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Lagrangean Based Methods for Solving Large-Scale Cellular Network Design Problems

FILIPE F. MAZZINI * and GERALDO R. MATEUS
Departamento de Ciência da Computação, Universidade Federal de Minas Gerais, Caixa Postal 702, CEP 30.123-970 Belo Horizonte, Minas Gerais, Brazil

JAMES MACGREGOR SMITH
Department of Mechanical and Industrial Engineering, 114 Marston Hall, University of Massachusetts Amherst, Amherst, MA 01003-5220, USA

Abstract. The cellular network design (CND) problem is formulated as a comprehensive linear mixed integer programming model integrating the base station location (BSL) problem, the frequency channel assignment (FCA) problem and the topological network design (TND) problem. A solution algorithm based on Lagrangean relaxation is proposed for solving this complex cellular network design problem. Pursuing the optimum solution through exact algorithms to this problem appears to be unrealistic considering the large scale nature and NP-hardness of the problem. Therefore, the solution algorithm strategy consists in computing effective lower and upper bounds for the problem. Lower bounds are evaluated through a Lagrangean relaxation technique and subgradient method. A Lagrangean heuristic is developed to compute upper bounds based on the Lagrangean solution. The bounds are improved through a customized branch and bound algorithm which takes in account specific knowledge of the problem to improve its efficiency. Thirty two random test instances are solved using the proposed algorithm and the CPLEX optimization package. The results show that the duality gap is excessive, so it cannot guarantee the quality of the solution. However, the proposed algorithm provides optimal or near optimal solutions for the problem instances for which CPLEX also provides the optimal solution. It further suggests that the proposed algorithm provides optimal or near optimal solutions for the other instances too. Finally, the results demonstrate that the proposed algorithm is superior to CPLEX as a solution approach for the CND problem.

Keywords: cellular network design, frequency channel assignment, base station location, optimization, algorithms

1. Introduction

The earliest cellular networks were comprised of a single base station with a large coverage area that was able to serve the entire mobile communication system. However, with the sharp increase of the demand of cellular services along with the limitations of the frequency spectrum use, a new structure of cellular networks developed. The first breakthrough was in deploying several base stations throughout the system area in order to decrease the coverage area of a base station and make frequency reuse possible.

Considering this new network configuration, new questions emerged. The first two questions were how many base stations should be deployed and where they should be placed. These questions have been addressed in the literature as the base station location problem (BSL) [1,2,18,19,21,24,36]. From a mathematical programming perspective, the problem can be formulated as the problem of selecting from a group of candidate base stations a subgroup of minimum cost that is able to cover the entire system area. Actually, this formulation is a particular instance of the well known facility location problem, where the facilities are the base stations. The facility location problem is NP-hard as shown by Mirchandani [41]. Therefore, the BSL is also NP-hard.

* Corresponding author.
E-mail: filipemazzini@hotmail.com

The second question to emerge was how to maximize the frequency reuse keeping at the same time the frequency interference under a threshold. Considering the second generation (2G) cellular networks as the GSM networks, which is based on Frequency Division Multiple Access (FDMA) and Time Division Multiple Access (TDMA), this question has been addressed in the literature as the frequency channel assignment (FCA) problem [3,7–9,14,18,19,25,27–30,35,43,44,46,48–51]. Again, from a mathematical programming viewpoint, the problem can be formulated as the problem of assigning frequency channels to the selected base stations subject to demand and frequency channel interference constraints. The objective function can vary according to the problem context. When the assignment is based on variable-size frequency spectrum, the objective usually intends to minimize the number of frequency channels used [3,25,27–29,48,50,51]. On the other hand, when the frequency spectrum size is fixed, the objective seeks to minimize the interference among frequency channels [7–9,27,46,48,51] or minimize the number of channels of unserved demand [14,30]. Hale [25] shows that the FCA problem is NP-hard exploiting it's equivalency to the graph coloring problem.

For the third generation (3G) cellular networks as the Universal Mobile Telecommunication System (UMTS) networks, the medium access is based on the Wideband Code Division Multiple Access (WCDMA) technology rather than on

TDMA or FDMA. In the CDMA technology, the whole spectrum is used for each connection rather than frequency channels and time slots. Despite the medium access technology, the problem of frequency spectrum reuse still exists in 3G networks since the frequency spectrum allocated is finite though it is wider than the one available for the second generation networks. Therefore, the interference problem also shows up in 3G networks, but now it is addressed by control power mechanisms [1,21,24].

The third question emerging from this new network configuration was how to connect the candidate base stations to the fixed telephone network. This question has been addressed in the literature as the topological network design (TND) problem [11,12,31]. Again according to the mathematical programming viewpoint, this problem can be formulated as the problem of defining the network topology of minimum cost that is able to connect the candidate base stations to the fixed telephone network. Some of the features that can be considered in the problem formulation are the use of hubs, different kinds of media, and link capacities. The topological network design problem is a classical one that has already been explored in other network contexts. Dutta [12] shows that the TND problem is NP-hard.

As introduced above, the cellular network design (CND) is a complex problem integrating the base station location (BSL) problem, the frequency channel assignment (FCA) problem and the topological network design (TND) problem. Therefore, the CND is an NP-hard problem since the BSL, FCA and TND problems are all NP-hard. It is easy to notice that you can always reduce the CND to any one of the BSL, FCA or TND problems. This is enough to prove the NP-hardness of the CND problem.

Numerous mathematical programming formulations have been proposed to solve separately each one of these problems, as well as corresponding solution techniques. However, it seems intuitive that both the FCA and the TND problems depend on the BSL problem solution, as well as their solutions. Previous computational analyses reported in [39] showed that there is a tradeoff between the BSL and FCA and between the BSL and TND problems. In order to show these relations, the authors solved the CND problem for a small instance. The problem was solved both separately and integratedly. In the first approach, the BSL problem was solved, and its solution was used as an instance for the FCA problem. In the BSL problem solution, 15 base stations were deployed. The FCA solution showed that this number was underestimated since 15% of the whole frequency channel demand was unserved. In the second approach, 16 base stations were deployed and there was no unserved frequency channel in the system. This computational result showed that the minimum BSL cost network was not able to serve the network demand. The paper also called the attention to the fact that although there were some base stations with available capacity in the first approach solution, it could not be used due to frequency channel interference. Considering the BSL and TND, the experiments showed that the set of base stations deployed was different for each approach and that the solution cost was higher for the first approach than for the second one. This result showed that the integrated approach was more cost-effective compared to the separate approach. In practice, that it is more cost-effective to address the relations among the subproblems since the cellular network companies solve them iteratively. They pursue the minimum network cost but also the quality of service. For instance, if the BSL solution is not able to provide quality of service, it is solved again with some extra constraints to force a capacity increase. However, it is more effective to address the problem tradeoff by an integrated formulation than by an iterative approach. Following this recent trend, research efforts have also been proposed [37,39,40] where the BSL and the FCA problems have been integrated into the same formulation.

An integrated approach to solve the cellular network design leads to a large scale linear mixed integer programming model with millions of variables and constraints. Unless the mathematical formulation has special structure, it appears to be unlikely that an exact solution can be obtained. Therefore, a typical approach is to look for solution techniques to provide effective bounds for the optimal solution. Considering that the cellular network design is a problem of minimizing costs, techniques such as linear relaxation [42], Lagrangean relaxation [15–17,22,23,42,45,47], Benders decomposition [6,32] as well as cutting plane methods [13] can be used to provide good lower bounds to the problem. On the other hand, heuristics [45] such as simulated annealing, tabu search, genetic algorithms as well as specific problem-based heuristics can be used to provide feasible solutions as well as upper bounds for the general problem. These bounds achieved by these latter methods can still be improved through tree search algorithms such as branch and bound.

This paper presents a solution algorithm to solve the cellular network design problem [39,40] for second generation networks. The problem is modeled as a linear mixed integer programming model that integrates the base station location (BSL), the frequency channel assignment (FCA) and the topological network design (TND) problems. The solution algorithm is based on a Lagrangean relaxation technique and subgradient method and consists in evaluating lower and upper bounds for the problem. The lower bound is derived by the solution of the Lagrangean dual problem and the upper bounds are computed based upon the Lagrangean problem solutions through a Lagrangean heuristic. These bounds are further improved through a customized branch and bound algorithm.

