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On the span in channel assignment problems: bounds, computing and counting

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Abstract

The *channel assignment problem* involves assigning radio channels to transmitters, using a small span of channels but without causing excessive interference. We consider a standard model for channel assignment, the *constraint matrix* model, which extends ideas of graph colouring. Given a graph $G=(V,E)$ and a length $l(uv)$ for each edge uv of G, we call an assignment $\phi : V \to \{1,\dots,t\}$ *feasible* if $|\phi(u)-\phi(v)| \geqslant l(uv)$ for each edge uv. The least t for which there is a feasible assignment is the *span* of the problem. We first derive two bounds on the span, an upper bound (from a sequential assignment method) and a lower bound. We then see that an extension of the Gallai-Roy theorem on chromatic number and orientations shows that the span can be calculated in $O(n!)$ steps for a graph with n nodes, neglecting a polynomial factor. We prove that, if the edge-lengths are bounded, then we may calculate the span in exponential time, that is, in time $O(c^n)$ for a constant c. Finally we consider counting feasible assignments and related quantities.

MSC: 05C15

1. Introduction

The *channel assignment problem* involves assigning radio channels to transmitters, using a limited range of channels but without causing interference. We consider a standard model for channel assignment, the *constraint matrix* or *weighted graph* model, which extends ideas of graph colouring, see for example [3,4,9]. Given a graph

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$G=(V,E)$ and a positive integral weight or length $l(uv)$ for each edge uv of G, we call an assignment $\phi: V \to \{1,\ldots,t\}$ *feasible* if $|\phi(u)-\phi(v)| \geqslant l(uv)$ for each edge uv. The nodes correspond to transmitters, and the lengths $l(uv)$ specify minimum channel separations to avoid interference. (Thus if u and v correspond to transmitters that are "close together" in some sense then $l(uv)$ will be large.) The least t for which there is a feasible assignment is the *span* of the problem, which we denote by $\mathrm{span}(G,l)$. When each edge length is 1 this is just the chromatic number $\chi(G)$.

We first discuss bounds on the span. In particular, we consider sequential methods for assigning channels, and see that the span is at most $\Delta_l(G)+1$, where the "weighted maximum degree" $\Delta_l(G)$ is the maximum over all nodes v of the sum of the weights of the edges incident with v. This upper bound of course corresponds to the bound $\chi(G) \leqslant \Delta(G)+1$. We give also a lower bound on the span, extending a result of Smith and Hurley [11], which corresponds to the bound $\chi(G) \geqslant |V|/\alpha(G)$. Here $\alpha(G)$ is the stability (or independence) number of G.

We next describe an extension of the Gallai-Roy theorem on chromatic number and orientations, following a result of Barasi and van den Heuvel [1]. This result shows that the span can be calculated in $O(n!)$ steps, neglecting a polynomial factor. We then consider the problem of calculating the span when the maximum edge-length is bounded. We give a recurrence which shows how to do this in exponential time, that is, in time $O(c^n)$ for a constant c, following an idea of Lawler [5] for the chromatic number. In particular we see that, if each edge-length is at most m, then we may calculate the span in $O((2m+1)^n)$ steps, neglecting a polynomial factor.

Finally we consider counting feasible assignments and related quantities. We see in particular that the number of feasible assignments agrees with a polynomial for sufficiently large numbers of available channels. See [13] for a discussion of such results.

2. Sequential assignment methods

Suppose that we want to colour the nodes of a graph with colours $1,2,\ldots$, and we have a given ordering on the nodes. Let us consider two variants of the greedy colouring algorithm. In the "one-pass" method, we run through the nodes in order and always assign the smallest available colour. In the "many-passes" method, we run through the nodes assigning colour 1 whenever possible, then repeat with colour 2 and so on. Both methods yield exactly the same colouring, and show that

$$\chi(G) \leqslant \Delta(G)+1, \tag{1}$$

since at most $\Delta(G)$ colours are ever denied to a node.

