ALGORITHMS FOR RADIO LINK FREQUENCY ASSIGNMENT: THE CALMA PROJECT

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The radio link frequency assignment problem occurs when a network of radio links has to be established. Each link must be assigned an operating frequency from a given domain. The assignment has to satisfy certain restrictions so as to limit the interference between links. The number of frequencies used is to be minimized.

Problems of this type were investigated within the CALMA project by a consortium consisting of research groups from Delft, Eindhoven, London, Maastricht, Norwich, and Toulouse. The participants developed optimization algorithms based on branch-and-cut and constraint satisfaction, and approximation techniques including a variety of local search methods, genetic algorithms, neural networks, and potential reduction. These algorithms were tested and compared on a set of real-life instances.

The explosive development in wireless communication has resulted in suitable frequencies becoming an increasingly scarce, and thereby expensive, resource. In a wireless communication network, it is therefore important to satisfy the demand of the clients using relatively few frequencies, which implies that several communication links may be assigned the same frequency. If communication links are physically close to each other, using the same— or almost the same—frequency may cause interference. In the many variants of the *frequency assignment problem*, a trade-off is sought between the number of frequencies used and the level of interference induced in the network.

The frequency assignment problem attracting most attention nowadays is the assignment of frequencies to transceivers in a terrestrial mobile cellular network, the GSM network being the most prominent one. Other frequency assignment problems occur in satellite-based cellular systems, fixed cellular telecommunication networks, and point-to-multipoint radio access systems. An overview of a broad range of frequency assignment problems can be found in the thesis by Koster (1999), and in the survey by Murphey et al. (1999). Eisenblätter (2000) describes basically all issues of frequency assignment in a GSM network in detail. The FAP website (Eisenblätter and Koster 2000), maintained by Eisenblätter and Koster, contains a large number of references and a history of wireless communication as well as a detailed description of various types of frequency assignment problems.

Frequencies can be assigned to communication links according to three general schemes: fixed channel assignment, dynamic channel assignment, and hybrid channel assignment. In fixed channel assignment, forecasted demand is used to determine the requirements of the network, and the frequencies are assigned to the communication links in advance. In dynamic channel assignment, an online assignment of frequencies to links is made so as to satisfy the current demand with low interference. For a survey of the various schemes, we refer the reader to Katzela and Naghshineh (1996). For the problem discussed in the present paper, a fixed channel assignment is made.

The radio link frequency assignment problem treated in this paper occurs in a military application. It is a type of problem related to fixed cellular communication networks, in that wireless connections have to be established between pairs of transceivers. This is in contrast to GSM networks in which connections are established between a transceiver and mobile units. Each transceiver in a GSM network operates on one frequency, so the assignment of frequencies is made to transceivers, whereas in our application an assignment is made to each pair of transceivers, i.e., to each radio communication link. The assignment has to comply with certain preferences, regulations, and physical characteristics of the transmitters. First, some links have preassigned frequencies. Second, when two links interfere, their frequencies must be far enough apart to ensure that communication is not distorted. Third, each link has a reverse link, and for technical reasons, the frequencies assigned to such parallel or "duplex" links must differ by a given distance. Some of the constraints of the first and second type may be violated at a certain cost. One variant of the problem is to find an assignment that minimizes the cost of violating the soft constraints. If all constraints can be satisfied, the second

variant is to find an assignment of zero cost that minimizes the number of frequencies used. The constraints described here are typical for many other types of frequency assignment problems; see, e.g., Koster (1999).

Our problem was investigated within the CALMA project of the EUCLID program. EUCLID, which stands for "EUropean Cooperation on the Long term In Defense," is a research program of West-European Departments of Defense. Three of its members—France, the Netherlands, and the United Kingdom—joined in the CALMA project, which had the purpose to investigate the use of "Combinatorial ALgorithms for Military Applications." They chose radio link frequency assignment as the subject of a pilot study, and specified three subprograms: "testing genetic algorithms, testing exact solution techniques, and testing approximate solution techniques" (Hajema et al. 1993). The project was granted to a consortium consisting of six research groups: the Centre d'Etudes et de Recherches de Toulouse (CERT) in France, the Technische Universiteit Delft (TUD), the Technische Universiteit Eindhoven (TUE) and the Universiteit Maastricht (UM) in the Netherlands, and King's College London (KCL) and the University of East Anglia (UEA), Norwich, in the United Kingdom.

In the period from December 1993 to December 1995, each of the groups contributed its expertise to the project. Together they developed and implemented a wide variety of optimization and approximation algorithms. The approximative approaches include neighborhood search methods such as simulated annealing, tabu search and variable-depth search, hyperneighborhood search methods like genetic algorithms, other search methods based on neural networks, incomplete optimization, and potential reduction methods. Such approaches all produce solutions that are hoped to be close to the optimum but that, in the present context, have no a priori quality guarantee. The participating groups also developed techniques for finding lower bounds on the optimum, based on linear programming, graph coloring, constraint satisfaction, and 0–1 quadratic programming. Lower bounds are used as a yardstick in two settings: measuring the quality of upper bounds found by approximation algorithms, and curtailing the search in enumerative optimization algorithms. Finally, all these approaches were tested and compared on a set of real-life instances.

An important goal of this paper is to provide a concise view of many approaches applied to the same data set. Because the instances are publicly available, it is possible to continue comparing algorithmic developments and computational results, even though the project has now ended. A problem with many other real-life frequency assignment instances is that they are available only to a limited number of researchers making a comparison of algorithmic work difficult.

The paper is organized as follows. Section 1 gives mathematical formulations of the problem under consideration and comments upon the test instances used in the project. Section 2 describes the approximation algorithms developed by the participants, §3 deals with lower bounding

techniques and optimization algorithms, and §4 gives a computational comparison of the various approaches. Section 5 reviews algorithmic work done after the project was finished. Section 6 summarizes our conclusions.

1. THE RADIO LINK FREQUENCY ASSIGNMENT PROBLEM

We are given a finite set L of radio links. Each link i is to be assigned a frequency f_i from a given finite domain D_i; we write $D = \cup_{i \in L} D_i$. A link i may have a preassigned frequency p_i, which may or may not be changed. Other restrictions are defined on pairs of links $\{i, j\}$: the frequencies of a pair of *interfering* links must be more than a given distance d_{ij} apart, and the frequencies of a pair of *parallel* links must differ by exactly d_{ij}. Each link belongs to exactly one parallel pair.