Nowadays, the research focus concerning cellular network design is mostly concentrated on 3G networks. However, despite this orientation there is still much to do concerning research on 2G networks. For instance, an integrated approach for the cellular network design has not been introduced so far. The cellular network companies use iterative approaches to solve the CND problem, which can not provide cost-effective solutions as the integrated approach does. In addition, the demand for 2G cellular network design is still active. For instance, 2G cellular networks are still being deployed in Brazil and all over the world. Moreover, the existing networks are constantly being expanded, and the framework proposed here is not only applied to the design of new networks, but it is also applied to the design of expanding networks as well. Finally, the framework introduced here can also be applied to the design of 3G cellular networks. Basically, the 2G and 3G network design differs from each other in the way that the frequency interference is addressed. In the 3G network design, the frequency channel assignment approach is replaced by the power control mechanism.

The remainder of the paper is organized as follows. Section 2 presents a mathematical formulation for the problem. Section 3 presents the Lagrangean solution algorithm. Section 4 presents the computational analyses, and section 5 presents a summary and conclusions.

2. Mathematical formulation

The mathematical formulation of the cellular network design (CND) problem was first introduced in [39,40], and it represents an effort to integrate the base station location, the frequency channel assignment and the network design problems into the same model.

Let

- I be the set of candidate base stations (BSs),
- H be the set of hubs,
- S be the set of switches,
- J be the set of points of demand,
- K be the set of frequency channels (FCs),
- N^i be the set of interfering BSs to BS i,

the cost coefficients are defined as:

- f_i the cost of selecting the BS i,
- c^1_{ih} the cost to connect the BS i to the hub h,
- c^2_{is} the cost to connect the BS i to the switch s,
- c^3_{hs} the cost to connect the hub h to the switch s,

the parameters are defined as:

- m_i the maximum number of points that can be served by BS i,
- n_i the maximum number of FCs that can be assigned to BS i,
- d_j the demand in number of communication channels of point j,
- e the number of communication channels carried by a FC,
- f_1 the minimum orthogonal frequency distance between adjacent FCs,

$$b_{ij} = \begin{cases} 1 & \text{if BS } i \text{ covers point } j, \\ 0 & \text{otherwise} \end{cases}$$

and the variables are defined as:

$$y_i = \begin{cases} 1 & \text{if the BS } i \text{ is selected} \\ 0 & \text{otherwise,} \end{cases}$$

$$t_{ih} = \begin{cases} 1 & \text{if the BS } i \text{ is connected to the hub } h \\ 0 & \text{otherwise,} \end{cases}$$

$$q_{is} = \begin{cases} 1 & \text{if the BS } i \text{ is connected to the switch } s \\ 0 & \text{otherwise,} \end{cases}$$

$$v_{hs} = \begin{cases} 1 & \text{if the hub } h \text{ is connected to the switch } s \\ 0 & \text{otherwise,} \end{cases}$$

$$z_{ik} = \begin{cases} 1 & \text{if the FC } k \text{ is assigned to BS } i \\ 0 & \text{otherwise,} \end{cases}$$

$0 \leqslant x_{ij} \leqslant 1$ the rate of point j served by BS i,

where $i \in I$, $h \in S$, $s \in S$, $j \in J$ and $k \in K$.

The objective of a cellular telephone operator when deploying a new cellular network or expanding an existing one is usually to minimize the network design costs. Such costs can be represented as the costs of base stations location and the costs to connect the located base stations to the fixed telephone network. Equation (1) represents the objective function of the CND model:

$$\min z = \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{h\in H} c^1_{ih} t_{ih} + \sum_{i\in I}\sum_{s\in S} c^2_{is} q_{is} + \sum_{h\in H}\sum_{s\in S} c^3_{hs} v_{hs}. \tag{1}$$

The CND problem is subject to different sets of constraints so that the feasibility of the BSL, FCA and TND problems can be guaranteed. For the BSL problem it is necessary to guarantee that every point of demand is served. In addition, a candidate base station that is not selected must not serve any point, and the number of points served by a selected base station must be kept under a threshold. These problem constraints are represented by the sets of equations (2) and (3). The set J of points of demand represents the service area of the cellular network. The parameters b_{ij} represent the coverage area of each base station. Finally, the linear variables x_{ij} define how much of a point j is served by a base station i. If $x_{ij} = 1$, the base station i serves completely (100%) the demand of the point j. If $0 < x_{ij} < 1$, the base station i serves only part of the demand of the point j. In this case, constraint (2) guarantees that another or other base stations serve the remaining demand not served by i. If $x_{ij} = 0$, the point j is not served by the base station i. The idea to make x_{ij} linear is to allow that a point j can be served by more than one base station. It makes the formulation more realistic.

$$\sum_{i\in I} b_{ij} x_{ij} \geqslant 1, \qquad \forall j \in J, \tag{2}$$

$$\sum_{j\in J} b_{ij} x_{ij} \leqslant m_i y_i, \quad \forall i \in I. \tag{3}$$

The feasibility of the TND problem is guaranteed when the connections between all selected base stations and the fixed network are established. Therefore, the set of constraints (4) guarantees for all base stations that if a base station i is selected, it must be connected at least to a hub or switch, while the set of constraints (5) guarantees for all hubs that if there is any selected base station connected to a hub h, it must be connected at least to a switch.

$$\sum_{h\in H} t_{ih} + \sum_{s\in S} q_{is} \geqslant y_i, \quad \forall i \in I, \tag{4}$$

$$\sum_{s\in S} v_{hs} \geqslant t_{ih}, \qquad \forall i \in I, \forall h \in H. \tag{5}$$

In the FCA problem, the feasibility depends on channel interference, demand and capacity constraints. The sets of FCA constraints are represented by equations (6)–(9). The set of constraints (6) guarantees that adjacent frequency channels assigned to the same base station are separated from each other by at least f_1 orthogonal frequency channels. The set of constraints (7) guarantees that if a frequency channel k is assigned to a base station i, it must not be assigned to any one of the base stations belonging to the interfering set N^i. The interfering sets N^i define for each base station i the other base stations that cause interference to its frequency channels. The set of constraints (8) imposes that the number of frequency channels assigned to a base station i is enough to serve the demand of the points served by that base station. The parameter e defines how many communication channels can be carried by a frequency channel. In case of FDMA technologies it is set to 1, while it can be set to 3 in case of some TDMA technologies. Finally, the set of constraints (9) guarantees that a frequency channel can be assigned to a base station i only if the base station is selected, and it still imposes a bound on the maximum number of frequency channels assigned to a base station i.

$$\sum_{k=l}^{l+f_1} z_{ik} \leqslant 1, \qquad \forall i \in I, \quad \forall l \in \{1, \ldots, |K| - f_1\}, \tag{6}$$

$$\sum_{l\in N^i} z_{lk} \leqslant (1 - z_{ik})|I|, \quad \forall i \in I,\ \forall k \in K, \tag{7}$$

$$e\sum_{k\in K} z_{ik} \geqslant \sum_{j\in J} d_j x_{ij}, \qquad \forall i \in I, \tag{8}$$

$$\sum_{k\in K} z_{ik} \leqslant n_i y_i, \qquad \forall i \in I. \tag{9}$$

The problem variables are defined as binary or linear by equations (10) and (11):

$$y_i, z_{ik}, t_{ih}, q_{is}, v_{hs} \in \{0, 1\}, \quad \forall i \in I, \forall k \in K, \forall h \in H, \forall s \in S, \tag{10}$$

$$0 \leqslant x_{ij} \leqslant 1, \quad \forall i \in I, \forall j \in J. \tag{11}$$

Two cuts were proposed in order to improve the quality of the problem lower bound. They are given by equations (12) and (13). Both cuts impose a lower bound on the number of selected base stations. As this number must be an integer, the two cuts avoid the integrality of the relaxed problem. Thus, an improvement was expected for the Lagrangean problem solution. However, a significant improvement was not achieved, so the cuts were no longer considered.

$$e\sum_{i\in I} y_i n_i \geqslant \sum_{j\in J} d_j, \tag{12}$$

$$\sum_{i\in I} m_i y_i \geqslant |J|. \tag{13}$$

Important aspects must be highlighted in the present formulation. The formulation considers coverage of the entire system area, null frequency channel interference in the assignment and channel demand totally satisfied. When the model gathers together all of these three features, the cellular network solution guarantees that every customer anywhere in the service area will be served by a channel without interference anytime it requests a channel. In addition, it allows a formulation with an objective function based only on cost terms. However, if the costs for deploying such networks becomes high, some of these restrictions can be partially relaxed. In the formulations presented in [7–9,27,46,48,51], the frequency channel interference is allowed in the final network configuration, while the problem objective aims to minimize it. In the formulations [14,30], the base station channel demand may not be totally satisfied. In such formulations, the objective function aims to minimize the demand not served, which is evaluated in a number of channels. Partial coverage may still be considered when the monetary resources are limited.

