Marking Sheet   Assignment 3   CS 370 Winter 2018

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Section: 1 (8:30pm)   2 (11:30pm) ✓   3 (1:30)

1. (a) Fourier (5) ________

   (b) Inverse Fourier (5) ________

........./10

2. (a) F_k (7) ________

   (b) F_{2k} (8) ________

........./15

3. (a) g, h (5) ________

   (b) G, H (5) ________

   (c) F (5) ________

   (d) F (butterfly) (5) ________

........./20

4. Edge Sharpening

........./15

5. train / bird

........./20

6. Image compression

........./20

Total: (100) ____________

Q1:

a)

$$f = (1,2,3,2)$$

$$W = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi i}{4}} = i$$

$$F_0 = \frac{1}{4}(f_1 + f_2 + f_3 + f_4) = 2$$

$$F_1 = \frac{1}{4}(f_1 + f_2 W^{-1} + f_3 W^{-2} + f_4 W^{-3}) = \frac{1}{4}(1 - 2i - 3 + 2i) = -\frac{1}{2}$$

$$F_2 = \frac{1}{4}(f_1 + f_2 W^{-2} + f_3 W^{-4} + f_4 W^{-6}) = \frac{1}{4}(1 - 2 + 3 - 2) = 0$$

$$F_3 = \frac{1}{4}(f_1 + f_2 W^{-3} + f_3 W^{-6} + f_4 W^{-9}) = \frac{1}{4}(1 + 2i - 3 - 2i) = -\frac{1}{2}$$

b)

$$F = (4, -1, 0, -1)$$

$$W = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi}{4}} = i$$

$$f_0 = f_1 + f_2 + f_3 + f_4 = 2$$

$$F_1 = f_1 + f_2 W^1 + f_3 W^2 + f_4 W^3 = 4 - i + 0 + i = 4$$

$$F_2 = f_1 + f_2 W^2 + f_3 W^4 + f_4 W^6 = 4 + 1 + 0 + 1 = 6$$

$$F_3 = f_1 + f_2 W^3 + f_3 W^6 + f_4 W^9 = 4 + i + 0 - i = 4$$

Q2:

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk}$$

a)

$$
\begin{aligned}
F_k &= \frac{1}{N}\left(f_0 + f_1 W^{-k} + f_2 W^{-2k} + \cdots + f_{N-1} W^{(N-1)k}\right) \\
&= \frac{1}{N}\left((-1)^0 + (-1)^1 W^{-k} + (-1)^2 W^{-2k} + \cdots + (-1)^{(N-1)} W^{(N-1)k}\right) \\
&= \begin{cases} \frac{1}{N}\cdot\left(\frac{1-(-1)^N W^{Nk}}{1-(-1)W^{-k}}\right), & if\ (-1)W^{-k} \neq 1 \\ \frac{1}{N}\cdot N, & if\ (-1)W^{-k} = 1, \quad k = \frac{N}{2} \end{cases} \\
&= \begin{cases} 1, & if\ k = \frac{N}{2} \\ \frac{1}{N}\cdot\left(\frac{1-1}{1-(-1)W^{-k}}\right), & otherwise \end{cases} \\
&= \begin{cases} 1, & if\ k = \frac{N}{2} \\ 0, & otherwise \end{cases}
\end{aligned}
$$

b)

$$
\begin{aligned}
F_k &= \frac{1}{N}\left(f_0 + f_1 W^{-k} + f_2 W^{-2k} + \cdots + f_{N-1} W^{-(N-1)k}\right) \\
&= -\frac{1}{N}\left(1 + W^{-k} + W^{-2k} + \cdots + W^{-\left(\frac{N}{2}-1\right)k}\right) + \frac{1}{N}\left(W^{-\left(\frac{N}{2}\right)k} + W^{-\left(\frac{N}{2}+1\right)k} + \cdots + W^{-(N-1)k}\right) \\
&= -\frac{1}{N}\cdot\frac{1-W^{-\left(\frac{N}{2}\right)k}}{1-W^{-k}} + \frac{W^{-\left(\frac{N}{2}\right)k}}{N}\cdot\frac{1-W^{-\left(\frac{N}{2}\right)k}}{1-W^{-k}} \\
&= \begin{cases} \frac{1}{N}\cdot\frac{N}{2} + \frac{1}{N}\cdot\frac{N}{2}, & if\ W^{-k} = 1, \quad k = 0 \\ -\frac{1}{N}\cdot\frac{1-W^{-\left(\frac{N}{2}\right)k}}{1-W^{-k}} + \frac{W^{-\left(\frac{N}{2}\right)k}}{N}\cdot\frac{1-W^{-\left(\frac{N}{2}\right)k}}{1-W^{-k}}, & if\ W^{-k} \neq 1 \end{cases} \\
&= \begin{cases} 1, & if\ \ k = 0 \\ \frac{W^{-\left(\frac{N}{2}\right)k}-1}{N}\cdot\frac{1-W^{-\left(\frac{N}{2}\right)k}}{1-W^{-k}}, & otherwise \end{cases} \\
&= \begin{cases} 1, & if\ \ k = 0 \\ -\frac{\left(1-W^{-\left(\frac{N}{2}\right)k}\right)^2}{N(1-W^{-k})}, & otherwise \end{cases}
\end{aligned}
$$

