A New Lower Bound for the Frequency Assignment Problem

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***Abstract*— We propose a new lower bound on the number of frequencies required to meet the frequency demands in a cellular network. It extends Gamst's work by devising a procedure of frequency insertion, which makes the best of unexploited frequency spaces between the assigned frequencies. Our lower bound is claimed to have much wider and easier real-world applicability due to its relaxed prerequisite condition. Furthermore, it is shown via an illustrative example to be much tighter than its previous counterpart.**

I. Introduction

THE recent demand explosion for mobile communication services, together with the finite frequency spectrum allocated to this service, makes the issue of frequency assignment ever more important for the design and operation of such systems. Existing studies on the frequency assignment problem (FAP), however, mostly focus on developing efficient heuristic algorithms [2], [3], [5], [6], [9], owing to its NP-completeness [1], [4]. The solutions thereby generated are often less than satisfactory, and what is worse there is no way of checking how far away they are from the optimum [7].

Studies on deriving lower bounds on the numbers of frequencies required for FAP's have thus arisen out of their capacity of indirectly checking the quality of the assignment solutions at hand. Of our particular interest is the lower bound stated as Lemma 9 in [7] obtained by Gamst. The bound, despite its relative tightness, is known to be of little practical value, since it is hard to locate the area in a real cellular system where its prerequisite condition is met. We extend this work by devising what we call the procedure of *frequency insertion*, which makes the best of unexploited frequency gaps. It will be shown that the new lower bound thus obtained not only broadens the applicability by relaxing the prerequisite condition, but also is tighter than its previous counterpart.

The FAP is originally defined for frequency-division multiple-access (FDMA) cellular systems, but still well fit for the core operation of frequency assignment in time-division multiple-access (TDMA) systems (see [10]). For exposition brevity, the development henceforth will be made assuming that the FAP is for an FDMA system. In Section II, we introduce the basic terminologies and notations. In Section III, a new lower bound is presented and compared with Lemma 9 in [7], followed by its verification. Comparative testing of two lower bounds is conducted via a simple but representative example in Section IV.

II. Terminologies and Notations

The terminologies and notations for FAP in [6] and [7] will mostly be used unless otherwise specified.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of cells in a cellular system. A *requirement vector* on X is an n-vector $M = (m_i)$ with nonnegative integer components. The component m_i represents the number of radio frequencies required for cell x_i. Radio frequencies are assumed to be evenly spaced, so they can be identified with the positive integers. A *compatibility matrix* on X is a symmetric $n \times n$ matrix $C = (c_{ij})$ with nonnegative integer entries. The value c_{ij} prescribes the minimum frequency separation required between frequencies assigned to cell x_i and cell x_j. If $c_{ij} = \nu$, cells x_i and x_j are said to be ν-compatible with each other. Then, a triple $P = (X, M, C)$ characterizes a FAP.[1]

An *admissible frequency assignment* for P will be a collection $F = (f_{il})$ of positive integers, $i = 1, \cdots, n, l = 1, \cdots, m_i$, such that

$$|f_{ik} - f_{jl}| \geq c_{ij}$$

for all indices i, j, k, l (except for $i = j, k = l$), where f_{il} is the frequency assigned to the lth requirement of cell x_i. For convenience, we assume that the requirements of each cell are ordered.

The *span* $S(F)$ of a frequency assignment F is the maximum frequency assigned to the system. That is,

$$S(F) = \max_{i,l} f_{il}.$$

The objective of the FAP is to find an admissible frequency assignment F with the minimum span $S_0(P)$. That is,

$$S_0(P) = \min\{S(F') \mid \text{ all admissible } F' \text{ for } P\}.$$

As done in others [6], [7], we shall obtain lower bounds on $S_0(P)$ by solving FAP's simpler than P defined on the following subsets of X.

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[1]The definition of the FAP in [7] considers two additional sets on X: *the set of preassigned frequencies* A and *the set of blocked frequencies* B. That is, the FAP is defined as the quintuple $P_5 = (X, M, C, A, B)$. Most lower bounds reported in [7] including Lemma 9, however, are generated from a relaxed FAP wherein $A = \emptyset$ and $B = \emptyset$. So A and B are excluded in our definition.

Definition 1: Let $P=(X,M,C)$ be an FAP and $\nu,\ \nu'$ be positive integers such that $\nu' > \nu$.