Another important formulation feature is that each frequency channel is addressed independently in the assignment rather than in the classical approach proposed by Lee [33] where the assignment is made through groups of channels. Finally, the mathematical formulation adopted is fairly general once no specific practical instance is being addressed. However, practical aspects like base station configuration can be handled by the model with no significant impact.

3. Lagrangean algorithm

The approach pursued to solve the CND problem consists in providing effective bounds for the problem solution. The lower bound is evaluated through a Lagrangean relaxation technique. The improvements of the lower bounds computed by the Lagrangean relaxation are achieved by a subgradient method. Since the Lagrangean relaxation seldom provides a feasible solution to the problem, a Lagrangean heuristic is formulated to provide an upper bound, as well as a feasible solution.

The bounds evaluated through the Lagrangean relaxation and heuristic give a conservative range where the solution lies. This range, usually called the duality gap, is reduced through a branch and bound algorithm. The solution approach is called a Lagrangean algorithm since it is based on the Lagrangean relaxation technique.

3.1. Lagrangean relaxation

The Lagrangean relaxation technique [15–17,22,23,42,45,47], originally developed to solve linear programming problems, has been successfully applied to solve large scale linear mixed integer programming problems. This solution technique consists basically in relaxing the sets of complicating constraints and penalizing them in the objective function by means of Lagrange multipliers. The idea of the technique is to make the problem easier to solve. Usually, the relaxed problem can be decomposed into nice subproblems of easy solution. The solution of the relaxed problem provides a lower bound for the original problem.

Many different relaxation trials were performed to make the CND an easy problem. In the CND model, there are two classes of complicating constraints. The first is the class of constraints that binds different kinds of variables avoiding the decomposition of the problem. This first class is comprised of constraints (3), (5), (8) and (9). The second class includes combinatorial constraints that hinder the decomposition of the problem into subproblems that can be solved in polynomial time. This second class is comprised of constraints (6) and (7). Therefore, the idea is to relax these complicating constraints.

Let $\alpha_i \geqslant 0, \forall i \in I$, be the set of multipliers associated with the set of constraints (3), $\beta_{ih} \geqslant 0$, $\forall i \in I$ and $\forall h \in H$, be the set of multipliers associated with the set of constraints (5), $\rho_{il} \geqslant 0, \forall i \in I$ and $\forall l \in \{1, \dots, |K| - f_1\}$, be the set of multipliers associated with the set of constraints (6), $\gamma_{ik} \geqslant 0$, $\forall i \in I$ and $\forall k \in K$, be the set of multipliers associated with the set of constraints (7), $\delta_i \geqslant 0, \forall i \in I$, be the set of multipliers associated with the set of constraints (8) and $\varepsilon_i \geqslant 0, \forall i \in I$, be the set of multipliers associated with the set of constraints (9). The Lagrangean problem can be stated as:

$$\begin{aligned}
&Z_{LR}(\alpha, \beta, \rho, \gamma, \delta, \varepsilon)\\
&= \min \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{h\in H} c^1_{ih} t_{ih} + \sum_{i\in I}\sum_{s\in S} c^2_{is} q_{is}\\
&\quad + \sum_{h\in H}\sum_{s\in S} c^3_{hs} v_{hs}\\
&\quad + \sum_{i\in I} \alpha_i \left(\sum_{j\in J} b_{ij} x_{ij} - m_i y_i \right)\\
&\quad + \sum_{i\in I}\sum_{h\in H} \beta_{ih} \left(t_{ih} - \sum_{s\in S} v_{hs} \right)\\
&\quad + \sum_{i\in I}\sum_{l=1}^{|K|-f_1} \rho_{il} \left(\sum_{k=l}^{l+f_1} z_{ik} - 1 \right)\\
&\quad + \sum_{i\in I}\sum_{k\in K} \gamma_{ik} \left(\sum_{l\in N^i} z_{lk} - |I| + z_{ik}|I| \right)\\
&\quad + \sum_{i\in I} \delta_i \left(\sum_{j\in J} d_j x_{ij} - \sum_{k\in K} e z_{ik} \right)\\
&\quad + \sum_{i\in I} \varepsilon_i \left(\sum_{k\in K} z_{ik} - n_i y_i \right)
\end{aligned} \tag{14}$$

subject to: (2), (4), (10) and (11).

For any given set of multipliers α, β, ρ,γ, δ and ε, the Lagrangean problem can be decomposed in four subproblems in that $Z_{LR}(\alpha, \beta, \rho, \gamma, \delta, \varepsilon) = Z_{LR_x}(\alpha, \delta) + Z_{LR_v}(\beta) + Z_{LR_{ytq}}(\alpha, \beta, \varepsilon) + Z_{LR_z}(\rho, \gamma, \delta, \varepsilon)$. The solution can be calculated by inspection for all of the four subproblems.

3.1.1. Subproblem $LR_x(\alpha, \delta)$

The subproblem LR_x is defined as:

$$Z_{LR_x}(\alpha, \delta) = \min \sum_{i\in I}\sum_{j\in J} (\alpha_i b_{ij} + \delta_i d_j) x_{ij} \tag{15}$$

subject to: constraints (2) and (11).

The subproblem LR_x can still be decomposed into J subproblems in that:

$$Z_{LR_x}(\alpha, \delta) = \sum_{j\in J} Z^j_{LR_x}(\alpha, \delta) \tag{16}$$

where LR^j_x is defined as:

$$Z^j_{LR_x}(\alpha, \delta) = \min \sum_{i\in I} (\alpha_i b_{ij} + \delta_i d_j) x_{ij} \tag{17}$$

subject to:

$$\sum_{i\in I} b_{ij} x_{ij} \geqslant 1, \tag{18}$$

$$0 \leqslant x_{ij} \leqslant 1, \quad \forall i \in I. \tag{19}$$

The solution of subproblem $Z^j_{LR_x}(\alpha, \delta)$ is given by setting $x_{i^*j} = 1$ for the base station i^* with minimum cost $(\alpha_i b_{ij} + \delta_i d_j)$ for all $i \in I$ and $x_{ij} = 0$ for all $i \in I \setminus \{i^*\}$.

3.1.2. Subproblem $LR_v(\beta)$

The subproblem LR_v is defined as:

$$Z_{LR_v}(\beta) = \min \sum_{h\in H}\sum_{s\in S} \left(c^3_{hs} - \sum_{i\in I} \beta_{ih} \right) v_{hs} \tag{20}$$

subject to:

$$v_{hs} \in \{0, 1\}, \quad \forall h \in H, \forall s \in S. \tag{21}$$

The solution of subproblem $Z_{LR_v}(\beta)$ is given by setting $v_{hs} = 1$ for all pairs $h \in H$ and $s \in S$ of hubs and switches such that the cost $(c^3_{hs} - \sum_{i\in I} \beta_{ih})$ is non-positive and $v_{hs} = 0$ otherwise.

3.1.3. Subproblem $LR_{ytq}(\alpha, \beta, \varepsilon)$

The subproblem LR_{ytq} is defined as:

$$\begin{aligned}
Z_{LR_{ytq}}(\alpha, \beta, \varepsilon) = \min \sum_{i\in I} \Bigg[(f_i - \alpha_i m_i - \varepsilon_i n_i) y_i + \sum_{s\in S} c^2_{is} q_{is}\\
+ \sum_{h\in H} (c^1_{ih} + \beta_{ih}) t_{ih} \Bigg]
\end{aligned} \tag{22}$$

subject to:

$$y_i, t_{ih}, q_{is} \in \{0, 1\}, \quad \forall i \in I, \forall h \in H, \forall s \in S \tag{23}$$

and constraints (4).

