A1:

a)

$$
\begin{aligned}
x_1 &= \frac{-b-\sqrt{b^2-4ac}}{2a} \\
&= \frac{-22.41-\sqrt{22.41^2-4\times 1.03\times 0.1123}}{2\times 1.03} \\
&= \frac{-22.41-\sqrt{502.20-4.12\times 0.1123}}{2.06} \\
&= \frac{-22.41-\sqrt{502.20-0.46267}}{2.06} \\
&= \frac{-22.41-22.399}{2.06} \\
&= \frac{-44.809}{2.06} \\
&= -21.751
\end{aligned}
$$

Relative error: $\left|\frac{(-21.752)-(-21.751)}{-21.752}\right| \times 100\% = 0.004597\%$

$$
\begin{aligned}
x_2 &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\
&= \frac{-22.41+22.399}{2.06} \\
&= \frac{-0.011}{2.06} \\
&= -0.0053398
\end{aligned}
$$

Relative error: $\left|\frac{(-0.0050123)-(-0.0053398)}{-0.0050123}\right| \times 100\% = 6.133\%$

b)

$$
x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}
$$

$$
\begin{aligned}
right &= \frac{c}{ax_1} \\
&= \frac{c}{a\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)} \\
&= \frac{2c}{\left(-b-\sqrt{b^2-4ac}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2c\left(-b+\sqrt{b^2-4ac}\right)}{\left(-b-\sqrt{b^2-4ac}\right)\left(-b+\sqrt{b^2-4ac}\right)} \\
&= \frac{-2bc+2\mathrm{c}\sqrt{b^2-4ac}}{\mathrm{b}^2-(b^2-4ac)} \\
&= \frac{-b+\sqrt{b^2-4ac}}{2a} \\
&= x_2 \\
&= left
\end{aligned}
$$

c)

x_1 is unchanged

$$x_1 = -21.751$$

Relative error: $\left|\frac{(-21.752)-(-21.751)}{-21.752}\right| \times 100\% = 0.004597\%$

$$
\begin{aligned}
x_2 &= \frac{c}{ax_1} \\
&= \frac{0.1123}{1.03 \times (-21.751)} \\
&= \frac{0.1123}{-22.403} \\
&= -0.0050127
\end{aligned}
$$

Relative error: $\left|\frac{(-0.0050127)-(-0.0050123)}{-0.0050123}\right| \times 100\% = 0.007980\%$

Q2:

a) The answer becomes an extremely negative number. (-4.063562×10^{13}).

b) the error in the first step is $e_1 = 0$, in the second step it is $e_2 = \left|0.5714285 - \frac{4}{7}\right| \leq \frac{1}{14000000}$.

Since $e_n + \frac{41}{14}e_{n-1} - 2e_{n-2} = 0$ for $n \geq 3$, let $a = \frac{41}{14}$ and $b = -2$, solve $x^2 + ax + b = 0$ gives $x_1 = -\frac{7}{2}$, $x_2 = \frac{4}{7}$. Hence $|e_{40}| = \left|c_1\left(-\frac{7}{2}\right)^{40} + c_2\left(\frac{4}{7}\right)^{40}\right| \approx \left|c_1\left(\frac{7}{2}\right)^{40}\right|$, which grows very big, therefore the final answer is far off from the correct answer.

c) assume $p_0^{approx} = p_0^{exact}$,

$$\begin{aligned} p_1^{approx} &= p_1^{exact}(1+\epsilon) \\ &= p_1^{exact} + e_1 \end{aligned}$$

Where $e_1 = p_1^{approx}\epsilon$.

$$\begin{aligned} p_2^{approx} &= -\frac{41}{14}p_1^{approx} + 2p_0^{approx} \\ &= -\frac{41}{14}p_1^{exact} - \frac{41}{14}e_1 + 2p_0^{exact} \\ &= p_2^{exact} - \frac{41}{14}e_1 \end{aligned}$$

$$\begin{aligned} p_3^{approx} &= -\frac{41}{14}p_2^{approx} + 2p_1^{approx} \\ &= \left(-\frac{41}{14}\right)p_2^{exact} - \left(-\frac{41}{14}\right)\frac{41}{14}e_1 + 2p_1^{exact} + 2e_1 \\ &= p_3^{exact} + \frac{41^2 + 2 \times 14^2}{14^2}e_1 \end{aligned}$$

by some induction steps similar as above, we know for $n \geq 3$,

$$p_n^{qpprox} = p_n^{exact} + xe_1$$

Where x is some function of n and $|x|$ increases as n increases.
Even though e_1 is small, when n gets big, the error $|xe_1|$ becomes large, and the answer becomes unreliable.