Now consider a constraint matrix problem (G,l). Define the *weighted degree* of a node v by $\deg_l(v)=\sum\{l(uv)\colon uv\in E\}$, and define the *maximum weighted degree* by $\Delta_l(G)=\max_v \deg_l(v)$. The above greedy methods generalise immediately.

Example. Let G be the 4-cycle C_4, with nodes a,b,c,d and edge lengths $l(ab)=1$ and $l(bc)=l(cd)=l(ad)=2$. Note that $\Delta_l=4$. The one-pass method assigns channels

1,2,4,6 to the nodes a, b, c, d respectively, with span 6. The many-passes method assigns channel 1 to nodes a and c, channel 2 to none of the nodes, and channel 3 to nodes b and d, with span 3.

In fact the many passes method always uses at most the channels $1, \ldots, \Delta_l + 1$, and so we may extend the inequality (1) as follows.

Proposition 2.1.

$$\mathrm{span}(G, l) \leqslant \Delta_l(G) + 1.$$

Proof. In order to show that the many passes method needs a span of at most the above size, suppose that it is about to assign channel c to node v. Let A be the set of neighbours u of v to which it has already assigned a channel $\phi(u)$. For each channel $j \in \{1, \ldots, c-1\}$ there must be a node $u \in A$ with $\phi(u) \leqslant j$ and $\phi(u) + l(uv) \geqslant j + 1$. Hence the intervals $\{\phi(u), \ldots, \phi(u) + l(uv) - 1\}$ for $u \in A$ cover $\{1, \ldots, c-1\}$. Thus

$$c - 1 \leqslant \sum_{u \in A} l(uv) \leqslant \deg_l(v) \leqslant \Delta_l(G),$$

and this completes the proof. □

There is a straightforward extension of (1), involving the "degeneracy" of a graph—see for example [12]. Given an ordering $\sigma = (v_1, \ldots, v_n)$ of the nodes, let $g(\sigma)$ be the maximum over $1 < j \leqslant n$ of the degree of node j in the subgraph induced by nodes $1, \ldots, j$. We call the minimum value of $g(\sigma)$ over all such orderings σ the *degeneracy* of G, and denote it by $\delta^*(G)$. We can compute $\delta^*(G)$ as follows. Find a node v of minimum degree, delete it and put it at the end of the order, and repeat. This shows that $\delta^*(G)$ equals the maximum over all induced subgraphs of the minimum degree, and that we can compute it and find a corresponding order in $O(n^2)$ steps.

If we colour the nodes of G in an order yielding the minimum above, then at each stage at most $\delta^*(G)$ colours are denied to a node. Hence

$$\chi(G) \leqslant \delta^*(G) + 1, \tag{2}$$

and further we can find a corresponding colouring quickly. (The quantity $\delta^*(G) + 1$ is sometimes called the *colouring number* of G.)

The inequality (2) does not extend to $\mathrm{span}(G, l)$. For, consider first the example where G consists of a triangle with one edge of length 2 and two of length 1 adjacent to a node v, and one pendant edge of length 2 attached to this node v: the span is 4, but in each induced subgraph there is a node with weighted degree at most 2. However, the inequality (2) does extend if we replace the degree of each node v not by its weighted degree $\deg_l(v)$ but by the sum of the values $2l(uv) - 1$ over all the nodes $u \neq v$ with $l(uv) \geqslant 1$. For, observe that if we have a feasible assignment for the graph without v and we wish to extend it to v, then the above sum bounds the number of channels denied to v—see Proposition 6 of [11].