The subproblem LR_{ytq} can be redefined as:

$$Z_{LR_{ytq}}(\alpha, \beta, \varepsilon) = \min \sum_{i \in I} (f_i - \alpha_i m_i - \varepsilon_i n_i + P_i) y_i \quad (24)$$

subject to:

$$y_i \in \{0, 1\}, \quad \forall i \in I, \quad (25)$$

where P_i is defined as:

$$P_i = \min \sum_{s \in S} c_{is}^2 q_{is} + \sum_{h \in H} (c_{ih}^1 + \beta_{ih}) t_{ih} \quad (26)$$

subject to:

$$\sum_{h \in H} t_{ih} + \sum_{s \in S} q_{is} \geqslant 1, \quad (27)$$

$$t_{ih}, q_{is} \in \{0, 1\}, \quad \forall h \in H, \forall s \in S. \quad (28)$$

The subproblem P_i is solved firstly by finding the switch s^* such that the cost c_{is} is minimum for all $s \in S$ and finding the hub h^* such that the cost $(c_{ih}^1 + \beta_{ih})$ is minimum for all $h \in H$. Finally, t_{ih^*} is set to 1 and t_{ih} is set to 0 for all $h \in H \setminus \{h^*\}$ if the cost $(c_{ih^*}^1 + \beta_{ih^*})$ is smaller than the cost c_{is^*}, and q_{is^*} is set to 1 and q_{is} is set to 0 for all $s \in S \setminus \{s^*\}$ otherwise. Once P_i is known for all $i \in I$, the subproblem $Z_{LR_{ytq}}(\alpha, \beta, \varepsilon)$ is solved by setting $y_i = 1$ for all base stations $i \in I$ such that the cost $(f_i - \alpha_i m_i - \varepsilon_i n_i + P_i)$ is non-positive and $y_i = 0$ otherwise. In the solution of the subproblem LR_{ytq}, if y_i is set to 0, the solution t_{ih^*} and q_{is^*} of the subproblem P_i is also set to 0.

3.1.4. Subproblem $LR_z(\rho, \gamma, \delta, \varepsilon)$

The subproblem LR_z is defined as:

$$\begin{aligned} &Z_{LR_z}(\rho, \gamma, \delta, \varepsilon) \\ &= \min \sum_{i \in I} \sum_{k \in K} \Big[(\gamma_{ik} |I| - \delta_i e + \varepsilon_i) z_{ik} + \gamma_{ik} \sum_{l \in N^i} z_{lk} \Big] \\ &\quad + \sum_{i \in I} \sum_{l=1}^{|K|-f_1} \rho_{il} \sum_{k=l}^{l+f_1} z_{ik} - |I| \sum_{i \in I} \sum_{k \in K} \gamma_{ik} \\ &\quad - \sum_{i \in I} \sum_{l=1}^{|K|-f_1} \rho_{il} \end{aligned} \quad (29)$$

subject to:

$$z_{ik} \in \{0, 1\}, \quad \forall i \in I, \forall k \in K. \quad (30)$$

The solution of subproblem $Z_{LR_z}(\rho, \gamma, \delta, \varepsilon)$ is given by setting $z_{ik} = 1$ for all pairs $i \in I$ and $k \in K$ of base stations and frequency channels such that the cost coefficient of z_{ik} is non-positive and $z_{ik} = 0$ otherwise.

3.2. Lagrangean dual

The LR problem gives, for any set of non-negative multipliers $\alpha, \beta, \rho, \gamma, \delta$ and ε, a lower bound for the CND problem. Therefore, the higher is the value of the LR solution, the better is the value of the lower bound for the CND problem. The problem of finding the set of non-negative multipliers that maximize the LR solution is called Lagrangean Dual (LD), which is defined as:

$$Z_{LD} = \max \ Z_{RL}(\alpha, \beta, \rho, \gamma, \delta, \varepsilon), \quad (31)$$

$$\alpha, \beta, \rho, \gamma, \delta, \varepsilon \geqslant 0. \quad (32)$$

The LD problem is a non-differentiable mathematical programming problem, for which the objective function is convex and piece-wise linear. The method used to solve the LD problem is the classical subgradient method [16,17,26,45]. This method consists of iteratively updating the values of the Lagrangean multipliers in that at the n_{th} iteration the multipliers are updated by the following equations:

$$\alpha^{n+1} = \max \{0, \alpha^n + p^n g_\alpha^n(x^n, y^n)\}, \quad (33)$$

$$\beta^{n+1} = \max \{0, \beta^n + p^n g_\beta^n(t^n, v^n)\}, \quad (34)$$

$$\rho^{n+1} = \max \{0, \rho^n + p^n g_\rho^n(z^n)\}, \quad (35)$$

$$\gamma^{n+1} = \max \{0, \gamma^n + p^n g_\gamma^n(z^n)\}, \quad (36)$$

$$\delta^{n+1} = \max \{0, \delta^n + p^n g_\delta^n(x^n, z^n)\}, \quad (37)$$

$$\varepsilon^{n+1} = \max \{0, \varepsilon^n + p^n g_\varepsilon^n(z^n, y^n)\}, \quad (38)$$

where g_α^n, g_β^n, g_ρ^n, g_γ^n, g_δ^n and g_ε^n are the subgradient vectors of the function $Z_{RL}(\alpha, \beta, \rho, \gamma, \delta, \varepsilon)$ in the nth iteration with respect to $\alpha, \beta, \rho, \gamma, \delta$ and ε, respectively. The max function is used in the equations (33)–(38) to guarantee the feasibility of the constraint (32). The subgradient vectors are defined as:

$$g_{\alpha_i}^n(x^n, y^n) = \sum_{j \in J} b_{ij} x_{ij}^n - m_i y_i^n, \quad \forall i \in I, \quad (39)$$

$$g_{\beta_{ih}}^n(t^n, v^n) = t_{ih}^n - \sum_{s \in S} v_{hs}^n, \quad \forall i \in I, \ \forall h \in H, \quad (40)$$

$$g_{\rho_{il}}^n(z^n) = \sum_{k=l}^{l+f_1} z_{ik}^n - 1, \quad \forall i \in I, \quad \forall l \in \{1, \dots, |K| - f_1\}, \quad (41)$$

$$g_{\gamma_{ik}}^n(z^n) = \sum_{l \in N^i} z_{lk}^n - |I| + z_{ik}^n |I|, \quad \forall i \in I, \ \forall k \in K, \quad (42)$$

$$g_{\delta_i}^n(x^n, z^n) = \sum_{j \in J} d_j x_{ij}^n - \sum_{k \in K} e z_{ik}^n, \quad \forall i \in I, \quad (43)$$

$$g_{\varepsilon_i}^n(z^n, y^n) = \sum_{k \in K} z_{ik}^n - n_i y_i^n, \quad \forall i \in I. \quad (44)$$

The step size is defined as:

$$p^n = \pi \frac{1.05UB - Z_{LR}(\alpha^n, \beta^n, \rho^n, \gamma^n, \delta^n, \varepsilon^n)}{\|g_\alpha^n\|^2 + \|g_\beta^n\|^2 + \|g_\rho^n\|^2 + \|g_\gamma^n\|^2 + \|g_\delta^n\|^2 + \|g_\varepsilon^n\|^2}, \quad (45)$$

where π is a scalar defined for $0 < \pi \leqslant 2$. The scalar π is used to control the method convergence speed since it controls the step size. The parameter UB is any upper bound for the CND problem that can be a known feasible solution or the maximum possible value for the problem objective function given by equation (1). It is premultiplied by 1.05 in order to speed up the convergence near the optimum region. The mathematical function $\|\cdot\|$ defines the Euclidean norm of a vector. This step size equation is the standard equation for the subgradient method.

At the first iteration, a lower bound is evaluated through the LR problem solution for an initial set of non-negative Lagrangean multipliers. For instance, the initial Lagrangean multipliers can be set to zero. Once the LR problem solution is known, for the given set of multipliers, the subgradients given by equations (39)–(44) and the step size given by equation (45) can be evaluated. Based on this information, the new values for the multipliers are evaluated by equations (33)–(38). The method iterates until it converges to the optimal LD problem solution or until it is ended by the convergence control strategy. In the latter case, the solution highest value found for the LD problem gives the LR problem lower bound. The convergence control follows the idea presented by Fisher [15,16] and Reeves [45]. The parameter π is initially set to 2, and it is halved after each 10 consecutive iterations up to a total of 60 iterations. Despite whether it converges or not, the subgradient method ends up with an improved lower bound to the LR problem.

3.3. Lagrangean heuristic

The upper bound (UB) for the CND problem is evaluated by the Lagrangean heuristic which is an heuristic developed specifically for the CND problem. The main idea of the heuristic consists in making the LR solution feasible performing the minimal number of modifications in its solution. The heuristic is invoked every time a better LR solution is obtained. When the LD converges, the best solution evaluated by the heuristic gives an upper bound for the CND problem.

