cases} \tag{L}$$

(The constant M allows for some constant additional interference caused by the transformation.) If property (L) holds, we could replace (i,j) by $D_{(i,j)}$ and get only a constant shift in the objective.

Let y^1 and y^2 be two optimal solutions of $P_{(i,j)}$ with

$$y^1_j - y^1_i = 0 \quad \text{and} \quad y^2_j - y^1_j = 2;$$

suppose they have the additional property

$$y^1_i = y^2_i. \tag{E}$$

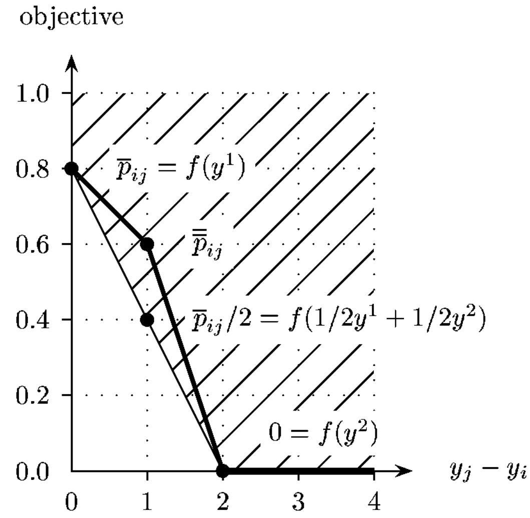

Figure 8: Mixed Interference with Large Adjacent Channel Interference.

But (L) and (E) together are not possible because f is not convex; a convex combination $1/2y^1 + 1/2y^2$ of y^1 and y^2 is a member of the convex hull of all feasible assignments but its objective value is less than the value of the optimal assignment

$$f(1/2y^1 + 1/2y^2) = 1/2f(y^1) + 1/2f(y^2) = M + \overline{p}_{ij}/2 < M + \overline{\overline{p}}_{ij},$$

a contradiction. Figure 8 illustrates the case $\overline{p}_{ij} = 0.8$ and $\overline{\overline{p}}_{ij} = 0.6$.

References

Aardal, K. and van Hoesel, C. (1995). A branch-and-cut algorithm for the frequency assignment problem. Technical Annex T-2.2.1 A, T.U. Eindhoven, Dept. of Math. and Computing Science, Eindhoven, The Netherlands.

Borndörfer, R., Grötschel, M., and Martin, A. (1995). Ansätze zur Lösung eines Frequenzzuweisungsproblems im Mobilfunk. Unpublished manuscript.

Box, F. (1978). A heuristic technique for assigning frequencies to mobile radio nets. *IEEE Transactions on Vehicular Technology*, VT-27(2):57–64.

Duque-Antón, M., Kunz, D., and Rüber, B. (1993). Channel assignment for cellular radio using simulated annealing. *IEEE Transactions on Vehicular Technology*, 42(1):14–21.

Gamst, A. (1986). Some lower bounds for a class of frequency assignment problems. *IEEE Transactions on Vehicular Technology*, VT-35(1):8–14.

Gamst, A. (1988). A resource allocation technique for FDMA systems. *Alta Frequenza*, LVII(2):89–96.

Grötschel, M. and Jünger, M. (1982). On the acyclic subdigraph polytope. Technical Report 82215-OR, Institut für Ökonometrie und Operations Research, Rheinische Friedrich-Wilhelms Universität, Bonn.

Haberland, D. (1995). Solving acyclic subdigraph problems arising in frequency assignment. Master's thesis, Technische Universität Berlin, Fachbereich Mathematik. (In Preparation).

Kaudewitz, A. and Kürner, T. (1995). Spezifikation eines Frequenzzuweisungsalgorithmus. Unpublished manuscript.

Metzger, B. (1970). Spectrum management technique. 38th National ORSA Meeting, Detroit.

Plehn, J. (1994). Applied frequency assignment. IEEE 0-7803-1927-3/94 pp. 853–857.

Quellmalz, A., Knälmann, A., and Krank, W. (1993). Simulated Annealing als Werkzeug zur Sendernetzplanung. *Rundfunktech. Mitteilungen*, 37:18–31.