

1 §Ω and Z

In this section, we want to answer why Z is a more efficient technology than the good 'ol Ω.

$$P(\mu_i) = e^{-\beta E_i} \left(\frac{Z_{N-1}(\beta)}{Z_N(\beta)} \right)$$

$$P(\mu_1, \mu_2, \mu_3, \dots) = \frac{e^{-\beta E_1} e^{-\beta E_2} e^{-\beta E_3} \dots}{Z(\beta)} = \frac{e^{-\beta E}}{Z(\beta)}$$

Let's zoom into the system to see what's really going in there. As usual, in thermal equilibrium, the probability of a microstate only depends on its energy.

$$P(\mu_i) = \frac{1}{\Omega_1(E_{\mu_i})}$$

$$P(\mu_1, \mu_2, \mu_3, \dots) = \frac{1}{\Omega_1(E_1)} \frac{1}{\Omega_1(E_2)} \frac{1}{\Omega_1(E_3)} \dots \quad (1)$$

$$= \frac{1}{\Omega_N(E)} \quad (2)$$

Notice that these equations already represent thermal equilibrium, since they are characterized only by energy. Thermal equilibrium is our one and only approximation, hence, one shouldn't expect to do more approximation in order to arrive at Z.

There are three ways to exploit the relation between (1) and (2):

Equation (1) to (2) may look like a tiny step in L^AT_EX, but it is actually a giant step. Rewrite Ω₁ as e^{S₁}, hence the relation becomes

$$\sum_i S_1(E_i) = S_N(E) = S_N(\sum_i E_i)$$

1. principle of least assumption ¹ S_N(E) must not depend on the spread of the E_i distribution, where mathematically you can freely rearrange your energy distribution as you wish

$$S_1(E_1) + S_1(E_2) = S_1(E_1 + E_2) + S_1(0)$$

From here I claim S₁ to be

$$S_1(x) = S_1(0) + \beta x$$

where β = ∂S₁ = constant.

¹a more mediocre and tedious derivation is given in the appendix

Hey, the entropy is automatically maximized, too (which means, it is redundant to further assume that entropy is maximized).

We can now write the probability as

$$P(\mu_i) = e^{-S_1(0) - \beta E_i}$$

2. vary N Back to one particle, we are now authorized to use this relation

$$\begin{aligned} p(\mu_i) &= \frac{\Omega_{N-1}(E - E_{\mu_i})}{\Omega_N(E)} \\ &= \exp(\log \Omega_{N-1}(E - E_{\mu_i}) - \log \Omega_N(E)) \\ &= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp\left(-\frac{\partial}{\partial E} \log \Omega_{N-1} E_i - \frac{1}{2} \partial^2 \log \Omega_{N-1} E_i^2 - \dots\right) \\ &= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp\left(-\beta E_i - \frac{1}{2} \frac{\partial \beta}{\partial E} E_i^2 - \dots\right) \\ &= \frac{\exp(-\beta E_i)}{Z_1(\beta, \mathcal{O}(E_i^2))} \end{aligned}$$

where β is just a label (until this point, you can safely assume that the). There are two completely different arguments: one, the multiplicity is maximized, two, the state only depends on energy

Another way to look at thermal equilibrium: we randomly sample a small element of our system, and yet it will still look exactly the same as its parent system (redundancy of information, rescaling, whatever).

Possible mathematical relations:

$$Z_{1R}(\beta, E_i) = \frac{\Omega_N(E)}{\Omega_{N-1}(E)} \frac{1}{\exp\left(+\frac{\beta}{2C_V T} E_i^2 - \dots\right)}$$

Now we begin to see why we need log. All of our arguments

$$\log Z_1 = \frac{\partial}{\partial N} \log \Omega_N(E)$$

The difference between the two description is that the former gives us the real-life particle description, why the latter only tells us about the smoothed out representation of the system.

In the Z description, we are instead viewing the system as a field ("N is not fixed", but we don't care).

2 APPENDIX

2.1 1. principle of least assumption: not even close to a harmonic oscillator

there are two styles to do the taylor expansion:

1. expand around average energy ϵ

$$\sum_i S_1(\epsilon) + \hat{\partial} S_1 \sum_i (E_i - \epsilon) + \hat{\partial}^2 S_1 \sum_i (E_i - \epsilon)^2 + \dots = S_N(E)$$

$$NS_1(\epsilon) + \hat{\partial} S_1(\epsilon) \cdot 0 + \hat{\partial}^2 S_1(\epsilon) \sigma_\epsilon + \dots = S_N(E)$$

$$NS_1(\epsilon) + \beta \cdot 0 + \frac{1}{2} \frac{\partial \beta}{\partial \epsilon} \sigma_\epsilon + \dots = S_N(E)$$

Unfortunately it is no longer obvious whether the higher order terms will die or not. in general it could be such that, for example $\frac{\partial \beta}{\partial \epsilon} \sigma_\epsilon = f(E)$.

2. expand around $E = 0$

$$\sum_i S_1(0) + \hat{\partial} S_1(0) \sum_i E_i + \hat{\partial}^2 S_1(0) \sum_i E_i^2 + \dots = S_N(E)$$

Now, here comes the crucial step in deriving Z , our **second approximation: we truncate our series by claiming that** (this would later means that the specific heat depends only on the variance of the energy!). One misconception would be to assume that $\frac{\partial \beta}{\partial \epsilon} = 0$.