ABM: Minecraft Redstone with Graph and Set Theory

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1 Notation

Standard

$\implies$ implies

$\iff$ if and only if

$\varnothing$ empty set

$\cup$ union

$\cap$ intersection

$A \subseteq B$ improper subset

$A \to B$ maps to

$A \setminus B$ difference of sets

$A \coloneqq B$ is defined to be

$A \times B$ Cartesian product (see Definition 3.3)

(a, b) ordered pair (see Definition 3.2)

$(n_1, n_2, \ldots, n_n)$ tuple (see Definition 3.2)

$\mathcal{P}(A)$ power set

$f : A \to B$ function mapping (see Definition 3.4)

B^A set of functions (see Definition 3.6)

π_1, π_2 coordinate projection functions (see Definition 3.5)

$F|_C$ function restriction (see Definition 3.7)

1_A indicator function (see Definition 3.8)

$\mathbb{N}_0$ set of natural numbers including zero

$\mathbb{Z}^+$ set of positive integers not including zero(see Definition 3.1)

$\mathbb{Z}^-$ set of negative integers not including zero (see Definition 3.1)

$\mathbb{Z}_0^-$ set of negative integers including zero (see Definition 3.1)

Non-Standard

max function mapping sets to their largest element (see Definition 3.9)

S set of possible signal strengths (see Definition 4.1)

S^+ set of positive signal strengths (see Definition 4.1)

S_0 set of zero signal strength (see Definition 4.1)

C set of supported redstone components (see Definition 4.3)

Φ_{G_r} function mapping C to every state update function (see Definition 4.5)

Ω function mapping C to every output function (see Definition 4.5)

Ψ_G propagation function (see Definition 4.2)

I_{G_r} input function (see Definition 4.6)

2 Introduction

In this model, we attempt to formalize Minecraft Redstone using only Zermelo-Fraenkel set theory with the axiom of choice.

3 Foundations

Definition 3.1. *(Number Set Notation)*

$$\mathbb{Z}^+ \coloneqq \mathbb{N}_0 \setminus \{0\} \tag{1}$$
$$\mathbb{Z}^- \coloneqq \{n \in \mathbb{Z} \mid n < 0\} \tag{2}$$
$$\mathbb{Z}_0^- \coloneqq \mathbb{Z}^- \cup \{0\} \tag{3}$$

Definition 3.2. *(Kuratowski Pair)*
Let a and b be any elements.
Then the ordered pair between them

$$(a, b) \coloneqq \{\{a\}, \{a, b\}\} \tag{4}$$

where for any elements $a_1, a_2, \ldots, a_n$,

$$(a_1, (a_2, \ldots, (a_{n-1}, a_n))) \coloneqq (a_1, a_2, \ldots, a_n) \tag{5}$$

and

$$((a_1, \ldots, (a_{n-2}, a_{n-1})), a_n) \coloneqq (a_1, a_2, \ldots, a_n) \tag{6}$$

Definition 3.3. *(Cartesian Product)*
Let A and B be any sets.
Then

$$A \times B \coloneqq \{(a, b)) \mid \exists a \in A, \exists b \in B\} \tag{7}$$

where for any sets $A_1, A_2, \ldots A_n$,

$$A_1 \times A_2 \times \cdots \times A_n \coloneqq \{(a_1, a_2, \ldots, a_n) \mid \exists a_1 \in A_1, \exists a_2 \in A_2, \ldots \exists a_n \in A_n\} \tag{8}$$

Definition 3.4. *(Functions)*
Let A and B be any sets.
Then a function $f : A \to B$ if

$$f \subseteq A \times B \tag{9}$$
$$\forall a \in A, \exists b \in B((a, b) \in f) \tag{10}$$

and

$$\forall a \in A \forall b, b' \in B((a, b) \in f \wedge (a, b') \in f \implies b = b') \tag{11}$$

Now let a function $f : A \to B$, then $f(x)$ for any $x \in A$ is defined such that

$$f(x) = y \iff (x, y) \in f \tag{12}$$

Next, let A and B be any sets where $A = A_1 \times A_2 \times \cdots \times A_n$, and let a function $f : A \to B$.
Then $f(x_1, x_2, \ldots, x_n)$ where $(x_1, x_2, \ldots, x_n) \in A$ is defined such that

$$f(x_1, x_2, \ldots, x_n) = f(x), x \in A \tag{13}$$

Definition 3.5. *(Coordinate Projection)*
Let A and B any sets.
Then the function

$$\pi_1 : A \times B \to A \tag{14}$$

where

$$\pi_1(a,b) = \bigcup\bigcap(a,b) \tag{15}$$

Furthermore, given A and B are any sets, then the function

$$\pi_2 : A \times B \to B \tag{16}$$

where

$$\pi_2(a,b) = \bigcup\{a \in \bigcup(a,b) \mid a \notin \bigcap(a,b)\} \tag{17}$$