3. Lower bounds

Consider the elementary lower bound on $\chi(G)$,

$$\chi(G) \geqslant |V|/\alpha(G). \tag{3}$$

Here the *stability number* (or independence number) $\alpha(G)$ is the maximum size of a stable set in G. As is well known, this inequality can be extended as follows. For each node v let α_v denote the maximum size of a stable set containing v. Then

$$\chi(G) \geqslant \sum_v 1/\alpha_v. \tag{4}$$

For, given any proper t-colouring of G, with colour sets $S_1,\dots,S_t$, we have $\alpha_v \geqslant |S_i|$ if $v \in S_i$, and so

$$\sum_v 1/\alpha_v = \sum_{i=1}^{t} \sum_{v\in S_i} 1/\alpha_v \leqslant \sum_{i=1}^{t} \sum_{v\in S_i} 1/|S_i| = t.$$

There are lower bounds for the span extending these ideas. Let r be a positive integer, and let us keep r fixed throughout. Consider an instance (G,l) of the constraint matrix problem. Call a subset U of nodes *r-assignable* if the corresponding subproblem has span at most r. Let $\alpha^{(r)}$ denote the maximum size of an r-assignable set. Similarly, for each node v let $\alpha_v^{(r)}$ denote the maximum size of an r-assignable set containing v. Then

$$\mathrm{span}(G,l) \geqslant r|V|/\alpha^{(r)} - (r-1), \tag{5}$$

and indeed [11]

$$\mathrm{span}(G,l) \geqslant r\sum_v 1/\alpha_v^{(r)} - (r-1). \tag{6}$$

Observe that (5) reduces to (3) and (6) reduces to (4) when $r=1$. The basic inequality (5) is crucial for example in [8]. The following result is a further natural slight extension of (6).

Let the index i always run through $1,\dots,r$. For each node v and each i, let $\alpha_{vi}^{(r)}$ denote the maximum size of an r-assignable set U containing v, such that there is a feasible assignment $\phi : U \to \{1,\dots,r\}$ with $\phi(v)=i$. For example, if G is the path with three nodes a,b,c (b in the middle) and both edges of length 2, then

$$\alpha_b^{(3)} = \alpha_{b1}^{(3)} = \alpha_{b3}^{(3)} = 3 \quad \text{and} \quad \alpha_{b2}^{(3)} = 1.$$

Proposition 3.1.

$$\mathrm{span}(G,l) \geqslant \sum_v \sum_i 1/\alpha_{vi}^{(r)} - (r-1). \tag{7}$$

Further, if

$$\alpha_{v1}^{(r)} > \alpha_v \ \textit{for each node } v \tag{8}$$

then this inequality is strict.

We make three comments before proving this result.

(i) Observe that $\alpha_{vi}^{(r)} \leqslant \alpha_v^{(r)}$, and so the bound (7) is always at least as good as (6). It reduces to (6) when r is 1 or 2.

(ii) The condition (8) must hold if G has at least one edge and each edge length is at most $r-1$. For, let S be a stable set containing v of size α_v: then there is a node $w \in V \setminus S$, and $S \cup \{w\}$ is r-assignable.

(iii) Consider the example introduced immediately before the Proposition, with span 3. For the purpose of illustration, let us take $r=3$. Then the lower bound in (7) is 2. But by (ii) above, the condition (8) holds, and so we may deduce from Proposition 3.1 with $r=3$ that the span is at least 3. (It is much simpler with $r=2$.)

Proof. Let $t = \mathrm{span}(G,l)$, and fix a feasible assignment $\phi : V \to \{1,\dots,t\}$. For each set I of integers let $\hat{I}$ denote $\phi^{-1}(I)$. For each v and i let I_{vi} denote the set $\{\phi(v)-i+1,\dots,\phi(v)+r-i\}$ of r consecutive integers, and let $\beta_{vi} = |\widehat{I_{vi}}|$. Then $1 \leqslant \beta_{vi} \leqslant \alpha_{vi}^{(r)}$. Let $\mathscr{I}$ denote the collection of sets $I = \{j,\dots,j+r-1\}$ of r consecutive integers such that $\hat{I} \neq \emptyset$. Then $|\mathscr{I}| \leqslant t+r-1$. Hence

$$\begin{aligned}\sum_v \sum_i 1/\alpha_{vi}^{(r)} &\leqslant \sum_v \sum_i 1/\beta_{vi}\\ &= \sum_v \sum_i \sum_{I\in\mathscr{I}} \mathbf{1}_{(I=I_{vi})}(1/|\hat{I}|)\\ &= \sum_{I\in\mathscr{I}} (1/|\hat{I}|) \sum_{v\in\hat{I}} \sum_i \mathbf{1}_{(I=I_{vi})}.\end{aligned}$$