Q3:

$$S(x) = \begin{cases} 28 + a_1 x + 9x^2 + x^3, & x \in [-3,-1] \\ a_2 + 19x + a_3 x^2 - x^3, & x \in [-1,0] \\ 26 + 19x + 3x^2 + a_4 x^3, & x \in [0,3] \end{cases}$$

a)

when $x = -1$, we have

$$\begin{aligned} 28 + a_1 x + 9x^2 + x^3 &= a_2 + 19x + a_3 x^2 - x^3 \\ 28 - a_1 + 9 - 1 &= a_2 - 19 + a_3 + 1 \\ a_1 + a_2 + a_3 &= 54 \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dx}(28 + a_1 x + 9x^2 + x^3) &= \frac{d}{dx}(a_2 + 19x + a_3 x^2 - x^3) \\ a_1 + 18x + 3x^2 &= 19 + 2a_3 x - 3x^2 \\ a_1 - 18 + 3 &= 19 - 2a_3 - 3 \\ a_1 + 2a_3 &= 31 \end{aligned}$$

When $x = 0$, we have

$$\begin{aligned} a_2 + 19x + a_3 x^2 - x^3 &= 26 + 19x + 3x^2 + a_4 x^3 \\ a_2 &= 26 \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dx}(a_2 + 19x + a_3 x^2 - x^3) &= \frac{d}{dx}(26 + 19x + 3x^2 + a_4 x^3) \\ 19 + 2a_3 x - 3x^2 &= 19 + 6x + 3a_4 x^2 \\ 19 &= 19 \end{aligned}$$

We have $\begin{cases} a_1 + a_2 + a_3 = 54 \\ a_1 + 2a_3 = 31 \\ a_2 = 26 \end{cases} \Longrightarrow \begin{cases} a_1 = 25 \\ a_2 = 26. \\ a_3 = 3 \end{cases}$

Since it is a natural spline, when $x = 3$ we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{d}{dx}(26 + 19x + 3x^2 + a_4 x^3)\right) &= 0 \\ 6 + 6a_4 x &= 0 \\ a_4 &= -\frac{1}{3} \end{aligned}$$

Hence $\begin{cases} a_1 = 25 \\ a_2 = 26 \\ a_3 = 3 \\ a_4 = \frac{1}{3} \end{cases}$ is the solution.

b)

by substitute $x = -3, -1, 0, 3$ into $S(x)$ we get $\begin{cases} S(-3) = 7 \\ S(-1) = 11 \\ S(0) = 26 \\ S(3) = 101 \end{cases}$.

the interpolation is

$$L(x) = 7 \times \frac{(x+1)x(x-3)}{(-3+1)(-3)(-3-3)} + 11 \times \frac{(x+3)x(x-3)}{(-1+3)(-1)(-1-3)} +$$

$$26 \times \frac{(x+3)(x+1)(x-3)}{(0+3)(0+1)(0-3)} + 101 \times \frac{(x+3)(x+1)x}{(3+3)(3+1)3}$$

$$= 7 \times \frac{(x+1)x(x-3)}{-36} + 11 \times \frac{(x+3)x(x-3)}{8} +$$

$$26 \times \frac{(x+3)(x+1)(x-3)}{-9} + 101 \times \frac{(x+3)(x+1)x}{72}$$

Q4:

$$S_1(x) = a_1 + b_1(x+1) + c_1(x+1)^2 + d_1(x+1)^3 \quad -1 \le x \le 1$$
$$S_2(x) = a_2 + b_2(x-1) + c_2(x-1)^2 + d_2(x-1)^3 \quad 1 \le x \le 2$$
$$S_3(x) = a_3 + b_3(x-2) + c_3(x-2)^2 + d_3(x-2)^3 \quad 2 \le x \le 5$$
$$S_4(x) = a_4 + b_4(x-5) + c_4(x-5)^2 + d_4(x-5)^3 \quad 5 \le x \le 7$$

a)

From second derivatives we know

$$\begin{cases} 2c_1 + 6d_1(0) = 0 \\ 2c_1 + 6d_1(2) = 2c_2 + 6d_2(0) \\ 2c_2 + 6d_2(1) = 2c_3 + 6d_3(0) \\ 2c_3 + 6d_3(3) = 2c_4 + 6d_4(0) \\ 2c_4 + 6d_4(2) = 0 \end{cases}$$

since $x_{1,2,3,4,5} = -1,1,2,5,7$ and $y_{1,2,3,4,5} = 2,1,4,3,-1$ respectively, we can see

$$\begin{cases} \Delta x_1 = 2 \\ \Delta x_2 = 1 \\ \Delta x_3 = 3 \\ \Delta x_4 = 2 \end{cases} \text{and} \begin{cases} \Delta y_1 = -1 \\ \Delta y_2 = 3 \\ \Delta y_3 = -1 \\ \Delta y_4 = -4 \end{cases}, \text{therefore} \begin{cases} y_1' = -\frac{1}{2} \\ y_2' = 3 \\ y_3' = -\frac{1}{3} \\ y_4' = -2 \end{cases}$$