1) A subset Q of X is said to be ν- *complete* if $c_{ij} \geq \nu$ for all $x_i, x_j \in Q$.
2) A subset pair (Q,Q') of X is said to be (ν,ν')-*complete* if Q is a ν-complete subset of X and Q' is a ν'-complete subset of Q.

III. A New Lower Bound for FAP

Listed below for self-completeness is an existing lower bound with a (ν,ν')-complete subset pair of X by Gamst [7].

Theorem 1—(Lemma 9 in [7]): Let $P=(X,M,C)$ be an FAP, and assume there exist a (ν,ν')-complete subset pair (Q,Q') of X such that

$$c_{ij} \geq \nu' > \nu, \qquad \text{for all } x_i \in Q' \text{ and } x_j \in Q. \tag{1}$$

If $\sum_{x_i \in Q|Q'} m_i > 0$, let

$$L_1 = 1 + \nu' \sum_{x_i \in Q'} m_i + \nu \left(\sum_{x_i \in Q|Q'} m_i - 1 \right).$$

Then, $S_0(P) \geq L_1$.

This theorem is of little practical use due to the difficulty of finding a (ν,ν')-complete subset pair satisfying Condition (1) that each cell in Q' is at least ν'-compatible with every cell in Q. We shall derive a new lower bound for P, when the strict Condition (1) is relaxed by allowing lower-than-ν' compatibilities for some cell pairs between Q' and Q.

A. Frequency Insertion

Let $P=(X,M,C)$ be an FAP with (ν,ν')-complete subset pair (Q,R) of X, where R is a maximal ν'-complete subset of Q. The aim is to derive a lower bound for P by considering an optimal assignment of $P'=(X,M',C')$, a relaxed version of P, wherein $M'=(m_i')$ and $C'=(c_{ij}')$ are defined as

$$m_i' = \begin{cases} m_i, & \text{if } x_i \in Q \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$c_{ij}' = \begin{cases} \nu' I_{[\nu',\infty)}(c_{ij}) + \nu I_{[\nu,\nu')}(c_{ij}), & \text{if } x_i \in R \text{ or } x_j \in R \\ \min\{\nu, c_{ij}\}, & \text{otherwise} \end{cases} \tag{3}$$

where $I_A(e)$ is an indicator function which takes its value as one when e is contained in A, zero otherwise. Then, $c_{ij}'=\nu'$ for $x_i \in R,\ x_j \in R,\ c_{ij}'=\nu$ for $x_i \in Q \mid R,\ x_j \in Q \mid R$, and $c_{ij}'=\nu'$ or ν for $x_i \in Q \mid R,\ x_j \in R$. Furthermore, each cell in $Q \mid R$ is ν-compatible with at least one cell among those in R, since R is the maximal ν'-complete subset of Q.

For the span-minimizing frequency assignment for FAP P', we incorporate the procedure of *frequency insertion* (PFI), which exploits most economically the unused frequency gaps within the current frequency-span. Initially the PFI assigns the frequencies separated each other with equi-distance ν', one for each requirement in R starting from the lowest numbered frequency, say frequency 1. Before moving on to the next phase of assigning frequencies to the requirements in $Q \mid R$, we need the following notations: Let q be a positive and ν_0 be a nonnegative integer such that $\nu' = q\nu + \nu_0,\ 0 \leq \nu_0 < \nu$.

Then for the requirements in $Q \mid R$, the frequency assignment can be differentiated into three (when $\nu_0 > 0$) or two (when $\nu_0 = 0$) kinds depending on the size of the frequency-span increase. The first kind is to naturally insert a frequency within the current span, thus called here the 0-*insertion*. For example, suppose that two frequencies $f,\ f+\nu'$ are assigned to two requirements of cells $x_i \in R$ and $x_j \in R$, respectively. Then, $(q-1)$ frequencies $f+\nu,\cdots,f+(q-1)\nu$ between f and $f+\nu'$ can be assigned to the requirements of cells in $Q \mid R$ which are ν-compatible with both cells x_i and x_j.

