Pure Maths Consolidation Notes

Jack Maguire

Contents

Part I

Book 1

Chapter 3

Inequalities

3.1 Basics

Very similar to solving a normal equation with one caveat - every time you multiply by -1, you need to flip the sign.

Example 3.1.1 $5 - 3x \geq 21$

$$\begin{aligned} 5 - 3x &\geq 21 \\ -3x &\geq 16 \\ 3x &\leq -16 \\ x &\leq -\frac{16}{3} \end{aligned}$$

Example 3.1.2 $17 + x \leq 32 + 3x \leq 21 + x$

$$\begin{aligned} 17 + x \leq 32 + 3x &\leq 21 + x \\ 17 \leq 32 + 2x &\leq 21 \\ -15 \leq 2x \quad &\leq -11 \\ -\frac{15}{2} \leq x \quad &\leq -\frac{11}{2} \end{aligned}$$

3.2 Set Notation

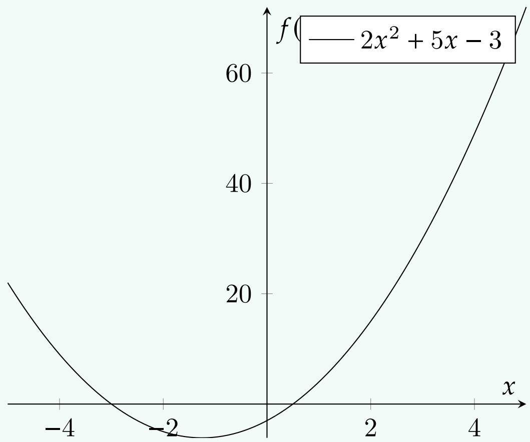
Whilst we can use $\leq$ and the other signs, we can also use set notation, which makes some easier, and is especially useful for quadratics.

$$\begin{aligned} x > 4 \quad &\rightarrow \quad \{x : x < 4\} \quad \equiv \quad x \in (4, \inf) \\ -15 \leq x \leq -11 \quad &\rightarrow \quad \{x : -15 \leq x \leq -11\} \quad \equiv \quad x \in [-15, -11] \end{aligned}$$

3.3 Quadratics

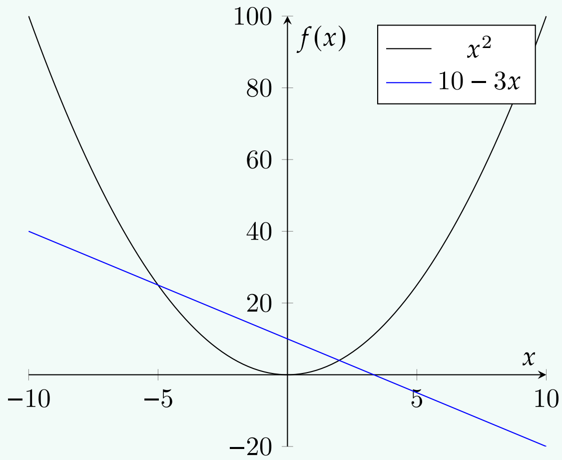
The difficulty comes with quadratics which have multiple x-solutions. We need to draw a graph, and then check which way around which we should give our answer - a single set, or a union of 2.

Example 3.3.1 $2x^2 + 5x - 3 < 0$



$$2x^2 + 5x - 3 < 0$$
$$(2x - 1)(x + 3) < 0$$
$$-3 < x < \frac{1}{2}$$
$$\left\{x : -3 < x < \frac{1}{2}\right\}$$

Example 3.3.2 $x^2 > 10 - 3x$



$$x^2 + 3x - 10 > 0$$
$$(x + 5)(x - 2) > 0$$
$$\{x : x < -5\} \cup \{x : x > 2\}$$

