Marking Sheet   Assignment 1   CS370 Winter 2018

Name: ....   ID:................

Section: 1 (8:30am)   2 (11:30am)   3 (1:30pm)

1. (a) roots by quadratic formula	(4)	________	
(b) alternate quadratic formula	(3)	________	
(c) roots by alternate formula	(3)	________	......../10
2. (a) Matlab function	(4)	________	
(b) stability analysis	(6)	________	
(c) number of steps	(5)	________	......../15
3. (a) Conditions and values	(6)	________	
(b) Lagrange	(4)	________	......../10
4. (a) Linear system	(8)	________	
(b) Spline coefficients	(8)	________	
(b) Graphic	(4)	________	........./20
5. (a) points	(4)	________	
(b) lines	(4)	________	
(c) splines	(12)	________	......../20

Total: (75) ____________

A1:

a)

$$\begin{aligned} x_1 &= \frac{-b-\sqrt{b^2-4ac}}{2a} \\ &= \frac{-22.41-\sqrt{22.41^2-4\times 1.03\times 0.1123}}{2\times 1.03} \\ &= \frac{-22.41-\sqrt{502.20-4.12\times 0.1123}}{2.06} \\ &= \frac{-22.41-\sqrt{502.20-0.46267}}{2.06} \\ &= \frac{-22.41-22.399}{2.06} \\ &= \frac{-44.809}{2.06} \\ &= -21.751 \end{aligned}$$

Relative error: $\left|\frac{(-21.752)-(-21.751)}{-21.752}\right| \times 100\% = 0.004597\%$

$$\begin{aligned} x_2 &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ &= \frac{-22.41+22.399}{2.06} \\ &= \frac{-0.011}{2.06} \\ &= -0.0053398 \end{aligned}$$

Relative error: $\left|\frac{(-0.0050123)-(-0.0053398)}{-0.0050123}\right| \times 100\% = 6.133\%$

b)

$$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\begin{aligned} right &= \frac{c}{ax_1} \\ &= \frac{c}{a\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)} \\ &= \frac{2c}{\left(-b-\sqrt{b^2-4ac}\right)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2c\left(-b + \sqrt{b^2 - 4ac}\right)}{\left(-b - \sqrt{b^2 - 4ac}\right)\left(-b + \sqrt{b^2 - 4ac}\right)} \\
&= \frac{-2bc + 2\mathrm{c}\sqrt{b^2 - 4ac}}{\mathrm{b}^2 - (b^2 - 4ac)} \\
&= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
&= x_2 \\
&= left
\end{aligned}
$$

c)

x_1 is unchanged

$$x_1 = -21.751$$

Relative error: $\left|\frac{(-21.752)-(-21.751)}{-21.752}\right| \times 100\% = 0.004597\%$

$$
\begin{aligned}
x_2 &= \frac{c}{ax_1} \\
&= \frac{0.1123}{1.03 \times (-21.751)} \\
&= \frac{0.1123}{-22.403} \\
&= -0.0050127
\end{aligned}
$$

Relative error: $\left|\frac{(-0.0050127)-(-0.0050123)}{-0.0050123}\right| \times 100\% = 0.007980\%$

Q2:

a) The answer becomes an extremely negative number. (-4.063562×10^{13}).

b) the error in the first step is $e_1 = 0$, in the second step it is $e_2 = \left|0.5714285 - \frac{4}{7}\right| \leq \frac{1}{14000000}$.

Since $e_n + \frac{41}{14}e_{n-1} - 2e_{n-2} = 0$ for $n \geq 3$, let $a = \frac{41}{14}$ and $b = -2$, solve $x^2 + ax + b = 0$ gives $x_1 = -\frac{7}{2}$, $x_2 = \frac{4}{7}$. Hence $|e_{40}| = \left|c_1\left(-\frac{7}{2}\right)^{40} + c_2\left(\frac{4}{7}\right)^{40}\right| \approx \left|c_1\left(\frac{7}{2}\right)^{40}\right|$, which grows very big, therefore the final answer is far off from the correct answer.

c) assume $p_0^{approx} = p_0^{exact}$,

$$\begin{aligned} p_1^{approx} &= p_1^{exact}(1+\epsilon) \\ &= p_1^{exact} + e_1 \end{aligned}$$

Where $e_1 = p_1^{approx}\epsilon$.

$$\begin{aligned} p_2^{approx} &= -\frac{41}{14}p_1^{approx} + 2p_0^{approx} \\ &= -\frac{41}{14}p_1^{exact} - \frac{41}{14}e_1 + 2p_0^{exact} \\ &= p_2^{exact} - \frac{41}{14}e_1 \end{aligned}$$

$$\begin{aligned} p_3^{approx} &= -\frac{41}{14}p_2^{approx} + 2p_1^{approx} \\ &= \left(-\frac{41}{14}\right)p_2^{exact} - \left(-\frac{41}{14}\right)\frac{41}{14}e_1 + 2p_1^{exact} + 2e_1 \\ &= p_3^{exact} + \frac{41^2 + 2 \times 14^2}{14^2}e_1 \end{aligned}$$

by some induction steps similar as above, we know for $n \geq 3$,

$$p_n^{qpprox} = p_n^{exact} + xe_1$$

Where x is some function of n and $|x|$ increases as n increases.
Even though e_1 is small, when n gets big, the error $|xe_1|$ becomes large, and the answer becomes unreliable.

