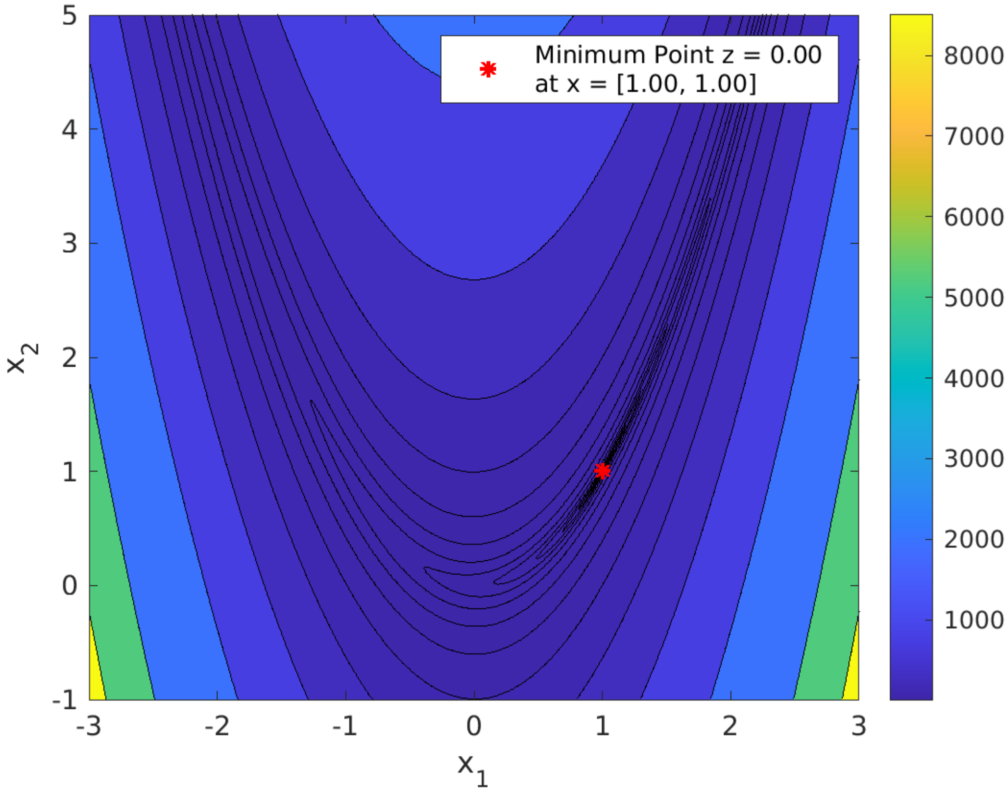
Problem 1: Root Solving

Problem Statement Consider the "Rosenbrock" function for x_1 and x_2:

$$f(\mathbf{x}) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

Part 1.1

Problem Statement Demonstrate the topology of the function for the ranges of $x_1 = [-33]$ and $x_2 = [-15]$



Part 1.2

Problem Statement Visually, where is the global minimum?

Looking specifically at the density of the countour lines, indicates a region around $[11]$ being the minimum.

Part 1.3

> **Problem Statement** Using calculus, demonstrate the answer is a local minimum with the sufficient and necessary conditions.

Second Order Necessary Condition: $\nabla f(\mathbf{x}_*) = \vec{0}$

Second Order Sufficient Condition: $\nabla^2 f(\mathbf{x}_*)$ is positive definite

Calculating the first gradient of the function yields a 2×1 vector:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \left[\begin{array}{c} 2\,x_1 - 400\,x_1\,(x_2 - {x_1}^2) - 2 \\ 200\,x_2 - 200\,{x_1}^2 \end{array} \right]$$

$$\left[\begin{array}{c} 2\,x_1 - 400\,x_1\,(x_2 - {x_1}^2) - 2 \\ 200\,x_2 - 200\,{x_1}^2 \end{array} \right] \Bigg|_{\mathbf{x}=[1,1]^T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

With the first order gradient having zeros in every index, the second order necessary condition is satisified. Taking the second gradient, using the first, yields a 2×2 matrix of second order parital derivatives:

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \left[\begin{array}{cc} 1200\,{x_1}^2 - 400\,x_2 + 2 & -400\,x_1 \\ -400\,x_1 & 200 \end{array} \right]$$

$$\texttt{chol}(\left[\begin{array}{cc} 1200\,{x_1}^2 - 400\,x_2 + 2 & -400\,x_1 \\ -400\,x_1 & 200 \end{array} \right] \Bigg|_{\mathbf{x}=[1,1]^T}) = \left[\begin{array}{cc} \sqrt{802} & -\frac{200\,\sqrt{802}}{401} \\ 0 & \frac{10\,\sqrt{2}\,\sqrt{401}}{401} \end{array} \right]$$

Seeing as the Cholesky factorization was completed successfully, the second order gradient evaluated at $[1,1]^T$ is positive definite, satisfying the second order sufficient condition.