But for each $v \in \hat{I}$ we have $\sum_i \mathbf{1}_{(I=I_{vi})} = 1$, and so the last quantity above equals

$$\sum_{I\in\mathscr{I}} (1/|\hat{I}|) \sum_{v\in\hat{I}} 1 = \sum_{I\in\mathscr{I}} 1 = |\mathscr{I}| \leqslant t+r-1.$$

Finally, suppose that the condition (8) holds. There is a node v_0 with $\phi(v_0)=1$. Then $\hat{I}_{v_0 r} = \{v\colon \phi(v)=1\}$, so

$$\beta_{v_0 r} = |\hat{I}_{v_0 r}| \leqslant \alpha_{v_0} < \alpha_{v_0 1}^{(r)} = \alpha_{v_0 r}^{(r)}.$$

Hence the first inequality displayed above is strict. □

4. Span and orientations

The Gallai-Roy Theorem (see for example [12]) relates the chromatic number $\chi(G)$ to the maximum length of a path (with no repeated nodes allowed) in an orientation of G. The theorem states that if D is an orientation of G with maximum directed path length $\rho(D)$, then

$$\chi(G) \leqslant 1 + \rho(D),$$

and further, equality holds for some acyclic orientation D. This theorem extends directly to the weighted graph case, that is to constraint matrix problems. The paper [1] focusses on the acyclic case, and discusses related algorithms: see also the survey [9], which gives a proof of the proposition below. We shall use the acyclic case in the next section. As usual, we let the length of a path be the sum of the lengths or weights of the edges.

Proposition 4.1. *Given (G, l) and an orientation D of G, let $\rho(D, l)$ denote the maximum length of a directed path. Then*

$$\mathrm{span}(G, l) \leqslant 1 + \rho(D, l),$$

and further, equality holds for some acyclic orientation D.

5. Computing the span

How quickly can we compute the span?

The problem is trivially solvable on bipartite graphs, since the span is just the maximum edge-length plus 1, see for example [10]. On the other hand, it is shown in [10] that unless $P = NP$, we cannot in polynomial time obtain a solution within a factor $\frac{4}{3}$ of the optimal in graphs which can be made bipartite by deleting a single node, even if the edge-lengths are restricted to 1 and 2.

Bipartite and "near-bipartite" graphs as above are rather special, so let us consider general n-node graphs. Since the problem is a generalization of graph colouring, unless $P = NP$ we cannot in polynomial time obtain a solution within a factor $n^{1/7-\varepsilon}$ of optimal for any $\varepsilon > 0$, see [2]. One might hope that the problem would be easy for graphs of bounded tree-width, but this is not true. It is shown in [10] that it is NP-hard to compute the span, even for graphs with treewidth at most 3, as long as we allow large edge-lengths. In contrast, for each fixed k, there is a fully polynomial time approximation scheme for finding the span on graphs of treewidth at most k.

Let us focus here on how quickly we can compute the span, when there is no restriction on the graph G. There are two natural cases to consider concerning the edge-lengths; namely when the maximum edge length is "small", and the most general case when it is not restricted. In practical problems we would not expect large edge-lengths.

In the latter case, when edge-lengths are not restricted, the best bound seems to come from Proposition 4.1. We may determine $\mathrm{span}(G, l)$ as follows. For an n-node graph G, we may run through all $n!$ linear orders on the nodes, and find the maximum path length $\rho(D, l)$ in the corresponding acyclic orientation D, in $O(n^2)$ arithmetic operations per linear order.