Some of the restrictions are *soft* and may be violated at a certain cost. The others are *hard* and may not be violated. Restrictions on the pairs of parallel links are always hard, and so are some of the interference and preassignment constraints. Nonnegative *interference costs* c_{ij} for violating soft interference constraints and nonnegative *mobility costs* m_i for changing soft preassigned frequencies are given. An assignment of frequencies is *complete* if every link in L has a frequency assigned to it. We denote by C and M the sets of all soft interference and mobility constraints, respectively.

The *minimum interference problem* is to find a complete assignment that satisfies all *hard* constraints at minimum cost:

$$\text{minimize} \sum_C c_{ij}\delta(|f_i - f_j| \leqslant d_{ij}) + \sum_M m_i\delta(|f_i - p_i| > 0),$$

subject to the hard constraints:

$|f_i - f_j| > d_{ij}$ for each pair of links $\{i, j\}$
  with a hard interference constraint,

$|f_i - f_j| = d_{ij}$ for each pair of parallel links $\{i, j\}$,

  $f_i = p_i$ for each link i with a hard
      preassigned frequency,

  $f_i \in D_i$ for each link i,

where $\delta(\gamma) = 1$ if condition γ is true and 0 otherwise.

If there exists a *feasible* assignment, i.e., a complete assignment of zero cost, then the *minimum spectrum usage problem* is to find a feasible assignment that satisfies *all* constraints and minimizes the number of distinct frequencies used:

$$\text{minimize} \ |\cup_i \{f_i\}|,$$

subject to the hard and soft constraints:

$|f_i - f_j| > d_{ij}$ for each pair of interfering links $\{i, j\}$,
$|f_i - f_j| = d_{ij}$ for each pair of parallel links $\{i, j\}$,
  $f_i = p_i$ for each link i with a preassigned frequency,
  $f_i \in D_i$ for each link i.

Both variants of the problem are generalizations of the graph coloring problem and thereby are NP-hard in the strong sense (Hale 1980).

We used two sets of test instances. One set of 11 real-life instances was provided by CELAR, the Centre d'ELectronique de l'ARmement in France. They range in size between 200 and 916 links, and the values of the interference and mobility costs vary widely. Six of these instances are feasible, i.e., there exists a complete assignment of zero cost, and the minimum spectrum usage problem has to be solved. Five instances are infeasible and only the minimum interference problem is to be solved.

A second set of 14 instances was made available by the group at TUD. They were randomly generated, but preserve the structure and main characteristics of the CELAR instances. They were released at the end of the project, and the groups were given one week to report their results. We refer to them as the *surprise* instances. Six are feasible, eight are not.

In practice, many problems of the type described above have two additional properties. First, any two parallel links are subject to the same interference constraints. Second, the frequency spectrum is split up into two bands, and the distance constraints are such that, when a link is assigned a frequency in one part, its parallel mate gets one in the other. In that case, we can ignore the two-way communication aspect: A solution for the problem involving one link of every parallel pair and using one part of the spectrum is easily extended to a solution for the entire problem. In our test instances, however, the interference constraints are not symmetric, the spectrum does not consist of two distinct bands, and we have to consider both links of each parallel pair explicitly.

Our test instances do have some special structure. For each pair of parallel links $\{i, j\}$, d_{ij} is equal to a constant $\bar{d}$ ($\bar{d} = 238$ for the CELAR instances), $D_i = D_j$, and for each $f \in D_i$ there is a unique $f' \in D_i$ such that $|f - f'| = \bar{d}$. See Cabon et al. (1999) for a further analysis of the test instances.

It may seem restrictive to consider possible interference only between pairs of radio links, but this is in fact common in most models. One exception is reported by Fischetti et al. (2000).

2. APPROXIMATION AND UPPER BOUNDS

Approximation algorithms seek to obtain good solutions in a reasonable amount of time. They provide upper bounds on the minimum solution value. Most of the methods to be discussed below apply some form of neighborhood or hyperneighborhood search; one of these uses a representation related to neural networks. We will also discuss an incomplete optimization method, based on truncated tree search, and a potential reduction method, which applies an interior point algorithm and rounding techniques to a binary quadratic formulation of the problem.

2.1. Local Search

The general idea of local search is to start with an initial solution and iteratively perform small transformations of the solution in an attempt to improve the objective value. The *neighborhood* of a solution is defined as the set of all solutions to which it can be transformed in one iteration or *move*. A mapping that specifies a neighborhood for each solution is called a *neighborhood function*.

A *search strategy* specifies the way in which at each move a solution from a neighborhood is selected. The basic *iterative improvement* strategy transforms the current solution into a neighboring solution of lower cost, and stops when no better neighbor exists. It can easily be trapped in a *local optimum* of poor quality. Search strategies that are intended to overcome this deficiency include simulated annealing, tabu search, and variable-depth search.

Various hybrid forms of algorithms have been proposed, which combine local search with constructive, enumerative, or iterative techniques. There exist constructive rules that apply local search to partial solutions, combinations of local search with partial enumeration or backtracking, and nested forms of local search. In the latter case, local search is applied at several levels. For example, a neighbor obtained at one level is subjected to local search at a second level before the search at the first level is resumed.

For a comprehensive discussion of local search techniques and their applications to problems in combinatorial optimization, we refer the reader to Aarts and Lenstra (1997). We now describe the various local search approaches taken in the CALMA project.

The participants took quite different approaches. TUE emphasized the development of neighborhood functions, which were subsequently used in any of the search strategies. In this way, the underlying mechanism of traversing the solution space could be tuned, and also the various strategies could be compared. In contrast, both CERT and KCL considered a plain local search technique and used little additional information to adapt it to the problem at hand.

2.1.1. Simulated Annealing.
Simulated annealing moves from a solution to a random neighbor. An improvement is always accepted. A deterioration is accepted with a certain probability: Given a solution of value z, the probability that a neighbor of value $z' > z$ is accepted is usually given by $\exp((z - z')/T)$, where T is a control parameter that decreases during the run. The algorithm stops when T reaches a termination value.