Definition 3.6. *(Set of Functions)*
Let A and B be any sets.
Then

$$B^A \coloneqq \{f \in \mathcal{P}(A \times B) \mid f : A \times B\} \tag{18}$$

Definition 3.7. *(Function Restriction)*
Let A, B, and C any sets where $C \subseteq A$, and let a function $f : A \to B$.
Then the function

$$f|_C : C \to B \tag{19}$$

where

$$f|_C(x) = f(x), x \in C \tag{20}$$

Definition 3.8. *(Indicator Function)*
Let X and A be any 2 sets where $A \subseteq X$.
Then the function

$$1_A : X \to \{0,1\} \tag{21}$$

where

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{22}$$

Definition 3.9. *(Max)*
Let Y be any non-empty totally ordered set.
Then the function

$$max : \mathcal{P}(Y) \to Y \tag{23}$$

where

$$\forall X \in \mathcal{P}(Y), \exists x \in X(max(X) = x \iff \forall y \in X(x \geq y)) \tag{24}$$

Definition 3.10. *(Vertices)*
Let V be a set.
Then V is a set of vertices if

$$V \neq \varnothing \tag{25}$$

Definition 3.11. *(Directed Edges)*
Let V be any set of vertices.
Then E is a set of directed edges if

$$E \subseteq (V \times V) \tag{26}$$

Definition 3.12. *(Graph)*
Let V be any set of vertices, and let E be any set of directed edges on V.
Then G is a digraph if

$$G = (V, E) \tag{27}$$

4 Redstone Model

Definition 4.1. *(Signal Sets)*

$$S \coloneqq \{0, 1, 2, ..., 15\} \tag{28}$$
$$S^{+} \coloneqq \{x \in S \mid x > 0\} \tag{29}$$
$$S_0 \coloneqq \{0\} \tag{30}$$

Definition 4.2. *(Propagation)*
Let $G = (V, E)$ be any digraph.
Then the function

$$\Psi_G : V \to \mathcal{P}(V) \tag{31}$$

where

$$\Psi_G(v) = \{v \in V \mid \exists (u, v) \in E\}$$

Definition 4.3. *(Components)*

$$C \coloneqq \{R_{1_c}, R_{2_c}, R_{3_c}, R_{4_c}, T_c\} \tag{32}$$

where

- R_{n_c} *for any* $n \in 1, 2, 3, 4$ *corresponds to a repeater with a delay of* n *ticks.*
- T_c *corresponds to a redstone torch.*

Definition 4.4. *(Redstone Digraph)*
Let

- $G = (V, E)$ *be any digraph*
- $\Sigma : V \times \mathbb{N}_0 \to \mathbb{N}_0$
- $\lambda : V \to C$
- $\mu : E \to S \times S$

where

- Σ *maps vertices and ticks to a numeric state at that tick.*
- λ *maps vertices to components.*
- μ *maps edges to the signal droppoff between the tail and head vertices where the output of the tail vertex is the input to the state of the head vertex.*
  - *note: a droppoff of* 15 *implies there is no signal going into the head vertex (i.e. its disconnected, and therefore floating, which defaults to zero in Minecraft), while a dropoff of* 0 *implies the vertices as components are touching,*

Then G_r *is a redstone digraph if*

$$G_r = (G, \Sigma, \lambda, \mu) \tag{33}$$

Definition 4.5. *(Behavior Functions)*
Let $G_r = (G, \Sigma, \lambda, \mu)$ *be any redstone digraph where* $G = (V, E)$*. Then the function*

$$\Phi_{G_r} : C \to \mathbb{N}_0^{V \times S \times \mathbb{N}_0 \times \mathbb{Z}^+} \tag{34}$$

and the function

$$\Omega : C \to S^{\mathbb{N}_0} \tag{35}$$

Where

- Φ_{G_r} *maps components to state "update" functions.*
  - *note: the update functions take in*
    - *the vertex of the component.*
    - *the back input.*
    - *the numeric internal state of the component at some tick* t*.*
    - *some tick* t*,*
      - *note: the tick* t *is a positive integer because the state at* $t = 0$ *should be manually defined.*
  - *note: the update functions map the input to the internal state of that vertex at the next tick.*
- Ω *maps components to "output" functions.*
  - *note: the output function takes in*
    - *the numeric internal state of the component*
  - *note: the output function maps the input to what output would be based on its state, i.e. it calculates the output for the current tick.*