When the maximum edge length is small we may hope to do better than $n!$ steps. For consider graph colouring. By repeatedly running through all stable sets in G, we may determine $\chi(G)$ in $O(3^n)$ steps (ignoring small polynomial factors); and further, as was pointed out by Lawler [5], if we consider only maximal stable sets, we need use only $O((1+3^{1/3})^n)$ steps. (This beats branch-and-bound methods based on contraction and deletion, see [6].) The general approach can be extended to determine the span.

Proposition 5.1. *Given (G,l) with maximum edge-length m, we can compute* $\mathrm{span}(G,l)$ *in* $O(n^2(2m+1)^n)$ *steps.*

The case $m=1$ corresponds to finding $\chi(G)$ in $O(3^n)$ steps: we do not seem to be able to take advantage of "maximality" here.

Let us describe the method. Let V denote the set of nodes of G. For each $S \subseteq V$ let

$$\partial_i S = \{v \in V \backslash S\colon \exists \text{ an edge } uv \text{ with } u \in S \text{ and } l(uv) \geqslant i\}.$$

For each nested family $A \supseteq B_1 \supseteq \cdots \supseteq B_{m-1}$ of m subsets of V and each non-negative integer t, let $F(A;B_1,\ldots,B_{m-1};t)$ be the set of all feasible assignments $\phi : A \to \{1,\ldots,t\}$ for the subproblem on A such that $\phi(v) \leqslant t-i$ whenever $v \in B_i$, for each $i=1,\ldots,m-1$. Let $f(A;B_1,\ldots,B_{m-1})$ be the least t such that $F(A;B_1,\ldots,B_{m-1};t)$ is non-empty. Thus the span is $f(V;\emptyset,\ldots,\emptyset)$. By definition, if $A=\emptyset$ then $F=\{\emptyset\}$ and $f=0$.

Claim. *For each non-empty $A \subseteq V$*

$$f(A;B_1,\ldots,B_{m-1}) = 1 + \min_S f(A\backslash S;B'_1,\ldots,B'_{m-1}), \tag{9}$$

where S runs over all stable subsets of $A\backslash B_1$; $B'_{i-1} = B_i \cup (A \cap \partial_i S)$ for each $i=2,\ldots,m-1$; and $B'_{m-1} = A \cap \partial_m S$. [Note that $A\backslash S \supseteq B'_1 \supseteq \cdots \supseteq B'_{m-1}$, as required for the domain of f.]

The method to calculate the span is brutal: we use the claim to tabulate all the values $f(A;B_1,\ldots,B_{m-1})$ in increasing order of the size of A. Note that for a given set A of size a, there are m^a points in the domain of f. The additional time to compute f for a given point with set A of size a is at most $cn^2 2^a$ for a constant $c>0$. Hence the total time taken is at most

$$cn^2 \sum_{a=0}^{n} \binom{n}{a} m^a 2^a = cn^2(2m+1)^n.$$

It remains only to prove the claim.

Proof of Claim. We show first that the left side is at most the right. Let S be a stable subset of $A \setminus B_1$, and let $f(A\setminus S;B'_1,\ldots,B'_{m-1}) = t-1$. We want to show that $f(A;B_1,\ldots,B_{m-1}) \leqslant t$. Let $\phi \in F(A\setminus S;B'_1,\ldots,B'_{m-1};t-1)$, and extend ϕ to $\hat{\phi} : A \to \{1,\ldots,t\}$ by setting $\hat{\phi}(v)=\phi(v)$ for each $v \in A\setminus S$ and $\hat{\phi}(v)=t$ for each $v\in S$. We must check that