CERT. An approach proposed by Bourret (1995) uses a binary solution representation. Each frequency–link combination gets a 0–1 decision variable x_{if} with $x_{if} = 1$ if link i is assigned frequency f and $x_{if} = 0$ otherwise. Exactly one such variable is turned on for each link:

$$\sum_{f \in D_i} x_{if} = 1 \text{ for each } i \in L. \tag{1}$$

All other constraints of the problems are relaxed, and their violation is penalized in the objective function. Indicator parameters are introduced to model the violation of the interference and mobility constraints:

$$w_{ij}^{fg} = \begin{cases} 0, & \text{if frequency } f \text{ of link } i \text{ interferes with} \\ & \text{frequency } g \text{ of link } j, \\ 1, & \text{otherwise,} \end{cases}$$

for $f \neq g$, and $w_{ii}^{ff} = 1$ if $f = p_i$, $w_{ii}^{ff} = 0$ otherwise. A *feasibility function* Φ is then defined by

$$\Phi(x) = \sum_{i, j \in L} \sum_{f \in D_i} \sum_{g \in D_j} \left(1 - w_{ij}^{fg}\right) x_{if} x_{jg}. \tag{2}$$

This function attains its minima in the feasible assignments. For the minimum interference problem, a relaxation is obtained by multiplying each term in Equation (2) by an appropriate coefficient (the cost for a soft constraint, a very large number for a hard constraint) and minimizing the resulting function subject to Equation (1). For the minimum spectrum usage problem, an *order preserving function* Ω is defined by

$$\Omega(x) = \sum_{f \in D, g \in D} \sum_{i \in L} x_{if} \sum_{j \in L} x_{jg}.$$

This is a surrogate objective function, which is intended to increase with the number of distinct frequencies used. A weighted sum of the feasibility and the order preserving functions is then minimized in this approximative approach.

For both problem types, a move in a neighborhood corresponds to changing the frequency of a single link. The chosen implementation has substantial running time and memory requirements. To cope with these difficulties, the author proposed to decompose the test instances into small subproblems and to treat these independently. In spite of various attempts to reduce the frequency domains and to discard links from the problem on heuristic grounds, the approach produced modest results.

TUE. The simulated annealing algorithms of Tiourine et al. (2000) use problem representations and neighborhood functions that are specific to the problem at hand. A graph representation of the problem is based on its similarity to certain graph coloring problems (Roberts 1989). The *interference graph* $G = (L, E)$ on the set L of links has an edge $\{i, j\} \in E$ if and only if i and j interfere. For a set τ of nonnegative integers, a τ-*coloring* is a frequency assignment satisfying $f_i \in D_i$ for all $i \in L$ and $|f_i - f_j| \notin \tau$ for all $\{i, j\} \in E$. Given a list of admissible frequencies for each link, a solution corresponds to a selection of one frequency from each list that respects the restrictions with adjacent nodes. This representation is used to enforce consistency of the domains of adjacent nodes with respect to the current assignment throughout the search; see Tsang (1993) and §3.2.

The authors developed different neighborhood functions and tested these using various search strategies. Because of the special structure of the test instances (see §1), they considered pairs of parallel links as atomic objects. In the minimum interference problem, a solution y is a neighbor of a solution x if x can be transformed into y by changing the frequency of a link with a nonzero contribution to the cost of x. This neighborhood is shown to be connected in the sense that, starting from an arbitrary solution and using moves of this type, an optimal solution can always be reached. Computational experiments were encouraging on small and medium-size problems, but the performance on larger instances was less satisfactory.

In the minimum spectrum usage problem, a neighbor of a solution x is obtained by removing a random seed frequency from the set of used frequencies. The links that were assigned this frequency are reallocated by a heuristic procedure. It first tries to use frequencies already used in x. If a link i cannot be assigned such a frequency without violating the interference constraints, forward probing is used to check if reallocation of the links interfering with i may resolve this infeasibility. Experiments with this rather wild neighborhood produced good results.

The implementation of these simulated annealing algorithms uses the cooling schedule proposed in Van Laarhoven et al. (1992). For each value of the control parameter T, a number of trials is performed, equal to the size of the largest neighborhood. T is then replaced by $T/(1 + [T \ln(1 + \Delta)/3\sigma)])$, where Δ controls the decrement rate of T and σ is the standard deviation of the solutions values generated for the current value of T.

2.1.2. Tabu Search. Tabu search always moves to the best neighbor. In this way, the cost of the solutions generated is not necessarily decreasing. To prevent the method from cycling, several recently visited solutions or the reversals of several recently performed moves are excluded or "put on the tabu list." A stopping criterion has to be defined, for example a maximum number of iterations without improvement.

KCL. Bouju et al. (1994) describe a tabu search algorithm which changes the frequency of a single link at each move. The size of the neighborhood is restricted to a certain percentage of the links that have the largest contribution to the total interference cost. When applied to the minimum interference problem, the algorithm performed quite poorly. For the minimum spectrum usage problem, they developed an extended algorithm that dynamically updates the frequency domains. Initially, each domain contains two frequencies only and the algorithm searches for an interference-free assignment. If no such solution is found, the frequency domains are extended and the search is repeated. The frequencies are added one by one in the order determined by the number of different frequency domains in which they occur. The approach shows encouraging results.

TUE. The tabu search algorithms of Tiourine et al. (2000) use the same representation and neighborhoods as their simulated annealing algorithms. They apply the backtracking mechanism of Nowicki and Smutnicki (1996) to restart the search from a neighbor of the best solution found when a stopping criterion is met. The moves for the minimum interference problem and the seed frequencies for the minimum spectrum usage problem are put on the tabu list. This algorithm worked well for the latter problem. Its poor performance for the former problem is attributed to the large size of the neighborhoods and the peculiar cost structure of the problem.

2.1.3. Variable-Depth Search. Variable-depth search was introduced as a highly problem-specific exchange algorithm for uniform graph partitioning (Kernighan and Lin 1970) and later for the traveling salesman problem (Lin and Kernighan 1973). The latter paper gives guidelines for possible extensions of the algorithm to other problems. In short, the method works as follows. Starting from an initial solution, it makes a sequence of small greedy moves. This process can be seen as the repeated application of some neighborhood function. It has to be ensured, though, that the moves within the same sequence are not reversed. Improving solutions encountered in such a sequence are registered. The sequence is typically terminated when it becomes too long or when no gain is expected from further moves. The next iteration starts from the best solution found in the sequence.

TUE. Tiourine et al. (2000) describe two algorithms of this type. Their algorithm for the minimum interference problem starts with a solution that is locally optimal with respect to moves that change the frequency of a single link. It selects a random link with a probability proportional to the cost incurred by the link. The link is assigned a random frequency from its domain. All links interfering with the new assignment are put on a list. A link is then drawn randomly from the list, it is assigned the locally best frequency from its domain, and it is in turn replaced on the list by its interfering links. No link may reenter the list during the same iteration. An iteration is terminated if an improvement is found, or otherwise randomly with a probability proportional to the deterioration of the solution value and the duration of an iteration. This strategy produced very good results, and even its average running time of 1.5 hours was quite competitive.