Once all 0-insertions are made wherever possible without increasing the current span, we proceed to the second kind of assignments for the remaining requirements in $Q \mid R$. This is to utilize the frequency gaps within the current span still unexploited by 0-insertions. Consider in the above example, the spacing between $f+(q-1)\nu$ and $f+\nu'$. If we push the frequency $f+\nu'$ by $(\nu-\nu_0)$ to the higher value of $f+(q+1)\nu = f+\nu'+(\nu-\nu_0)$, a frequency of $f+q\nu$ can be additionally assigned to a requirement in $Q \mid R$ which is ν-compatible with both cells x_i and x_j. This assignment of incrementally widening a fractional frequency space and thus extending the current span by that amount is called the $(\nu-\nu_0)$-*insertion*. Obviously this kind of insertions cannot take place when $\nu_0 = 0$. Thus for exposition brevity, the development henceforth will be made assuming the inclusive case of $\nu_0 > 0$.

Finally for each remaining, if any, requirement of the cell, say x_i, in $Q \mid R$, one of the frequencies $\{f_{jk}+\nu \mid c_{ij}' = \nu, x_j \in R, k=1,\cdots,m_j\}$ can be inserted in between by making a frequency room of its own. This is always possible as there is at least one cell $x_j \in R$ such that $c_{ij}=\nu$. Each of these assignments increases the current span by ν, and we call it the ν-*insertion*.

When the PFI takes the 0- and $(\nu-\nu_0)$-insertions a_0 and $a_{\nu-\nu_0}$ times, respectively, it generates an admissible frequency assignment F, the span of which is given by

$$\begin{aligned} S(F) &= 1 + \nu' \left(\sum_{x_i \in R} m_i - 1 \right) + (\nu-\nu_0) a_{\nu-\nu_0} \\ &\quad + \nu \left(\sum_{x_i \in Q|R} m_i - a_0 - a_{\nu-\nu_0} \right) \\ &= 1 + \nu' \left(\sum_{x_i \in R} m_i - 1 \right) + \nu \sum_{x_i \in Q|R} m_i \\ &\quad - \nu a_0 - \nu_0 a_{\nu-\nu_0}. \end{aligned} \tag{4}$$

The first two terms of (4) represent the length of the span initially set for the requirements in R, inside which all 0-insertions are made. The third and last terms indicate, respectively, the portions of the span extended by $(\nu-\nu_0)$- and ν-insertions.

Since a_0 and $a_{\nu-\nu_0}$ are the only decision variables in this expression, the *best* frequency insertion strategy is to rearrange the requirements in Q so that $\nu a_0 + \nu_0 a_{\nu-\nu_0}$ may be maximized.

B. New Lower Bound

The optimal assignment for P' must be *compact* so that its span cannot be decreased simply by shortening some of the gaps between the assigned frequencies. In other words, at optimality, there is not any unnecessary spacing between any two neighboring assigned frequencies. Moreover, any compact frequency assignment for P' can always be reordered to an admissible one having the span less than or equal to the given compact assignment, which can be generated by the PFI. This assertion is validated in the Appendix along with an illustrative example. Therefore, in finding the optimal assignment for P', the focus is placed only on those assignments that the PFI can generate. From the form of the $S(F)$ for an arbitrary admissible assignment F generated by the PFI, the objective is now set at determining the values, a_0^* and $a_{\nu-\nu_0}^*$, which maximize $\nu a_0 + \nu_0 a_{\nu-\nu_0}$.

A difficulty with this objective, however, is that both a_0 and $a_{\nu-\nu_0}$ depend on the initialization of the PFI, i.e., the initial frequency assignment of the requirements in R. Considering there exist as many as $\frac{\left(\sum_{x_i\in R} m_i\right)!}{\Pi_{x_i\in R} m_i!}$ possible initializations, it is computationally prohibitive to obtain the exact values of a_0^* and $a_{\nu-\nu_0}^*$ by enumerating all such cases. Instead we shall be satisfied with obtaining an upper bound on $\nu a_0 + \nu_0 a_{\nu-\nu_0} = (\nu-\nu_0)a_0 + \nu_0(a_0 + a_{\nu-\nu_0})$, by deriving upper bounds on a_0 and $a_0 + a_{\nu-\nu_0}$. The development hitherto made leads to the following result.

