ASE387P Optimal Spacecraft Trajectories

Homework 1

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Problem 1: Computing

Problem Statement Using the following equation complete parts 1-5:

$$P = \cos\left(z^2 - x_1 x_2 x_3\right)$$

Part 1.1

Problem Statement Using a symbolic manipulator, automate the computation of the first derivative of P with respect to $\mathbf{x}$, $\frac{\partial P}{\partial \mathbf{x}}$ or $P_{\mathbf{x}}$. Assuming $z = 4$. Show the resulting $P_{\mathbf{x}}$ for $\mathbf{x} = [1, 2, 3]^T$.

Take the partial derivative of P with respect to each index of $\vec{x}$ results in the following vector $P_{\mathbf{x}}$:

$$P_{\mathbf{x}} = \begin{bmatrix} x_2 x_3 \sin\left(z^2 - x_1 x_2 x_3\right) \\ x_1 x_3 \sin\left(z^2 - x_1 x_2 x_3\right) \\ x_1 x_2 \sin\left(z^2 - x_1 x_2 x_3\right) \end{bmatrix}$$

Plugging in the values for $\mathbf{x} = [1, 2, 3]^T$, results in the following vector: $P_{\mathbf{x}} = [-3.26413, -1.63206, -1.08804]^T$

Part 1.2

Problem Statement Use complex step differentiation to achieve the same gradient as in Part 1. Show the result and absolute error with respect to previous results.

$$f' = \frac{\text{imag}\left(f\left(x_0 + hi\right)\right)}{h}$$

Using the above equation to calculate the partial derivative, $P_{\mathbf{x}}$, a step of h in the imaginary direction is taken for each x_i and then evaluated above. Using an $h = 10^{-20}$, resulted in the follow vector $P_{\mathbf{x}} = [-3.26413, -1.63206, -1.08804]^T$ and an absolute error of 0 (I believe this is due to how Julia shortens values). Not until an $h = 10^{-11}$ does the error term become on the order of 10^{-16}.

Part 1.3

Problem Statement Use a 3, 5, and 7 point central difference stencil to approximate the same gradient using finite differentiating, at a step size of $h = 10^{-4}$. Show formulas used, the approximated gradients, and their absolute errors with respect to Part 1.

$$f'_{\text{3pt}} = \frac{-f(x_0 - h) + f(x_0 + h)}{2h}$$
$$f'_{\text{5pt}} = \frac{f(x_0 - 2h) - 8f(x_0 - h) + f(x_0 + h) - f(x_0 + 2h)}{12h}$$
$$f'_{\text{7pt}} = \frac{-f(x_0 - 3h) + 9f(x_0 - 2h) - 45f(x_0 - h) + 45f(x_0 + h) - 9f(x_0 + 2h) + f(x_0 + 3h)}{60h}$$

Using the above equations for each of the stencils, the following gradients and errors were calculated.

$$f'_{\text{3pt}} = [-3.2641264694904, -1.6320633081884, -1.0880422145226]^T \,; \quad \varepsilon_{\text{3pt}} = 1.9750 \times 10^{-7}$$
$$f'_{\text{5pt}} = [-3.2641266653377, -1.6320633326697, -1.0880422217761]^T \,; \quad \varepsilon_{\text{5pt}} = 3.3971 \times 10^{-12}$$
$$f'_{\text{7pt}} = [-3.264126665337, -1.6320633326698, -1.0880422217775]^T \,; \quad \varepsilon_{\text{7pt}} = 2.6353 \times 10^{-12}$$

Part 1.4

Problem Statement Use a symbolic manipulator to automate the computation of the second derivative of P with respect to $\mathbf{x}$, assuming z = 4. Show the resulting $P_{x_1 x_1}$ or $\frac{\partial^2 P}{\partial \mathbf{x}^2}$ evaluated at $\mathbf{x} = [1, 2, 3]^T$

Symbolically taking the partial derivative of P_{x_1} results in the following equation for $P_{x_1 x_1}$:

$$P_{x_1 x_1} = -{x_2}^2 {x_3}^2 \cos\left(z^2 - x_1 x_2 x_3\right)$$

Evaluating the equation for $\mathbf{x} = [1, 2, 3]^T$ yields this analytical derivative value $P_{x_1 x_1} = 30.206575046752288$.

Part 1.5

Problem Statement Use the complex step method to differentiate the results of Part 1 in order to approximate the Hessian results in Part 4. Show the result and the absolute error with respect to Part 4.

By substituting P_{x_1} into the complex derivative formula for f gives us a numerical solution: $P_{x_1x_1} = 30.206575046752285$ with an error of $3.552713678800501e-15$ for an $h = 10^{-20}$.

Problem 2

Problem Statement Consider the following initial conditions for a two-body orbit. μ is the standard gravitational parameter. Units are generic length and time units (LU and TU).

$$\mu = 1\,\frac{LU^3}{TU^2}; \quad \mathbf{r}_0 = \begin{bmatrix} 1 \\ 1 \\ 0.1 \end{bmatrix} LU; \quad \mathbf{v}_0 = \begin{bmatrix} 0.1 \\ 0.7 \\ 0.3 \end{bmatrix} \frac{LU}{TU}; \quad \mathbf{x}_0 = \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix}$$

Part 2.1

Problem Statement What is the period of the orbit in TU.

Part 2.2

Problem Statement Use a variable step ODE solver with tight tolerances to compute and plot 1 period of the orbit in the xy-plane. Verify periodicity by reporting the norm of the vector difference between the beginning and end states. Change ODE tolerances to reduce this error to under 10^{-13}. Report the order/type of the integrator used and the tolerances needed.

Part 2.3

Problem Statement Using the final tolerance from above, compute the first order state transition matrix for the computed 1 period orbit using:

1) Complex Step Method
2) Variational Method
3) Finite Difference with 3 Point Stencil
4) Finite Difference with 5 Point Stencil

Part 2.4

Problem Statement Report the full 6x6 matrix for Part 2.3.1.

Part 2.5

Problem Statement Report the Frobenius norm of the matrix in Part 2.4.

Part 2.6

Problem Statement Compute the Frobenius norm of the matrix difference of the Parts 2.3.2-2.3.4 with respect to Part 2.3.1.

Part 2.7

Problem Statement Experiment with fixed vs variable step integrators, at different tolerances. Report on the accuracy of the STMs by repeating Part 2.6, treating the complex STM as truth. Discuss the results.

Part 2.8

Problem Statement Compute the full Hessian, $\frac{\partial^2 KE}{\partial \mathbf{x}_0^2}$, of the kinetic energy KE at $t = t_0 + 5\,TU$ with respect to the initial state using:

1) Variational Method plus Chain Rule
2) Finite Difference of Your Choice (Show Formula)

Part 2.9

Problem Statement Report the full 6×6 matrix for Part 2.8.1.

Part 2.10

Problem Statement Report the Frobenius norm of the matrix from Part 2.9.