Chapter 5

Coordinate Geometry

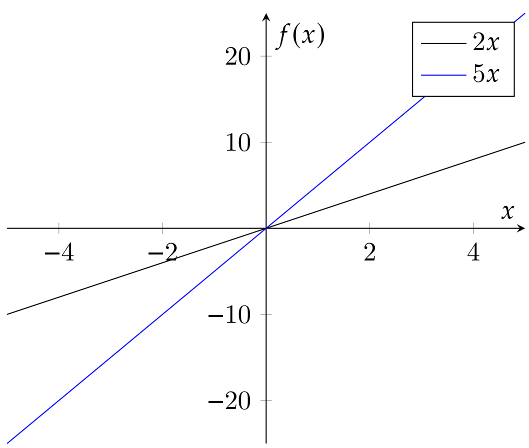
A line is a function which is straight, and they have operations that can be performed on them. If the function is not of first order, then they can have multiple gradients. All lines follow $y = mx + c$, where m is the gradient, and c is the y-intercept.

5.1 Characteristics of Lines

5.1.1 Gradient

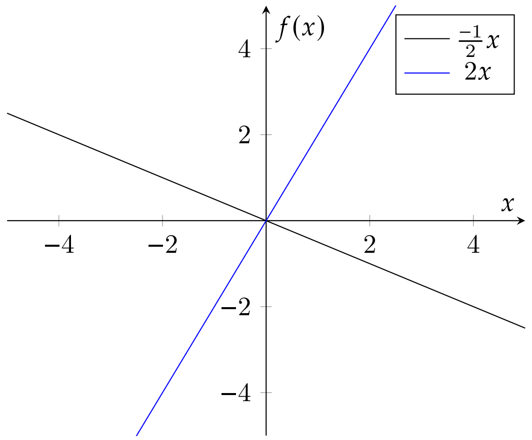
Definition 5.1.1: Gradient

The gradient of a line represents how shallow or deep it is.



Definition 5.1.2: Perpendicular Lines

One line is perpendicular to another if $m_a = \frac{-1}{m_b}$, ie they meet at $90°$



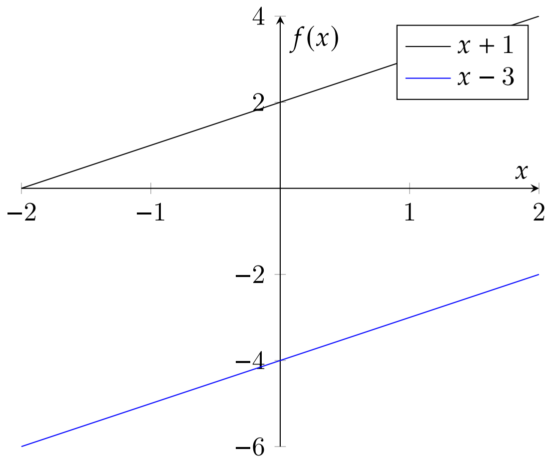
5.1.2 Y-Intercept

Definition 5.1.3: Y Intercept

The Y-Intercept is where the line intersects the y-axis.

Definition 5.1.4: Parallel Lines

One line is parallel to another if the gradients are identical, but the y-intercepts are different



5.1.3 Lines between points

Definition 5.1.5: Midpoint

The midpoint of a line is the average between 2 points.

Question 1: Midpoint between 2 points

Example 5.1.1 Between $(1, 4)$ and $(5, 1)$

$$\begin{aligned} &= \left(\frac{1+5}{2}, \frac{4+1}{2}\right) \\ &= (3, 2.5) \end{aligned}$$

Definition 5.1.6: Distance

To find the distance between 2 points, use Pythagoras.

Question 2: Distance between 2 points

Example 5.1.2 Between $(1, 4)$ and $(5, 1)$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c &= \sqrt{a^2 + b^2} \\ &= \sqrt{(5-1)^2 + (4-1)^2} \\ &= \sqrt{4^2 + 3^2} \qquad\qquad = 5 \end{aligned}$$

5.2 Solving Line Equations

Mainly just a question of plugging in formulas and knowing the above. One other useful thing is this:

$$y - y_1 = m(x - x_1)$$

This equation can be used to construct a line at (x_1, y_1) given gradient m. This can also be used to find the y intercept.

