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The Well Tempered Term Project: A Musical Ode-yssey

I remember the first time I heard someone mention the difference between a C♯ and D♭. It was my violin friend. I had been playing piano for a few years, so I thought he was crazy, since on a (regular) piano there's only one key for both. (And if the notes were different, who's good enough to hear the difference, or play one instead of the other?) Eventually I began to learn more about musical tuning systems, and I no longer think my friend is crazy, at least not for that.

This paper seeks to outline the history of musical tuning systems in broad strokes, beginning with Pythagorean tuning and ending with modern equal temperament, though alternative tuning systems are occasionally proposed and even used today. Accompanying this paper is a diagram visualizing pitch intervals in each key while tuned in one fixed key. Also included is the code to generate this diagram[1] and an audio comparison of various notes and keys.[2]

Before diving into tuning systems, we start with a bit of background. We note that we are mainly studying the 12-note scale, common in Western cultures. Following Barbour (1972), by a *tuning* we mean a system in which all "intervals can be expressed in rational numbers," and by a *temperament* we mean a system that is not a tuning—that is, at least one irrational interval is used (p. xii). The *diatonic scale* consists of seven notes in each octave with spacing as of the white keys on a piano, while the *chromatic scale* consists of all twelve notes in an octave. In referring to a *consonant* interval or chord, we mean one that is pleasant or harmonious to the ear,

[1] https://github.com/Burt360/tuning-systems
[2] https://www.youtube.com/watch?v=OB2uVhkyDGU

while *dissonant* means unpleasant or harsh. The main tuning systems we will discuss are Pythagorean tuning, just intonation, meantone temperament, and equal temperament.

And now we begin with the Greek Pythagoras (c. 570–495 BC). Before this time the development of tuning systems is unclear, while his contribution marks a decided and documented advancement in music theory. The tradition goes that Pythagoras was experimenting on a "monochord," a device with a single string and adjustable endpoints so that the pitch of a played note can be varied and precisely measured. (Of course, multiple strings may be set up, allowing one to set each string to a different length.) In his experimenting, he discovered that basic ratios of string lengths produced consonant chords; for example, the ratio 2:1 produces an octave, or the ratio 3:2 produces a "fifth" (such as from C to G) (Rees, 2009).

This gave rise to the "Pythagorean tuning," in which all seven notes of the diatonic scale are derived using only these two intervals. For example, one may start with an initial note F and obtain C, G, D, A, E, B. However, this also results in major thirds that are a little sharp (Ellis & Mendel, 1969, p.12). One can see this in Figure 1: in the base key of C, the fifth is a perfect interval (on the vertical red line), while the major third is somewhat sharp. One may also use this system of octaves and perfect fifths to obtain the entire chromatic scale, though the Greeks made modifications if they did so (Barbour, 1972, p. 1).

However, one can observe in the diagram that some intervals—the third, sixth, and seventh—deviate from perfect consonant intervals in the base key. Claudius Ptolemy (c. 100–170 AD) suggested that a tuning system should not solely depend on mathematical theory, as Pythagoras thought, but also on what is pleasing to the ear. Ptolemy held that ratios of the form $(n + 1)/n$ yielded intervals that were pleasant both to the listening ear and to the mathematical mind. This pattern leads to one variety of what is today known as "just intonation" (Barbour, 1972, p. 2). It should be noted that the vertical red lines in the diagram correspond to this tuning, since these intervals are one possible “ideal” to obtain the consonance for which one would hope in a musical tuning.

Just intonation didn’t gain favor after Ptolemy’s time. In fact, the math seems unknown through to the Middle Ages. On the other hand, it seems Pythagorean tuning continued to be the dominant theory into the 1500s. The musical styles of the time (such as Gregorian chant and organum) didn’t require much harmony, and the intervals they relied on were the more consonant ones in Pythagorean tuning (Barbour, 1972, pp. 2–3).

However, as time went on, additional intervals were used, and people began to view major thirds as consonant rather than dissonant intervals, so that the major thirds and sixths of Pythagorean tuning were perceived as more harsh and dissonant (Barbour, 1972, p. 3). As much of the music consisted of singing, it’s possible that singers subconsciously adjusted their pitches to more “just” ratios—those coinciding with just intonation (Ellis & Mendel, 1969, p. 12; Barbour, 1972, p. 3). But theorists began to innovate to find systems without the dissonances. For example, one attempt was somewhat a combination of Pythagorean tuning and just intonation, presented as an adjustment to Pythagorean tuning by Ramos de Pareja (c. 1440–1522). It’s possible there was much experimentation even though little survives to our day (Barbour, 1972, p. 4).

Beginning in the later 1500s, it seems more history of tuning systems was recorded, but the trends leading to today’s tuning paradigm also become messier across so many instruments and cultures. Meantone temperament, proposed by Francisco de Salinas (c. 1513–1590), was based on perfect major thirds and flattened fifths (Barbour, 1972, p. x; Ellis & Mendel, 1969, p. 12), and seems to have been for some time a dominant temperament on keyboard instruments, such as the organ (Barbour, 1972, p. 11).