$$\hat{\phi} \in F(A;B_1,\ldots,B_{m-1};t). \tag{10}$$

Let uv be an edge with $u \in S$ and $v \in A \setminus S$. Thus $\hat{\phi}(u) = t$ and $\hat{\phi}(v) \leqslant t-1$. If $l(uv) = i \in \{2,\ldots,m\}$, then $v \in \partial_i S \subseteq B'_{i-1}$, and so $\hat{\phi}(v) = \phi(v) \leqslant (t-1)-(i-1) = t-i$. Thus in each case $\hat{\phi}(u) - \hat{\phi}(v) \geqslant l(uv)$. Since S is stable and ϕ is feasible for the subproblem on $A \setminus S$, it now follows easily that $\hat{\phi}$ is feasible for the subproblem on A. If $v \in B_1$ then $\hat{\phi}(v) = \phi(v) \leqslant t-1$ since $S \subseteq A \setminus B_1$; and if $v \in B_i$ for some $i \in \{2,\ldots,m-1\}$ then $v \in B'_{i-1}$ and so $\hat{\phi}(v) = \phi(v) \leqslant t-i$ by our choice of ϕ. Now we see that indeed (10) holds.

Conversely, let us show that the right side is at most the left. Let $f(A; B_1,\ldots,B_{m-1}) = t$. Let $\phi \in F(A; B_1,\ldots,B_{m-1}; t)$. Let S be the stable set $\phi^{-1}(\{t\})$. Then S must be non-empty by the minimality of t, and $S \subseteq A \setminus B_1$ since $\phi(v) \leqslant t-1$ for each $v \in B_1$. If $t=1$ then $S=A$ and the result holds, so let us assume that $t \geqslant 2$. Define $\phi' : A \setminus S \to \{1,\ldots,t-1\}$ by setting $\phi'(v) = \phi(v)$ for each $v \in A \setminus S$. It suffices for us to check that

$$\phi' \in F(A \setminus S; B'_1,\ldots,B'_{m-1}; t-1). \tag{11}$$

Clearly ϕ' is feasible for the subproblem on $A \setminus S$. Let $i \in \{2,\ldots,m-1\}$ and let $v \in B'_{i-1} = B_i \cup (A \cap \partial_i S)$. If $v \in B_i$ then $\phi'(v) = \phi(v) \leqslant t-i = (t-1)-(i-1)$ by the condition on ϕ, and if $v \in \partial_i S$ then the same inequality holds, since S is non-empty and ϕ is feasible for A. Finally, if $v \in B'_{m-1} = A \cap \partial_m S$, then as before $\phi'(v) = \phi(v) \leqslant t - m = (t-1)-(m-1)$. Thus (11) holds, which completes the proof. □

6. Counting assignments

Given a graph G, for each positive integer t let $f(t)$ be the number of (proper) t-colourings of G. Thus for example if G consists of two adjacent nodes then $f(t) = t(t-1)$. It is well known and easy to see that there is a unique polynomial $p(x)$ defined for all real x which agrees with f on the positive integers: this is the *chromatic polynomial* of G.

Does this result extend to the constraint matrix problem? Let G be a graph with n nodes, and with edge lengths l as usual. For each positive integer t let $f(t)$ be the number of feasible assignments from V to $\{1,\ldots,t\}$. If each edge-length is 1 then this is the just the chromatic polynomial of the graph G.

For example, let G consist of two adjacent nodes u and v with $l(uv) = 3$. Then it is easy to check that $f(t)$ agrees with the polynomial $p(t) = (t-2)(t-3)$ for each $t \geqslant 2$, but $f(1) = 0$ and $p(1) = 2$. Thus there is no "feasible assignment counting polynomial". However, there is nearly one.

Proposition 6.1. *Given (G,l) where G has n nodes, there is a monic polynomial $p(x)$ of degree n such that $f(t) = p(t)$ for all sufficiently large integers t.*

This result was shown independently in [13] by methods based on counting hyperplane arrangements, and in the unpublished manuscript [7] by elementary methods. Let us use the methods in [7] to extend the above result.

Consider a graph $G = (V,E)$ with length vector l. The *defect* of an assignment ϕ on the edge $e = uv$ is the larger of 0 and $l(uv) - |\phi(u) - \phi(v)|$. The *defect vector* defect(ϕ) of ϕ is the vector of these defects indexed by the edges of G. Thus ϕ is feasible if and only if its defect vector is $\mathbf{0}$.