Theorem 2: Let $P = (X, M, C)$ be an FAP having a (ν,ν')-complete subset pair (Q, R) of X with a maximal ν'-complete subset R of Q, from which an FAP $P' = (X, M', C')$ is defined with (2) and (3). Let α, β be some upper bounds on a_0 and $a_0 + a_{\nu-\nu_0}$, respectively, for P'. Define

$$L_2 = 1 + \nu'\left(\sum_{x_i\in R} m_i - 1\right) + (\nu-\nu_0)(\beta-\alpha) + \nu\left(\sum_{x_i\in Q|R} m_i - \beta\right).$$

Then $S_0(P) \ge S_0(P') \ge L_2$.

Note that when $\nu_0 = 0$, only 0- and ν-insertions can take place for the requirements in $Q \mid R$ so that β is replaced by α in L_2.

C. Maximal Flow Problem for Upper Bound on a_0

Two maximal flow problems (MFP's), $\mathcal{P}_1$ and $\mathcal{P}_2$, are defined for the purpose of generating upper bounds on a_0 and $a_0 + a_{\nu-\nu_0}$, respectively.

Let S be the subset of R such that each cell in S is ν-compatible with at least one cell in $Q \mid R$. For convenience, re-index the cells in Q as follows: $S = \{x_1, x_2, \cdots, x_s\}$, $Q \mid R = \{x_{r+1}, x_{r+2}, \cdots, x_{r+t}\}$, where s, r, t are the cardinalities of the sets $S, R, Q \mid R$, respectively. Furthermore, define S_h be the subset of S such that each cell in S_h is ν-compatible with the cell x_{r+h}. That is, $S_h = \{x_i \in S \mid c'_{i,r+h} = \nu\}$. Then $\bigcup_{h=1}^t S_h = S$.

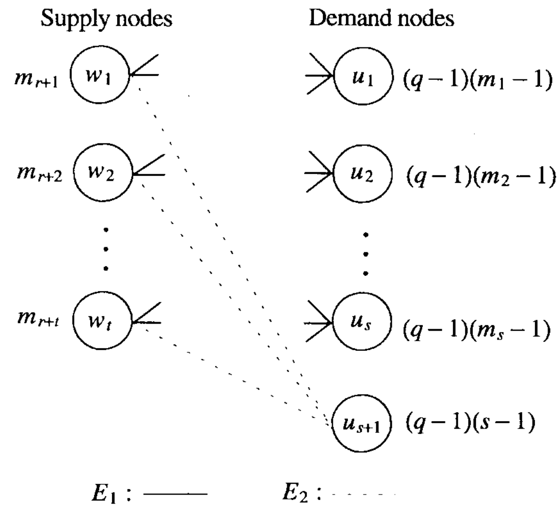

Fig. 1. The maximal flow problem $\mathcal{P}_1$.

The capacitated network in Fig. 1, on which MFP $\mathcal{P}_1$ is defined, arises from the assumption that at the initialization of the PFI, the requirements of the same cell in R are assigned frequencies spaced ν' each other. In addition it is assumed that we assign frequencies to the requirements of the cells as ordered above, i.e., the cells in S first and then in $R \mid S$. We only consider the former portion of the resulting initial span associated with S wherein 0-insertions can take place. In that portion, there are two kinds of frequency spacings, intra- and inter-cell. Inside each such spacing, we can assign frequencies, spaced ν each other, to $(q-1)$ requirements in $Q \mid R$ by 0-insertions. Cell $x_j \in S$ with m_j requirements is associated with $(m_j - 1)$ intra-cell spacings, rendering the capacity of demand node $u_j \in U$ to be $(q-1)(m_j-1)$. There are a total of $(s-1)$ inter-cell spacings, for which an extra demand node, indexed as $j = s+1$, with capacity $(q-1)(s-1)$ is defined. And the capacity of supply node $w_h \in W = \{w_1, w_2, \cdots, w_t\}$ represents the number of requirements of cell x_{r+h}, hence is set to m_{r+h}.

Let $E = E_1 \cup E_2$ be the set of arcs joining nodes in W and U, where

$$E_1 = \{(w_h, u_j) \mid c'_{r+h,j} = \nu, j = 1, \cdots, s, h = 1, \cdots, t\}$$
$$E_2 = \{(w_h, u_{s+1}) \mid |S_h| \ge 2,\ h = 1, \cdots, t\}.$$

The capacities of the arcs in E_2 are all set at $(q-1)(|S_h|-1)$, whereas the capacities of those in E_1 all unlimited.