Q3:

$$S(x)=\begin{cases} 28+a_1x+9x^2+x^3, & x\in[-3,-1] \\ a_2+19x+a_3x^2-x^3, & x\in[-1,0] \\ 26+19x+3x^2+a_4x^3, & x\in[0,3] \end{cases}$$

a)

when $x=-1$, we have

$$\begin{aligned} 28+a_1x+9x^2+x^3 &= a_2+19x+a_3x^2-x^3 \\ 28-a_1+9-1 &= a_2-19+a_3+1 \\ a_1+a_2+a_3 &= 54 \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dx}(28+a_1x+9x^2+x^3) &= \frac{d}{dx}(a_2+19x+a_3x^2-x^3) \\ a_1+18x+3x^2 &= 19+2a_3x-3x^2 \\ a_1-18+3 &= 19-2a_3-3 \\ a_1+2a_3 &= 31 \end{aligned}$$

When $x=0$, we have

$$\begin{aligned} a_2+19x+a_3x^2-x^3 &= 26+19x+3x^2+a_4x^3 \\ a_2 &= 26 \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dx}(a_2+19x+a_3x^2-x^3) &= \frac{d}{dx}(26+19x+3x^2+a_4x^3) \\ 19+2a_3x-3x^2 &= 19+6x+3a_4x^2 \\ 19 &= 19 \end{aligned}$$

We have $\begin{cases} a_1+a_2+a_3=54 \\ a_1+2a_3=31 \\ a_2=26 \end{cases} \Longrightarrow \begin{cases} a_1=25 \\ a_2=26. \\ a_3=3 \end{cases}$

Since it is a natural spline, when $x=3$ we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{d}{dx}(26+19x+3x^2+a_4x^3)\right) &= 0 \\ 6+6a_4x &= 0 \\ a_4 &= -\frac{1}{3} \end{aligned}$$

Hence $\begin{cases} a_1=25 \\ a_2=26 \\ a_3=3 \\ a_4=\frac{1}{3} \end{cases}$ is the solution.

b)

by substitute $x = -3, -1,0,3$ into $S(x)$ we get $\begin{cases} S(-3) = 7 \\ S(-1) = 11 \\ S(0) = 26 \\ S(3) = 101 \end{cases}$.

the interpolation is

$$L(x) = 7 \times \frac{(x+1)x(x-3)}{(-3+1)(-3)(-3-3)} + 11 \times \frac{(x+3)x(x-3)}{(-1+3)(-1)(-1-3)} +$$

$$26 \times \frac{(x+3)(x+1)(x-3)}{(0+3)(0+1)(0-3)} + 101 \times \frac{(x+3)(x+1)x}{(3+3)(3+1)3}$$

$$= 7 \times \frac{(x+1)x(x-3)}{-36} + 11 \times \frac{(x+3)x(x-3)}{8} +$$

$$26 \times \frac{(x+3)(x+1)(x-3)}{-9} + 101 \times \frac{(x+3)(x+1)x}{72}$$

Q4:

$$
\begin{aligned}
S_1(x) &= a_1 + b_1(x+1) + c_1(x+1)^2 + d_1(x+1)^3 && -1 \le x \le 1 \\
S_2(x) &= a_2 + b_2(x-1) + c_2(x-1)^2 + d_2(x-1)^3 && 1 \le x \le 2 \\
S_3(x) &= a_3 + b_3(x-2) + c_3(x-2)^2 + d_3(x-2)^3 && 2 \le x \le 5 \\
S_4(x) &= a_4 + b_4(x-5) + c_4(x-5)^2 + d_4(x-5)^3 && 5 \le x \le 7
\end{aligned}
$$

a)

From second derivatives we know

$$
\begin{cases}
2c_1 + 6d_1(0) = 0 \\
2c_1 + 6d_1(2) = 2c_2 + 6d_2(0) \\
2c_2 + 6d_2(1) = 2c_3 + 6d_3(0) \\
2c_3 + 6d_3(3) = 2c_4 + 6d_4(0) \\
2c_4 + 6d_4(2) = 0
\end{cases}
$$

since $x_{1,2,3,4,5} = -1,1,2,5,7$ and $y_{1,2,3,4,5} = 2,1,4,3,-1$ respectively, we can see

$$
\begin{cases}
\Delta x_1 = 2 \\
\Delta x_2 = 1 \\
\Delta x_3 = 3 \\
\Delta x_4 = 2
\end{cases}
\text{and}
\begin{cases}
\Delta y_1 = -1 \\
\Delta y_2 = 3 \\
\Delta y_3 = -1 \\
\Delta y_4 = -4
\end{cases}, \text{therefore}
\begin{cases}
y_1' = -\frac{1}{2} \\
y_2' = 3 \\
y_3' = -\frac{1}{3} \\
y_4' = -2
\end{cases}
$$