For the minimum spectrum usage problem, an iteration of variable-depth search starts by selecting a random used frequency. Each link that is assigned that frequency gets another used frequency from its domain. Infeasibilities resulting from this reassignment are resolved using a variant of the variable-depth search algorithm for the minimum interference problem. This algorithm performed well.

2.2. Genetic Algorithms

Genetic algorithms also apply local search, but now with *hyperneighborhoods*, where a *set* of solutions is trans-

formed into a new set of solutions. These methods usually mimic mechanisms from evolution theory and typically encode solutions as bitstrings. Starting from a *population* of *parent* solutions, at each iteration its *offspring* is determined by applying *genetic operators*. The binary *crossover* operator aims at propagating the characteristics of good solutions from one generation to the next. The unary *mutation* operator is randomly applied to offspring to ensure a diversity of solutions in the population. It is hoped that the entire process will evolve toward better solutions.

UEA (Chardaire et al. 1995, Rayward-Smith et al. 1995). An early implementation at UEA used GAmeter, their toolkit for genetic algorithms. It applies textbook binary representations, crossover and mutation operators, and produced quite poor results. The introduction of problem-specific operators and data structures significantly improved the performance. One such crossover operator tries to propagate constraints satisfied by parents down to their offspring, another is designed to preserve spectrum usage. Specialized mutation operators implement a variety of ideas that boil down to the application of standard local search to the offspring, so that the overall procedure can be viewed as bilevel search. These refinements led to good results for some of the test instances, but the overall solution quality is not uniform.

UM. The problem-specific and highly tuned genetic algorithm of Kolen (1999) appeared to be one of the most effective approximation techniques for the minimum interference problem. Doing away with the traditional binary representation, random mutations, and crossovers, he interpreted the principles underlying genetic algorithms at a conceptual level and developed sophisticated subroutines for the mutation and the crossover operators.

Solutions are represented by vectors of frequencies. At each iteration a new population of a fixed size is constructed by applying a crossover operator to pairs of solutions. A crossover produces one new solution, which is subjected to a mutation operator to ensure its local optimality. Each solution in the parent population is selected in turn for the crossover; its mate is determined randomly with a probability inversely proportional to its value.

The mutation operator applies iterative improvement, changing the frequency of a single link at each move. The crossover is somewhat more complex. For a pair of parent solutions in the population, its offspring is a solution to the original instance of the problem with the domain of each link restricted to the frequencies assigned to it in one of the parent solutions. The algorithm that competed in the CALMA project (Rayward-Smith et al. 1995) used a constructive heuristic for this problem. A later version uses a cutting plane algorithm; see §5.1 (see Kolen 1999).

This approach produced excellent results. The refinement that is to be discussed in Section 5.1 runs much faster.

2.3. Neural Networks

Abstracted from their biological background, neural networks represent *connectionist* models of computation.

Computations are performed by a network of richly inter-connected processors, each performing a relatively simple task. Each processor receives signals from its neighboring processors and calculates a certain response, which in turn is propagated through the network. The computations are performed in parallel, and eventually the network should stabilize in some state. The network is *trained* by adjusting the weights of the interconnections.

KCL. KCL obtained promising results using GENET, a generic connectionist tool that simulates neural networks on a sequential computer (Tsang and Wang 1992). Originally developed for constraint satisfaction problems, GENET was adapted by Vom Scheidt (1995) to the problem at hand.

An instance of the minimum interference problem is represented by a network, each link corresponding to a cluster of vertices, with one vertex for each frequency in its domain and with an edge between two vertices when the frequencies in question interfere. The mobility and interference costs define weights of the vertices and edges, respectively. A complete frequency assignment now corresponds to a subset of vertices, one from each cluster; its cost is the sum of the weights over the selected vertices and the edges induced by them. The global constraint that exactly one vertex must be selected from—or be *active* in—each cluster is the major departure from classical neural networks.

At each iteration, one vertex in each cluster is active. Each vertex calculates its response depending on the weights of the adjacent active vertices and the incident edges. In each cluster, the vertex with the maximum response becomes active and the other vertices become inactive. When a local maximum is reached, GENET makes it less attractive by decreasing the weights of the active vertices and of the edges between them. The entire process can be viewed as a bilevel search strategy. At one level, iterative improvement modifies the solution by turning vertices on and off. At another level, the training mechanism changes the instance by adjusting the weights. Good results are reported for the smaller instances.

Major modifications were necessary to apply GENET to the minimum spectrum usage problem. The objective function cannot be evaluated locally anymore and has to be calculated externally after each iteration. Details on how the weights are updated to discourage poor local optima are reported neither in Bouju et al. (1994) nor in Vom Scheidt (1995).

Another approach to the minimum spectrum usage problem amounts to solving a sequence of feasibility problems. Initially, each domain is restricted to two frequencies and the minimum interference problem is solved. The domains are then extended and the process is iterated until an interference-free assignment is found. Dimitropoulos (1998) suggests that the domain extension procedure may follow the same tactics as the tabu search implementation of KCL.

2.4. Incomplete Optimization

Optimization algorithms usually apply some form of tree search; see §3. By limiting the search on heuristic grounds, one may gain speed, but at the expense of losing the optimality guarantee of the solution obtained.

The *partial constraint satisfaction* algorithm of CERT (Bensana and Schiex 1995) is an example of this approach. It seeks to maximize the number of satisfied constraints. Its performance is unsatisfactory.

2.5. Potential Reduction

Kamath et al. (1990) and Karmarkar et al. (1991) proposed an interior point algorithm for binary feasibility problems. One first formulates the problem as an optimization problem by dropping the integrality conditions and introducing a nonconvex potential function, whose minimizers are feasible solutions to the original problem. An interior point method is then used to obtain approximate solutions to the new problem. This algorithm starts with a point in the interior of the feasible region and generates a sequence of interior points with decreasing objective value. For that purpose, at each iteration a descent direction for the potential function is determined using its quadratic approximation defined on an inscribed ellipsoid in the feasible region around the last generated point. The algorithm proceeds as long as a substantial reduction in the value of the potential function is found in that direction and no feasible integer solution has been generated by a rounding scheme. When a local minimum is encountered, the potential function is modified in some way and the entire process is restarted. Computation times are mainly influenced by the rate of convergence of the interior point method applied and by the density of the Hessian of the function.