Definition 4.6. *(Input)*
Let $G_r = (G, \Sigma, \lambda, \mu)$ *be any redstone digraph where* $G = (V, E)$. *Then the function*

$$I_{G_r} : V \times \mathbb{Z}^+ \times \{1, 2\} \to S \tag{36}$$

where

$$I_{G_r}(v, t, i) = \begin{cases} 0 & \textit{if } \Psi_G(v) = \varnothing \\ max(\{\Omega(\lambda(u))(\Sigma(u, t)) - \pi_1(\mu(u, v)) \in S \mid u \in \Psi_G(v)\} \cup \{0\}) & \textit{if } i = 0 \wedge \Psi_G(v) \neq \varnothing \\ max(\{\Omega(\lambda(u))(\Sigma(u, t)) - \pi_2(\mu(u, v)) \in S \mid u \in \Psi_G(v)\} \cup \{0\}) & \textit{otherwise} \end{cases} \tag{37}$$

Note

- v *is the vertex to get the input of.*
- t *is time in ticks to get the input at.*
- i *is used to multiplex between the back input and side input of a vertex.*

Definition 4.7. *(State Update)*
Let $G_r = (G, \Sigma, \lambda, \mu)$ *be any redstone digraph where* $G = (V, E)$.
Then, for any vertex $v \in V$ *at an arbitrary tick* $t \in \mathbb{N}_0$,

$$\Sigma(v, t + 1)|_{V \times \mathbb{Z}^+} = \Phi_{G_r}(\lambda(v))(v, I_{G_r}(v, t, 1), \Sigma(v, t), t) \tag{38}$$

Note: the state mapping isn't defined for $t = 0$ *to allow the implementation to define a custom mapping for all vertices at* $t = 0$.

5 Redstone Component Output Functions

Definition 5.1. *(Redstone Torch)*
Let $G_r = (G, \Sigma, \lambda, \mu)$ *be any redstone digraph where* $G = (V, E)$. *Next let*

$$\phi(v, i, \sigma, t) = i \tag{39}$$

and

$$\omega(\sigma) = 15 * 1_{S_0}(\sigma) \tag{40}$$

Then $\Phi_{G_r}(T_c) = \phi$ *and* $\Omega(T_c) = \omega$.

Definition 5.2. *(Repeater(s))*
Let

- $G_r = (G, \Sigma, \lambda, \mu)$ *be any redstone digraph where* $G = (V, E)$.
- $t \in \mathbb{N}_0$.
- $i_i = max(\{I_{G_r}(u, t, 2) \mid u \in \Psi_G\} \cup \{0\})$

Next, for $n \in \{1, 2, 3, 4\}$, *let*

$$N_n(v, i, \sigma, t) = \begin{cases} 1 & \textit{if } (i \in S^+) \wedge [(\sigma \geq 2n) \vee (\sigma = 0)] \\ \sigma + 1 & \textit{if } 0 < \sigma < n \\ \sigma + 1 & \textit{if } (i \in S_0) \wedge (n \leq \sigma < 2n) \\ n & \textit{if } (i \in S^+) \wedge (\sigma = 2n - 1) \\ 0 & \textit{otherwise} \end{cases} \tag{41}$$

$$L_n(v, i, \sigma, t) = \begin{cases} n & \textit{if } n <= \sigma < 2n \\ 0 & \textit{otherwise} \end{cases} \tag{42}$$

$$\phi_n(v, i, \sigma, t) = \begin{cases} L_n(v, i, \sigma, t) & \textit{if } (i_i > 0) \vee (I_{G_r}(v, t, 2) > 0) \\ N_n(v, i, \sigma, t) & \textit{otherwise} \end{cases} \tag{43}$$

and

$$\omega_n(\sigma) = \begin{cases} 15 & \textit{if } n \leq \sigma < 2n \\ 0 & \textit{otherwise} \end{cases} \tag{44}$$

Then, $\Phi_{G_r}(R_{n_c}) = \phi_n$ *and* $\Omega(R_{n_c}) = \omega_n$*.*

6 Applications

Proof. Clock proof

- Let $G_r = (G, \Sigma, \lambda)$ be a redstone digraph where $G = (V, E)$.
- Let $V = \{t_1, t_2, t_3\}$ and $E = \{(t_1, t_2), (t_2, t_3), (t_3, t_1)\}$.
- Let $\Sigma(t_1, 0) = 0, \Sigma(t_2, 0) = 15, \Sigma(t_3, 0) = 0$.
- Let $\forall v \in V(\lambda(v) = T_c)$.
- Let $\mu(v_1, v_2) = (2, 15)$, $\mu(v_2, v_3) = (2, 15)$, $\mu(v_3, v_1) = (2, 15)$.