Chapter 6

Circles

Circles all follow a formula: $(x - a)^2 + (y - b)^2 = r^2$. (a, b) is the centre of the circle, and r is the radius. Questions might ask you to describe a circle given a formula - just rearrange till you can get to the formula. When rearranging from the formula, be careful to preserve all solutions, as y is squared so you need to preserve negative values of y.

6.1 Intersections between lines and circles

Substitute into the circle equation. Make sure to check the discriminant or graph to check how many solutions exist.

Question 3: Intersection between $y = 2x + 1$ and $(x - 3)^2 + (y + 1)^2 = 64$

$$(x - 3)^2 + (2x + 1 + 1)^2 = 64$$
$$x^2 - 6x + 9 + 4x^2 + 8x + 4 = 64$$
$$5x^2 + 2x - 51 = 0$$

$$2^2 - 4 * 5 * -51 > 0 \therefore \text{2 Intersections}$$

$$(5x + 17)(x - 3) = 0$$
$$x = 3, -3.4$$
$$y = 2x + 1$$
$$y = 7, -5.7$$

$$= (3, 7)(-3.4, -5.8)$$

Chapter 8

Binomial Expansion

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Chapter 12

Differentiation

Chapter 13

Integration

Chapter 14

Logs & Exponents

14.1 Logarithms

Definition 14.1.1: Logarithms

$a^x = y \therefore \log_a y = x$

Exponential Fact	Logarithm Fact
$10^3 = 1000$	$\log_{10} 1000 = 3$
$5^4 = 625$	$\log_5 625 = 4$
$36^{\frac{1}{2}} = 6$	$log_{36} 6 = \frac{1}{2}$
$2^{-3} = \frac{1}{8}$	$log_2 \frac{1}{8} = 3$

14.1.1 Log Laws

Definition 14.1.2: Log & Index Laws

Laws of Indices	Laws of Logs
$a^x * a^y = a^{x+y}$	$\log_a x + \log_a y = \log_a mn$
$a^x \div a^y = a^{x-y}$	$\log_a x - \log_a y = \log_a \frac{x}{y}$
$(a^x)^y = a^{xy}$	$\log_a x * \log_a y = \log_a m^n$

Question 4: $\log_4 \frac{x}{x-1} = \log_4 3 + log_4 2$

$$\begin{aligned}\log_4 \frac{x}{x-1} &= \log_4 6\\ \frac{x}{x-1} &= 6\\ x &= 6x - 6\\ &\dots\end{aligned}$$

Question 5: $\log_7 4x = \log_7 \frac{1}{x-6} + 1$

$$\begin{aligned}\log_7 \frac{4}{\frac{1}{x-6}} &= 1\\ \log_7 4x(x-6) &= 1\\ \log_7 4x^2 - 24x &= 1\\ 4x^2 - 24x &= 7\\ &\dots\end{aligned}$$

Question 6: $2^x = 75$

$$\begin{aligned}x &= \log_2 75\\ x &= 6.23\end{aligned}$$

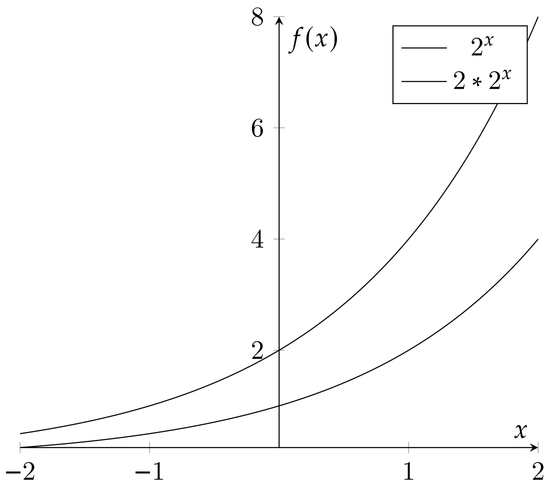
Question 7: $5^{2x} - 6\left(5^x\right) = 0$

$$\begin{aligned}\text{let } y &= 5^x\\ y^2 - 6y - 7 &= 0\\ (y-7)(y+1) &= 0\\ \text{Logarithms always positive} \therefore y &= 7\\ 5^x &= 7\\ x &= log_5 7\\ x &= 1.21\end{aligned}$$