Since $c_n = \frac{3y_n' - 2s_n - s_{n+1}}{\Delta x_n}$, $d_n = \frac{s_n + s_{n+1} - 2y_n'}{\Delta x_n^2}$, we get

$$\begin{cases} c_1 = \frac{-\frac{3}{2} - 2s_1 - s_2}{2} \\ c_2 = \frac{9 - 2s_2 - s_3}{1} \\ c_3 = \frac{-1 - 2s_3 - s_4}{3} \\ c_4 = \frac{-6 - 2s_4 - s_5}{2} \end{cases} \text{and} \begin{cases} d_1 = \frac{s_1 + s_2 + 1}{4} \\ d_2 = \frac{s_2 + s_3 - 6}{1} \\ d_3 = \frac{s_3 + s_4 + \frac{2}{3}}{9} \\ d_4 = \frac{s_4 + s_5 + 4}{4} \end{cases}$$

Substitute all into the original gives

$$\begin{cases} 2\frac{-\frac{3}{2}-2s_1-s_2}{2}+6\frac{s_1+s_2+1}{4}(0)=0 \\ 2\frac{-\frac{3}{2}-2s_1-s_2}{2}+6\frac{s_1+s_2+1}{4}(2)=2\frac{9-2s_2-s_3}{1} \\ 2\frac{9-2s_2-s_3}{1}+6\frac{s_2+s_3-6}{1}(1)=2\frac{-1-2s_3-s_4}{3} \\ 2\frac{-1-2s_3-s_4}{3}+6\frac{s_3+s_4+\frac{2}{3}}{9}(3)=2\frac{-6-2s_4-s_5}{2} \\ 2\frac{-6-2s_4-s_5}{2}+6\frac{s_4+s_5+4}{4}(2)=0 \end{cases}$$

Simplify gives

$$\begin{cases} 2s_1+s_2=-\frac{3}{2} \\ s_1+6s_2+2s_3=\frac{33}{2} \\ 3s_2+8s_3+s_4=26 \\ 2s_3+10s_4+3s_5=-20 \\ s_4+2s_5=-6 \end{cases}$$

b)

$$\begin{cases} 2s_1+s_2=-\frac{3}{2} \\ s_1+6s_2+2s_3=\frac{33}{2} \\ 3s_2+8s_3+s_4=26 \\ 2s_3+10s_4+3s_5=-20 \\ s_4+2s_5=-6 \end{cases}$$

Solve to get $\begin{cases} s_1=-\frac{2285}{1248} \\ s_2=\frac{1349}{624} \\ s_3=\frac{6689}{2496} \\ s_4=-\frac{1201}{624} \\ s_5=-\frac{2543}{1248} \end{cases}$

Since for any S_n we have $\begin{cases} a_n=y_n \\ a_n+b_n(\Delta x_n)+c_n(\Delta x_n)^2+d_n(\Delta x_n)^3=y_{n+1} \\ b_n=s_n \\ b_n+2c_n(\Delta x_n)+3d_n(\Delta x_n)^2=s_{n+1} \end{cases}$

Recall $\begin{cases} \Delta x_1 = 2 \\ \Delta x_2 = 1 \\ \Delta x_3 = 3 \\ \Delta x_4 = 2 \end{cases}$ and $\begin{cases} y_1 = 2 \\ y_2 = 1 \\ y_3 = 4 \\ y_4 = 3 \\ y_5 = -1 \end{cases}$, substitute and solve gives

$$\begin{cases} a_1 = y_1 \\ a_1 + b_1(\Delta x_1) + c_1(\Delta x_1)^2 + d_1(\Delta x_1)^3 = y_2 \\ b_1 = s_1 \\ b_1 + 2c_1(\Delta x_1) + 3d_1(\Delta x_1)^2 = s_2 \end{cases} \Longrightarrow \begin{cases} a_1 = 2 \\ b_1 = -\dfrac{2285}{1248} \\ c_1 = 0 \\ d_1 = \dfrac{1661}{4992} \end{cases}$$

$$\begin{cases} a_2 = y_2 \\ a_2 + b_2(\Delta x_2) + c_2(\Delta x_2)^2 + d_2(\Delta x_2)^3 = y_3 \\ b_2 = s_2 \\ b_2 + c_2(\Delta x_2) + d_2(\Delta x_2)^2 = s_3 \end{cases} \Longrightarrow \begin{cases} a_2 = 1 \\ b_2 = \dfrac{1349}{624} \\ c_2 = \dfrac{1661}{832} \\ d_2 = -\dfrac{2891}{2496} \end{cases}$$

$$\begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \Longrightarrow \begin{cases} a_3 = 4 \\ b_3 = \dfrac{6689}{2496} \\ c_3 = -\dfrac{615}{416} \\ d_3 = \dfrac{91}{576} \end{cases}$$

$$\begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \Longrightarrow \begin{cases} a_4 = 3 \\ b_4 = -\dfrac{1201}{624} \\ c_4 = -\dfrac{47}{832} \\ d_4 = \dfrac{47}{4992} \end{cases}$$