The Experimental Basis of Electromagnetism: Part II - Electrostatics

1948 Proc. Phys. Soc. 60 27

(http://iopscience.iop.org/0959-5309/60/1/306)

REFERENCES

BODEMANN, 1929, *Ann. Phys., Lpz.*, **3**, 614.
CASE, 1921, *Phys. Rev.*, **17**, 388.
CREW, 1926, *Phys. Rev.*, **28**, 1265.
DE BOER, 1935, *Electron Emission and Adsorption Phenomena* (Cambridge), p. 362.
FINEMAN and EISENSTEIN, 1946, *J. Appl. Phys.*, **17**, 643.
HEADRICK and LEDERER, 1936, *Phys. Rev.*, **50**, 1094.
HUBER and WAGENER, 1942, *Z. tech. Phys.*, **23**, 1.
JACOBS, 1946, *J. Appl. Phys.*, **17**, 596.
JOHNSON, J. B , 1946, *Phys. Rev.*, **66**, 352.
MERRITT, 1921, *Phys. Rev.*, **17**, 525.
NEWBURY, 1929, *Phys. Rev.*, **34**, 1418.
NISHIBORI and KAWAMURA, 1940, *Phys.-Math. Soc. Japan*, **22**, 378.
POMERANTZ, 1946, *Phys. Rev.*, **70**, 33.
RAMANADOFF, 1931, *Phys. Rev.*, **37**, 884.
REIMANN, 1934 *Thermionic Emission* (Chapman and Hall), p. 228.
WRIGHT, D. A., 1947, *Proc. Roy. Soc.*, A, **190**, 394.

The Experimental Basis of Electromagnetism: Part II—Electrostatics

BY N. R. CAMPBELL AND L. HARTSHORN

MS. received 28 *March* 1947

ABSTRACT. This paper continues the argument of a previous paper with the same title. It explores the foundations of the science of electrostatics as practised in the modern laboratory, admitting as evidence only those experiments that experience has proved to be practicable. The basis of the subject is found in alternating currents and the laws of capacitance, which lead to the conception of electric potential energy. The various experiments on the mechanical forces between electrified bodies fit in well with this conception, and we are led to the conclusion that the soundest procedure in the investigation of any system of conductors and dielectrics is to represent it by its equivalent capacitance network. Coulomb's Law and point charges play no part in our scheme, but they are discussed because of the prominent part that they play in classical theory. The familiar equations for point charges, which are the premises from which that theory starts, can be regarded as definitions of what is meant by " point charge ", or as approximations to the laws governing the forces on small charged bodies, which become true in a limiting case. One incidental advantage of our treatment is that the somewhat elusive notion of " earth " in the usual expositions is replaced by more definite conceptions.

§ 1. ALTERNATING CURRENTS

THE general principles to be followed in this enquiry were outlined in a previous paper (1946), here called Part I, which established the magnitudes characteristic of D.C. circuits, viz. current, voltage, resistance etc. Our present object is to show that the working principles of electrostatics as practised in the modern laboratory can be soundly based on real experiments, and in this way to indicate the true nature of the experimental foundations of the subject. It is a commonplace among experimental workers that the magnitudes that appear in electrostatics are today almost invariably measured by alternating current methods, which are far more accurate than those used by the pioneers in the subject; we naturally therefore look to alternating currents for our basic conceptions.

In one of the circuits from which we started, let the battery be replaced by an A.C. generator (say audio-frequency). Current indicators in series will not generally read the same current, according to their D.C. calibration; nor will voltage indicators in parallel read the same voltage. If the indicators are of the kind generally employed in modern practice—but not necessarily otherwise—most of the direction-sensitive indicators will read nothing at all, while the reversible indicators will continue to agree among themselves. If we had started from such an arrangement, we should simply have ignored the direction-sensitive indicators as not indicators at all.

But, even then, if we had tried to calibrate the reversible indicators by addition in the manner described in Part I, we should have failed, because the necessary laws of addition are not true in general. They are true if the circuit elements are limited to a narrow class (that of "pure" elements all of the same kind, see § 2); and this fact is used in the calibration of A.C. instruments according to the best modern practice. But, though we might by such methods measure independently magnitudes corresponding to current and voltage, it would be doubtful whether we should be wise to call them by those names. For there would be no negative currents or voltages; and the laws from which "current" and "voltage" derive so much of their significance would not be true; neither the sum of the currents flowing to a node nor the sum of the voltages round a circuit would be zero.

However, if we include cathode-ray oscillographs among the indicators, we shall find that they, though direction-sensitive, continue to give an indication; but they indicate that the state of the system is not constant, but is varying cyclically. If we calibrate these instruments by D.C. for current and voltage, and by some form of stroboscope for time, and if we then regard only those values of current and voltage that are simultaneous, we shall find that simultaneous current and voltage possess over a wide range the properties of direct current and voltage; in particular they obey the laws from which these terms derive their significance. Moreover, we shall find that the behaviour of all the instruments that give steady indications in an A.C. circuit can be completely explained in terms of their reactions to different direct currents and of the presence of some kind of inertia that causes them to read a mean value of a rapidly varying influence. Electrostatic and dynamometer instruments agree among themselves because they all read nearly the same mean; rectifier instruments diverge because they read a different mean.

These are the considerations that justify the conceptions of instantaneous and RMS current, voltage and power, and that underlie the calibration of RMS instruments in terms of D.C. in standardizing laboratories. Certain elaborate precautions have to be observed; but since they do not involve any of the laws we are about to discuss, they need not detain us.

§ 2. PURE RESISTORS AND PURE CAPACITORS

When the A.C. generator is substituted for the battery, none of the laws concerning circuit elements set forth in Part I remains generally true. In particular, if equality of impedance or admittance means equivalence in respect of a reversible indicator in an A.C. circuit, impedance is not generally additive in series or admittance in parallel; consequently impedance and admittance are not, like resistance and conductance, measurable by addition. But it is possible by certain tests to select, from the whole class of elements that may form part of an A.C. circuit, two sub-classes that possess characteristics that are additive and are therefore measurable independently; they will be termed respectively "pure resistors" and "pure capacitors". (In a later part of this enquiry it will be pointed out that there are no even approximately pure inductors.)

This statement needs some explanation. There is no circuit element that satisfies completely all (or perhaps any of) the tests that will be prescribed in their most stringent form. But, if elements are selected that very nearly satisfy all the tests, certain simple laws are found to be very nearly true of them; departures from the laws are concealed by experimental error in all but the most accurate experiments, and even in those are distinguishable from experimental error only by careful analysis. Accordingly we define a pure element as one of which these simple laws are true, and attribute departures from purity in actual elements to a combination of elements of different kinds, each of which would be pure in isolation. We find that the behaviour of actual elements can be explained on this hypothesis. The statements that follow must be interpreted in view of this procedure which, of course, is adopted with minor modifications in many branches of physics.