Question 8: $3^{2x+1} = 2^{5x}$

$$\begin{aligned}
\log_e 3^{2x+1} &= \log_e 2^{5x} \\
(2x+1)\log_e 3 &= (5x)\log_e 2 \\
2(\log_e 3)x + \log_e 3 &= 5(log_e 2)x \\
\log_e 3 &= 5(\log_e 2)x - (2log_e 3)x \\
\log_e 3 &= x(5log_e 2 - 2loge_3) \\
x &= \frac{log_e 3}{5log_e 2 - 2loge_3} \\
x &= 0.866
\end{aligned}$$

14.2 Exponentials



Graphs with anything to the power of x always have a y-intercept of 1, because anything to the power of 0 is equal to 1.

14.2.1 Euler's Number - e

Definition 14.2.1: e

e is defined as having the following characteristics:

- $y = e^x$, $\frac{\Delta y}{\Delta x} = e^x$
- $y = e^{kx}$, $\frac{\Delta y}{\Delta x} = ke^{kx}$

Note

$$\log_e x \equiv \ln x$$

Question 9: $y = Ae^{kx}$, and passes through $(0,6)$ and $1,9$. Find A and k

$$6 = Ae^0$$
$$A = 6$$

$$9 = 6e^k$$
$$e^k = \frac{9}{6}$$
$$k = \ln\frac{9}{6}$$
$$k = 0.41$$

14.3 Modelling With Exponentials

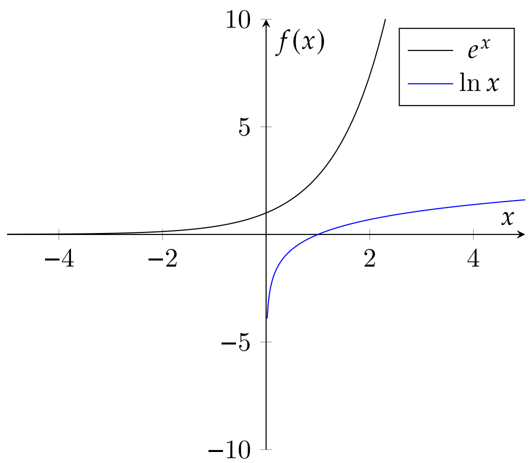
All Exponentials can be written with e, which makes calculus much easier.

Given $y = Ae^{kx}$, if $k > 0$ we get exponential growth, and if $k < 0$ then we get exponential decay.

When modelling with exponentials, A typically represents the initial population, and k the growth/decline factor.

14.4 Logarithm Graphs

As you might expect, logarithm graphs are just exponential graphs reflected in $y = x$.



14.5 Logs & Non-Linear Data

When given non-linear data, we can use logs to extrapolate from it and find an equation. Usually, we can take logs of the y equivalent and then plot x vs $\log y$.

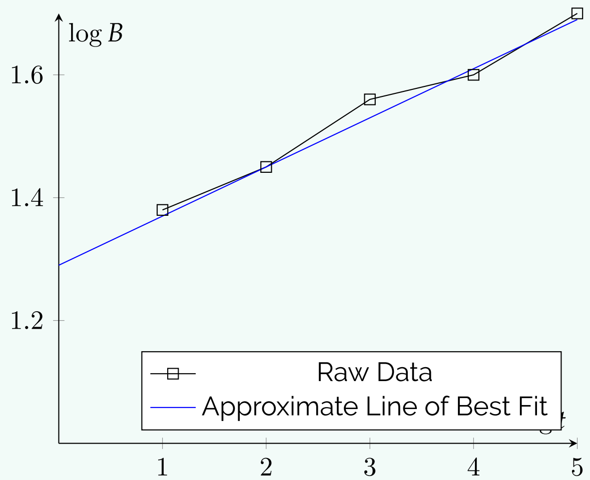
Question 10: $B = Ak^t$, **where** t **represents days and** B **represents the population, find** k **and** A

