Q1:

a)

$$P = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \end{matrix}$$

Since there is no dead end page,

$$Q = P$$

And

Finally

$$M = \alpha Q + \frac{1-\alpha}{7}\vec{e}\vec{e}^T$$

$$= \begin{matrix} \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \\ \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \\ \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \\ \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \\ \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \\ \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \alpha + \frac{1-\alpha}{7} \\ \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \alpha + \frac{1-\alpha}{7} & \frac{1}{2}\alpha + \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} & \frac{1-\alpha}{7} \end{matrix}$$

Is the google matrix.

b)

after 15 iterations, $\vec{x} = \begin{bmatrix} 0.21028926 \\ 0.05125640 \\ 0.13096160 \\ 0.10953409 \\ 0.06849849 \\ 0.22735764 \\ 0.20210252 \end{bmatrix}$

iterations as below:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.14286	0.14286	0.19446	0.26026	0.21366	0.18989	0.20336	0.23055	0.21048	0.2022	0.20858	0.2195	0.21017	0.20731	0.21029
0.14286	0.08214	0.05634	0.04537	0.04537	0.04933	0.05439	0.05081	0.04898	0.05002	0.0521	0.05056	0.04993	0.05042	0.05126
0.14286	0.14286	0.11705	0.12802	0.15132	0.13152	0.1231	0.13097	0.141	0.1317	0.12862	0.13222	0.13621	0.13197	0.13096
0.14286	0.08214	0.08214	0.10408	0.13204	0.11223	0.10213	0.10785	0.11941	0.11088	0.10736	0.11007	0.11472	0.11075	0.10953
0.14286	0.08214	0.05634	0.05634	0.06566	0.07755	0.06913	0.06483	0.06727	0.07218	0.06855	0.06706	0.06821	0.07018	0.0685
0.14286	0.20357	0.28098	0.22615	0.19818	0.21403	0.24602	0.22241	0.21268	0.22017	0.23303	0.22205	0.21868	0.22219	0.22736
0.14286	0.26429	0.21268	0.17978	0.19376	0.22545	0.20188	0.19258	0.20018	0.21285	0.20175	0.19853	0.20208	0.20718	0.2021

c)

15	16	17
0.21029	0.21468	0.21041
0.05126	0.05054	0.05032
0.13096	0.13259	0.13415
0.10953	0.1108	0.11267
0.0685	0.06798	0.06852
0.22736	0.22233	0.22124
0.2021	0.20108	0.2027

by watching the 15^{th} and 16^{th} column, in the first row, 0.210 does not equal to 0.215, therefore the Ranking vector $\vec{x}$ does not match $M\vec{x} = \vec{x}$ up to at least 3 significant digits. To reach 3 significant digits there should be around 24 iterations.

Q2:

Let matrix $O = P + \frac{1}{R}ed^T$. As discussed in class, columns of O sums to 1.

Since columns of $\begin{bmatrix} \frac{1}{R} & \cdots & \frac{1}{R} \\ \vdots & \ddots & \vdots \\ \frac{1}{R} & \cdots & \frac{1}{R} \end{bmatrix}$ also sum to 1, columns of $\frac{1}{R}ee^T$ sum to 1.

Hence for each column of M, $sum = \ \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1$ as required.

As discussed in class, matrix O is the probability matrix, replacing entries of zero columns with 1/R.

Hence for each entry of O, $0 \leq [o_{ij}] \leq 1$,

Then $[q_{ij}] = [o_{ij}] + \frac{1-\alpha}{R} > 0$.

Since columns of M sum to 1, each entry of M is strictly greater than 0, then for any entry, the entry is equal to 1 minus all other entries in that column, which must be less than 1, $[q_{ij}] < 1$.

Hence, we can conclude that Google matrix is a Markov matrix.

Q3:

$$A = LU$$

$$AA^T \cdot \vec{x} = \vec{b}$$

Substitute gives

$$LU(LU)^T \cdot \vec{x} = \vec{b}$$
$$LUU^TL^T \cdot \vec{x} = \vec{b}$$

Then we can solve $L\vec{a} = \vec{b}$ in $\mathrm{O}(n^2)$ time. Now we know $\vec{a} = UU^TL^T \cdot \vec{x}$.