The distinction between pure resistors and pure capacitors is that the former do, and the latter do not, pass current in a D.C. circuit. (Pure capacitors thus violate the most fundamental property of D.C. circuits, namely that the path of the current lies wholly in conductors.) One test of purity applicable to both sub-classes is that the laws of addition in parallel and series must actually be obeyed. This test, since it involves a plurality of elements, can be applied only to the sub-class as a whole, not to individual members of it. It might fail completely for this reason; for impure elements might be additive, if they were all impure in the same way. Actually the test is very useful, because impure elements usually differ in their impurity. Further, the test, applied to capacitors, limits the sub-class to what we shall call (for a reason that will appear presently) "closed" capacitors (in common parlance, screened condensers). Other elements that are called pure capacitors according to convention—which we shall follow—are not additive.

Accordingly it is desirable to have other tests. Another class test is that the equivalence of two elements of the same sub-class should be independent of the indicator (i.e. of the nature of the mean it indicates) and of the generator (i.e. of its frequency and wave form). But this test, though satisfied by pure capacitors, is not fully satisfied by pure resistors; in ordinary language, their resistance varies slightly with the frequency.

There is a very perfect individual test for pure resistors. It is that the current through the resistor and the voltage across it, measured by cathode-ray oscillographs, must be "in phase"; in particular, the maxima and minima of current must be respectively simultaneous with the maxima and minima of voltage; this test can be applied without measuring anything, and involves only the relations of greater and less. The corresponding test for pure capacitors is less easily applied and its consideration therefore is postponed. The best individual test for capacitors is that which distinguishes them from resistors, namely that they pass no direct current; but this is not conclusive, since it does not include dielectric loss and series resistance or inductance. The class tests for capacitors are therefore more important than the class tests for resistors.

The magnitudes characteristic of pure resistors that are additive in series and parallel turn out to be the same as D.C. resistance and conductance, at least at low frequencies; no separate name for them is required. On the other hand, new names are required for the corresponding magnitudes characteristic of capacitors; we shall call them elastance and capacitance. In modern terminology these magnitudes are distinguished from impedance and admittance; but, from our present standpoint, when we are considering independent measurement alone, they are the same magnitude according to our meaning of that expression (Part I, p. 645). For, whatever the frequency and wave-form, the ratio of the impedances or admittances of two pure capacitors is the same as the ratio of their elastances or capacitances.

The measurement of capacitance by the parallel connection of pure closed capacitors plays as important a part in the most refined calibration of pure capacitors as does series connection in the calibration of D.C. resistors. The measurement of capacitance is slightly less accurate than that of resistance, because (for a reason that will appear presently) the conditions in which the law of addition is true cannot be realized so precisely. For the same reason, and perhaps for others also, elastance plays a much less important part than capacitance in actual measurement; but it should be noted that, in so far as it is an independently measurable magnitude, it must be the reciprocal of capacitance, if the same standard is used to define both units.

§ 3. CAPACITANCE AS A DERIVED MAGNITUDE

If we measure by cathode-ray oscillographs both the current I flowing through a pure capacitor and the voltage V across it, then we find that I is proportional to the simultaneous dV/dt, and that this relation is independent of the particular generator used and of the way in which V varies. The constant of proportionality depends on the capacitor, and is therefore a derived magnitude; we may write the law

$$I = S \cdot C \cdot dV/dt \qquad \ldots\ldots (3.1)$$

C turns out to be the same magnitude as the capacitance measured by addition, when the capacitor is closed and obeys the law of addition. It we choose suitably, with respect to the units of current, voltage and time, the closed capacitor to which unit capacitance is to be assigned, the scale factor S may be made 1. This is the convention always adopted; we shall assume its adoption in what follows.

Equation (3.1) is true even if the capacitor is not closed, so long as it satisfies the tests of purity other than that of addition. Accordingly, as indicated already, we shall regard capacitors obeying (3.1) as pure, even if they are not closed. It follows from (3.1) that, if there is a definite phase difference between I and V—which implies that both of them are sinusoidal—then it is 90°. This corresponds to the phase test for pure resistors; and since a phase difference of 90° implies that the maxima of one quantity coincides with the minima of the other, can be applied without measurement. But if the capacitor is not entirely pure, and the phase difference not exactly 90°, the distinction between a leading and a lagging current, which will turn out to be important, will arise. It is therefore doubtful whether the test can actually be applied in a form in which it differs from the test whether (3.1) is true.

(3.1) cannot be established with great accuracy by oscillograph measurements, if only because the deduction of a derivative from a set of disparate values is always liable to considerable error. But on the assumption that (3.1) is true, it is possible to devise bridge networks fed by generators of sinusoidal wave-form (so that the relation between V and dV/dt is known) that permit C to be determined, both relative to other C's and to the conductances of pure resistors. The consistency of the results obtained with such networks confirms with high accuracy the truth of (3.1), and the identity of C in that law with capacitance measured by addition in closed capacitors.

Just as the most refined measurement of resistance depends both on the law of addition and on Ohm's law, so the most refined measurement of capacitance depends both on addition and on (3.1). Bridges are also useful in judging equality in independent measurement; it is worth noting that, when they are used for this purpose, the wave-form of the generator need not be sinusoidal.

§ 4. CAPACITANCE AND GEOMETRICAL FORM: PERMITTIVITY

On examination of their structure, capacitors prove to consist essentially of a pair of conductors ("plates") separated by a gap filled with a non-conducting material, or dielectric and generally narrow compared with the dimensions of the plates. When the plates approximate sufficiently nearly to concentric spheres, coaxial infinite cylinders or infinite parallel planes, experiment shows that there are simple relations between capacitance and geometrical form. In the usual notation they are

$$C = K_1 \,.\, r_1 r_2/(r_2 - r_1) \qquad \text{(spheres)} \qquad \ldots\ldots(4.1)$$

$$C = K_2 l/(\log r_2 - \log r_1) \qquad \text{(cylinders)} \qquad \ldots\ldots(4.2)$$

$$C = K_3 A/d \qquad \text{(planes)} \qquad \ldots\ldots(4.3)$$

K_1, K_2, K_3 vary with the dielectric; each is therefore a derived magnitude requiring a scale factor. But it turns out that they all vary together, so that—within experimental error—$K_1 = 2K_2 = 4\pi K_3$. Accordingly, if we write

$$K_1 = S \,.\, \kappa; \quad K_2 = \tfrac{1}{2} S \,.\, \kappa; \quad K_3 = 1/4\pi \,.\, S \,.\, \kappa. \qquad \ldots\ldots(4.4)$$

a single magnitude κ (permittivity) characteristic of the dielectric and a single scale factor will suffice. Experiment shows further that the independence of permittivity and geometrical form is complete, so that if a value for κ is assigned by convention to one dielectric, it can be determined for any other by measuring the capacitance of a capacitor of any form, filled first uniformly with the standard dielectric and then uniformly with the other.

Somewhat more complicated formulae have also been established experimentally when the dielectric, though not uniform, is distributed in uniform layers. They involve different κ's for the different layers, but they need no special consideration here.

All these formulae are special cases of a general rule for determining capacitance from the geometry of the capacitor and the nature of the dielectric. The form in which it is given here depends historically on laws not mentioned so far. But the discovery of the rule by guesses based on empirical laws such as (4.1), (4.2), (4.3) would have been no more remarkable than the discovery of the Ampère-Neumann rule (Part I, (11), (11.1)) by a similar process.