Days	Population
1	24
2	28
3	36
4	40
5	50

Example 14.5.1 Solving Graphically

$$
\begin{aligned}
B &= A \quad *k^t \\
\\
y &= xm \quad + c \\
\log B &= t \log k + A
\end{aligned}
$$

t	B	$\log B$
1	24	1.38
2	28	1.45
3	36	1.56
4	40	1.60
5	50	1.70



From the line of best fit, we can see that the y-intercept is 1.29 and using non-shown calculations, we can see that the gradient is 0.08

$$
\begin{aligned}
1.3 &= \log k \\
k &= 10^{1.3} \\
k &= 19.95
\end{aligned}
$$

$$
\begin{aligned}
0.08 &= \log A \\
A &= 10^{0.08} \\
A &= 1.20
\end{aligned}
$$

$k = 19.95$ and $A = 1.20$

Part II

Other Bits

Chapter 1

Transforming & Sketching Graphs

1.1 Graph Appearances

1.1.1 Transformations

Definition 1.1.1: Translation

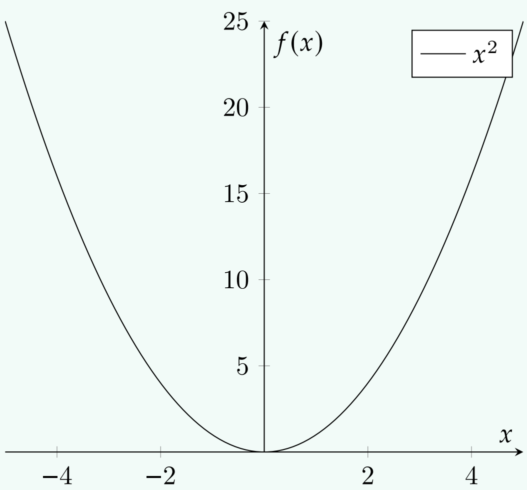
- The graph of $f(x-a)$ is the graph of $f(x)$ translated right by a units.
- The graph of $f(x)+b$ is the graph of $f(x)$ translated upwards by b units.

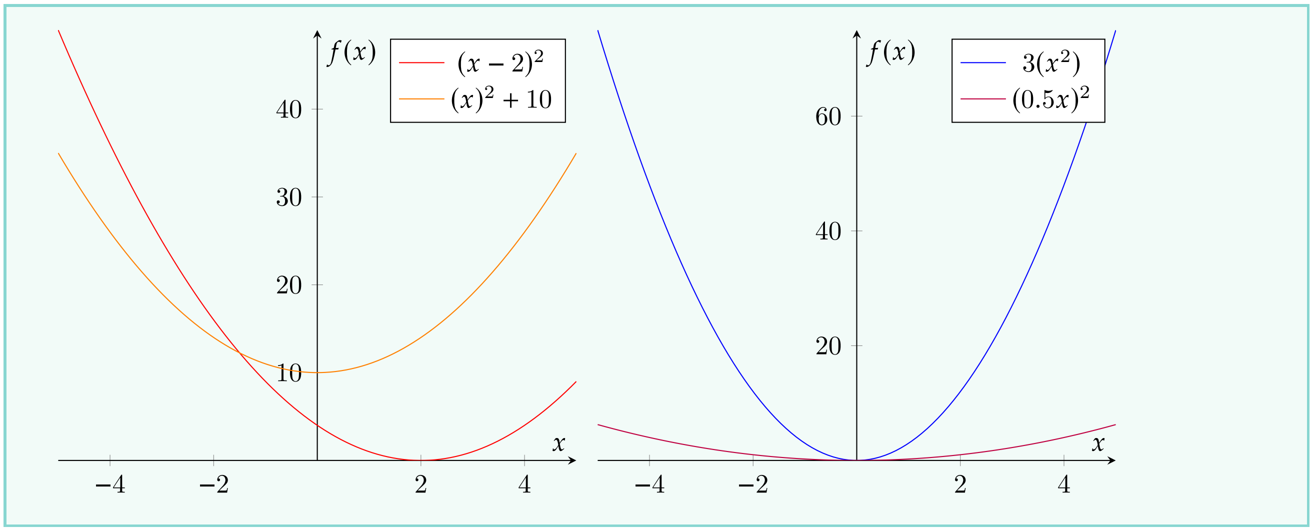
Definition 1.1.2: Scaling

Note
Never say shrink: always say stretch by a factor e where $|e| < 1$

- The graph of $cf(x)$ is the graph of $f(x)$ stretched vertically by a factor of c.
- The graph of $f(dx)$ is the graph of $f(x)$ stretched horizontally by a factor of d^{-1}.