The strategy defined to make the LR solution feasible can be divided into two phases. In the initial phase, which is further divided in two stages, feasibility concerning network capacity is guaranteed. Though the FCA also interferes in the capacity of a network, in the first phase, the network capacity matters only if the network is able or not to serve all points and their demand based on the capacity parameters m_i and n_i. In the first stage, the algorithm guarantees that each base station, selected or not, does not serve a number of points greater than its capacity. It is done by re-allocating points from base stations which capacity is exceeded to base stations with some available capacity. Moreover, a point is re-allocated only between base stations that cover such point. In the second stage, the algorithm tries to re-allocate the points served by base stations that are not selected to selected base stations. The idea is to minimize the number of base stations selected. If it is not possible to re-allocate the points served by a base station not selected, this base station is selected. In the first phase, the capacity of a base station i can be limited by the constraint $m_i y_i - \sum_{j\in J} b_{ij}x_{ij} \geqslant 0$ ($cap = 1$) or $en_i y_i - \sum_{j\in J} x_{ij}d_j \geqslant 0$ ($cap = 2$). The former constraint is the capacity constraint for the BSL problem. The latter constraint is formulated in order to guarantee that a base station demand does not exceed its capacity in terms of frequency channels despite interference restrictions. The idea is to make the heuristic solution in the first phase a good candidate to become feasible considering the FCA problem. Since it is not known in advance which one is tighter, both capacities should be considered. The heuristic capacity 1 ($cap = 1$) is considered first. If the solution satisfy this capacity for all base stations, it considers the capacity 2 ($cap = 2$). If it succeeds again, the solution is feasible concerning network capacity. A solution is infeasible when no more re-allocation is possible and there is still a base station or more than one with the capacity exceeded. The algorithm is given below.

Step 1. Let $cap = 1$, $stage = 1$ and $P^i = \{j \in J \mid x_{ij} > 0\}$, $\forall i \in I$.

Step 2. Case:

1. $cap = 1$ and $stage = 1$, compute the costs $c_i = m_i - \sum_{j\in J} b_{ij}x_{ij}$, $\forall i \in I$.
2. $cap = 1$ and $stage = 2$, compute the costs $c_i = m_i y_i - \sum_{j\in J} b_{ij}x_{ij}$, $\forall i \in I$.
3. $cap = 2$ and $stage = 1$, compute the costs $c_i = en_i - \sum_{j\in J} x_{ij}d_j$, $\forall i \in I$.
4. $cap = 2$ and $stage = 1$, compute the costs $c_i = en_i y_i - \sum_{j\in J} x_{ij}d_j$, $\forall i \in I$.

Step 3. For all $i \in I$, let $i \in F$ (feasible) if $c_i \geqslant 0$ and $i \in NF$ (not feasible) otherwise.

Step 4. If $NF \neq \emptyset$, choose $n \in NF \mid c_n = \min c_i$, $\forall i \in NF$, and make $NF = NF \setminus \{n\}$. Otherwise:

1. If $stage = 1$, let $stage = 2$ and return to step 2.
2. If $stage = 2$:
   (a) If $cap = 1$, let $stage = 1$, $cap = 2$ and return to step 2.
   (b) If $cap = 2$, end. **Feasible solution found**.

Step 5. If $stage = 1$, compute the number of feasible BS that cover each point, that is compute sum_i_j such as $sum\_i_j = \sum_{i\in F} b_{ij}$, $\forall j \in J$.

Step 6. While $c_n < 0$:

1. If $stage = 1$, select $p \in P^n \mid sum\_i_p = \min\ sum\_i_j$, $\forall j \in P^n$, otherwise select $p \mid p = \min j \in P^n$.
2. Make a copy F^* of F ($F^* = F$).
3. If $F^* \neq \emptyset$, select the BS $f \in F^* \mid c_f = \min c_i$, $\forall i \in F^*$, and make $F^* = F^* \setminus \{f\}$. Otherwise:
   (a) If $stage = 2$, let $y_n = 1$ and return to step 2.
   (b) If $stage = 1$, end. **Feasible solution not found**.
4. If BS f covers the point p, that is if $b_{fp} = 1$ and (($c_f \geqslant 1$ and $cap = 1$) or ($c_f \geqslant d_p$ and $cap = 2$)):
   (a) Make $P^n = P^n - \{p\}$ and $P^f = P^f \cup \{p\}$.
   (b) If $stage = 1$, make $c_n = c_n + 1$ and $c_f = c_f - 1$.
   (c) If $stage = 2$, make $c_n = c_n + d_p$ and $c_f = c_f - d_p$.

   Otherwise, return to step 6.3.

Step 7. Make $F = F \cup \{n\}$ and return to step 4.

When the first phase is completed, the solution is feasible concerning the BSL problem and there is a good probability that the problem can become feasible concerning the FCA problem. Before proceeding to the second phase, a simple algorithm is built to insure TND feasibility. Basically, the algorithm connects each hub used and disconnected from the fixed network to a network switch. The hub-switch connection is the minimum cost one.

The second phase consists in recovering the solution feasibility considering the FCA problem. This second phase checks that there is no frequency channel assigned to non-selected base stations, there is no interference among frequency channels, the number of frequency channels assigned to a selected BS is not greater than its capacity and, finally, the number of frequency channels assigned to a base station is adequate to satisfy its demand.

The first condition is achieved simply releasing the frequency channels assigned to base stations not selected. In the second condition, the adjacent channel and co-channel interference problems are solved separately. In the former problem, the algorithm goes through the sorted list of frequency channels assigned to each base station and releases the frequency channels that are closer or equal to an orthogonal distance of f_1 frequency channels from its previous neighbor. The algorithm is stated as follows.

Step 1. Let $F^i = \{k \in K \mid z_{ik} = 1\}$ be the set of FC assigned to BS i, $\forall i \in I$.

Step 2. Select $F^i \mid i = \min l, \forall l \in I$.

Step 3. While $F^i \neq \emptyset$:

1. Select the FC $k_1 \mid k_1 = \min k, \forall k \in F^i$, and make $F^i = F^i \setminus \{k_1\}$.
2. Select the FC $k_2 \mid k_2 = \min k, \forall k \in F^i$.
3. If the frequency distance $d = k_2 - k_1 \leqslant f_1$:
   (a) Make $z_{ik_2} = 0$ and $F^i = F^i \setminus \{k_2\}$.
   (b) If $F^i = \emptyset$ go to step 4, otherwise return to step 4.2.

Step 4. Make $I = I \setminus \{i\}$.

Step 5. If $I = \emptyset$, end. Otherwise, return to step 2.

The co-channel interference is eliminated as follows. First, a set E^k of base stations that uses a frequency channel k is evaluated for all frequency channels. Second, the frequency channel k is reserved to the base station i with the lowest capacity in E^k, which is measured considering the frequency channels demand and availability. The other base stations belonging to E^k and interfering to i that are assigned the frequency channel k have it released. These base stations are removed from E^k as well as base station i. This second step is repeated until E^k is empty, and it is carried out for all frequency channels $k \in K$. The algorithm is stated as follows.

Step 1. Let $E^k = \{i \in I \mid z_{ik} = 1\}$ be the set of BS i that are assigned the FC k, $k \in K$.

Step 2. Select $E^k \mid k = \min l, \forall l \in K$, and compute the cost $c_i = \sum_{k \in K} e z_{ik} - \sum_{j \in J} x_{ij} d_j$, $\forall i \in E^k$.

Step 3. While $E^k \neq \emptyset$:

1. Select $N^i \mid c_i = \min c_l, \forall l \in E^k$, make $E^k = E^k \setminus \{i\}$ and let $S = \emptyset$.
2. While $E^k \cap N^i \neq \emptyset$:
   (a) Select $\bar{i} \mid \bar{i} = \min l, \forall l \in E^k \setminus S$.
   (b) If $\bar{i} \subset N^i$, make $z_{\bar{i}k} = 0$ and $E^k = E^k \setminus \{i\}$. Otherwise, make $S = S \cup \{\bar{i}\}$.

Step 4. Make $K = K \setminus \{k\}$. If $K = \emptyset$, end, otherwise return to step 2.

The third condition is easily accomplished simply releasing the extra number of frequency channels assigned to the selected base stations. Finally, to guarantee an equilibrium between availability and required demand it is necessary to reconfigure the FCA. Once the assignments of the LR solution were made based on costs, they are irregular and waste part of the spectrum. Therefore, the algorithm strategy consists in rearranging some assignments and making new ones considering the interference constraints and the demand of each base station. First, the selected base stations are divided in two sets of feasible (F) and not feasible (*NF*) base stations according to the constraint (8). Second, the base station n with the highest frequency channel deficit is selected. Its frequency channels are rearranged starting from the lowest one and aiming to avoid distances greater than f_1 between consecutive channels. In the rearrangement, frequency channels are released and others are assigned. The assignment of a new channel implies in its release for interfering base stations that are using it. The algorithm allows that a frequency channel can be released from base stations belonging to both feasible and not feasible sets. It increases the probability of finding a feasible solution to the CND problem. The release is not allowed only for base stations that were already addressed by the algorithm. When frequency channels are being assigned and released, the feasibility status of base stations may change in both directions. The rearrangement is carried out until the base station n moves from the not feasible set to the feasible set. The algorithm ends when the not feasible set is empty. It fails to find a feasible solution when no more frequency channel can be assigned to a base station belonging to the *NF* without making base stations already visited by the algorithm infeasible. The algorithm is described as follows.