The rule depends on a geometrical theorem that can be stated with sufficient accuracy and generality for our purpose thus*:—

Let, $s_1 \ldots s_k, \ldots s_n$ be a set of closed non-intersecting surfaces. Let each point outside these surfaces be associated with a quantity κ, characteristic of it. Then it is possible to find a variable v having the following properties:—

(1) v is constant over each surface s;

(2) at all points outside the surfaces s, v is continuous and finite or zero, being constant at all points sufficiently distant from the surfaces;

(3) at each point at which κ is continuous, v satisfies the differential equation

$$\frac{d}{dx}\left(\kappa\frac{dv}{dx}\right)+\frac{d}{dy}\left(\kappa\frac{dv}{dy}\right)+\frac{d}{dz}\left(\kappa\frac{dv}{dz}\right)=0; \qquad \ldots\ldots(4.5)$$

(4) at any surface at which κ changes discontinuously, $\kappa \,.\, dv/d\boldsymbol{n}$ has the same value on either side of the surface, $\boldsymbol{n}$ being the normal to it.

It follows from the properties of v that, if determinate values v_1, v_2 are assigned to two of the surfaces s_1, s_2, while the values for the other surfaces are left undetermined, the quantity

$$\int \kappa \,.\, dv/d\boldsymbol{n} \,.\, dS = 4\pi\phi_e, \qquad \ldots\ldots(4.6)$$

where dS is an element of surface, $\boldsymbol{n}$ is the outward normal, and the integral is taken over the surface, has the same modulus but opposite signs for the surfaces s_1, s_2, and is zero for any other of the surfaces.

In applying this purely geometrical theorem to a capacitor, s_1, s_2 are identified with the surfaces of the plates, each plate including all the conductors connected conductively to it. v_1, v_2 are chosen arbitrarily. The value of κ associated with each point is the permittivity of the medium occupying it. If any point is occupied by a conductor insulated from both plates, κ for it is infinite; v is then constant over the surface of that conductor and ϕ_e for it is zero. Then v is in general determined everywhere; the quantity ϕ_e is everywhere proportional to $(v_1 - v_2)$; and the rule for capacitance is

$$C = S\left|\frac{\phi_e}{v_1 - v_2}\right| \qquad \ldots\ldots(4.7)$$

where ϕ_e refers to the plates and S is the scale factor of (4.4).

This is the most general relation between capacitance and geometrical form; formulae (4.1), (4.2), (4.3) are special cases, but they do not provide a satisfactory test of it, because the capacitors to which they refer cannot be realized fully. However, they can be realized approximately; the rule is then found to give a "correction" that agrees with measurement. If the rule is applied to capacitors

* The reader will note that it is assumed that there are no surface or volume charges in the dielectric. The modification required if there are such charges is considered below in § 16. But it should be pointed out that the same assumption is made in § 5. He may note further that no reference is made to the vector D, of which ϕ_e is the flux. The reason for this omission will appear in later parts of this enquiry.

differing widely from these simple forms, mathematical difficulties often prevent the prediction of a definite value with which measurement can be compared. The direct evidence for the accurate and universal validity of the rule is therefore not complete; but it is very strong—stronger than that for any other "electrostatic" relation.

§ 5. THE FIELD EQUATIONS: ELECTRIC FIELD STRENGTH

In § 4 v is a mere mathematical variable, convenient (but perhaps not absolutely necessary) in arriving at formulae (such as (4.1–3)) applicable to experiment; it does not appear at all in the formulae. But further inquiry shows that v, or more accurately grad v, has great physical significance.

In Part I the magnitude H was introduced to express the fact that there is a group of phenomena, including the deflection of a compass needle or of a cathode ray, that are characteristic of the space about a current-carrying circuit, and are determined by the product of the current in that circuit by a geometrical function G. There is another group of phenomena similarly characteristic of the space about conductors between which voltages are maintained; like the "magnetic phenomena", they form a group, because if one of them is the same at two points in such a space (or spaces), then each of them is the same at those points. The group includes again the deflection of a cathode-ray; it includes also the Kerr electro-optical effect and the breakdown of a dielectric.

Suppose that we identify the surfaces of the conductors with the surfaces s, assign values of v to them so that $v_k - v_l$ is the voltage V_{kl} maintained between s_k and s_l, and calculate v throughout the space between them by means of the theorem of § 4. Then we find that the phenomena are determined by grad v in the sense that they are the same wherever grad v is the same. Accordingly

$$\mathscr{E} = \operatorname{grad} v \qquad \ldots\ldots(5.1)$$

is a defined magnitude determining these electric phenomena in the same way as H determines the magnetic phenomena; it may be fitly called the electric field strength. If there are only two conductors between which a voltage is maintained, if V is that voltage, and if the dielectric is uniform, the similarity between E and H may be made even more evident by writing

$$\mathscr{E} = VF, \qquad \ldots\ldots(5.2)$$

where F, like G, is a function depending only on the geometry of the system.

In view of these facts we shall term the propositions of § 4 collectively the *field equations*. Further, we shall assume—for this is obviously the easiest way—that the v's are to be assigned to the conductors in accordance with the voltages, by assigning $v = 0$ to one conductor (zero plate) and to the others their voltages to this conductor.

In some circumstances—it is unnecessary here to inquire what they are—it is possible to establish that the voltage between the zero plate and a probe introduced into the space between the conductors is equal to v, calculated from the field equations. Then v, as well as grad v, has physical significance. Measurements of probe voltages are inaccurate and have little evidential value. On the other hand, $\mathscr{E}$ can be determined with considerable accuracy from the deflection of a cathode-ray, by the use of a theorem that will be examined in § 15. It is possible that the future development of electron optics may provide evidence for the field equations more stringent than that derived from the measurement of capacitance; but for the present they must be based primarily on such measurement.

v of this section is closely related to the v that was introduced in Part I to explain the properties of conducting media. It follows from the field equations and from the properties of v set forth in § 13 of Part I that, if the dielectric of a capacitor is replaced by a medium whose conductivity is everywhere proportional to the permittivity of the dielectric, and if the plates are still maintained at the same voltage, then the v of the conductor is everywhere the same as the v of the capacitor. This relation is sometimes used to determine the distribution of v in a capacitor of complicated form. But it is more important because there is no sharp distinction between dielectrics and conductors; there are media in which it would be possible to determine both equipotentials as indicated in § 13 of Part I, and the electric field strength by one of the group of phenomena mentioned earlier in this section; more generally we find that the properties of most dielectrics can be explained by attributing to them both permittivity and conductivity. It is an experimental fact of great practical importance that, in isotropic media, the distribution of v is the same whether it is derived from conductive or from electrostatic phenomena. Perhaps the strongest evidence on this point is the well-known fact, frequently established with great precision, that the "loss tangent", and therefore σ/κ, for any uniform medium is independent of geometrical form.

§ 6. CLOSED CAPACITORS: SCREENS

The exceptions to the statement in § 4, that v is determined everywhere by the assignment of v_1, v_2, are very important.