Then solve $U\vec{c} = \vec{a}$ in $\mathrm{O}(n^2)$ time. Now we know $\vec{c} = U^TL^T \cdot \vec{x}$

Then solve $U^T\vec{d} = \vec{c}$ in $\mathrm{O}(n^2)$ time. Now we know $\vec{d} = L^T \cdot \vec{x}$

And finally solve $L^T\vec{x} = \vec{d}$ in $\mathrm{O}(n^2)$ time.

In total it took $O(n^2)$ time.

Q4:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 6 \\ 2 & 2 & 2 \\ 2 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 6 \\ 0 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ -\dfrac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 6 \\ 0 & 1 & 5 \\ 0 & -2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ -\dfrac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 6 \\ 0 & -2 & 7 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -4 & -2 & 6 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ -\dfrac{1}{2} & -\dfrac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 6 \\ 0 & -2 & 7 \\ 0 & 0 & \dfrac{17}{2} \end{bmatrix}$$

So $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} -4 & -2 & 6 \\ 0 & -2 & 7 \\ 0 & 0 & \frac{17}{2} \end{bmatrix}$.

Solve $A\vec{x} = \vec{b}$:

$$PA\vec{x} = P\vec{b}$$

$$LU\vec{x} = P\vec{b}$$

Let $\vec{y} = U\vec{x}$, then

$$\begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ -\dfrac{1}{2} & -\dfrac{1}{2} & 1 \end{bmatrix}\vec{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \vec{y} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 4 \\ 7 - \left(-\frac{1}{2}\right) \times 4 \\ 2 - \left(-\frac{1}{2}\right) \times 4 - \left(-\frac{1}{2}\right) \times \left(7 - \left(-\frac{1}{2}\right) \times 4\right) \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ \frac{17}{2} \end{bmatrix}$$

Now solve $\vec{y} = U\vec{x}$ for $\vec{x}$:

$$\begin{bmatrix} -4 & -2 & 6 \\ 0 & -2 & 7 \\ 0 & 0 & \frac{17}{2} \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 9 \\ \frac{17}{2} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{4} \times \left(4 - 6 - (-2) \times \left(\left(-\frac{1}{2}\right) \times (9 - 7)\right)\right) \\ -\frac{1}{2} \times (9 - 7) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Q5:

a)

$$A = \begin{bmatrix} 2 & 1 & 0 & & \cdots & 0 & 0 \\ 1 & 4 & 1 & & & 0 & 0 \\ 0 & 1 & 4 & 1 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & & 1 & 4 & 1 \\ 0 & 0 & 0 & & & 1 & 2 \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 & 0 \\ 0 & 1 & 0 & & & 0 & 0 \\ \vdots & 0 & 1 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & 1 & 0 & \vdots \\ 0 & 0 & & & 0 & 1 & 0 \\ 0 & 0 & \cdots & & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & & \cdots & 0 & 0 \\ 1 & 4 & 1 & & & 0 & 0 \\ 0 & 1 & 4 & 1 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & & 1 & 4 & 1 \\ 0 & 0 & 0 & & & 1 & 2 \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & 0 & & & 0 & 0 \\ \vdots & 0 & 1 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & 1 & 0 & \vdots \\ 0 & 0 & & & 0 & 1 & 0 \\ 0 & 0 & \cdots & & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & & \cdots & 0 & 0 \\ 0 & 4-\frac{1}{2} & 1 & & & 0 & 0 \\ 0 & 1 & 4 & 1 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & & 1 & 4 & 1 \\ 0 & 0 & 0 & & & 1 & 2 \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & 0 & & & 0 & 0 \\ \vdots & \left(4-\frac{1}{2}\right)^{-1} & 1 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & 1 & 0 & \vdots \\ 0 & 0 & & & 0 & 1 & 0 \\ 0 & 0 & \cdots & & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & & \cdots & 0 & 0 \\ 0 & 4-\frac{1}{2} & 1 & & & 0 & 0 \\ 0 & 0 & 4-\left(4-\frac{1}{2}\right)^{-1} & 1 & & 0 & 0 \\ & & \ddots & & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & & 1 & 4 & 1 \\ 0 & 0 & 0 & & & 1 & 2 \end{bmatrix}_{n\times n}$$