Suppose that the maximum edge-length is m. Let $\mathscr{D} = \mathscr{D}(G,l)$ denote the set $\{0,1,\ldots,m\}^E$, which contains all possible defect vectors. For each $\mathbf{d}\in\mathscr{D}$ and positive integer t, let $f_{\mathbf{d}}(t)$ be the number of assignments $\phi\in\{1,\ldots,t\}^V$ with defect$(\phi)=\mathbf{d}$. We are interested in particular in the behaviour of $f_{\mathbf{0}}(t)$, the number of feasible assignments.

Proposition 6.2. *Given (G,l) where G has n nodes, there is a monic polynomial $p_{\mathbf{0}}(x)$ of degree n such that the number $f_{\mathbf{0}}(t)$ of feasible assignments with t available channels agrees with $p_{\mathbf{0}}(t)$ for all sufficiently large integers t.*

Indeed, suppose that the maximum edge-length is m. Then for any possible defect vector $\mathbf{d}\in\mathscr{D}$, there is a polynomial $p_{\mathbf{d}}(t)$ such that the number $f_{\mathbf{d}}(t)$ of assignments $\phi\in\{1,\ldots,t\}^V$ with defect$(\phi)=\mathbf{d}$ *satisfies $f_{\mathbf{d}}(t)=p_{\mathbf{d}}(t)$ for all integers $t\geqslant(m-1)(n-1)$. The polynomial $p_{\mathbf{d}}(t)$ has degree at most n, with equality if and only if $\mathbf{d}=\mathbf{0}$.*

It follows for example from this result, that for any non-negative integer b, the number of assignments with exactly b violated constraints agrees with a polynomial when the number t of available channels is sufficiently large; namely the polynomial which is the sum of the polynomials $p_{\mathbf{d}}(t)$ over all $\mathbf{d}\in\mathscr{D}$ with exactly b strictly positive entries. See also [13].

Proof. For each $1\leqslant k\leqslant n$, let Π_k denote the set of ordered partitions of V into k non-empty blocks, and let G_k denote the set of all functions from $\{1,\ldots,k-1\}$ to $\{1,\ldots,m\}$. For each $1\leqslant k\leqslant n$, $\pi\in\Pi_k$, $g\in G_k$ and positive integer t, we let $A(\pi,g,t)$ denote the set of all assignments $\phi\in\{1,\ldots,t\}^V$ such that there exist integers c_i $(i=1,\ldots,k)$ satisfying the following three conditions:

- for each $i=1,\ldots,k$, $\phi(v)=c_i$ for each v in block i of π,
- $1\leqslant c_1<c_2<\cdots<c_k\leqslant t$,
- for each $i=1,\ldots,k-1$,

$$c_{i+1}-c_i=g(i) \quad \text{if } g(i)<m$$

- and

$$c_{i+1}-c_i\geqslant m \quad \text{if } g(i)=m.$$

Observe that, given π and g as above, there is a vector $\mathbf{d}\in\mathscr{D}$ with the property that defect$(\phi)=\mathbf{d}$ for each $\phi\in A(\pi,g,t)$ (and each t such that $A(\pi,g,t)$ is non-empty). Let us denote this vector by defect(π,g).

We may write $f_{\mathbf{d}}(t)$ as

$$\sum_{k=1}^{n}\sum_{\pi\in\Pi_k}\sum_{g\in G_k}\mathbf{1}_{\text{defect}(\pi,g)=\mathbf{d}}|A(\pi,g,t)|.$$

The three sums above are each over sets which do not depend on t. Let $1 \leqslant k \leqslant n$, $\pi \in \Pi_k$ and $g \in G_k$: we shall investigate $|A(\pi, g, t)|$.