(*a*) If a conductor, insulated from both plates, encloses one but not the other, then v is completely indeterminate unless something is added to the rule. The obvious addition, based on the fact that ϕ_e for a conductor insulated from both plates is zero, is that, if ϕ_e' is the value of ϕ_e for the enclosed plate, then $-\phi_e'$ is its value for the inner surface of the enclosing conductor, $+\phi_e'$ for its outer surface, and $-\phi_e'$ for the unenclosed plate. The experimental facts that justify this addition will be stated later. With this addition, which will be assumed in what follows, v in this case is determinate everywhere.

(*b*) If one of the plates completely encloses the other, then the assignment of v to the enclosed plate and to the inner surface of the enclosing plate, though it determines v everywhere within the enclosing plate, does not determine v anywhere outside it. The corresponding experimental fact is that changes in the voltages between conductors inside the enclosing plate (including the inner surface of that plate) produce no change of voltage between any pair of conductors outside the enclosing plate (including the outer surface of that plate), and no change of the field strength outside that plate; and conversely, interchanging "inside" and "inner" with "outside" and "outer".

This deduction from the field equations can be tested with great sensitivity; since it is true, some variations from the rule of capacitance given by the field equations as stated in § 4, which might otherwise be possible, are excluded. But the fact is even more important practically than theoretically. Thus, it is used in radio-frequency experiments when the observer, in order to isolate himself from irrelevant disturbances, encloses himself and all his apparatus inside a metallic cage. (In view of later discussions, it is well to point out here that the efficiency of the cage for this purpose does *not* depend on its being connected to any other conductor, whether inside or outside the cage.)

Again, it makes it possible to limit the conductors that have to be taken into account in any electrostatic problem to a definite group. If one of a group of conductors encloses all the rest, then any conductor outside the group—and outside the enclosing conductor—can be ignored. This limitation and isolation by a "screen" is necessary both in theoretical arguments and in all experiments except the very roughest. Accordingly it will always be assumed hereafter, unless the contrary is stated explicitly, that the system under discussion is a group within and including a surrounding screen. Of course an isolating screen need not in practice be absolutely complete; the metallic cage above mentioned is often made of wire gauze or netting. The effect of a small aperture tends to zero with the ratio of its area to the whole area of the screen, and is often quite inappreciable.

Other closely related deductions from (*b*) are that the capacitance of a capacitor of which one plate (screen) completely surrounds the other is independent of any body outside it, and that the capacitance of a pair of such capacitors, with their screens and their plates respectively connected together, is the sum of their individual capacitances. If one plate does not surround the other, capacitance will not in general be additive in parallel, because the field between the plates of one capacitor affects the field between the plates of the other. This is, of course, why capacitors that obey the laws of addition were called "closed" in §2.

If a capacitor could not be "closed", and obey the laws of addition, unless the screen *completely* surrounded the plate, a closed capacitor would be an unattainable ideal. For, in order that the plate of a screen-and-plate capacitor should be connected to anything outside the screen, e.g. another plate, there must be an aperture in its screen through which a lead may pass. The aperture itself, if small enough, may be unimportant; but the lead passing through it is another matter. It is a question of fact, not to be decided by anything stated so far, whether the departure from ideal closure, represented by the passage of a lead through the scr.en, produces a failure of the law of addition.

The answer depends, of course, on the sensitivity of the criterion of equality and on the size and location of the lead. If the most sensitive means for judging equality is used, and the capacitors have ordinary terminals, then the law of addition is found to be not *generally* true, although it is true for capacitors exceeding (say) $0{\cdot}5\,\mu$F. in value, provided that reasonable precautions are taken in choosing and arranging the leads. It is, however, possible so to design capacitors that the leads connecting them together make no detectable contribution to their capacitance. For example, the plate terminals may take the form of sockets mounted in small apertures in the screen, so that the face of the socket is flush with the surface of the screen. If the sockets of two such capacitors register with one another when the screens are placed in contact, then they can be connected together by a double-ended plug, which is inserted in one socket before the capacitors are brought together, and which becomes totally enclosed when the connection is complete. When capacitors of this type are used, the law of addition can be verified with the highest accuracy over a very large range of values; indeed, it is probably the most accurately known of all the quantitative laws of electrostatics. Moreover (see §12 below) there are methods of assigning capacitances to ordinary standard capacitors that are also additive in respect of the most sensitive criteria of equality; and these capacitances agree within experimental error with those calculated from the rule of capacitance. Accordingly there are actual capacitors completely equivalent to ideal closed capacitors, having no leads through their screens; and in basing the laws of electrostatics on the properties of ideal closed capacitances, we shall not be departing from our principle of relying only on real experiments and real conceptions.

There is a corresponding deduction from (*a*), namely that elastance is additive in series if, and in general only if, the plate common to the two capacitors completely encloses the other plate of one of the capacitors, and is completely enclosed by the other plate of the other. This condition is not easily realized, even if the leads are ignored, and elastance, even if it proved accurately additive in the presence of leads, would not be mportant. Actually the departures from accurate additivity of elastance are of the same nature as those just discussed; they are often concealed by experimental error; and even when they are not concealed, they may be abolished by the methods of §12 below.

§7. ELECTROMETERS

We now return to the main theme, namely the coordination of the laws of current circuits.

We note that electrostatic voltmeters and electrometers are essentially closed capacitors in which the plate is movable relative to the screen, so that their capacitance varies with the motion of the plate, and the voltage V to be measured is applied between plate and screen. (If such voltmeters were used to investigate (3.1), account would have to be taken of the variation of their capacitance, which is part of C, with V.) Experiment shows that their behaviour is consistent with the law

$$F_q = S \cdot dC/dq \cdot V^2, \qquad \ldots\ldots(7.1)$$

where F_q is the force tending to increase the coordinate q, the voltage being maintained independent of q by means of a battery or generator, and S (which is positive) is the same for all voltmeters that obey (7.1), so that it is a scale factor that does not need to be supplemented by a derived magnitude.

The only voltmeters in which the law can be established accurately are those in which C can be calculated accurately from the field equations; but, in some of these, it can be established as accurately as any of the laws previously mentioned in this Part. By assigning values to S in (4.4), (4.7), (7.1), and to κ of a standard dielectric, the units of C, V, I, κ can be fixed in terms of the units of length, area, time and force. They could actually be fixed by this method with an accuracy little less than that with which they are now fixed by other methods. In "electrostatic absolute units", κ is made unity for vacuum, S of (4.4), (4.7) are made unity, and S of (7.1) is made $\frac{1}{2}$. (The reason for this last choice will appear presently.)

§8. THE CHARGE IN A CAPACITOR

We have said that (7.1) is true only if V is made independent of q by connecting plate and screens through a battery or generator. Electrometers are sometimes used idiostatically with plate and screen insulated from each other; then V varies with their relative position. We shall now show that the law corresponding to (7.1) *for this case* can be inferred from the laws already stated.