We are now in a position to state the following result.

Theorem 3: The maximal flow from W to U for MFP $\mathcal{P}_1$ is an upper bound on a_0, i.e., on the maximum number of 0-insertions attainable for FAP P' with the PFI.

Proof: It suffices to show that, for any set of 0-insertions made for P' with the PFI, there always corresponds a feasible flow $\mathcal{F} = (\sigma_{hj})$ for $\mathcal{P}_1$, the total flow value, $\sum_{hj} \sigma_{hj}$, of which equals the total number of 0-insertions of that set. Here σ_{hj} denotes the flow value from w_h to u_j.

Let $F_R = (f_{il})\ i = 1, 2, \cdots, r,\ l = 1, 2, \cdots, m_i$ be the initial frequency-assignment for the requirements in R made by the PFI. We reindex the cells in S and their requirements such that f_{im_i} and f_{sm_s} are the largest frequencies among those assigned to the cell $x_i \in S$ and to S, respectively. And let the function $n(f_{il})$ indicate the smallest frequency larger than f_{il} among those assigned to R.

Each individual 0-insertion made for a requirement in $Q \mid R$ corresponds to some unit flow of $\mathcal{F}$: Consider the 0-insertion made for a requirement of x_{r+h} to the frequency space between f_{il} and $n(f_{il})$. Within the framework of MFP $\mathcal{P}_1$, this implies sending one unit of flow from w_h to u_{s+1} if $l = m_i$, and from w_h to u_i otherwise. This also hints why the MFP $\mathcal{P}_1$ has the bipartite structure consisting of three kinds of entities: supply nodes w_h, demand nodes u_j and arcs in between.

Now we show how this group of unit flows, each corresponding to some 0-insertion, are constrained in association with each of the three entities of the bipartite MFP.

1) *Capacity of a Supply Node:* Each 0-insertion made for a requirement of x_{r+h} corresponds to an individual unit flow originating from w_h. So the total flow from w_h, $\sum_j \sigma_{hj}$, cannot exceed m_{r+h} for $h = 1, 2, \cdots, t$.
2) *Capacity of a Demand Node:* By the same reasoning above, the total flow to u_i, $\sum_h \sigma_{hi}$, equals the number of requirements in $Q \mid R$ for which frequencies in the set T_i are assigned by 0-insertion, where T_i is given by

$$T_i = \begin{cases} \{f \mid f_{il} < f < n(f_{il}), l = 1, 2, \cdots, m_i - 1\}, \\ \qquad\qquad\qquad\qquad \text{for } i = 1, 2, \cdots, s \\ \{f \mid f_{jm_j} < f < n(f_{jm_j}), j = 1, 2, \cdots, s - 1\}, \\ \qquad\qquad\qquad\qquad \text{for } i = s + 1. \end{cases}$$

So, $\sum_h \sigma_{hi}$ cannot exceed $(q-1)(m_i - 1)$ for $i = 1, 2, \cdots, s$ and $(q-1)(s-1)$ for $i = s+1$.
3) *Arc Capacity:* Let i_h be the index of the cell, to a requirement of which $\max_{i \in S_h} f_{i,m_i}$ is assigned. Then, the total flow from w_h to u_{s+1}, $\sigma_{h,s+1}$, equals the number of requirements of the x_{r+h}, for which frequencies in the set $\{f \mid f_{im_i} < f < n(f_{im_i}), x_i \in S_h, i \neq i_h\}$ are assigned by 0-insertion, hence cannot be greater than $(q-1)(|S_h| - 1)$. □

As with a_0, we can find an upper bound on $a_0 + a_{\nu - \nu_0}$ by solving the MFP $\mathcal{P}_2$ which is defined the same as MFP $\mathcal{P}_1$ except for the capacities of demand nodes and arcs in E_2. The capacity of a demand node $u_j \in U$ is set to $q(m_j - 1)$ for $j = 1, \cdots, s$, and to $q(s-1)$ for $j = s+1$. The capacity of arc $(w_h, u_{s+1}) \in E_2$ is set to $q(|S_h| - 1)$.