Since $c_n = \frac{3y_n' - 2s_n - s_{n+1}}{\Delta x_n}$, $d_n = \frac{s_n + s_{n+1} - 2y_n'}{\Delta x_n^2}$, we get

$$
\begin{cases}
c_1 = \frac{-\frac{3}{2} - 2s_1 - s_2}{2} \\
c_2 = \frac{9 - 2s_2 - s_3}{1} \\
c_3 = \frac{-1 - 2s_3 - s_4}{3} \\
c_4 = \frac{-6 - 2s_4 - s_5}{2}
\end{cases}
\text{and}
\begin{cases}
d_1 = \frac{s_1 + s_2 + 1}{4} \\
d_2 = \frac{s_2 + s_3 - 6}{1} \\
d_3 = \frac{s_3 + s_4 + \frac{2}{3}}{9} \\
d_4 = \frac{s_4 + s_5 + 4}{4}
\end{cases}
$$

Substitute all into the original gives

$$\begin{cases} 2\dfrac{-\frac{3}{2}-2s_1-s_2}{2}+6\dfrac{s_1+s_2+1}{4}(0)=0 \\ 2\dfrac{-\frac{3}{2}-2s_1-s_2}{2}+6\dfrac{s_1+s_2+1}{4}(2)=2\dfrac{9-2s_2-s_3}{1} \\ 2\dfrac{9-2s_2-s_3}{1}+6\dfrac{s_2+s_3-6}{1}(1)=2\dfrac{-1-2s_3-s_4}{3} \\ 2\dfrac{-1-2s_3-s_4}{3}+6\dfrac{s_3+s_4+\frac{2}{3}}{9}(3)=2\dfrac{-6-2s_4-s_5}{2} \\ 2\dfrac{-6-2s_4-s_5}{2}+6\dfrac{s_4+s_5+4}{4}(2)=0 \end{cases}$$

Simplify gives

$$\begin{cases} 2s_1+s_2=-\dfrac{3}{2} \\ s_1+6s_2+2s_3=\dfrac{33}{2} \\ 3s_2+8s_3+s_4=26 \\ 2s_3+10s_4+3s_5=-20 \\ s_4+2s_5=-6 \end{cases}$$

b)

$$\begin{cases} 2s_1+s_2=-\dfrac{3}{2} \\ s_1+6s_2+2s_3=\dfrac{33}{2} \\ 3s_2+8s_3+s_4=26 \\ 2s_3+10s_4+3s_5=-20 \\ s_4+2s_5=-6 \end{cases}$$

Solve to get $\begin{cases} s_1=-\frac{2285}{1248} \\ s_2=\frac{1349}{624} \\ s_3=\frac{6689}{2496} \\ s_4=-\frac{1201}{624} \\ s_5=-\frac{2543}{1248} \end{cases}$

Since for any S_n we have $\begin{cases} a_n=y_n \\ a_n+b_n(\Delta x_n)+c_n(\Delta x_n)^2+d_n(\Delta x_n)^3=y_{n+1} \\ b_n=s_n \\ b_n+2c_n(\Delta x_n)+3d_n(\Delta x_n)^2=s_{n+1} \end{cases}$

Recall $\begin{cases} \Delta x_1 = 2 \\ \Delta x_2 = 1 \\ \Delta x_3 = 3 \\ \Delta x_4 = 2 \end{cases}$ and $\begin{cases} y_1 = 2 \\ y_2 = 1 \\ y_3 = 4 \\ y_4 = 3 \\ y_5 = -1 \end{cases}$, substitute and solve gives

$$\begin{cases} a_1 = y_1 \\ a_1 + b_1(\Delta x_1) + c_1(\Delta x_1)^2 + d_1(\Delta x_1)^3 = y_2 \\ b_1 = s_1 \\ b_1 + 2c_1(\Delta x_1) + 3d_1(\Delta x_1)^2 = s_2 \end{cases} \Longrightarrow \begin{cases} a_1 = 2 \\ b_1 = -\dfrac{2285}{1248} \\ c_1 = 0 \\ d_1 = \dfrac{1661}{4992} \end{cases}$$

$$\begin{cases} a_2 = y_2 \\ a_2 + b_2(\Delta x_2) + c_2(\Delta x_2)^2 + d_2(\Delta x_2)^3 = y_3 \\ b_2 = s_2 \\ b_2 + c_2(\Delta x_2) + d_2(\Delta x_2)^2 = s_3 \end{cases} \Longrightarrow \begin{cases} a_2 = 1 \\ b_2 = \dfrac{1349}{624} \\ c_2 = \dfrac{1661}{832} \\ d_2 = -\dfrac{2891}{2496} \end{cases}$$

$$\begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \Longrightarrow \begin{cases} a_3 = 4 \\ b_3 = \dfrac{6689}{2496} \\ c_3 = -\dfrac{615}{416} \\ d_3 = \dfrac{91}{576} \end{cases}$$

$$\begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \Longrightarrow \begin{cases} a_4 = 3 \\ b_4 = -\dfrac{1201}{624} \\ c_4 = -\dfrac{47}{832} \\ d_4 = \dfrac{47}{4992} \end{cases}$$