Example 1.1.1 Basic Graph Transformations





1.1.2 Combining Transformations

Question 11: Combining Transformations

$$y = f(-2x)$$

This is obtained from $f(x)$ by doing the following:

1. Flip horizontally.
2. Stretch horizontally by a factor of 0.5.

Question 12: Combining Transformations

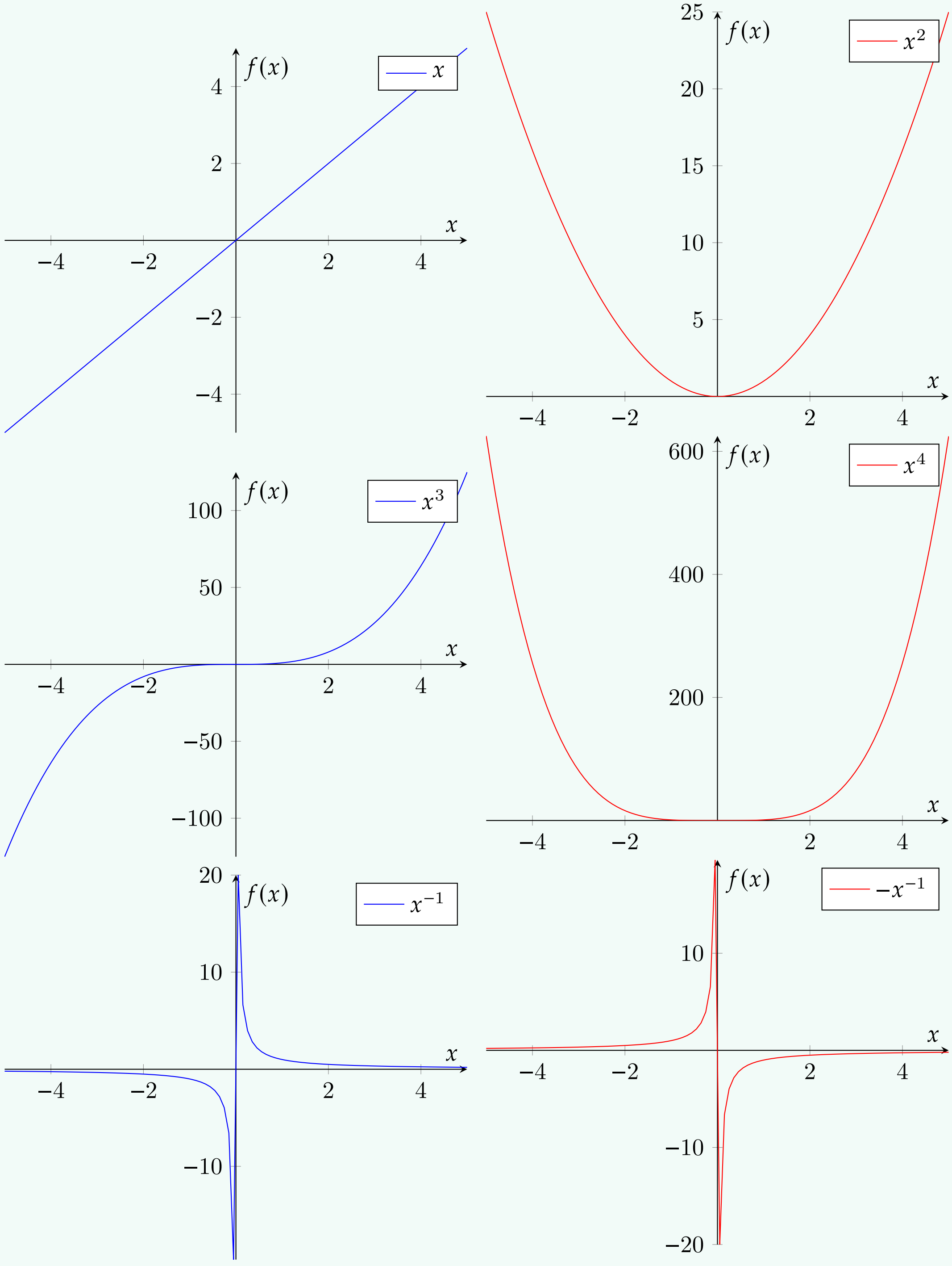
$$y = cf(\frac{1}{a} * (x-b)) + d$$

This is obtained from $f(x)$ by doing the following:

1. Shift to the right b units.
2. Stretch horizontally by a factor of a.
3. Stretch vertically by a factor of c.
4. Shift upwards by d units.

1.1.3 Graph Shapes

Example 1.1.2 Basic Power Graphs



These are all of the basic graph shapes, and can be transformed just like $y = x^2$ above.

1.2 Solving Using Graphs

Find one or more $y = \ldots$ equation, plot it, find the x position of any intercepts. If only one equation, find intersections with $y = 0$.

Question 13: Solving $5 = 6x + 8$

$$5 = 6x + 8 \tag{1.1}$$

$$-3 = 6x \tag{1.2}$$