TUD (Warners et al. 1997a). A creative adaptation of this technique to the radio link frequency assignment problem uses a specially structured binary quadratic formulation. Let $x_{if} = 1$ if link i is assigned frequency f and $x_{if} = 0$ otherwise. The minimum interference problem is now to

minimize $x^T Q x$,

subject to $\sum_{f \in D_i} x_{if} \geq 1$, $i \in L$,

$x_{if} \in \{0, 1\}$, $i \in L, f \in D_i$,

where Q is a matrix representing the interference and mobility costs.

For the minimum spectrum usage problem, the formulation is slightly modified so as to incorporate an upper bound on the number of frequencies used. Variables z_f are introduced, with $z_f = 1$ if frequency f is blocked and $z_f = 0$ otherwise. The objective $x^T Q x$, which contains terms $x_{if} x_{jg}$ for incompatible simultaneous assignments of f to i and of g to j, is extended to include terms $x_{if} z_f$. In this way, a solution of zero cost corresponds to an assignment without

interference and without use of blocked frequencies. A constraint $\sum_{f \in D} z_f \geqslant |D| - n$ is added to ensure that no more than n frequencies are used.

Approximate solutions to both problem types are now obtained by relaxing all integrality constraints and introducing a weighted potential function of the form $x^T Q x - \sum_j \log w_j s_j$, where s_j is the slack of the jth constraint. In contrast to the model resulting from a straightforward implementation of Karmarkar's (1991) approach, these potential functions have sparse Hessians, which allows for the use of sparse matrix techniques and hence facilitates efficient implementations.

A nice feature of the approach is that it generates many integral solutions. These are obtained while rounding the fractional solutions to the relaxed problem to integer solutions with the same value of $x^T Q x$. The availability of many alternative solutions may be useful when a secondary objective has to be taken into account.

The method obtained fairly good results but has substantial memory requirements. This is because of the size of the models, which have up to 32,000 variables. In some cases, various preprocessing routines could be applied to reduce the number of variables and large instances could be solved.

The authors extended their approach to a wider class of quadratic binary optimization problems, whose constraint matrices have specific properties (Warners et al. 1997b).

3. OPTIMIZATION AND LOWER BOUNDS

Algorithms that solve the problem to optimality must apply some form of enumeration of the solution space, using lower bounds on the minimum solution value as well as logical arguments, so as to curtail the search process. A lower bound is usually obtained by relaxing the problem, that is, by discarding some of its constraints and solving the remaining, simpler problem. We will discuss lower bounding techniques based on linear programming, graph coloring, constraint satisfaction, and 0–1 quadratic programming.

3.1. Branch-and-Bound

One way to obtain a lower bound is to take an integer linear programming formulation of the problem and to drop the integrality requirements on the variables. The bound provided by this *linear relaxation* of the problem is usually too weak. *Polyhedral lower bounds* (Aardal and Van Hoesel 1996, 1999, Nemhauser and Wolsey 1988) are obtained by identifying additional linear inequalities that strengthen the linear relaxation, and in the best case are necessary in the linear description of the convex hull of feasible solutions. If we would know the linear description completely, then we could solve the problem as a linear programming problem, which is computationally easy. Obtaining such a complete description is, however, as hard as solving the problem itself. We therefore settle for certain families of linear inequalities that define high-dimensional faces of the

part of the convex hull of solutions where we expect to find an optimal solution. Because the formulation should not grow too big, we add inequalities only if they are violated by the current fractional solution. An algorithm that at each iteration adds violated linear inequalities to the current formulation and resolves the linear programming problem based on the extended formulation is called a *cutting plane algorithm*. In *branch-and-cut*, we apply a cutting plane algorithm in every node of the tree.

Other lower bounds can be based on combinatorial arguments. Branch-and-bound algorithms generally benefit from preprocessing techniques that reduce the size of the problem instance and from heuristics that generate feasible solutions in each node of the tree.

TUD, TUE. Aardal et al. (1998) developed an optimization algorithm for the minimum spectrum usage problem based on branch-and-bound, using a *vertex packing* formulation. Information about the structure of an instance is extracted from the *interference graph* $G = (L, E)$ introduced in §2.1. We assume that the pairs of parallel links are given by $\{i, i+1\}$ with i odd.

The interference graph formulation of the problem is a graph coloring problem with the additional restrictions that not all colors are admissible for all vertices, and that certain distance requirements have to be satisfied. This formulation is not convenient to write down as an integer programming problem, but it gives important structural information that is useful in solving the problem.

A formulation that leads more naturally to a mathematical programming formulation is associated with an extended *graph of variables* $G' = (V', E')$, which is very similar to the network used by GENET; see §2.3. There is a vertex $v_{if} \in V'$ for each combination of a link $i \in L$ and a frequency $f \in D_i$. There is an edge $\{v_{if}, v_{jg}\} \in E'$ if $|f - g| \leqslant d_{ij}$. An optimal solution is a subset $S \subset V'$ such that for every link $i \in L$ exactly one vertex v_{if} belongs to S, such that S forms a *vertex packing* or *independent set* in G' (i.e., no two vertices of S are adjacent), and such that S is of minimum cardinality.

Let $x_{if} = 1$ if link i is assigned frequency f and $x_{if} = 0$ otherwise. Let $y_f = 1$ if frequency $f \in D$ is used and $y_f = 0$ otherwise. The vertex packing formulation of the problem is now as follows:

Minimize $\sum_{f \in D} y_f,$

subject to $\sum_{f \in D_i} x_{if} = 1 \quad$ for all $i \in L,$

$x_{if} + x_{jg} \leqslant 1 \quad$ for all $i, j \in L$: $|f - g| \leqslant d_{ij},$

$x_{if} + x_{jg} \leqslant 1 \quad$ for all $i, j \in L$: $j = i+1,$

$\qquad\qquad i$ odd, $|f - g| \neq \bar{d},$

$x_{if} \leqslant y_f \quad$ for all $i \in L, f \in D_i,$ \qquad\qquad (3)

$x_{if} \in \{0, 1\} \quad$ for all $i \in L, f \in D_i,$

$y_f \in \{0, 1\} \quad$ for all $f \in D.$

The constraints (3) can be converted to vertex packing constraints by using the complement $\bar{y}_f = 1 - y_f$ of the variables y_f. This substitution reformulates constraints (3) as

$$x_{if} + \bar{y}_f \leqslant 1 \quad \text{for all } i \in L, f \in D_i. \tag{4}$$

Clearly, one could extend G' by adding vertices corresponding to the variables $\bar{y}_f$ for $f \in D$, and edges corresponding to the constraints (4).

