Q1:

$$u''(t) + c_1\big(u'(t) - v'(t)\big) + k_1\big(u(t) - v(t)\big) + k_2 u(t) = \sin(t)$$
$$v''(t) + c_2\big(u(t) - v(t)\big) - c_1\big(u'(t) - v'(t)\big) = 0$$

$$\begin{cases} u(0) = 1 \\ v(0) = 2 \\ u'(0) = 0 \\ v'(0) = 0 \end{cases}$$

a)

Rewrite first two equations gives

Let $p(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}$ and $q(t) = \begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}$,

Then $p'(t) = \begin{bmatrix} u'(t) \\ u''(t) \end{bmatrix}$, $q'(t) = \begin{bmatrix} v'(t) \\ v''(t) \end{bmatrix}$, $p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $q(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Since

$$u''(t) = -c_1 u'(t) + c_1 v'(t) - (k_1 + k_2)u(t) + k_1 v(t) + \sin(t)$$
$$\begin{bmatrix} u'(t) \\ u''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(k_1 + k_2) & -c_1 \end{bmatrix}\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & c_1 \end{bmatrix}\begin{bmatrix} v(t) \\ v'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$
$$p'(t) = \begin{bmatrix} 0 & 1 \\ -(k_1 + k_2) & -c_1 \end{bmatrix} p(t) + \begin{bmatrix} 0 & 0 \\ k_1 & c_1 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

Similarly,

$$v''(t) = -c_2 u(t) + c_2 v(t) + c_1 u'(t) - c_1 v'(t)$$
$$\begin{bmatrix} v'(t) \\ v''(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -c_2 & c_1 \end{bmatrix}\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ c_2 & -c_1 \end{bmatrix}\begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}$$
$$q'(t) = \begin{bmatrix} 0 & 0 \\ -c_2 & c_1 \end{bmatrix} p(t) + \begin{bmatrix} 0 & 1 \\ c_2 & -c_1 \end{bmatrix} q(t)$$

Therefore the new system of first order ordinary differential equations is:

$$\begin{cases} p'(t) = \begin{bmatrix} 0 & 1 \\ -(k_1 + k_2) & -c_1 \end{bmatrix} p(t) + \begin{bmatrix} 0 & 0 \\ k_1 & c_1 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} \\ q'(t) = \begin{bmatrix} 0 & 0 \\ -c_2 & c_1 \end{bmatrix} p(t) + \begin{bmatrix} 0 & 1 \\ c_2 & -c_1 \end{bmatrix} q(t) \\ p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ q(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{cases}$$

Q3:

Assume y_{n-1}, and y_n are accurate.

$$\begin{aligned} y_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\ &= y_{n-1} + 2hy'(t_n) \\ &= 2hy'(t_n) + y(t_n) - y'(t_n) \cdot h + y''(t_n) \cdot \frac{h^2}{2} - y'''(t_n) \cdot \frac{h^3}{3!} + \cdots \\ &= y(t_n) + y'(t_n) \cdot h + y''(t_n) \cdot \frac{h^2}{2} - y'''(t_n) \cdot \frac{h^3}{3!} + \cdots \end{aligned}$$

So we get

$$y(t_{n+1}) - y_{n+1} = 2 \cdot y'''(t_n) \cdot \frac{h^3}{3!} + 2 \cdot y''''(t_n) \cdot \frac{h^5}{5!} + \cdots$$

Therefore $Error = 2 \cdot y^{(3)}(c) \cdot \frac{h^3}{3!}$ For some $c \in [t_n, t_n + h]$, and hence $Error \in O(h^3)$.

Q4:

$$y_{n+1}^* = y_n + \frac{3hf(t_n, y_n)}{4}$$
$$y_{n+1} = y_n + \frac{h}{3}\left[f(t_n, y_n) + 2f\left(t_n + \frac{3h}{4}, y_{n+1}^*\right)\right]$$

By the test equation,

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{3}\,[-\lambda y_n + 2(-\lambda y_{n+1}^*)] \\
&= y_n + \frac{h}{3}\left[-\lambda y_n + 2\left(-\lambda\left(y_n + \frac{3h(-\lambda y_n)}{4}\right)\right)\right] \\
&= y_n - \frac{h}{3}\lambda y_n - \frac{h}{3}\cdot 2\lambda y_n - \frac{h}{3}\cdot 2\lambda\frac{3h(-\lambda y_n)}{4} \\
&= y_n\left(1 - \lambda h + \frac{\lambda^2 h^2}{2}\right)
\end{aligned}$$

And computed

$$\hat{y}_{n+1} = \hat{y}_n\left(1 - \lambda h + \frac{\lambda^2 h^2}{2}\right)$$

Error:

$$\begin{aligned}
e_{n+1} &= e_n\left(1 - \lambda h + \frac{\lambda^2 h^2}{2}\right) \\
&= e_{n-1}\left(1 - \lambda h + \frac{\lambda^2 h^2}{2}\right)^2 \\
&= \cdots \\
&= e_0\left(1 - \lambda h + \frac{\lambda^2 h^2}{2}\right)^{n+1}
\end{aligned}$$

$$|e_{n+1}| = |e_0| \cdot \left|1 - \lambda h + \frac{\lambda^2 h^2}{2}\right|^{n+1}$$

To make the method stable, we need $\left|1 - \lambda h + \frac{\lambda^2 h^2}{2}\right| < 1$.
since $\left(1 - \lambda h + \frac{\lambda^2 h^2}{2} > -1\right) \equiv (4 - 2\lambda h + \lambda^2 h^2 > 0) \equiv (3 + (1 - \lambda h)^2 > 0)$ which is always true,

$\left(1 - \lambda h + \frac{\lambda^2 h^2}{2} < 1\right) \equiv (\lambda^2 h^2 < 2\lambda h) \equiv (\lambda h < 2) \equiv \left(h < \frac{2}{\lambda}\right)$, which is only true when $h < \frac{2}{\lambda}$, this method is stable only when $h < \frac{2}{\lambda}$.