Suppose that a direction-sensitive ammeter is permanently connected in series with one of the terminals of the capacitor. While the capacitor is a circuit element, let us record I, the current flowing through the ammeter, as a function of time; and let us define a quantity Q, the *charge in the capacitor*, by

$$Q = \int_0^V I \, . \, dt, \qquad \ldots\ldots(8.1)$$

where the limits of the integral indicate that it is to be taken between an instant when the voltage across the capacitor is zero and an instant when it is V. Then from (3.1), with $S=1$

$$Q = CV. \qquad \ldots\ldots(8.2)$$

Comparison of (4.7) and (8.2) shows that

$$|Q| = |\phi_e|, \qquad \ldots\ldots(8.3)$$

accordingly in so far as Q, defined by (8.2), receives physical significance from the considerations of this section, ϕ_e receives the same significance.

When the terminals of a pure capacitor are not connected in a circuit, $I=0$. If (3.1) is true of these circumstances (an assumption not used so far), then V, and therefore by definition Q, must remain constant. Experiment shows that this condition may be approached very nearly in suitable circumstances; any change in V can be attributed to the presence of a conductor between the terminals whose conductance (representing a slight impurity) is so small that it cannot be detected in the range of frequencies within which C is most conveniently measured.

Let us now connect the terminals of the charged capacitor by a resistor of resistance R. Then, if Ohm's law is true when the appearance of a voltage across the resistor is not associated with the presence of a generator in the circuit of which it forms part (again a new assumption), we must have

$$I = -V/R; \qquad dI/dt = (-1/R)\,\frac{dV}{dt} = -I/CR \qquad \ldots\ldots(8.4)$$

$$\int_0^V Idt = \int_{I=0}^{I=-V/R} -CRdI = CV. \qquad \ldots\ldots(8.5)$$

This relation is confirmed by experiment, i.e. Q is the same whether it is estimated by charging or by discharging, and our assumptions are therefore justified. From these experiments we conclude that Q or ϕ_e is a significant magnitude, characteristic of the state of a capacitor, having a constant terminal-voltage, when its terminals are insulated.

We expect then that, when the terminals of the capacitor are insulated, and its configuration changes, Q will still remain constant. If so, in order to ascertain the behaviour of the electrometer when its terminals are insulated, we may substitute for V in (7.1) in terms of Q from (8.2), and perform the differentiation with respect to q subject to the constancy of Q. We thus have

$$F_q = -S \, . \, \frac{d}{dq}\left(\frac{1}{C}\right) . \, Q^2. \qquad \ldots\ldots(8.6)$$

This law is confirmed by measurements; but, owing to the difficulty of maintaining the terminals truly insulated, it cannot be established as accurately as (7.1). Its truth is, however, a great part of the evidence for the propositions about Q.

§ 9. ENERGY IN A CAPACITOR

(8.6) is the law that would follow from general dynamical principles, if it were known that a pure insulated capacitor is a conservative system and that its potential energy is SQ^2/C. We therefore inquire whether there is any other support for these suggestions, taking into account equation (6) of Part I.

From (3.1)

$$\int_{t_1}^{t_2} IV dt = \tfrac{1}{2}C(V_2^2 - V_1^2), \qquad \ldots\ldots(9.1)$$

where V_1, V_2 are the voltages at t_1, t_2. It follows that, even if (6) were true of the instantaneous power converted into heat in a pure capacitor, the total energy and mean power dissipated in a complete cycle or in any number of cycles, would be zero. Experiment shows that it is zero or, more accurately, that any power dissipated in a capacitor in such circumstances can be associated with a departure of the capacitor from complete purity that can be detected in other ways. But it shows also that, in a pure capacitor, no energy is dissipated in even part of a cycle throughout which IV is of the same sign, although then, if (6) were true, power would be drawn from the generator and dissipated. Moreover, when a charged capacitor is disconnected from the generator by which it has been charged and is discharged through a resistor, V and I being observed during the discharge, power is dissipated in the resistor according to (6), being apparently drawn from the capacitor.

All this is consistent with the storage in the charged capacitor of potential energy W_Q, where, if the units of I, V are chosen to make the scale factor in (6) unity,

$$W_Q = \tfrac{1}{2}Q^2/C. \qquad \ldots\ldots(9.2)$$

Accordingly (8.6) will be identical with the law to be expected on general principles, so long as the units are chosen so that $S = \frac{1}{2}$. This is the reason for that choice in (7.1).

The potential energy can also be written

$$W_V = \tfrac{1}{2}CV^2. \qquad \ldots\ldots(9.3)$$

But it must *not* be concluded that the force tending to increase q, when V is constant, is $-dW_V/dq$. For, when V is constant, the capacitor alone is not a conservative system, because it can exchange energy with the generator by which V is maintained. The fact that F_q in these circumstances is given by (7.1) shows that the energy drawn from the generator in maintaining V is twice as great as, and opposite in sign to, the change in potential energy.

§ 10. GENERALIZATION OF CAPACITANCE AND CHARGE

The capacitors that we have considered so far ("simple" capacitors) are essentially pairs of conductors. Any conductor other than the pair that is relevant at all, in the sense that its removal would effect the capacitance of the pair, must be insulated from the pair, so that it is effectively part of the dielectric. But sometimes we have to consider systems of more than two conductors, each pair of which is, or may be, maintained at a different non-zero voltage. The main, if not the only, examples of such systems offered in the older text-books are electrometers used heterostatically; but today more important and interesting examples are furnished by telephone cables and thermionic-valve circuits.

A study of the field equations suggests that the magnitudes *capacitance* and *charge* may have some relevance to such systems. Thus, suppose that we have N conductors, distinguished by suffixes $1, \ldots m, \ldots n, \ldots N$, of which one is a screen surrounding all the rest; and that we consider the simple capacitor of which one plate is the conductor 1, characterized by v_1, and the other all the remaining conductors, characterized by v_0. If ϕ_m is the integral of (4.6) taken over m, and C_1 is the capacitance of this simple capacitor, then we must have

$$\phi_1 = -\Sigma\,\phi_m \qquad \ldots\ldots(10.1)$$

$$C_1 = \left|\frac{\phi_1}{v_1 - v_0}\right| = \Sigma\left|\frac{\phi_m}{v_1 - v_0}\right| \qquad \ldots\ldots(10.2)$$

where the summation excludes 1; and, if we define C_{1m} by

$$C_{1m} = \left|\frac{\phi_m}{v_1 - v_0}\right| \qquad \ldots\ldots(10.3)$$

we must have

$$C_1 = \Sigma\,C_{1m}. \qquad \ldots\ldots(10.4)$$

Suppose now that we divide the conductors differently, taking 2 as one plate and all the others as the other plate. The capacitance C_2 of this simple capacitor will have components C_{2m} defined analogously to (10.3). By selecting in turn each of the conductors as one plate of the simple capacitor, we can find all the C_{mn}. It is now found to be a consequence of the field equations that

$$C_{mn} = C_{nm}. \qquad \ldots\ldots(10.5)$$

A more general consequence is that, if we assign to the conductors the values $v_1, \ldots v_m, \ldots v_n$, which may all be different, then, if ϕ_m and C_{mn} are defined as before,

$$\phi_m = \Sigma\,C_{mn}(v_m - v_n), \qquad \ldots\ldots(10.6)$$

$$\Sigma\phi_m = 0, \qquad \ldots\ldots(10.7)$$

where the summations extend over all the conductors. If we write $Q_m = \phi_m$, $v_m - v_n = V_{mn}$, these become

$$Q_m = \Sigma\,C_{mn}V_{mn}; \quad \Sigma\,Q_m = 0. \qquad \ldots\ldots(10.8)$$

This clearly suggests that the C_{mn} (which will be termed *mutual capacitances*) have the same physical significance as capacitances, and that, if the V_{mn} are identified with the voltages between mn, the Q_m have the physical significance of charges. Indeed, the system of conductors should be equivalent to a set of terminals $1, \ldots m, n, \ldots$, interconnected by simple closed capacitors of capacitance C_{mn}. The only difference between the system and a set of terminals interconnected by real simple closed capacitors is that each C_{mn} is determined, not by the configuration of two conductors only (namely the plate and its screen), but by the configuration of all the conductors.