Suppose that there are $r=r(g)$ indices i such that $g(i)=m$, and let the sum of the other values $g(i)$ be $s=s(g)$. To obtain numbers c_i as above, we must choose non-negative integers $b_1, \dots, b_r$ and set the "big gaps" to $m+b_1, \dots, m+b_r$, and we must choose an initial gap b_0 and a terminal gap b_{r+1} : thus we choose $r+2$ non-negative integers $b_0, b_1, \dots, b_r, b_{r+1}$ such that

$$t-1 = s + b_0 + (m+b_1) + \cdots + (m+b_r) + b_{r+1},$$

that is

$$b_0 + \cdots + b_{r+1} = t - 1 - s - mr = t - \gamma,$$

where $\gamma = \gamma(g)$ is defined by $\gamma = s + mr + 1$. Hence $A(\pi, g, t)$ is non-empty if and only if $t \geqslant \gamma$; and if $t \geqslant \gamma$ then $|A(\pi, g, t)|$ is the number of ways of choosing $r+2$ non-negative integers summing to $t-\gamma$, and this equals

$$\binom{t-\gamma+r+1}{r+1} = \frac{(t-\gamma+r+1)_{(r+1)}}{(r+1)!},$$

a polynomial in t of degree $r+1$. Thus we obtain

$$f_{\mathbf{d}}(t) = \sum_{k=1}^{n} \sum_{\pi \in \Pi_k} \sum_{g \in G_k} \mathbf{1}_{\mathrm{defect}(\pi,g)=\mathbf{d}} \mathbf{1}_{t \geqslant \gamma} \frac{(t-\gamma+r+1)_{(r+1)}}{(r+1)!}.$$

Let $p_{\mathbf{d}}(x)$ be the polynomial

$$p_{\mathbf{d}}(x) = \sum_{k=1}^{n} \sum_{\pi \in \Pi_k} \sum_{g \in G_k} \mathbf{1}_{\mathrm{defect}(\pi,g)=\mathbf{d}} \frac{(x-\gamma+r+1)_{(r+1)}}{(r+1)!}.$$

We shall show that

$$\mathbf{1}_{t \geqslant \gamma}(t-\gamma+r+1)_{(r+1)} = (t-\gamma+r+1)_{(r+1)} \tag{12}$$

for all integers $t \geqslant (m-1)(n-1)$. This will show that $f_{\mathbf{d}}(t) = p_{\mathbf{d}}(t)$ for all integers $t \geqslant (m-1)(n-1)$, as we wished to show.

Let us then prove (12). It is certainly true if $t \geqslant \gamma$. Further, both sides equal 0 if $t = \gamma-1, \gamma-2, \dots, \gamma-r-1$. Thus (12) holds for each integer $t \geqslant \gamma - r - 1$. But

$$\begin{aligned} \gamma &= s + mr + 1 \\ &\leqslant (m-1)(k-1-r) + mr + 1 \\ &= (m-1)(k-1) + r + 1 \\ &\leqslant (m-1)(n-1) + r + 1. \end{aligned}$$

Thus (12) holds for all integers $t \geqslant (m-1)(n-1)$, as required.

To complete the proof, note that always $r \leqslant n-1$, and if $r=n-1$ then defect$(\pi, g)=\mathbf{0}$. Hence the degree of $p_{\mathbf{d}}(x)$ is at most n, and is at most $n-1$ unless $\mathbf{d}=\mathbf{0}$. Further, we get a contribution of a polynomial of degree n with leading coefficient $1/n!$ when π is one of the $n!$ trivial ordered partitions of V into singletons and $r=n-1$. Hence $p_{\mathbf{0}}$ is monic of degree n. $\square$

The lower bound $(m-1)(n-1)$ for t in the above proposition cannot be improved (in terms of m and n). For consider the complete graph K_n with n nodes, with each edge-length m. Let $p(x)$ be the polynomial

$$p(x) = (x-(m-1)(n-1))_{(n)}$$

of degree n. From the proof above, $f_{\mathbf{0}}(t) = p(t)$ for integral $t \geqslant (m-1)(n-1)$, but for integral $t < (m-1)(n-1)$ we have $f_{\mathbf{0}}(t) = 0$ and $p(t) \neq 0$.

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