Step 1. Let $F^i = \{k \in K \mid z_{ik} = 1\}$ be the set of FC assigned to BS i, $\forall i \in I$. Compute the costs $c_i = \sum_{k \in K} e z_{ik} - \sum_{j \in J} d_j x_{ij}$, and let $i \in F$ (feasible) if $c_i \geqslant 0$ and $i \in NF$ (not feasible) otherwise, $\forall i \in I \mid y_i = 1$.

Step 2. If $NF \neq \emptyset$, select $n \in NF \mid c_n = \min c_i, \forall i \in NF$, make $NF = NF \setminus \{n\}$, make a copy K^* of K ($K^* = K$) and let $\bar{F} = \emptyset$. Otherwise, end. **Feasible solution found**.

Step 3. While $c_n < 0$:

1. If $K^* \neq \emptyset$, select $k \mid k = \min l, \forall l \in K^*$, otherwise, end. **Feasible solution not found**.

2. If $F^n \neq \emptyset$:
   (a) Select $\bar{k} \mid \bar{k} = \min l, \forall l \in F^n$. Compute the distance $d = \bar{k} - k$.
   (b) Case $0 < d \leqslant f_1$, make $F^n = F^n \setminus \{\bar{k}\}$, $c_n = c_n - e$, $z_{n\bar{k}} = 0$ and return to step 3.2(a). Case $d = 0$, make $K^* = K^* \setminus \{k, \ldots, k + f_1\}$, $F^n = F^n \setminus \{\bar{k}\}$, $\bar{F} = \bar{F} \cup \{k\}$ and return to step 3.1. Otherwise, go on.
3. Let $S^1 = S^2 = \emptyset$.
4. While $N^n \setminus S^1 \neq \emptyset$:
   (a) Select the BS $i \mid i = \min l, \forall l \in N^n \setminus S^1$, and make $S^1 = S^1 \cup \{i\}$.
   (b) If $k \subset F^i$:
   - If $0 \leqslant c_i < e$:
     (i) $K^* = K^* \setminus \{k\}$.
     (ii) While $S^2 \neq \emptyset$:
       A. Select $\bar{i} \mid \bar{i} = \min l, \forall l \in S^2$, make $F^{\bar{i}} = F^{\bar{i}} \cup \{k\}$ and $c_{\bar{i}} = c_{\bar{i}} + e$.
       B. If $c_{\bar{i}} \geqslant 0$ and $\bar{i} \in NF$, make $F = F \cup \{\bar{i}\}$ and $NF = NF \setminus \{\bar{i}\}$.
       C. Make $S^2 = S^2 \setminus \{\bar{i}\}$ and $z_{\bar{i}k} = 1$.
     (iii) Return to step 3.1.
   - Otherwise
     (i) Make $F^i = F^i \setminus \{k\}$, $c_i = c_i - e$.
     (ii) If $c_i < 0$ and $i \in F$, make $F = F \setminus \{i\}$ and $NF = NF \cup \{i\}$.
     (iii) Make $S^2 = S^2 \cup \{i\}$ and $z_{ik} = 0$.
5. Make $z_{n,k} = 1$, $c_n = c_n + e$, $\bar{F} = \bar{F} \cup \{k\}$ and $K^* = K^* \setminus \{k, \ldots, k + f_1\}$.

Step 4. Make $F = F \cup \{n\}$, $F^n = F^n \cup \bar{F}$ and return to step 2.

If the algorithm succeeds in the first and second phases, a new feasible solution is obtained for the CND problem. If this new feasible solution is lower than the current upper bound, the upper bound is updated.

3.4. Branch and bound algorithm

The branch and bound algorithm developed for the CND problem is based on the bounds provided by the Lagrangean relaxation and the Lagrangean heuristic algorithms. The objective here is to reduce the duality gap providing tighter guarantees of the solution quality. Guarantees of optimality are not usually a reality due to the NP-hardness of the problem and the excessively large number of integer variables.

The key point when a branch and bound algorithm is developed for a particular problem is the fact that the knowledge about the problem can be used to improve the algorithm efficiency. In the CND problem, it is easily noticed that the variables t, q, x and z are strongly related to the variable y. A base station i cannot be connected to a hub or switch, or it cannot serve a point of demand, or it can not be assigned a frequency channel if it is not selected. Thus, if some variables y_i are marked in a subproblem of the tree search, the solution space is strongly reduced. Based on this problem specific feature, the strategy adopted consists in marking only the variables y in the tree search. Consequently, the algorithm proposed cannot give any optimality guarantees of the solution, although it is expected to provide optimal or near optimal solutions.

Some algorithm features are as follows. The choice of the branching node is based on the depth-first strategy. The backtracking step is carried out when the lower bound exceeds the upper bound at a tree node or when the node is infeasible. A node is infeasible if the capacity of the base stations that were not marked to 0 is not enough to serve the network demand. Let L be the subset of BS not marked to 0 yet. The node is feasible if $\sum_{i \in L} n_i \geqslant \sum_{j \in J} d_j$ and $\sum_{i \in L} m_i \geqslant |J|$, otherwise it is infeasible. The forward branching rule, suggested by Reeves [45], is as follows:

1. Let $(\alpha_i, \varepsilon_i, y_i)$ represent the multipliers and variables associated with the best lower bound found at the branching tree node.
2. Let the relaxed constraints related to the multipliers and variables $(\alpha_i, \varepsilon_i, y_i)$ given by $\alpha_i(\sum_{j \in J} b_{ij} x_{ij} - m_i y_i)$ and $\varepsilon_i(\sum_{k \in K} z_{ik} - n_i y_i)$ for all $i \in L$.
3. Choose out of these constraints the one with the highest value. Set y_i associated to this constraint to 1.

A final remark is related to the initial node. In the tree root node, the computational effort spent is larger than that spent in a tree leaf node. The idea is to allow a greater number of iterations of the subgradient method in order to obtain good initial bounds. An improvement in the initial bounds can reduce considerably the searching time, and the extra computational effort spent in the root node is insignificant compared to the total time spent in the tree nodes. In the root node, the parameter π of equation (45) is initially set to 2, and it is halved when the solution Z_{RL} does not improve for 60 consecutive iterations. The subgradient algorithm stops when $\pi \leqslant 5.00 \times 10^{-4}$.

4. Computational analyses

In this section, many instances of the CND problem are solved using the Lagrangean algorithm and the CPLEX optimization package [10]. The objective of the computational analyses is to evaluate the solution quality of the proposed Lagrangean algorithm.

The Lagrangean algorithm is implemented using the C ANSI programming language, and all the sparse matrices are handled by vectors of linked lists. The code is compiled with the compiler gcc (GNU compiler gcc-2.96). The Lagrangean algorithm tests are run on a personal computer (Pentium III, clock of 500 MHz and RAM memory of 256 MB) running the Linux operating system (Linux Mandrake release 8.0). The input for the CPLEX solver is built using the AMPL [20]. The

CPLEX tests are run on a SUN Workstation (UltraSPARC-II, clock of 248 MHz and RAM memory of 512 MB) running Solaris operation system (SunOS 5.5.1).

The tests were run for 32 problem instances that were divided into three sets of small, medium and large scale instances. The instances used for tests are not extracted from real cases but were created by an Instance Generator Algorithm [39]. The common parameters for all instances are as follows. The number of available frequency channels is 333. The maximum number of frequency channels that can be assigned to each base stations is 47. The medium access protocol considered is the Time Division Multiple Access protocol (TDMA) [44]. Each frequency channel is able to carry 3 communication channels ($e = 3$). The minimum orthogonal distance between adjacent frequency channel is 6. These data were obtained from a cellular service provider that operates a cellular network at Belo Horizonte city, Brazil. The other common parameters were estimated due to cellular operator privacy politics or due to design reasons. The channel demand of a point lies in the intervals [1,3] and [3,6] for points of low and high traffic pattern, respectively. The maximum number of points that a base station can serve is $m_i = \sum_j b_{ij}, \forall i \in I$. Finally, the number of hubs and switches is 4 and 2, respectively.

The parameters that differ from one instance to the others are the number of points of demand, which defines the size of the service area, the number of candidate base stations, and the total demand originated from the points of demand. The number of candidate base stations is given approximately by $|I| \approx \eta \sum_{j \in J} d_j / e \times 47$ where the quantity 47 is the common base station capacity in number of frequency channels and $\eta \in [1, 2]$.