Further, we should expect each Q_m to be equal to some integral $\int Idt$, where I is a current that flows to or from it during its charging or discharging. It is unnecessary for our purpose to define these relations generally; but it may be noted that Q_m should be $\int_0^\infty Idt$, where I is the current that flows in a circuit connecting the charged m to all the other conductors connected together. However, one matter requires more detailed attention. Hitherto, when Q has been the charge in a capacitor, it has been unnecessary to pay attention to the sign of Q, and the sign of I in the integral to which it is equal. Now Q_m is the charge on a single body, and, as (10.8) shows, it must sometimes be of one sign and sometimes of the other; it is important to lay down rules by which its sign can be ascertained. Formally, in order to solve all problems, these rules should be given in terms of the red and black terminals of ammeters and voltmeters discussed in Part I. But, since there is really no difficulty in the matter for anyone practised in electrical experiments, it suffices to say that the charge in a simple closed capacitor is to be regarded as positive if the voltage of the plate relative to the screen is positive, and that, when we write

$$Q=\int_0^\infty Idt, \qquad \ldots\ldots(10.9)$$

I must be the positive current flowing during discharge from the plate to the screen.

§ 11. PROPERTIES OF A COMPLEX CAPACITOR

All these propositions about a complex capacitor (for so we shall term a set of conductors such as is considered in the preceding section) are so far mere expectations, based on suggestions derived from the form of the field equations. There is a major difficulty in relating any of them to experiment. It is that, while the reasoning that leads to the propositions involves the assumption that the conductors are all isolated, any possible experiment requires leads between them. However, if we suppose that the presence of the leads does not destroy the equivalence between a complex capacitor and a set of terminals connected by simple closed capacitors, but merely modifies the mutual capacitances represented by the capacitances of those simple capacitors, then some experimental tests can be applied.

Thus, the propositions indicate that the formula corresponding to (9.3) should be

$$W_V=\tfrac{1}{2}\Sigma\, C_{mn}V_{mn}^2, \qquad \ldots\ldots(11.1)$$

where the summation is taken over all pairs without permutations. Since

$$V_{mn}=-V_{nm}; \quad V_{mo}=V_{mn}+V_{no} \qquad \ldots\ldots(11.2)$$

there are only $N-1$ independent V_{mn}; by solving the $N-1$ independent equations (10.8), we can obtain them as linear functions of the Q_m. Substituting these values in (11.1), we have

$$W_Q=\tfrac{1}{2}\Sigma p_{mn}Q_mQ_n, \qquad \ldots\ldots(11.3)$$

where the p_{mn} are functions of the C_{mn} subject to certain interrelations that need not be considered here. In an electrometer used heterostatically with insulated terminals, F_q should be $-dW_Q/dq$; accordingly (11.3) predicts certain laws relating the forces on the electrometer to the charges on its parts, in which the constants are functions of the mutual capacitances. It is possible, as in many similar instances in various branches of physics, to apply tests that assume merely that these constants are indeed constants, and that do not require any knowledge of their actual values; if these tests are satisfied, some evidence will be obtained for the conclusions of this section without any detailed knowledge of the mutual capacitances.

Qualitative tests of this nature applied to electrometers are partially, but not completely, satisfied; it is almost certain that the discrepancies arise, not from any failure of the conclusions of this paragraph, but from the difficulty of taking into account *all* the mutual capacitances and of otherwise fulfilling exactly the conditions postulated. Perhaps better evidence of this nature is derived from valve-circuit theory, where again important conclusions can be reached, independent of accurate knowledge of the mutual capacitances. But the accuracy possible in all such tests is very low.

§12. MEASUREMENT OF MUTUAL CAPACITANCE: SCREENING

However, if a set of conductors is indeed equivalent to a network of closed capacitors, representing the mutual capacitances as modified by the leads, the capacitances of these capacitors can be measured by apparatus of the kind shown in figure 1.

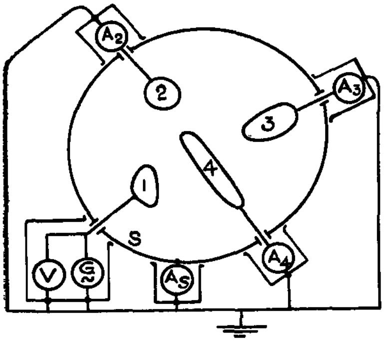

Figure 1 Scheme for the measurement of the mutual capacitance C_{mn} of any pair of conductors m, n

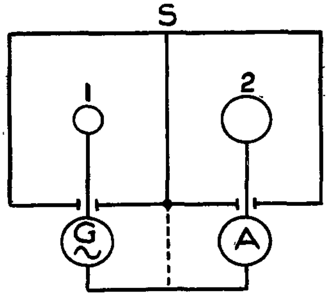

Figure 2.

Here 1, 2, 3, 4 are conductors within the screen S. G is a generator with one terminal connected to 1; A_2, A_3, A_4, A_S are ammeters with one terminal connected respectively to 2, 3, 4, S; all the connecting leads are as thin as possible. The other terminals of the generator and ammeters are all connected together. The ammeters are all of low impedance, so that the voltage between 1 and all the other bodies is V, measurable by a voltmeter across the generator, and there is no voltage between any pair of these bodies.

If the generator and instruments had no effect beyond supplying and measuring current and voltage, it would be desirable to place them within S, in order that the leads should be as short as possible. Examination of the equivalent network would then show that the only current I_m through A_m is the current through the capacitor C_{1m}; consequently we should have

$$I_m = C_{1m} \, . \, dV/dt. \qquad \ldots\ldots(12.1)$$

Since I_m is measurable by A_m and V by the voltmeter, C_{1m} could be determined. (10.4) would be necessarily true in virtue of Kirchhoff's First Law and the method of measurement. C_{2m} etc. could be measured by substituting the other bodies in turn for 1; but experiment alone could determine whether $C_{mn} = C_{nm}$.