$$\tag{1.3}$$

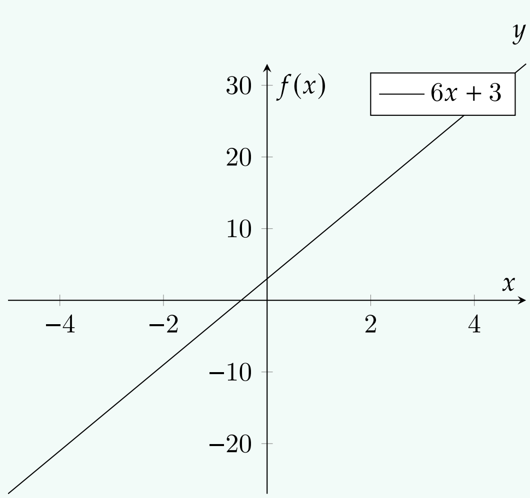
Example 1.2.1 Using Algebra

$$6x = -3 \tag{1.4}$$

$$x = \frac{6}{-3} \tag{1.5}$$

$$x = -\frac{1}{2} \tag{1.6}$$

Example 1.2.2 Using a Graph



Intersects at $-\frac{1}{2}$.

Whilst this might seem less useful for basic equations, this can become much more useful for more complicated questions like below.

Question 14: Finding the intersection of $y = 3x^2 - 2x - 21$ and $y = (x-3)(x+3)$

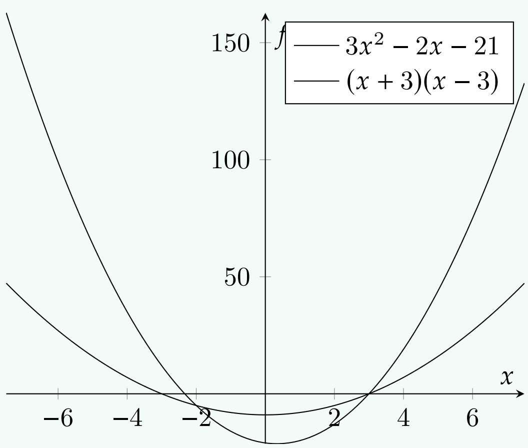
Example 1.2.3 Using Algebra

1. Set equal to each other
2. Simplify
3. Check how many roots exist using the discriminant
4. Work out all roots (possibly using factor theorem which can take a while)

$$3x^2 - 2x - 21 = (x-3)(x+3) \tag{1.7}$$
$$3x^2 - 2x - 21 = x^2 - 9 \tag{1.8}$$
$$2x^2 - 2x - 12 = 0 \tag{1.9}$$
$$x^2 - x - 6 = 0 \tag{1.10}$$
$$(x-3)(x+2) = 0 \tag{1.11}$$
$$x = 3, -2 \tag{1.12}$$

Example 1.2.4 Using a Graph

1. Plot
2. Check intersections.



Intersects at $x = -2, 3$.

