String Quantization

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1 Discrete phonons

Suppose we have a chain of masses m, each linked by a string of length a. Each mass can be displaced about the equilibrium position vertically, by an amount x_n.

Then we have the following equations of motion:

$$m\ddot{x_n} = k\left(x_{n+1} + x_{n-1}\right) - 2kx_n$$

We can work in the fourier basis, which decouples the motion of the masses into modes of vibrations labelled by the wavenumber $q \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$x_q = e^{i(qan - \omega(q)t)} \tag{1}$$

$$\omega_q^2 = -2k/m\cos(qa) - 2k/m \tag{2}$$

1.1 Lagrangian and Hamiltonian Style

It is nice to work in the Lagrangian and Hamiltonian scheme in preparation for quantization. Let's set up a finite string of N masses, each mass m, separated by distance l in equilibrium. The displacement of the ith mass is ϕ_i.

The Hamiltonian is KE +PE:

$$H = \frac{1}{2}\sum_{i=0}^{N-1}(m(\dot{\phi_i}^2) + k(\phi_i - \phi_{i-1})^2) \tag{3}$$

$$\frac{1}{2}\sum_{i=0}^{N-1} l\left(\frac{m}{l}\dot{\phi_i}^2 + kl^2(\phi_i - \phi_{i-1}/l)^2\right) \tag{4}$$

The lagrangian is:

$$L = \frac{1}{2}\sum_{i=0}^{N-1}(m(\dot{\phi_i}^2) - k(\phi_i - \phi_{i-1})^2) \tag{5}$$

$$\frac{1}{2}\sum_{i=0}^{N-1}\left(l\frac{m}{l}\dot{\phi_i}^2 - kl^2(\phi_i - \phi_{i-1}/l)^2\right)) \tag{6}$$

In the limit N goes to infinity, l to 0 , we can define the ratio: $l/m = \mu$, $kl = T$. The lagrangian density, $\mathcal{L} = L/l$:

$$\mathcal{L} = \int dx\frac{1}{2}\left(\mu\dot{\phi}^2 - T\frac{\partial\phi}{\partial x}^2\right) \tag{7}$$

The principle of least action guarantees that the variation of the action is 0:

$$\delta\int d^4x\mathcal{L}(\phi, \partial_{x^\mu}\phi) = 0 \tag{8}$$

$$\int d^4x\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu(\phi) \tag{9}$$

$$\int d^4(\frac{\partial\mathcal{L}}{\partial\phi}\delta - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{10}$$

The last term is the 4-divergence, which can be converted to the difference between the spatial integrals at the initial and final times, and therefore vanishes.

We therefore have the condition:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{11}$$

Applying equation (11) to the string, we obtain the famous string equation:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{12}$$

where $v^2 = \frac{T}{\mu}$

1.2 Classical Source Term

1.3 Canonical Momenta

We can compute the following canonical momenta:

$$\frac{\partial L}{\partial \dot{\phi}} = \mu\dot{\phi} - T\partial_x\phi v \tag{13}$$

$$\frac{\partial L}{\partial \partial_x\phi} = \frac{\mu}{v}\dot{\phi} - T\partial_x\phi \tag{14}$$

We see the canonical momenta is associated with the momentum of the transverse motion of the wave, i.e. the larger $\dot{\phi}$ is, the larger the canonical momentum.

Some redefinition : for convenience purposes, we redefine the field $\phi\sqrt{T}\phi$, so that T disappears from the equations. The field will have different units

2 Decoupling the String into independent Oscillators

2.1 Normal Coordinates

As with any wave equation, canonical quantization lends itself more nicely in a fourier basis. We make the following guess:

$$\phi_n = \frac{1}{\sqrt{L}} e^{i(k_n x - \omega_n t)} \tag{15}$$

Where we choose

- $\omega_n > 0$
- $k_n = 2n\pi/L$ with $n \in I$(periodic Boundary Conditions)
- $\omega_n^2 = v^2 k_n^2$ (Wave Equation)

We note that there are both positive and negative $\omega and k_n$ solutions. By Plancherel's theorem, the general solution can be written as follows.

$$\phi(x,t) = \sum_{n=-\infty}^{\infty} c_n(a_n\phi_n + a_n * \phi_n *) \quad \phi(x,t) = \sum_{n=-\infty}^{\infty} c_n(a_n(t)e^{ik_n x} + a_n(t) * e^{-ik_n x}) \tag{16}$$

Where $a_n(t) = a(0)e^{-i\omega_n t}$ We have indeed successfully decoupled the string into independent oscillators, since the coefficient $a(t)$ satisfy:

$$\ddot{a_n} + \omega_n^2 a_n = 0$$

The complex conjugate is there to ensure real solutions for the field.

We also have the following orthogonality identities:

$$: \int_0^L \phi_n \phi_m = \delta_{n,-m} e^{-2i\omega_n t} \tag{17}$$

$$\int_0^L \phi_n * \phi_m = \delta_{n,m} \tag{18}$$

2.2 Hamiltonian

The new kinetic energy, potential energy and Hamitonian are:

$$T = \frac{1}{2v^2} \int dx \dot{\phi}^2 \tag{19}$$

$$U = \frac{1}{2} \int dx \frac{\partial \phi}{\partial x}^2 \tag{20}$$

$$H = T + U = \sum_{-\infty}^{\infty} c_n^2 (\frac{\omega^2}{v^2} + k_n^2) a_n^2(t) + c_n c_{-n} (k_n^2 - \frac{\omega_n^2}{v^2}) Re(a_n a_{-n}) \tag{21}$$

$$\tag{22}$$

Using the $c_{-n} = -c_n$, we can rewrite:

$$H = \sum_n 2c_n^2 \frac{\omega_n^2}{v^2} a_n(t) a_n(t)* \tag{23}$$

We work in natural units, where $h = c = 1$. We can make a dimensionless by defining $c_n = \sqrt{\frac{v^2}{2\omega_n}}$. THe hamiltonian takes the form of:

$$H = \sum_n \omega_n a_n a*_n$$

Remarks:

- The number a and a* have a lot of similarity to raising and lowering operators
- To make real a real dynamical variable, we can define $q_n = \frac{1}{\sqrt{2\omega_n}}(a_n + a_n*)$
- We can then define $p_n = \frac{\mathrm{dq}}{\mathrm{dt}} = -i(\frac{\omega_n}{2})^{1/2}(a_n - a_n*)$
- The hamiltonian takes the form: $H = \frac{1}{2}\sum_n (p_n^2 + \omega_n^2 q_n^2)$
- Re-expressing $q_n = \sqrt{2\omega_n} q_n + i\sqrt{\frac{2}{\omega_n}} p_n$
- $p_n = \sqrt{2\omega_n} q_n - i\sqrt{\frac{2}{\omega_n}} p_n$

3 Canonical Quantization

We are now ready to quantize the string. The canonical commutation relations are:

$$[q_n, q_m] = [p_m, p_n] = 0 \tag{24}$$
$$[q_n, p_m] = i\hbar\delta_{nm} \tag{25}$$

Those commutation relations immediately imply commutation relations for the raising lowering operators:

$$[a_n, a_m*] = \delta_{mn}$$