But the hypothesis is not true. The generator and instruments are really part of the system, having mutual capacitances with each other and with 1, 2, 3, 4, S. In order that (12.1) should be true in spite of these mutual capacitances, the generator and ammeters are placed outside S, and connected to the conductors within S by leads passing through apertures in S; and each is surrounded by its own screen connected to a common terminal (see § 14 below). This is a simple example of the art of screening, which forms an important part of A.C. technique. We shall not discuss it fully here; for one of us has discussed it in a recent publication (Hartshorn 1940). But some statement of the underlying principle is desirable

The principle involves more than that of the metallic cage mentioned in § 6. The object of the screens is not to isolate the generator and instruments completely from each other and from bodies inside S; but, firstly, to limit the extension in space of the electric fields of the bodies within the several screens and, secondly, to direct the capacitance currents associated with those fields to appropriate terminals of the equivalent network; or, in other words, not so much to abolish capacitance currents as to direct unwanted capacitance currents into paths where they are innocuous. This function can be explained with reference to figure 2.

The introduction of any conductor S between two others 1, 2, will decrease the mutual capacitance C_{12} to (say) C'_{12}; but it will also introduce two new mutual capacitances C_{1S}, C_{2S}. The total capacitance of the network between 1, 2 will become $C'_{12} + C_{1S}C_{2S}/(C_{1S} + C_{2S})$, and this will always be greater than C_{12}. If, as in figure 2, S surrounds both 1 and 2, C'_{12} will be very small and tend to zero with the area of the aperture in S; but the total capacitance will still be greater than C_{12} in the absence of S. Accordingly S in itself increases rather than decreases the mutual reaction of 1, 2. But if, as shown by the dotted lines, S is connected to the common terminal of the generator and ammeters, the current through C_{1S} is diverted from the ammeter to the dotted connection, and the current through C_{2S} is abolished, because there is no voltage driving it.

The action of the screens in figure 1 can be understood in the light of this simple example. But that example is merely an illustration of the principle involved; it must not be supposed that any feature of figure 1 is characteristic of all applications of the principle. Thus, it is not always desirable that each element should have its own screen, or that all screens should be connected to a common terminal. The application of the principle is so complex that it is scarcely possible to state any practical rule that is valid without limitation.

Experiments are not often made in exactly the way indicated in figure 1; but the various bridge and resonance methods for the measurement of mutual capacitances often (called "direct" or "partial" capacitances by communication engineers, who are chiefly concerned with them) are extensions of the same principle, and depend upon (12.1) just as bridge methods for simple capacitors depend upon (3.1). These methods lead to consistent values in accordance with §§ 10, 11; in particular, such experiments fully establish the fact that C_{mn}, though dependent on the configuration of all the conductors, is not dependent on their voltages. The relation $C_{mn}=C_{nm}$ can also be established with high accuracy, provided that, in performing the experiment, the generator connected to m and the ammeter connected to n have their positions in the network interchanged without any disturbance of the leads that penetrate S. The values obtained often agree well with those calculated from the field equations. The measurements made by Rosa and Dorsey (1907) on the complex capacitor (a central cylinder with guard-ring extensions), used in their determination of **c**, is an outstanding example. Such experiments provide the best evidence for the general theory of complex capacitors, i.e. in traditional language, "systems of conductors".

Finally, precision methods of measuring the capacitance of standard capacitors are essentially methods of measuring mutual capacitance. It is therefore impossible in principle that they should lead to entirely definite values. For, as we have seen, the mutual capacitance of two conductors depends on all the other conductors with which it forms a definite group in the sense of § 6; moreover, measurable mutual capacitances are always modified by the presence of leads; the very purpose of a standard capacitor forbids that it should always be associated with the same conductors or joined to the same leads. However, the margin of uncertainty can be made very small, e.g. 2 parts in a hundred thousand for a capacitance exceeding 0·1 μF.; and, as indicated in § 6, the rules of addition in parallel are satisfied by some values within this margin.

In some exceptional circumstances it is possible to assign values to elastances within a comparable margin of uncertainty, and to show that elastance is additive in series. But it will not usually be possible to prove this with high accuracy by the use of standard capacitors; for, as usually constructed, they do not satisfy sufficiently nearly the condition, stated in § 6, for the ideal addition of elastance.

§ 13. ABSOLUTE CHARGE

In the foregoing discussion the charge on a body has appeared as a property that it possesses in virtue of its relation to other bodies with which it forms a capacitor. On the other hand, in the classical theory of electrostatics, charge appears as an "absolute" property of a body, independent of its relation to any other body, and moreover, as one that can be measured without the knowledge of any of the algebraic laws involving charge, and, in particular, of those algebraic laws that concern capacitors. We have to inquire on what facts such a conception of absolute charge is, or could be, based.

Suppose that a closed capacitor is charged by connecting the plate to the screen through a battery, and that the plate is then discharged by connecting it to the screen through an ammeter. If I is the current through the ammeter during the discharge, the charge acquired by the plate during the charging is given by $Q=\int_0^\infty I dt$. Suppose now that the plate, after having been charged, is removed from the screen in which it was charged and, remaining always insulated, is transferred to the interior of another screen. If it is now connected to this screen through an ammeter, $\int_0^\infty Idt$ should have the same value as before, whatever the form of the second screen, wherever the plate is placed within this screen, whether the first screen is insulated from the second, or whether a voltage, zero or non-zero, is maintained between them by a conductor or battery. In classical electrostatics this proposition is stated as a consequence of the theorem that the charge on a conductor resides wholly on the outside of it; but it does not follow from anything in our previous discussion. If it is true, it must be established by direct experiment. If it proves to be the fact that, whenever a body, charged as part of a capacitor, simple or complex, is discharged to an enclosing shield, $\int_0^\infty Idt$ turns out to be the same as the charge that it acquired as part of the capacitor, then *charge* will have a significance independent of any particular capacitor, though still not independent of the general properties of capacitors.

The experiment would be difficult to perform with any accuracy, and the test has probably never been applied. But an equivalent test may be devised by noting that the law $Q = CV$ is inseparable, according to our exposition, from $Q = \int_0^\infty Idt$. If, instead of discharging the body to the second screen, we measure the capacitance of the closed capacitor, formed by the body and the second screen, and also the voltage across the capacitor, then we should find that, though C and V depend on the form of the second screen and on the position of the charged body within it, CV is independent of both and equal to the charge that the body acquires in the first screen.

However, the test is still not convenient, since it requires the measurement of a different capacitance each time a charge is to be measured. The celebrated "Faraday cage" is essentially a device for overcoming this difficulty. It is illustrated in figure 3, which shows diagrammatically the complete electrical system involved in any attempt to perform the experiment accurately. Here the internal screen 1 is the "cage" into which the charged body 3 is introduced, so that 1 and 3 form a closed capacitor of which the product VC is required. An outer screen 2 surrounds 1, and a voltmeter V_{12} is connected between them; this arrangement provides another closed capacitor whose capacitance C_{12} and voltage V_{12} are accurately measurable. Now the complete capacitor 3, 1, 2 has that special form for which elastance is strictly additive (§ 6), and which is therefore completely represented by two simple capacitors in series, C_{13} and C_{12} in figure 3. Thus, when this complex capacitor is charged, which is the case when the body 3 enters the cage 1, each of these simple capacitors receives the same charge. Consequently, if V_{12} is originally zero, the value that it will acquire when the body 3 bearing charge Q enters the cage 1 is given by

$$Q = C_{12}V_{12}, \qquad \ldots\ldots(13.1)$$

where C_{12} includes the capacitance of the voltmeter. Accordingly a knowledge of C_{12} alone is required, and this is independent of the form of 3 and of its position within the cage.