4.1. Instance set I

The instance set I represents the set of small instances. Table 1 gives the configuration and size of the instances. The first column gives the instance names, the second column gives the number of candidate base stations, the third column gives the number of points and the fourth column gives the total communication channel demand. As can be noticed, the number of variables and constraints given by the fifth and sixth columns is relatively large compared to the instance sizes.

Table 1
Instance set I configuration.

I	Parameters			Number of variables	Number of constraints
	I	J	d		
I_1	2	36	119	760	1372
I_2	3	49	158	1175	2053
I_3	4	64	221	1642	2736
I_4	5	91	265	2070	3376
I_5	6	100	323	2632	4095

Table 2 gives the results of the Lagrangean algorithm and CPLEX optimization package. Column two up to column six report the results computed by the Lagrangean algorithm. They are the lower bound, the upper bound, the duality gap, which is measured based on the Lagrangean bounds as $dgap = 100(UB - LB)/UB$, the number of branch and bound nodes searched and the solution time respectively. Column seven up to column eleven report the results computed by CPLEX. They represent the linear relaxation lower bound, the upper bound, the integrality gap, which is measured based on the CPLEX bounds as $igap = 100(UB - LB)/UB$, the number of branch and bound nodes searched and the solution time respectively. Column twelve reports the gap between the Lagrangean upper bound and the CPLEX upper bound as $gap = 100(UB_{LA} - UB_{CPLEX})/UB_{CPLEX}$.

The Lagrangean algorithm results show that the duality gap is under 10% for all instances but instance I_3. It means that the Lagrangean algorithm provides effective solutions for the set of instances I. The $dgap$ for instance I_3 will be discussed later.

CPLEX found the optimal solution for all instances in short running times. Comparing the optimal solution to the Lagrangean upper bound it can be realized that the Lagrangean algorithm found the optimal solution for three out of five instances and found near optimal solutions for the other two instances. This result reinforces the quality of the Lagrangean solution.

Many authors [38,42,45] have already shown that the Lagrangean relaxation lower bound is greater or equal to the linear relaxation lower bound. They are equal when the Lagrangean problem has the integrality property, and the Lagrangean relaxation lower bound is greater than the linear relaxation lower bound otherwise. Unfortunately, the Lagrangean problem defined in section 3 has the integrality property, so the lower bounds should be equal. The results of table 2 confirm the expectations. The small differences between the lower bounds arise due to convergence problems of the subgradient method next to the optimal solution.

Concerning the instance I_3, both the integrality and duality gaps are high. As the Lagrangean relaxation lower bound is bounded by the linear relaxation lower bound due to the

Table 2
Lagrangean algorithm and CPLEX results: Instance set I.

I	Lagrangean algorithm					CPLEX					*gap*
	LB	*UB*	*dgap*	Nodes	$t(s)$	*LB*	*UB*	*igap*	Nodes	$t(s)$	
I_1	344.01	368	6.51	4	136.12	354.00	354	0.00	0	0.36	3.90
I_2	698.79	734	4.80	10	150.27	728.00	734	0.82	5	6.60	0.00
I_3	685.12	1073	36.15	21	39.00	715.33	1073	33.33	15	17.05	0.00
I_4	1379.90	1418	2.69	16	29.01	1306.00	1418	7.90	3	6.50	0.00
I_5	1385.01	1515	8.58	25	67.54	1457.74	1478	1.37	110	33.83	2.50

Lagrangean integrality property, the duality gap could not be good. In addition, the Lagrangean upper bound is optimal. Thus, the high duality gap does not imply that the Lagrangean solution quality is not good.

As a final remark, it is clear that the time efficiency of CPLEX for small instances is much better than the Lagrangean algorithm time efficiency. It is a natural result since CPLEX has an extremely optimized code as shown in [34] and since the Lagrangean algorithm is developed for solving large instances, where CPLEX is not competitive.

4.2. Instance set II

The instance set II represents the set of medium size instances. Table 3 gives the configuration and size of the instances. The definition of each column is the same presented for table 1. Although these instances are classified as medium size instances concerning the size of real cellular networks, the number of variables and constraints is already considerably large for a linear mixed integer programming problem.

Table 4 reports the results of the Lagrangean algorithm and the CPLEX optimization package. The meaning of each column is the same previously defined for table 2. The running time is arbitrarily limited in 86400 s for both the Lagrangean algorithm and the CPLEX package.

The Lagrangean results show that the duality gap is very wide for all instances of the set II. The average value is around 36%. Therefore, it is not possible to use the duality gap to get much useful information concerning the solution quality. However, the algorithm is expected to provide good upper bounds for this set of instances considering the optimal or near optimal solutions evaluated for the set of instances I. Although the algorithm execution has been interrupted for instances I_{13}–I_{16}, the duality gaps for these problems are around the average measured for the whole set. Thus, the speculation about the solution quality is still valid.

CPLEX did not conclude the branch and bound search for all instances of the set due to time limitations. Therefore, there is no guarantee of optimality for the evaluated upper bounds. Comparing the best feasible solutions computed by the Lagrangean algorithm and by CPLEX indicates that the Lagrangean algorithm is superior to CPLEX for nine out the eleven total instances. In the best case, the Lagrangean upper bound is 24% below the CPLEX upper bound.

Table 4 also shows that both the duality and integrality gaps are high. Furthermore, the duality gap is below the integrality gap for nine out of the eleven instances of the set. Once the Lagrangean lower bound is upper bounded by the linear lower bound, the duality gap cannot be smaller than the integrality gap. Thus, the CPLEX solutions are not optimal. However, it is not expected a significant reduction in the integrality gap due to the problem's weak formulation. According to some authors [27], the FCA constraints when relaxed with respect to integrality provide a poor lower bound to the problem. This suggests that although the duality gap is high for this set of instances, the Lagrangean upper bound should be optimal or near optimal.

Table 3
Instance set II configuration.

I	Parameters			Number of variables	Number of constraints
	I	J	d		
I_6	9	144	492	4333	6119
I_7	12	196	691	6426	8198
I_8	16	256	855	9534	10934
I_9	19	324	1110	12624	13016
I_{10}	22	400	1390	16253	15061
I_{11}	25	484	1646	20591	17707
I_{12}	29	576	1970	26523	19899
I_{13}	33	676	2343	33526	22710
I_{14}	40	784	2657	44968	27504
I_{15}	45	900	3058	55791	30943
I_{16}	49	1024	3521	66827	33739

4.3. Instance set III

Instance set III represents the set of large scale instances. Table 5 gives the instances configurations and their sizes. The definition of each column is the same presented for table 1. The number of variables and constraints for this set is significant. Actually, the largest instances of this set overestimate the size of real cellular networks.

Table 6 reports the results of the Lagrangean algorithm and CPLEX. The meaning of each column is the same previously defined for table 2. The running times are again limited to 86400 s. The CPLEX upper bound as well as the integrality gap and the gap between the Lagrangean and CPLEX

Table 4
Lagrangean algorithm and CPLEX results: Instance set II.

I	Lagrangean algorithm					CPLEX					*gap*
	LB	*UB*	*dgap*	Nodes	$t(s)$	*LB*	*UB*	*igap*	Nodes	$t(s)$	
I_6	2067.06	2534	18.42	53	36.59	1898.80	2502	24.11	453256	86400	1.28
I_7	2060.41	3281	37.20	239	88.73	1968.46	3279	39.97	20215	86400	0.06
I_8	2465.23	4050	39.13	1647	552.93	2560.52	4068	37.06	3174	86400	−0.44
I_9	2896.38	4517	35.88	5069	2136.61	2890.25	5947	51.40	482	86400	−24.00
I_{10}	3622.94	6067	40.28	8743	804.70	3906.32	6834	42.84	779	86400	−11.22
I_{11}	4228.15	6606	36.00	29571	13127.07	4384.50	7400	40.75	435	86400	−10.73
I_{12}	5053.95	8212	38.46	70825	39564.84	5382.14	8245	34.72	513	86400	−0.40
I_{13}	6164.76	9265	33.46	118353	86400.00	6351.51	10917	41.82	779	86400	−17.83
I_{14}	6899.05	11578	40.41	71373	86400.00	7187.70	13459	46.60	672	86400	−13.98
I_{15}	7956.88	13588	41.44	70710	86400.00	8180.32	14256	42.65	674	86400	−4.69
I_{16}	9169.72	14922	38.55	54427	86400.00	9614.24	16115	40.34	742	86400	−7.40

upper bounds are not available due to running time limitations.