Q3:

$$f = (1, 0, 2, 0, -1, 0, -2, 0)$$

a)

$$W = e^{\frac{2\pi i}{8}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$g = \left(\frac{1-1}{2}, \frac{0+0}{2}, \frac{2-2}{2}, \frac{0+0}{2}\right) = (0,0,0,0)$$

$$h = \left(\frac{1-(-1)}{2}, \frac{0-0}{2} W^{-1}, \frac{2-(-2)}{2} W^{-2}, \frac{0-0}{2} W^{-3}\right) = (1,0,-2i,0)$$

b)

$$W_G = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi}{4}} = i$$

$$G_0 = \frac{1}{4}(g_1 + g_2 + g_3 + g_4) = 0$$

$$G_1 = \frac{1}{4}\left(g_1 + g_2 W_G^{-1} + g_3 W_G^{-2} + g_4 W_G^{-3}\right) = 0$$

$$G_2 = \frac{1}{4}\left(g_1 + g_2 W_G^{-2} + g_3 W_G^{-4} + g_4 W_G^{-6}\right) = 0$$

$$G_3 = \frac{1}{4}\left(g_1 + g_2 W_G^{-3} + g_3 W_G^{-6} + g_4 W_G^{-9}\right) = 0$$

$$W_H = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi i}{4}} = i$$

$$H_0 = \frac{1}{4}(h_1 + h_2 + h_3 + h_4) = \frac{1}{4}(1 + 0 + (-2i) + 0) = \frac{1}{4} - \frac{1}{2} i$$

$$H_1 = \frac{1}{4}(h_1 + h_2 W_H^{-1} + h_3 W_H^{-2} + h_4 W_H^{-3}) = \frac{1}{4}(1 + 0 + (-2i)(-1) + 0) = \frac{1}{4} + \frac{1}{2} i$$

$$H_2 = \frac{1}{4}(h_1 + h_2 W_H^{-2} + h_3 W_H^{-4} + h_4 W_H^{-6}) = \frac{1}{4}(1 + 0 + (-2i)(1) + 0) = \frac{1}{4} - \frac{1}{2} i$$

$$H_3 = \frac{1}{4}(h_1 + h_2 W_H^{-3} + h_3 W_H^{-6} + h_4 W_H^{-9}) = \frac{1}{4}(1 + 0 + (-2i)(-1) + 0) = \frac{1}{4} + \frac{1}{2} i$$

c)

$$F = \left(0, \frac{1}{4} - \frac{1}{2} i, 0, \frac{1}{4} + \frac{1}{2} i, 0, \frac{1}{4} - \frac{1}{2} i, 0, \frac{1}{4} + \frac{1}{2} i\right)$$

d)

$$f = (1, 0, 2, 0, -1, 0, -2, 0)$$

Since

$$W_4 = e^{\frac{2\pi i}{4}} = i, \qquad W_4^{-3} = -i$$

$$\begin{array}{cccccc}
f_{000} & 1 & 0 & 0 & 0 & F_{000} \\
f_{001} & 0 & 0 & 0 & 0 & F_{100} \\
f_{010} & 2 & 0 & 0 & 0 & F_{010} \\
f_{011} & 0 & 0 & 0 & 0 & F_{110} \\
f_{100} & -1 & 1 & \frac{1}{2} - i & \frac{1}{4} - \frac{1}{2}i & F_{001} \\
f_{101} & 0 & 0 & 0 & \frac{1}{4} - \frac{1}{2}i & F_{101} \\
f_{110} & -2 & 2W_4^{-3} = -2i & \frac{1}{2} + i & \frac{1}{4} + \frac{1}{2}i & F_{011} \\
f_{111} & 0 & 0 & 0 & \frac{1}{4} + \frac{1}{2}i & F_{111}
\end{array}$$

$$F = \left(0, \frac{1}{4} - \frac{1}{2}i, 0, \frac{1}{4} + \frac{1}{2}i, 0, \frac{1}{4} - \frac{1}{2}i, 0, \frac{1}{4} + \frac{1}{2}i\right)$$

Q4:

$$f(x,y) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi kx}{N}\right)\cos\left(\frac{2\pi ly}{N}\right)$$

$$f^* = f - \alpha\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

Compute the partials:

$$\frac{\partial f}{\partial x} = -\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi ly}{N}\right)\sin\left(\frac{2\pi kx}{N}\right)\cdot\frac{2\pi k}{N}$$

$$\frac{\partial^2 f}{\partial x^2} = -\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi ly}{N}\right)\cos\left(\frac{2\pi kx}{N}\right)\cdot\left(\frac{2\pi k}{N}\right)^2$$

$$\frac{\partial f}{\partial y} = -\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi kx}{N}\right)\sin\left(\frac{2\pi ly}{N}\right)\cdot\frac{2\pi l}{N}$$

$$\frac{\partial^2 f}{\partial y^2} = -\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi kx}{N}\right)\cos\left(\frac{2\pi ly}{N}\right)\cdot\left(\frac{2\pi l}{N}\right)^2$$

Assume $g(k,l) = F_{k,l}\cos\left(\frac{2\pi kx}{N}\right)\cos\left(\frac{2\pi ly}{N}\right)$

Then

$$\begin{aligned}
f^* &= f - \alpha\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) - \alpha\left(-\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\cdot\left(\frac{2\pi k}{N}\right)^2 - \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\cdot\left(\frac{2\pi l}{N}\right)^2\right) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) + \alpha\sum_{k=0}^{N-1}\left(\left(\frac{2\pi k}{N}\right)^2\sum_{l=0}^{N-1} g(k,l)\right) + \alpha\sum_{k=0}^{N-1}\left(\frac{2\pi^2(N-1)(2N-1)}{3N}\sum_{l=0}^{N-1} g(k,l)\right) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) + 2\alpha\frac{2\pi^2(N-1)(2N-1)}{3N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) \\
&= \frac{4\alpha\pi^2(N-1)(2N-1)+3N}{3N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cos\left(\frac{2\pi kx}{N}\right)\cos\left(\frac{2\pi ly}{N}\right)
\end{aligned}$$