Corollary 4: Let α and β be the upper bounds on a_0 and $a_0 + a_{\nu - \nu_0}$, obtained by solving MFP's $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, when the PFI is employed for FAP P'. If $\sum_{x_i \in Q|(R|S)} m_i > 0$, we have, from Theorem 1, the following lower bound,

$$L_1 = 1 + \nu' \sum_{x_i \in R|S} m_i + \nu \left(\sum_{x_i \in Q|(R|S)} m_i - 1 \right).$$

This, with the lower bound L_2 from Theorem 2, then satisfies $L_2 \geq L_1$.

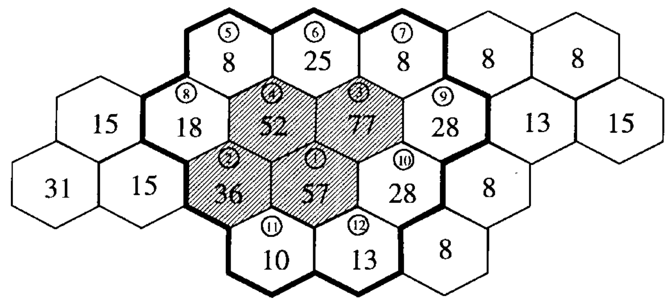

Fig. 2. An example.

Proof: From Theorems 1 and 2, we have

$$L_2 - L_1 = (\nu - \nu_0) \left[(q-1) \left(\sum_{x_i \in S} m_i - 1 \right) - \alpha \right] + \nu_0 \left[q \left(\sum_{x_i \in S} m_i - 1 \right) - \beta \right].$$

The right-hand side of the last equation is not smaller than zero, since the inequalities, $\alpha \leq (q-1)(\sum_{x_i \in S} m_i - 1)$ and $\beta \leq q(\sum_{x_i \in S} m_i - 1)$, obviously hold from the sums of the demand capacities in MFP's $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. □

IV. Example

Consider an FAP defined for a cellular radio network which consists of regular hexagonal cells with nonhomogeneous requirements, as shown in Fig. 2. The encircled number in each cell represents the cell index, while the other number the size of the requirements of the cell. We assume that $c_{ii} = 5$ and for $i \neq j$:

$$c_{ij} = \begin{cases} 0, & \text{if } d_{ij} > 3 \\ 1, & \text{if } \sqrt{3} < d_{ij} \leq 3 \\ 2, & \text{if } d_{ij} \leq \sqrt{3} \end{cases}$$

where d_{ij} is the distance between the centers of x_i and x_j. Distance is here normalized to be unity between the centers of any two adjacent cells. Then, the 12-cell subset enclosed by bold line is 1-complete, and the shaded 4-cell subset is 2-complete. Note that this FAP is the same as that of Fig. 5 in [7], except that the compatibility constraints on adjacent channels are further tightened here.

The tightest lower bound with Theorem 1 is $L_1 = 1 + 2 \times 134 + 181 = 450$, which is obtained by taking $Q = \{x_1, x_2, x_3, x_4, x_6, x_9, x_{10}, x_{12}\}$, $Q' = \{x_1, x_3\}$ and $\nu = 1$, $\nu' = 2$. Note here the difficulty of finding the sets, Q and Q', arising from that they may not be maximal.

To apply Theorem 2, we set $Q = \{x_1, x_2, \cdots, x_{12}\}$, $R = S = \{x_1, x_2, x_3, x_4\}$ and $\nu = 1$, $\nu' = 2$, from which MFP $P' = (X, M', C')$ is defined with (2) and (3). Noting $q = 2$ and $\nu_0 = 0$, we have MFP $\mathcal{P}_1$ as shown by Fig. 3, the solution of which gives $\alpha = 121$, our upper bound on a_0. This is the case that all of the requirements in $Q \mid R = \{x_5, x_6, \cdots, x_{12}\}$ except for 17 ones of cell 6 or 9 are assigned frequencies by 0-insertion. Then, the lower bound obtained from Theorem 2 is $L_2 = 1 + 2 \times 221 + (138 - 121) = 460$.

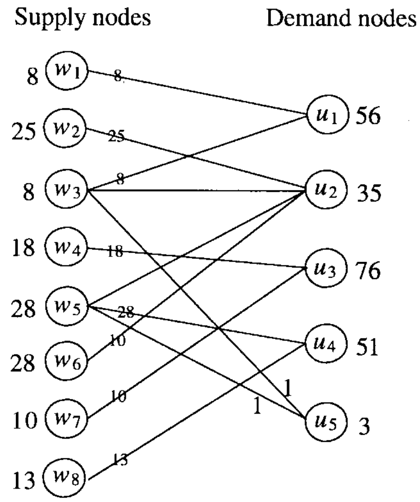

Fig. 3. MFP $\mathcal{P}_1$ for the example.