On the other hand, there is evidence that string instruments, such as those in the lute and viol families, took a different path. At least at first, Pythagorean tuning held sway (Barbour, 1972, p. 4; Ellis & Mendel, 1969, p. 12). This makes sense, as the strings are commonly tuned in perfect fifths, the interval on which Pythagorean tuning is based (along with the octave). However, these instruments had fixed pitches due to the use of frets, and some intervals were very imperfect. In fact, even an octave might not sound at all like an octave (Barbour, 1972, p. 8). (This situation isn't pictured in Figure 1.) Hence, an alternate system became necessary.

In particular, various theorists attempted equal temperament, in which all semitones (half-steps, or the distance, for example, from C to C♯) are exactly equal. Such an idea may have been proposed as early as 335 BC by the Greek Aristoxenus of Tarentum (Barbour, 1972, p. 2; Ellis & Mendel, 1969, p. 12), but it evidently wasn't used much if at all until the string instrument difficulties. One proponent was Salinas, though we have no record of an exact method from him of constructing the intervals. Another was Vincenzo Galilei (1520–1591), who proposed equal semitones using the ratio 18:17. This wouldn't quite achieve a perfect octave, but very nearly—a ratio of about 1.986:1 rather than 2:1—so that minor adjustments would yield the octave (Barbour, 1972, pp. 6, 8). While the predominant formula or exact method of construction of the time isn't clear, it seems apparent that this became the dominant system on string instruments thereafter. One can see in Figure 1 that most of the intervals are slightly off from just intonation, but the differences are uniform across keys, and no key suffers extreme deviations from perfect intervals, as other systems do. (See, for example, the sixth in the key of E in either just intonation or meantone temperament.)

Now we see that for some time, meantone temperament was used on keyboard instruments and equal temperament on strings. This created problems whenever the two groups played together, so that they tended to play separately rather than suffer the disharmonies. However, in the 1600s the unfretted violin family of string instruments began its rise to popularity, alleviating the difficulties of playing in two tuning systems. As a result, strings and keyboards were able to play together—but as a result the adoption of equal temperament on keyboards may also have been delayed (Barbour, 1972, p. 8).

But eventually meantone temperament gave way to equal temperament on keyboards, in addition to the other instrument families already using equal temperament. According to one 1880 analysis, this transition began around 1844 on harpsichords and 1854 on organs (Ellis, p. 550). If the switch truly did take place from meantone directly to equal, the change would have been jarring on any given instrument. It's likely, instead, that a variety of other minor systems were used during the transition, and it wasn't unheard of for musicians to make small adjustments to their instruments to mitigate dissonances (Barbour, 1972, pp. 5, 13). Between these two departures from meantone and related temperaments, keyboard instruments would have arrived at tunings substantially in agreement with exact equal temperament.

The only component that remained was the mathematical construction of equal temperament. As mentioned, Aristoxenus and Salinas proposed the concept but we have no record of how they intended to construct the necessary intervals. Vincenzo proposed equal rational semitones, but they didn't combine to produce an octave. The French polymath Marin Mersenne proposed as the ratio between semitones $\sqrt[4]{2/(3 - \sqrt{2})}$ ("Marin Mersenne," 2023), but this too doesn't produce an octave (it yields about 2.006:1 instead of 2:1). It seems the exact ratio, $\sqrt[12]{2}$, was known in China even before Aristoxenus (Ellis & Mendel, 1969, p. 12), though their musical scales didn't require it as does the 12-tone system (Barbour, 1972, p. 7). And Simon Stevin (1548–1620), a Flemish mathematician, wrote about the ratio around the year 1600 but it wasn't published until the end of the 1800s (Malina, 2010). It's unclear exactly how this final formula became the dominant approach, but it seems likely that it was a gradual rather than sudden shift as the math and methods of construction became known.

As alluded to in the introduction, the arrival and spread of equal temperament didn't mark the end of alternative tuning systems. Some systems use more than 12 tones to the octave—19, 31, even 53. The system of 53 pitches was proposed by the ancient Greeks after Pythagoras, and in the 1800s a harmonium employing the pitches was designed and constructed. The 19- and 31-pitch systems were devised nearer the 1600s, and at least a harpsichord was devised for each, while later mathematicians and musicians continued to study and comment on the systems. However, as one might imagine, these systems haven't been as influential as the 12-tone system, at least in Western cultures, likely in part due to the difficulty of constructing such instruments and in part due to the difficulty of playing them (Barbour, 1972, pp. 8–9).