Because the domains D_i are large, the above formulation contains many variables. When applying branch-and-bound, the size of the instances and the strength of the linear relaxation of the formulation are possibly the two most crucial issues.

Preprocessing. The number of variables in the CELAR instances varies between approximately 8,000 and 32,000. To reduce the size of the instances, one can apply various preprocessing techniques. One obvious way of reducing the number of variables is to make use of the restriction that, for i odd, if link i is assigned the frequency $f \in D_i$, link $i + 1$ has to be assigned the unique frequency $f' \in D_i$ for which $|f - f'| = \bar{d}$. We can therefore replace all variables $x_{i+1,f}$, i odd, by variables $x_{if'}$. This substitution reduces the number of variables by a factor two. We refer to Aardal et al. (1998) for various other preprocessing techniques.

Polyhedral Lower Bounds. The classes of inequalities used to strengthen the linear programming relaxation of the vertex packing formulation are all variants of the general class of *clique inequalities*. A clique is a complete subgraph of an undirected graph. Note that in any clique in the interference graph G, any frequency can be used at most once. Clique inequalities have the form $\sum_{i \in C} z_i \leqslant 1$, where the variables z_i are associated with the vertices of the graph under consideration and where C is a clique of that graph. A clique inequality is known to define a facet of the vertex packing polytope if and only if the clique in question is maximal (Padberg 1973). Given the sizes of G' and of its maximal cliques, it is not realistic to derive clique inequalities from G'. We observe, however, that a clique in the much smaller interference graph translates into a clique in the graph of variables, and therefore search for maximal cliques in G. We refer to Aardal et al. (1998) for further details on the identification of violated clique inequalities and on the implementation of the branch-and-cut algorithm.

Lower Bounds from Graph Coloring. For some of the minimum spectrum usage instances a restricted *chromatic number* dominates the polyhedral bound. Consider G and relax the problem by setting $d_{ij} = 1$ for each edge $\{i, j\} \in E$. The minimum number of colors needed to color the vertices of the modified interference graph such that adjacent vertices receive different colors and each vertex i is restricted to receive a color from its domain D_i, is a lower bound on the optimal solution value. Aardal et al. (1998) describe an enumerative algorithm for determining this number.

An algorithm incorporating the above ingredients solved all feasible CELAR instances to optimality in a reasonable amount of time (Aardal et al. 1998). The implementation uses MINTO, a software package for integer linear programming. The application of this kind of approach is restricted to problems where a strong linear relaxation is available, which is not yet the case for the minimum interference problem.

3.2. Constraint Satisfaction

Constraint satisfaction is a specialized backtracking algorithm for integer feasibility problems. In the present context, we wish to find an assignment of frequencies to links respecting a number of constraints. A constraint satisfaction algorithm assigns frequencies to links in an order specified by *variable* and *value selection* rules. If a feasible solution is found, the algorithm terminates. Otherwise, it performs backtracking to the nearest active node with a feasible alternative frequency. It applies *consistency enforcing* techniques to limit the search and to detect infeasibility of a partial assignment. Two of these techniques, *forward checking* and *arc consistency*, appear to be very useful. Given a partial assignment, they seek to reduce the size of the domains of the free links. Forward checking removes all frequencies from these domains that are inconsistent with the partial assignment made. Arc consistency ensures that every frequency in the domain of a free link i has a support in the domain of another free link j if i and j are bound by a constraint. In other words, it will remove a frequency from the domain of i if it cannot be assigned to i to satisfy the constraint with j. For a detailed overview of the field of constraint satisfaction we refer the reader to Mackworth (1987) and Tsang (1993).

UM. The algorithm of Kolen and Van Hoesel (1995) solves the minimum spectrum usage problem as a sequence of feasibility problems. The number of frequencies used is bounded from above, initially by the value of the best known solution. If the problem is proved to be infeasible, then the best known solution is optimal. Otherwise, the right-hand side of the objective constraint is reduced and the algorithm is resumed. Note that the problem is only strengthened this way. The algorithm does not have to be restarted from scratch, because all nodes fathomed in the previous run will retain their status.

The algorithm uses the same preprocessing step as the polyhedral methods. The variable selection rule is determined after preprocessing. For that purpose, the variables are removed one by one from the problem in such a way that at each step, the variable involved in the least number of constraints is removed; the variable selection order is the reverse of the order of removal. The value selection rule takes a random frequency already used in the partial assignment, or otherwise the frequency present in most of the domains of the free links.

The authors also proposed to curtail the search tree by a lower bound based on the chromatic number of the underlying interference graph, thereby stressing the similarity of

their approach to depth-first branch-and-bound. They succeeded in proving optimality of solutions for all benchmarks of the minimum spectrum usage problem, albeit at the expense of considerable running times.

3.3. A 0–1 Quadratic Programming Relaxation

To obtain a lower bound for the minimum interference problem, TUE considered a relaxation that, instead of assigning a frequency to each link, decides whether or not to adhere to its preassigned frequency. For this relaxation, it is easy to define an objective function that covers most of the incurred costs and is subject to integrality constraints only. Denote the interference cost of a pair of links $\{i, j\}$ by K_{ij} if i and j are set to their preassigned frequencies:

$$K_{ij} = \begin{cases} c_{ij}, & \text{if } |p_i - p_j| \leqslant d_{ij}, \\ 0, & \text{otherwise}, \end{cases}$$

and by K_{ifj} if i is set to f while j is set to its preassigned frequency:

$$K_{ifj} = \begin{cases} c_{ij}, & \text{if } |f - p_j| \leqslant d_{ij}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $x_i = 1$ if link i is set to its preassigned frequency and $x_i = 0$ otherwise. The relaxation is then to

$$\text{minimize } \sum_{i, j \in L} K_{ij} x_i x_j + \sum_{i \in L} m_i (1 - x_i)$$

$$+ \sum_{i \in L} (1 - x_i) \min_{f \in D_i \setminus \{p_i\}} \sum_{j \in L \setminus \{i\}} K_{ifj} x_j,$$

subject to $x_i \in \{0, 1\}$ for all $i \in L$.