V. Concluding Remarks

We have proposed a new lower bound for the FAP $P = (X, M, C)$ with a (ν, ν')-complete subset pair of X, which extends Lemma 9 in [7].

Two significant enhancements are made with this lower bound. The first is in its wide and easy applicability. Note that Gamst's lower bound requires searching through all the *non*-maximal complete subsets so as to find the ones satisfying the condition. So, it would take excessive time for complex real-world cellular networks. On the other hand, ours requires finding another maximal complete subset contained in a maximal complete subset, which can be carried out easily by employing an existing clique finding algorithm like in [8]. The second enhancement is in its tightening effect, as has been well demonstrated by the above example.

Appendix

The Appendix validates the assertion that any compact assignment can be reordered to the one the PFI can generate.

Let $F = (f_{il})$ be a compact assignment for the relaxed FAP $P' = (X, M', C')$ with a maximal ν'-complete set R. Define the following two sets of individual assignments whose positions are subject to change:

$$A = \{f_{il} : x_i \in Q \mid R \text{ is } \nu'\text{-compatible with either } x_p(f_{il}) \text{ or } x_n(f_{il})\},$$
$$A' = \{f_{il} : x_i \in Q \mid R, f_{il} < \min_{x_j \in R} f_{jl} \text{ or } f_{il} > \max_{x_j \in R} f_{jl}\}$$

where the function $x_p(f)[x_n(f)]$ indicate the cell to which the largest [smallest] frequency smaller (larger) than f among assigned to R is assigned.

A compact assignment is iteratively updated by performing the following reordering step: Select an assignment $f_{il} \in A \cup A'$, remove it from the current position and insert it just after an assignment f^*, where $f^* \in B = \{f_{jl} : x_j \in R, x_i$ is ν-compatible with both x_j and $x_n(f_{jl})\}$ if $B \neq \emptyset$, $f^* \in B' = \{f_{jl} : x_j \in R, x_i$ is ν-compatible with $x_j\}$, otherwise.

The removing and inserting operations are performed without producing any unnecessary frequency gaps so that the compactness is preserved throughout. Note that removing an assignment in $A \cup A'$ decreases the span by at least ν, while inserting it increases the span by at most ν. Therefore, the whole span is not increased by each reordering step. Furthermore the compact assignment finally obtained by this iterative procedure has the same structure as the one that the PFI generates.

An Illustrative Example

Consider an FAP $P' = (X, M', C')$, as shown in Fig. 4 where encircled numbers indicate cell indices.

Assume that a compact assignment F for P' is given as (A.1), shown at the bottom of the page.

We have $A = \{f_{43}, f_{31}, f_{32}\}$ and $A' = \{f_{41}, f_{42}\}$ from $R = \{x_1, x_2\}$ and $c'_{14} = c'_{32} = 5$. Select f_{43} from $A \cup A'$, and move its position just after f_{21}. Then the compact assignment is updated as (A.2), shown at the bottom of the page.

Now, $A = \{f_{31}, f_{32}\}$ and $A' = \{f_{41}, f_{42}\}$, from which we select f_{31}, and move its position just after f_{11}. Then,

$$\begin{array}{lcccccccccccccc}
 & & & & \uparrow \text{remove} & & & & & \downarrow \text{insert} & & & & & \\
f_{il} & f_{41} & f_{42} & f_{11} & f_{43} & f_{12} & f_{13} & f_{14} & f_{21} & f_{31} & f_{51} & f_{22} & f_{32} & f_{52} & f_{23} \\
\text{frq.} & 1 & 3 & 8 & 13 & 18 & 23 & 28 & 33 & 35 & 37 & 39 & 41 & 43 & 45
\end{array} \tag{A.1}$$