$l_2 = \frac{1}{2}$, $l_3 = \left(4-\frac{1}{2}\right)^{-1}$, from the calculation steps we get

$$l_n = (4 - l_{n-1})^{-1}$$

$d_1 = 2$, $d_2 = 4 - \frac{1}{2}$, from the calculation steps we get

$$d_n = 4 - d_{n-1}^{-1}$$

b) once we know the previous value, by the recurrence formula, we need constant time to compute the next. So the cost of computing LU decomposition is

$$2 \times \overbrace{1 + 1 + \cdots + 1}^{n\ terms} \in \mathrm{O}(n)$$

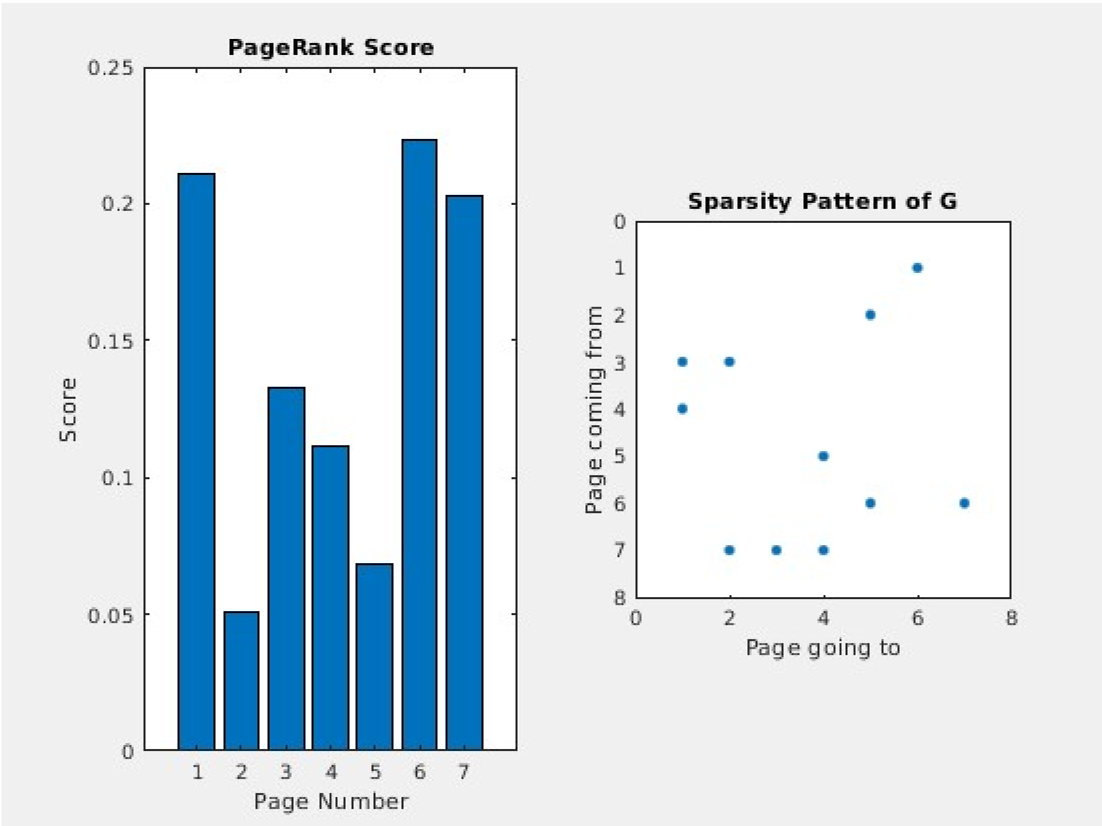
c)

since in L and U matrices, each row only contains two non-zero entries. Therefore solving $L\vec{y} = \vec{b}$ and $U\vec{x} = \vec{y}$ require steps $\overbrace{1 + 2 + 2 + \cdots + 2}^{n\ terms} \in \mathrm{O}(n)$. Therefore solving $A\vec{x} = \vec{b}$ requires $\mathrm{O}(n)$ time, we get the derivatives at all points in $\mathrm{O}(n)$ time. By now we know all the coordinates and derivatives at those points. Using formulas $a_n = y_n, b_n = s_n, c_n = \frac{3y_n' - 2s_n - s_{n+1}}{\Delta x}, d_n = \frac{s_n + s_{n+1} - 2y_n'}{\Delta x^2}$, which take $\mathrm{O}(1)$ time for each interval, we can calculate the spline in $\mathrm{O}(n)$ time.