The average duality gap evaluated by the Lagrangean algorithm concerning instances I_{17}–I_{25} is around 36% while it is above 39% concerning instances I_{26}–I_{32}. Therefore, the duality gap does not provide too much information about the quality of solutions. However, as the duality gap range for instances I_{17}–I_{25} is near the same for instances of set II, the solution is still expected to be optimal or near optimal for these instances.

CPLEX is inefficient for solving this set of problem instances. It could not compute a feasible solution for any one of the sixteen instances of this set. For instances from I_{29}–I_{32} the problem could not even be loaded by the CPLEX due to machine memory limitations. Therefore, the only information provided by the CPLEX package for the set of instances III is the linear lower bound.

Comparing the Lagrangean and CPLEX lower bounds, it can be noticed that the larger is the problem size, the greater is the distance between them. This result shows that the subgradient method has convergence problems for large instances making the algorithm unstable in these cases. In the worst case, the distance achieves 8%. Two possible approaches to solve this problem are readjust the convergence parameters of the algorithm or to use a more efficient algorithm such as the volume algorithm [4,5].

The first consequence of the non-optimal Lagrangean problem solution is the ineffectiveness of the duality gap. This can be clearly noticed taking a look at instances I_{26}–I_{32}. Second, the poor lower bound drastically weakens the efficiency of the branch and bound algorithm. In effect, the worse the lower bound, the more difficult it is to prune the tree branches. Thus, branch and bound, which is based on a depth-first strategy, cannot diversify its search into the time limit and thereby reducing the probability of improving the upper bound. This problem becomes even more significant for large problem instances where the computing times for a single tree node are significant. Although it is not shown in table 6, the Lagrangean algorithm could not improve the upper bound through the branch and bound algorithm for the entire set of instances. The best feasible solution was always found at the root node making the computational effort provided by branch and bound algorithm worthless.

Table 5
Instance set III configuration.

I	Parameters			Number of variables	Number of constraints
	I	*J*	*d*		
I_{17}	50	1296	4389	81768	34656
I_{18}	65	1600	5402	125979	44891
I_{19}	92	2304	7776	243039	63546
I_{20}	100	2704	9089	304205	69301
I_{21}	116	3136	10685	402932	79999
I_{22}	133	3600	12167	523776	92192
I_{23}	151	4096	13788	669491	104611
I_{24}	170	4624	15727	843528	126840
I_{25}	192	5184	17524	1060232	133056
I_{26}	212	5776	19420	1296076	146868
I_{27}	235	6400	21687	1583285	162757
I_{28}	257	7056	23858	1900131	178083
I_{29}	280	7744	26039	2280328	194784
I_{30}	310	8464	28662	2747848	215544
I_{31}	337	9216	31113	3240600	234332
I_{32}	368	10000	33633	3827208	255824

5. Conclusions

In this article, a solution approach to the NP-hard large scale cellular network design problem is proposed. The problem is formulated as a comprehensive linear mixed integer programming model including the base station location, the frequency channel assignment and the topological network design problems.

Table 6
Lagrangean algorithm and CPLEX results: Instance set III.

I	Lagrangean algorithm					CPLEX		
	LB	*UB*	*dgap*	Nodes	$t(s)$	*LB*	Nodes	$t(s)$
I_{17}	11844.14	16442	27.96	56016	86400	12446.57	26	86400
I_{18}	15191.72	22705	33.09	39429	86400	15929.90	786	86400
I_{19}	23340.52	37971	38.53	26242	86400	24584.57	25	86400
I_{20}	27047.25	42714	36.68	27395	86400	28981.75	4	86400
I_{21}	32366.17	49060	34.03	18433	86400	34973.51	4	86400
I_{22}	39199.96	62771	37.55	16983	86400	41924.30	3	86400
I_{23}	45383.58	72961	37.80	17500	86400	49268.02	3	86400
I_{24}	52623.73	85751	38.63	12242	86400	55842.37	0	86400
I_{25}	60676.48	99269	38.88	10298	86400	66478.93	1	86400
I_{26}	71264.30	119515	40.37	8961	86400	77117.16	0	86400
I_{27}	82741.85	135755	39.05	6674	86400	90129.27	0	86400
I_{28}	94216.70	160778	41.40	5931	86400	102630.43	0	86400
I_{29}	103752.88	178911	42.00	0	86400	*	*	*
I_{30}	120190.35	201756	40.43	0	86400	*	*	*
I_{31}	136475.65	242378	43.70	0	86400	*	*	*
I_{32}	150535.47	265895	43.38	0	86400	*	*	*

* Problem not solved due to memory machine limitations.

A Lagrangean algorithm is introduced aiming to provide effective bounds for the problem. The lower bound is evaluated through a Lagrangean relaxation technique and the upper bound is evaluated through a corresponding Lagrangean heuristic. A customized branch and bound algorithm is developed in order to refine the Lagrangean bounds. The goal of the branch and bound algorithm consists in reducing the branching variables based on the problem knowledge.

The computational analyses are based on 32 problem instances that are solved by the Lagrangean algorithm and by the CPLEX optimization package. The results show that both the duality and integrality gaps are excessive. Thus, these gaps do not add too much valuable information about the quality of the upper bound evaluated by the Lagrangean algorithm. As the optimal solution is not known for most of the instances, a precise evaluation of the quality of the solutions is not possible.

Considering the instance set I, for which the optimal solutions are known, the Lagrangean algorithm provides optimal or near optimal solutions. Therefore, considering the instance set II, for which the optimal solutions are not known, the Lagrangean algorithm is supposed to provide optimal or near optimal solutions. This speculation is valid at least for the instances for which the algorithm converges. However, even for the instances that have their execution aborted, the solution quality should not diverge from the rest of the instance set once the duality gap is stable for all the instances of the set. Keeping the focus on the duality gap, it can be said that the quality of the Lagrangean upper bound for the instance set III should be also effective. It should be true even for the instances with duality gap values above the average because the increase in the duality gap is associated with the lower bound that is worsened as the problem size increases.

Although the evaluation of the solution quality is based on speculation, the comparison between CPLEX and the Lagrangean algorithm can provide something concrete. First, when CPLEX provides the optimal solution, the Lagrangean algorithm provides the optimal or a near optimal solution. Second, when CPLEX provides an upper bound rather than the optimal solution, the Lagrangean algorithm provides a better upper bound. Finally, when CPLEX cannot provide even a feasible solution, the Lagrangean algorithm provides an upper bound as well as a duality gap.

The drawback of such a comprehensive formulation for the cellular network design problem is the ineffectiveness of the duality gap in providing precise information about the solution optimality. Thus, some further work may still be pursued in order to improve the results achieved. Some alternative methods such as cutting plane and others represent possible alternatives. Another possibility for future work consist in comparing the presented results with the results provided by other related works based on a common data set. A third possibility consists in applying this methodology to a real data set supported by a national cellular telephone operator. Finally, a fourth possibility consists in modeling the problem as a multi-objective one. The first part of the objective function aims to minimize the setup costs, while the second part aims to minimize the frequency channel interference. The latter approach is an interesting formulation for the problem once cellular operators allow channel interference in their networks.

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Filipe F. Mazzini was born in Belo Horizonte, Brazil, in 1975. He received his B.S. degree in mechanical engineering and his M.S. degree in computer sience, both from Universidade Federal de Minas Gerais, Brazil. His research interests are focused on optimization and its applications.
E-mail: filipemazzini@hotmail.com

Geraldo Robson Mateus is Full Professor in Computer Science at Universidade Federal de Minas Gerais, Belo Horizonte, Brazil. He received his Ph.D. and M.S. in computer science from Universidade Federal do Rio de Janeiro, Brazil, in 1980 and 1986, respectively. He spent 1991 and 1992 at the University of Ottawa, Canada, as a visiting researcher. His research interests span network optimization, combinatorial optimization, algorithms and telecommunications. He is a Member of INFORMS, IFORS and SIAM. He has published over 100 scientific papers and is a leader of several national and international projects.
E-mail: mateus@dcc.ufmg.br

James MacGregor Smith received his Bachelors and Masters of Architecture from the University of California at Berkeley and his Ph.D. in mechanical and industrial engineering from the University of Illinois at Champaign-Urbana. He is also a registered architect. His academic and professional interests include the design of algorithms for combinatorial optimization and stochastic network problems, the optimal design of facilities, networks, and the design and analysis of manufacturing systems. Dr. Smith has carried out sponsored research for the National Science Foundation, the Digital Equipment Corporation, IBM, and the U.S. Army Natick Research and Development Laboratories. He is a member of the INFORMS, ACM, SIAM, The Mathematical Programming Society and Sigma Xi.
E-mail: jmsmith@ecs.umass.edu