In this paper we have overviewed the development in tuning theory from Pythagoras's system based on perfect fifths, to just intonation and meantone temperament, and on through modern equal temperament based not on perfect ratios of frequencies but solely on a mathematical construction using exponents (or equivalently radicals or logarithms). Of course, we've omitted many systems and variations even among those that are known, and we've barely scratched the surface of the math involved both for the sake of the reader and for that of the author. Perhaps my biggest takeaways from researching and writing about this topic are respect for the many minds that worked on music theory and mathematics that are difficult even with today's advancements in notation and understanding, and wonder at how imperfect tuning systems can still yield beautiful music.

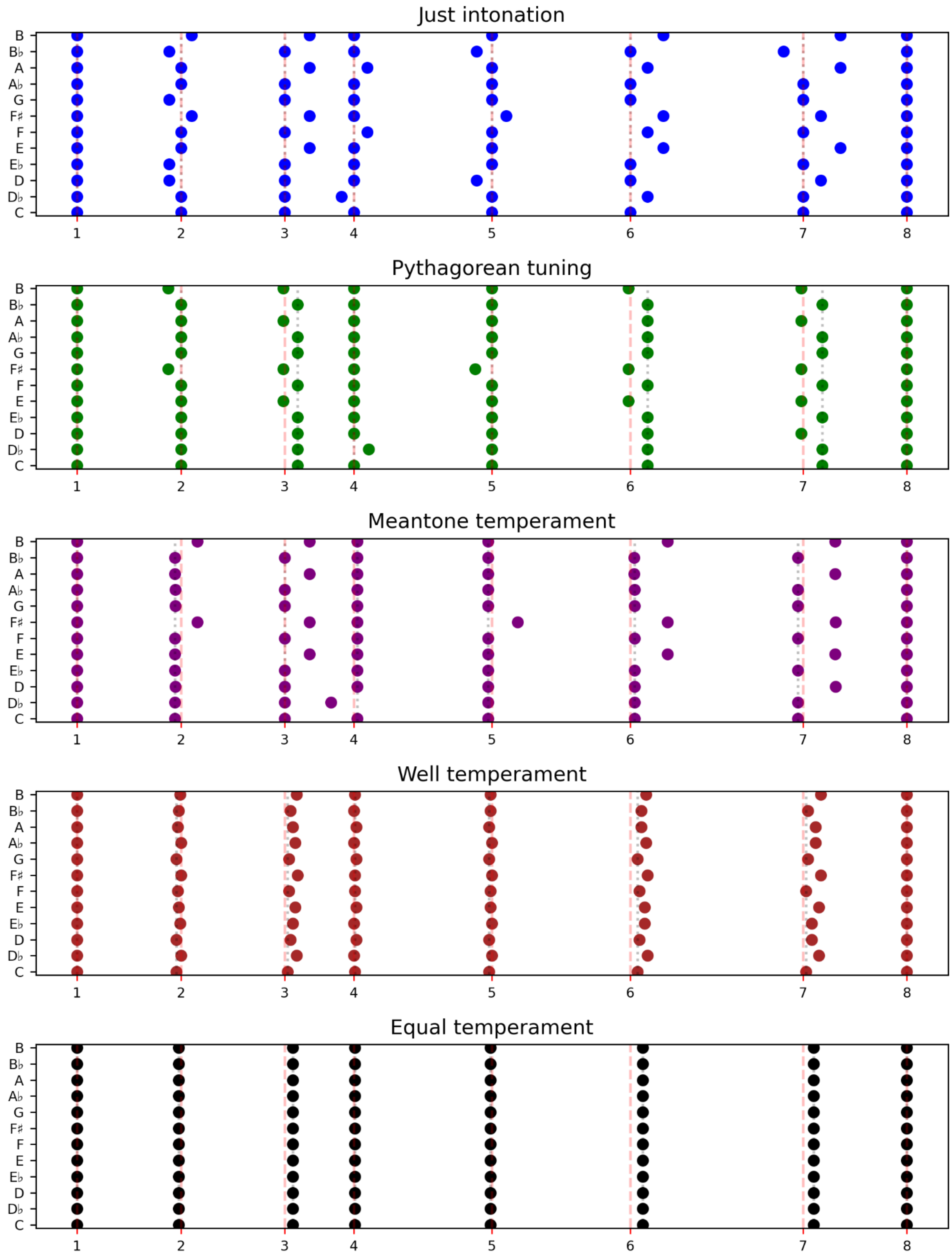
Just intonation
Pythagorean tuning
Meantone temperament
Well temperament
Equal temperament
B
B♭
A
A♭
G
F♯
F
E
E♭
D
D♭
C
1
2
3
4
5
6
7
8

Figure 1: The intervals resulting from playing in one key while tuned to another key in each tuning system. Along the bottom are the numbers 1–8, marking the notes in major scale (the "white keys" on a piano). Each plot uses C major as the tuned key, and the left axes mark the key in which the intervals are played. The red vertical lines mark "just" intervals, or those formed by perfect ratios of frequencies—these are often considered most pleasant to the ear. The gray vertical lines mark the position of the note in the original key, demonstrating the deviation as the intervals are played in other keys.

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