$$\begin{array}{lcccccccccccccc}
 & & & & \downarrow \text{insert} & & & & & \uparrow \text{remove} & & & & & \\
f_{il} & f_{41} & f_{42} & f_{11} & f_{12} & f_{13} & f_{14} & f_{21} & f_{43} & f_{31} & f_{51} & f_{22} & f_{32} & f_{52} & f_{23} \\
\text{frq.} & 1 & 3 & 8 & 13 & 18 & 23 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42
\end{array} \tag{A.2}$$

$$\begin{array}{lcccccccccccccc}
f_{il} & f_{41} & f_{42} & f_{11} & f_{31} & f_{12} & f_{13} & f_{14} & f_{21} & f_{43} & f_{51} & f_{22} & f_{32} & f_{52} & f_{23} \\
\text{frq.} & 1 & 3 & 8 & 10 & 13 & 18 & 23 & 28 & 30 & 32 & 34 & 36 & 38 & 40
\end{array} \tag{A.3}$$

$$\begin{array}{lcccccccccccccc}
f_{il} & f_{11} & f_{31} & f_{12} & f_{32} & f_{13} & f_{14} & f_{21} & f_{43} & f_{42} & f_{51} & f_{22} & f_{41} & f_{52} & f_{23} \\
\text{frq.} & 1 & 3 & 6 & 8 & 11 & 16 & 21 & 23 & 25 & 27 & 29 & 31 & 33 & 35
\end{array} \tag{A.4}$$

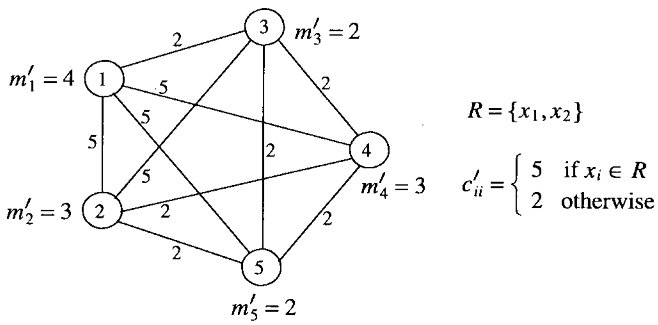

Fig. 4. An example FAP for the validation.

the updated compact assignment becomes (A.3), shown at the bottom of the previous page.

Completing the round of iterative reordering steps with every elements in $A \cup A'$ yields the assignment (A.4) shown at the bottom of the previous page, which can certainly be generated by the PFI.

References

[1] W. K. Hale, “Frequency assignment: Theory and applications,” *Proc. IEEE*, vol. 68, no. 12, pp. 1497–1514, 1980.
[2] F. Box, “A heuristic technique for assigning frequencies to mobile radio nets,” *IEEE Trans. Veh. Technol.*, vol. VT-27, no. 2, pp. 57–64, 1978.
[3] J. A. Zoellner and C. L. Beall, “ A breakthrough in spectrum conserving frequency assignment technology,” *IEEE Trans. Electromag. Compat.*, vol. EMC-19, no. 3, pp. 313–319, 1977.
[4] M. R. Garey and D. S. Johnson, “The complexity of near-optimal graph coloring,” *J. ACM*, vol. 23, pp. 43–49, 1976.
[5] R. D. Dutton and R. C. Brigham, “A new graph coloring algorithm,” *Computer J.*, vol. 24, no. 1, pp. 85–86, 1981.
[6] A. Gamst and W. Rave, “On the frequency assignment in mobile automatic telephone systems,” in *Proc. GLOBECOM’82, IEEE*, 1982.
[7] A. Gamst, “Some lower bounds for a class of frequency assignment problems,” *IEEE Trans. Veh. Technol.*, vol. VT-35, no. 1, pp. 8–14, 1986.
[8] C. Bron and J. Kerbosch, “Finding all cliques of an undirected graph (Algorithm 457),” *Commun. ACM*, vol. 16, no. 9, pp. 575–577, 1973.
[9] K. N. Sivarajan, R. J. McEliece, and J. K. Ketchum, “Channel assignment in cellular radio,” in *39th IEEE VTC*, 1989.
[10] K. Raith and J. Uddenfeldt, “Capacity of digital TDMA systems,” *IEEE Trans. Veh. Technol.*, vol. 40, no. 2, 1991.

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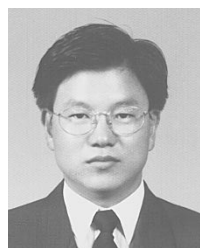

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