Q6:

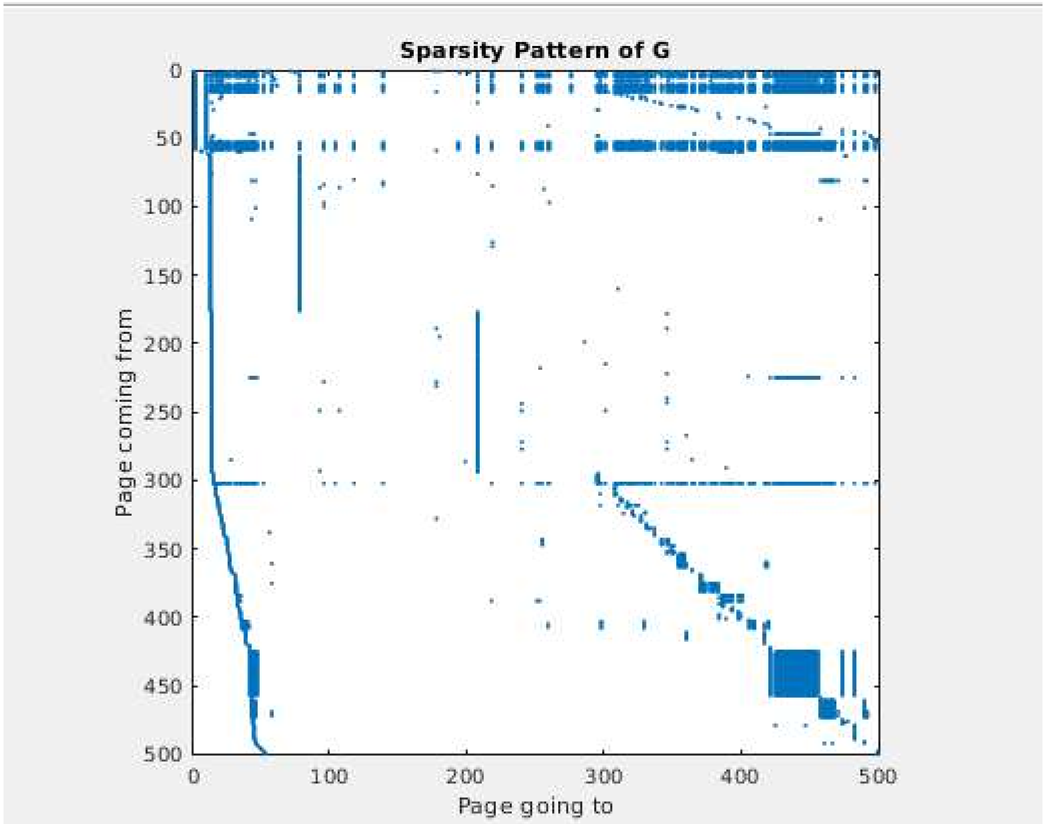
b)

the order of importance is 6,1,7,3,4,5,2 in decreasing order.



c)

1: https://uwaterloo.ca/support
2: https://www.facebook.com/university.waterloo
3: https://uwaterloo.ca/admissions
4: https://uwaterloo.ca/about
5: http://uwaterloo.ca/support
6: http://uwaterloo.ca/admissions
7: http://uwaterloo.ca/about
8: https://ar-ar.facebook.com/university.waterloo
9: https://bg-bg.facebook.com/university.waterloo
10: https://uwaterloo.ca
11: https://uwaterloo.ca/hire
12: https://uwaterloo.ca/offices-services
13: https://uwaterloo.ca/faculties-academics
14: http://www.mozilla.org/firefox
15: http://www.google.com/chrome
16: http://windows.microsoft.com/en-US/internet-explorer/products/ie/home
17: http://uwaterloo.ca/hire
18: https://twitter.com/uWaterloo
19: https://www.youtube.com/uwaterloo
20: https://instagram.com/uofwaterloo



d)

alpha	Number of Iterations
0.15	7
0.35	12
0.55	20
0.75	41
0.95	225

The number of iterations increases as alpha increases, with an increasing acceleration.

As alpha increases, the weight of following links increases, then the algorithm needs more iterations to follow the links, and finally converge to the page ranking vector within tolerance.