Chapter 2

Polynomial Division, Factor Theorem & Cubics

2.1 Polynomial Division

Any expression $\frac{f(x)}{g(x)}$ can be expressed as $g(x) \operatorname{rem} r(x)$. For example, $\frac{11}{4} \equiv 2 \operatorname{rem} 3$.

There are 2 main methods of dividing polynomials - Long Division and Synthetic. Synthetic is usually considered easier, but a question might ask for Long Division, so learn both.

Question 15: $x^3 - 17x + 6 \quad \div \quad x - 3$

Example 2.1.1 Synthetic Division

Synthetic Division isn't very hard, but has steps you need to carefully follow. Firstly, copy all of the coefficients into the top row, and the negative of the divisor constant into the left column in the row below. For the first coefficient, copy it directly to the bottom. Then, for all of the others follow these steps:

1. Multiply the result from the order above by the negative of the constant, and copy it to the middle row.
2. Add that to the original and place the result in the bottom row.

$$\begin{array}{c|cccc} & 1 & 0 & -17 & 6 \\ 3 & & 3 & 9 & -24 \\ \hline & 1 & 3 & -8 & -18 \end{array}$$

$$\therefore \frac{x^3 - 17x + 6}{x - 3} = x^2 + 3x - 8 \operatorname{rem} -18$$

Example 2.1.2 Long Division

Long Division involves a few main steps, which you repeat for every term of the dividend. It starts with laying out the equation in the box as you would for normal long division, and then you do the following:

1. Divide the dividend term by the divisor term an order below, and add that to the result at the top
2. Copy the dividend term a row below
3. Multiply the next dividend term by the negative of the constant in the divisor.
4. Treat what you've written down as long subtraction.
5. Copy the rest of the row down into the results bit.

Continue until you get to a lone constant, and that is the remainder

$$
\begin{array}{r|rrrr}
 & & x^2 & +3x & -8 \\
\hline
x-3 & x^3 & & -17x & +6 \\
 & -x^3 & +3x^2 & & \\
\hline
 & & 3x^2 & -17x & \\
 & & -3x^2 & +9x & \\
\hline
 & & & -8x & +6 \\
 & & & 8x & -24 \\
\hline
 & & & & -18
\end{array}
$$

$$\therefore \frac{x^3 - 17x + 6}{x-3} = x^2 + 3x - 8 \,\text{rem}\, -18$$

2.2 Factor Theorem

The factor theorem is a method of solving cubic equations.

Definition 2.2.1: Factor Theorem

Given $f(x)$, if $f(y)$ is a solution, then $(x-y)$ is a factor of $f(x)$

To solve an equation using factor theorem, usually we need to follow a few steps:

1. Work through factors of the constant until you find one that is a factor of the whole equation.
2. Create a trial factorising function.
3. Expand the faux-factorised function, and equate coefficients.
4. Write a proper expanded function.
5. Properly factorise.

Question 16: Given one solution is an integer, solve $2x^3 + x^2 - 18x + 9$

Example 2.2.1 Solving with Factor Theorem

$$\begin{aligned} f(x) &= 2x^3 + x^2 - 18x - 9 \\ f(1) &= 2 + 1 - 18 - 9 \neq 0 \\ f(-1) &= 2 + 1 + 18 - 9 \neq 0 \\ f(3) &= 2 * 27 + 9 - 54 - 9 = 0 \\ f(3) &= 0 \quad \therefore (x-3) \text{ is a factor} \end{aligned}$$

$$\begin{aligned} f(x) &= (x-3)\left(2x^2 + bx + 3\right) \\ &= \ldots - 6x^2 + bx^2 + \ldots \\ b - 6 &= 1 \\ b &= 7 \end{aligned}$$

$$\begin{aligned} f(x) &= (x-3)\left(2x^3 + 7x + 3\right) \\ &= (x-3)(2x+1)(x+3) \\ \therefore x &= -3, \frac{1}{2}, 3 \end{aligned}$$

Part III

Book 2

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Chapter 2

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