Where the initial state, components, and dropoff were taken directly from a 3 torch clock circuit in Minecraft, assuming that $t = 0$ is the moment the last piece of redstone was placed to complete the circuit, and that the circuit stabalized (no components changed state between some arbitrary t and $t + 1$) before the circuit was completed.
Next let $v = t_3$ and $t = 1$.
Then

$$\begin{aligned}
\Sigma(v, t+1) &= \Phi_{G_r}(\lambda(v))(v, I_{G_r}(v, t, 1), \Sigma(v, t), t) \implies \\
\Sigma(v, t) &= \Phi_{G_r}(\lambda(v))(v, I_{G_r}(v, t-1, 1), \Sigma(v, t-1), t-1) \implies \\
\Sigma(t_3, 1) &= \Phi_{G_r}(\lambda(t_3))(v_3, I_{G_r}(t_3, 1-1, 1), \Sigma(t_3, 1-1), 1-1) \\
&= \Phi_{G_r}(\lambda(t_3))(v_3, I_{G_r}(t_3, 0, 1), \Sigma(t_3, 0), 0) \\
&= \Phi_{G_r}(T_c)(t_3, I_{G_r}(t_3, 0, 1), 0, 0) \\
&= \Phi_{G_r}(T_c)(t_3, max(\{\Omega(\lambda(t_2))(\Sigma(t_2, 0)) - \pi_1(\mu(t_2, t_3))\} \cup \{0\}), 0, 0) \\
&= \Phi_{G_r}(T_c)(t_3, max(\{\Omega(T_c)(15) - \pi_1(2, 15)\} \cup \{0\}), 0, 0) \\
&= \Phi_{G_r}(T_c)(t_3, max(\{0 - 2\} \cup \{0\}), 0, 0) \\
&= \Phi_{G_r}(T_c)(t_3, 0, 0, 0) \\
&= 0
\end{aligned}$$

Next let $v = t_3$ and $t = 2$.
Then

$$\begin{aligned}
\Sigma(t_3, 2) &= \Phi_{G_r}(\lambda(t_3))(t_3, I_{G_r}(t_3, 2-1, 1), \Sigma(v_3, 2-1), 2-1) \\
&= \Phi_{G_r}(\lambda(t_3))(t_3, I_{G_r}(t_3, 1, 1), \Sigma(v_3, 1), 1) \\
&= \Phi_{G_r}(T_c)(t_3, max(\{\Omega(\lambda(t_2))(\Sigma(t_2, 1)) - \pi_1(\mu(t_2, t_3))\} \cup \{0\}), 0, 1)
\end{aligned}$$

where

$$\begin{aligned}
\Sigma(t_2, 1) &= \Phi_{G_r}(\lambda(t_2))(t_1, I_{G_r}(t_2, 1-1, 1), \Sigma(t_2, 1-1), 1-1) \\
&= \Phi_{G_r}(T_c)(t_2, I_{G_r}(t_2, 0, 1), \Sigma(t_2, 0), 0) \\
&= \Phi_{G_r}(T_c)(t_2, max(\{\Omega(\lambda(t_1))(\Sigma(t_1, 0)) - \pi_1(\mu(t_1, t_2))\} \cup \{0\}), 15, 0) \\
&= \Phi_{G_r}(T_c)(t_2, max(\{\Omega(T_c)(15) - \pi_1(2, 15)\} \cup \{0\}), 15, 0) \\
&= \Phi_{G_r}(T_c)(t_2, max(\{0 - 2\} \cup \{0\}), 0, 1), 15, 0 \\
&= \Phi_{G_r}(T_c)(t_2, 0, 15, 0) \\
&= 0
\end{aligned}$$

then

$$\begin{aligned}
&\Phi_{G_r}(T_c)(t_3, max(\{\Omega(\lambda(t_2))(\Sigma(t_2, 1)) - \pi_1(\mu(t_2, t_3))\} \cup \{0\}), 0, 1) = \\
&= \Phi_{G_r}(T_c)(t_2, max(\{\Omega(T_c)(15) - \pi_1(2, 15)\} \cup \{0\}), 0, 1) \\
&= \Phi_{G_r}(T_c)(t_2, max(\{0 - 2\} \cup \{0\}), 0, 1) \\
&= \Phi_{G_r}(T_c)(t_2, 0, 0, 1) \\
&= \Phi_{G_r}(T_c)(t_2, 0, 0, 1) \\
&= 15
\end{aligned}$$

The output of v_3 at $t = 2$ is then

$$\begin{aligned}
\Omega(\lambda(t_3))(\Sigma(t_3, 2)) &= \Omega(T_c)(15) \\
&= 0
\end{aligned}$$

then for $t = 1$

$$\begin{aligned}
\Omega(\lambda(t_3))(\Sigma(t_3, 1)) &= \Omega(T_c)(0) \\
&= 15
\end{aligned}$$

and finally for $t = 0$

$$\begin{aligned}
\Omega(\lambda(t_3))(\Sigma(t_3, 0)) &= \Omega(T_c)(\Sigma(t_3, 0)) \\
&= 15
\end{aligned}$$

Hence, it is then proven that for t_3,

$t = 0$	$t = 1$	$t = 2$
15	15	0

$\square$