We will interpret the objective function for a link k. If the decision is made not to use the preassigned frequency p_k, in other words if $x_k = 0$, then the second and the third term

occur for k. The second term is simply the mobility cost of k. The third term is the minimum cost of interference that occurs between k and all other links that are set to their preassigned frequencies. Suppose now that k is set to its preassigned frequency: $x_k = 1$. The first term gives the cost of interference between k and all other links fixed to their preassigned frequencies. The variable x_k is also considered in the third term, determining the cost of the cheapest alternative frequency for each link whose frequency is different from its preassigned one. The only cost that is not covered by this objective function concerns the interference between links neither of which is set to its preassigned frequency. Because all costs in the problem are nonnegative, an optimal solution to the relaxation yields a lower bound for the minimum interference problem.

This approach works well if, first, a significant number of links have a preassigned frequency and at least some of these cannot be changed, and second, if the mobility costs are not much smaller than the interference costs. For the few instances that satisfy these assumptions, a preprocessing technique and an efficient enumeration scheme incorporated in a branch-and-bound framework were used to obtain lower bounds of a reasonable quality. See Tiourine et al. (2000) for details.

4. COMPUTATIONAL COMPARISON

Tables 1 and 2 present the computational results obtained for the two sets of test instances mentioned in §1. For each approach, we indicate the quality of the reported solutions and an average running time for the classes of feasible and infeasible instances. The reader is invited to compare these results with the qualifications given in the text.

The list of computer equipment used should help to correct running times for differences in hardware performance. However, for a balanced comparison between the running

Table 1. Computational results for CELAR instances.

| Method | Group | Minimum Spectrum Usage | | | | | | | Minimum Interference | | | | | |
		1	2	3	4	5	11	Time	6	7	8	9	10	Time
SA (simulated annealing)	TUE	2	0	0	0	0	2	1	7%	65%	5%	0%	0%	310
TS (tabu search)	TUE	2	0	0	0	—	0	5	—	—	—	—	—	—
VDS (variable-depth search)	TUE	2	0	0	0	0	2	6	4%	0%	14%	0%	0%	85
SA (simulated annealing)	CERT	4	0	0	0	—	10	41	44%	1299%	70%	2%	0%	42
TS (tabu search)	KCL	2	0	0	0	0	2	40	170%	1804%	566%	8%	1%	111
GENET (see §2)	KCL	0	0	0	0	0	2	2	14%	27%	40%	—	—	20
GA (genetic algorithm)	UEA	6	0	2	0	i	10	24	1%	386%	134%	3%	0%	120
GA (genetic algorithm)	UM	—	—	—	—	—	—	—	1%	0%	0%	0%	0%	hrs§
PCS (partial constraint satisfaction)	CERT	4	0	6	0	0	—	28	86%	2563%	246%	47%	12%	6
PR (potential reduction)	TUD	0	0	2	0	0	—	3+	29%	—	—	4%	1%	10+
BC (branch-and-cut)	TUD, TUE	0*	0*	0*	0*	0*	0*	10+	—	—	—	—	—	—
CS (constraint satisfaction)	UM	0*	0*	0*	0*	0*	0*	hrs	—	—	—	—	—	—
0–1 QP (0–1 quadratic programming)	TUE	—	—	—	—	—	—	—	—	—	—	−5%	−1%	hrs
Best solution		16*	14*	14*	46*	792*	22*		3389†‡	343592	262	15571†	31516†	

Table 2. Computational results for surprise instances.

Method	Group	Minimum Spectrum Usage									Minimum Interference								
		1	2	3a	4a	8	9	10	14	Time	3	4	5	6	7	11	12	13	Time
SA	TUE	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
TS	TUE	0	0	0	0	2	4	4	2	3	—	—	—	—	—	—	—	—	—
VDS	TUE	—	—	—	—	—	—	—	—	—	0%	0%	1%	2%	0%	14%	0%	10%	63
SA	CERT	4	4	—	—	—	—	—	—	22	28%	—	—	127%	23%	—	—	—	81
TS	KCL	0	2	—	—	6	4	—	4	28	—	—	409%	20%	303%	3034%	186%	1911%	min
GENET	KCL	0	0	—	—	4	4	—	—	5	—	—	—	—	—	—	—	—	—
GA	UEA	2	2	i	i	—	10	i	6	9	156%	346%	33%	288%	39%	884%	29%	387%	20
GA	UM	—	—	—	—	—	—	—	—	—	—	—	0%	5%	0%	—	0%	125%	hrs§
PCS	CERT	6	2	0	0	—	12	0	14	6	—	—	3305%	919%	293%	2219%	217%	827%	5
PR	TUD	0	0	—	—	—	12	—	2	5	—	—	49%	—	—	—	—	—	20
BC	TUD, TUE	0*	0*	—	—	—	16	—	18	min	—	—	—	—	—	—	—	—	—
CS	UM	0*	0*	0*	0*	—	0*	0*	2	hrs	—	—	—	—	—	—	—	—	—
0–1 QP	TUE	—	—	—	—	—	—	—	—	—	—	—	—	—	−53%	—	—	—	min
Best solution		18*	14*	380*	394*	18‡	18*	394*	8†‡		39	46	221	4123†‡	4324	3080‡	11827	10110‡	

Minimum spectrum usage		Average running times		Computer hardware	
2	number of frequencies above best known solution	41	minutes of computation time	CERT	SUN SPARC 10
i	infeasible solution is reported	min	minutes of computation time	TUD	HP9000/720
		hrs	hours of computation time	TUE	SUN SPARC 4
Minimum interference		+	preprocessing time is not included	KCL	DEC Alpha 3000 (130 MHz)
6%	deviation from best known solution	§	improved version (Kolen 1999)		
			runs in minutes	UM (GA)	DEC OSF/1 AXP
				UM (CS)	PC 486
—	method not applicable, or no solution reported			UEA	DEC Alpha (133 MHz)
*	optimality proved within CALMA project				
†	optimality proved after CALMA project (Koster et al. 1999, Pasechnik 1998)				
‡	best known solution improved after CALMA project (Kolen 1999, Pasechnik 1998, Warners 1998)				

times needed by the different methods more detailed information about the relative speeds of the machines used would be needed.

5. FURTHER DEVELOPMENTS

The literature reports on the design of several local search algorithms for variants of the problem considered in this paper. For example, Castelino et al. (1996) compare implementations of iterative improvement, tabu search and genetic algorithms for a combat net radio link frequency assignment problem, and Hao et al. (1998) use tabu search to attack frequency assignment problems occurring in mobile radio networks. Also worth mentioning are the papers by Hurley et al. (1997), describing software for a range of frequency assignment problems, and by Allen et al. (1999), focusing on lower bounds.

Below, we will briefly describe further developments inspired by the CALMA project. We recommend the FAP website (Eisenblätter and Koster 2000), where references on frequency assignment problems are maintained.

5.1. Approximation: Genetic Algorithms

Crisan and Mühlenbein (1998) successfully applied evolutionary search to some of the easier minimum spectrum usage instances. The approach is essentially an efficient implementation of a standard genetic algorithm.

Kolen (1999) further refined the genetic algorithm of UM (see §2.2) by developing an optimization routine for the crossover. This sort of gene engineering method constructs an optimal offspring by solving an integer linear programming formulation of the crossover problem using a cutting plane algorithm (Kolen 1999). The cutting planes that are added to a linear relaxation of the problem correspond to *three-cycle inequalities*. The algorithm terminates when an integer solution is obtained or no further violated inequalities are found. In the latter case, the remaining problem is solved by CPLEX, a commercial branch-and-bound code. Essential for the success of this approach is the extensive preprocessing applied to the integer programming formulation so as to eliminate dominated links and redundant constraints and to reduce the domains.

5.2. Approximation: Guided Local Search

Guided local search is a bilevel search strategy of the sort embedded in GENET (see §2.3). The objective function is extended by penalties associated with so-called *features* of a solution. The penalties are only charged for those features that are *active*. The penalties are initialized at zero, and as soon as the search algorithm reaches a local optimum,

the penalties corresponding to some of the more expensive active features are increased, in the hope that the search process will escape from the local optimum. For the frequency assignment problem, a feature is for instance a certain soft constraint, and it is active if the constraint is violated.

Various types of search engines can be used within this framework. Voudouris and Tsang (1998) use a fast tabu search routine, and obtain results that are very close to the best results ever reported. The approach seems to be generic in the sense that it is applicable to a range of other partial constraint satisfaction problems. Lau and Tsang (1998) used a genetic algorithm and were less successful in terms of both solution quality and running time.

5.3. Approximation: Gradient Descent

Warners (1998) describes a gradient descent method for the minimum spectrum usage problem, which is faster than the potential reduction method of TUD (see §2.5). The two algorithms differ in the way a descent direction is determined. Whereas the older method uses a computationally involved technique based on second-order derivatives, the new method confines itself to the use of a gradient.

Pasechnik (1998) further developed this approach. For surprise instance 14, he found a solution of value 8, matching a known lower bound. Both Pasechnik (1998) and Warners (1998) improved the best known solution for surprise instance 8.

5.4. Approximation: Ant Colonies

Maniezzo et al. (1999) and Maniezzo and Carbonaro (2000) applied yet another breed from the family of heuristics. The ant colony paradigm is basically a randomized constructive approach, where the way in which a partial solution is extended depends on the immediate cost incurred and on the experience gathered during previous attempts. The authors only considered the minimum interference objective. They relaxed the problem by allowing equality in the interference constraints, which makes it hard to assess the approach in the present context.

5.5. Optimization: Tree Decomposition

Koster et al. (1999) recently proposed a *dynamic programming* algorithm for the minimum interference problem. It decomposes the problem into smaller subproblems and extends these gradually to the overall problem. The decomposition is based on the observation that the optimal choice of a frequency for a link only depends on the frequencies assigned to the links interfering with it. In general, if S is a separating vertex set of the interference graph G and if an optimal assignment for S is known, then the optimal assignments for all connected components separated by S only depend on S and can be computed independently of each other. The problem has to be solved for every feasible assignment for S, because an optimal assignment for S is not known beforehand.

The algorithm starts by computing a *tree decomposition* of G (Robertson and Seymour 1986). It yields a cover of G by a set of subgraphs corresponding to the nodes in a tree. The vertices of such a subgraph form a separating vertex set. Two nodes in the tree are connected via a path if their subgraphs intersect. Based on this decomposition, the initial subproblems and the order in which they are extended are determined. The initial subproblems are given by the subgraphs corresponding to the leaf nodes in the tree decomposition. Subsequently, the subproblems are extended by adding new nodes to them or by merging two subproblems.

All feasible solutions are evaluated for each subproblem. Extensive preprocessing is applied to reduce its size, and bounds are used to curtail its solution space. An upper bound for a subproblem is calculated by subtracting from the best known overall upper bound the values of the lower bounds of the solved subproblems that are vertex disjoint from the given one. Similarly, a lower bound is derived from the lower bounds of the subproblems that are completely contained in the given subproblem.

This algorithm has been successfully applied to the CELAR instances, solving three of them to optimality and obtaining strong lower bounds for the others.

6. CONCLUSIONS

Radio link frequency assignment problems form a relatively new class of practical problems, to which many of the techniques of combinatorial optimization can be applied. The algorithmic work by the six groups involved in the CALMA project leads to the following conclusions.

(1) The instances of the minimum interference problems appear to be harder than those of the minimum spectrum usage problem. Each class contains its notorious instances. Most (but not all) algorithms developed perform better on the latter class.

(2) Standard *local search* has benefits for everyone. On one hand, straightforward codes that incorporate little structural information give reasonable solutions in moderate running times. On the other hand, more sophisticated implementations that employ problem-dependent information in their neighborhood function and search strategy produce better results, often in less time.

(3) With *genetic algorithms*, the distinction becomes more marked. The approach must be tuned to the problem at hand to make it work, and then it can work very well.

(4) For methods based on (hyper)neighborhood search, it is worthwhile to investigate *hybrid variants*, in particular, those that embody some form of bilevel search.

(5) The *potential reduction* method is a new element in the array of approximation algorithms. It evidently has great potential.

(6) In case of *optimization*, the general and tailored approaches are drawn even farther apart. Complete enumeration of all possible solutions is obviously out of the question. In order to find provably optimal solutions,

algorithms must incorporate highly specialized elimination rules, based, e.g., on polyhedral techniques or consistency arguments.

(7) Branch-and-bound and constraint satisfaction are two sides of the same coin. Exploring the combination of concepts and techniques from *mathematical programming* and *artificial intelligence* is a promising topic of investigation.

(8) Overall, there is a strong positive correlation between the amount of problem-specific information used, the extent to which mathematical insight is exploited, the development and implementation effort required, and the quality of the results obtained.

On the basis of a broader experience, we venture to suggest that the validity of our conclusions is not restricted to the radio link frequency assignment problem but extends to many difficult combinatorial problems arising in planning and design.

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