It is now quite practicable to apply the test. 3 is given a known charge Q by charging a closed capacitor, of which 3 forms the plate and 4 the screen, to a voltage V_{34} by a battery between 3 and 4. 3 is now introduced into 1. If V_{12} were originally zero, it should now be given by

$$C_{12}V_{12} = C_{34}V_{34}. \qquad \ldots\ldots(13.2)$$

If this relation is always fulfilled, we have a way of measuring the charge on a body independently of the capacitor in which it received its charge.

V_{12} need not be zero originally. If it were not, then (13.1), (13.2) should be true, if δV_{12} is substituted for V_{12}, where δV_{12} is the change of V_{12} when 3 is introduced. This observation is important, because it leads to a method of measuring Q that does not require the prior establishment of methods of measuring C and V, or indeed of any other magnitude. Suppose that the voltmeters V_{12}, V_{34} are originally uncalibrated. We make some mark on the scale of V_{34}, and assert that, when its pointer stands at this mark, we are going to say that 3, being within 4, has the charge 1. We transfer 3 to within the cage, and, V_{12} having been originally zero, note the reading of V_{12}, and call this 1. We discharge 3 to the cage (by virtue

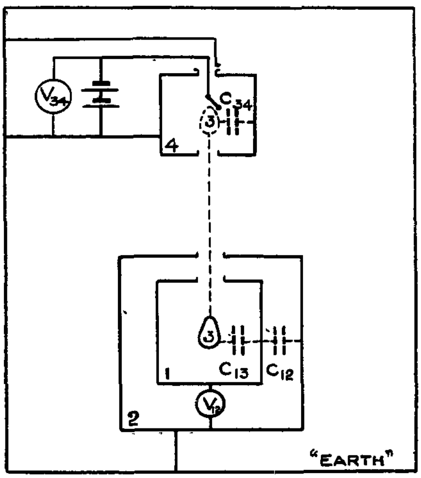

Figure 3. The "Faraday Cage" Experiment.

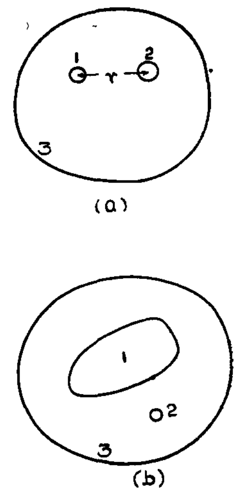

Figure 4. The problem of three charged bodies Coulomb's Law is an approximation for case (*a*) and $F_r = -Q_2 \frac{dv}{dr}$ an approximation for case (*b*).

of principles already laid down, this should not change the reading of V_{12} and experiment confirms this), put 3 back into 4, charge it so that again it has charge 1, and transfer it again to the cage. The reading of V_{12} now we mark 2. Repeating the process, we get points on the scale of V_{12} marked 3, 4, ... In order to ascertain whether these marks on V_{12} provide a consistent scale of charge, we change the voltage V_{34} or the body 3, or both, until we find that, when 3 charged in 4 is transferred to 1, it causes V_{12} to change from 0 to 2. Then, repeating this process, we find that the second transfer changes V_{12} from 2 to 4. If the rules of addition, tested by a sufficient variety of such experiments, are always fulfilled, the calibrated scale of V_{12} will measure charge, and we shall have established charge as an independently measurable magnitude; we shall have gone a long way to providing evidence for the fundamental postulate of classical electrostatics. Of course nowadays this independent measurement of Q is of little importance; it is easier and more accurate to measure Q as the derived magnitude CV. But it is remarkable that no experiment of this kind, so necessary as a starting point for classical theory, appears in either the standard treatises or the histories. However, this becomes less surprising when we observe that all the laws of physical addition, even those of length and mass, have almost always been accepted purely intuitively. They are so fundamental that the underlying assumptions are made unconsciously. It is, however, desirable that they should be brought to light.

But, though we have now dissociated Q from any magnitude characteristic of capacitors, we have not rendered it entirely independent of them. It still is significant only in connection with the charging of capacitors. Two further experiments are needed to establish complete independence. One arises from the discovery that bodies that have not been charged as part of a capacitor may produce a reading of V_{12} when they are introduced into the cage. In order to establish that the magnitude charge may fitly characterize such bodies, we must show that the introduction into the cage of n bodies, each of which when introduced singly gives the same value of δV_{12}, gives a reading of $n\delta V_{12}$.

For the second experiment, we require two similar bodies 3 as shown in figure 3, and we break the connections of 2 and 4 to earth, and connect these screens together. We then have in figure 4 a complex capacitor consisting of three simple closed capacitors in series, namely those shown dotted in figure 3. We connect the two bodies 3 by a battery in series with an ammeter, and measure $\int Idt$, where I is the current that flows during the charging of the complex capacitance. We now remove 3 from within 1, and introduce into the cage 1 the body 3 which has been charged within 4. We shall find that the charge on 3 so measured is $\int Idt$ measured during the charging. We have then proved that the transport of a charged body between the two positions 3 in figure 2 has precisely the same effect, in respect of this apparatus, as the passage of a current I between those two positions, so long as the charge on the body is $\int Idt$, and *vice versa*. This leads us to inquire whether the transport of a charge is equivalent to the passage of a current in respect of other apparatus, e.g. that for measuring the magnetic field produced by a current. We find that it is, so long as the path traversed by the charge during its transport is the same as the path of the current. When we have proved that, we have proved that charge has a significance wider than that of electrostatic experiments, all of which involve capacitors; we shall have proved, in the language of our ancestors, the "identity of all forms of electricity".

§ 14. "EARTH"

Earth is a conception hardly less fundamental in classical theory than *charge*; but the explanations offered of what constitutes earth* are usually very vague, and

* We leave on one side problems concerning the "atmospheric gradient" or old-fashioned telegraphy with an "earth return". For in them "earth" undoubtedly means The Earth. But it is perhaps worth noting that there was a period (roughly that of the 1870's) when electrical nomenclature was becoming standardized and telegraphy was the only form of electrical engineering. The present uses of "earth" are probably derived in some measure from "earth returns".

sometimes demonstrably false. In the absence of a formal definition the meaning of earth can be ascertained only by examining the statements that are made about it.

Thus *earth* is the thing to which zero potential is assigned in conventional discussions of "systems of conductors". Since differences of potential are voltages, it is the thing from which voltages are measured, i.e. the "zero plate" of § 5. Now if there is a terminal to which many more conductors are connected than to any other, it is clearly convenient to take that conductor as the zero plate. In apparatus containing many screens through which leads pass, and which are therefore connected together, such as that of figure 1, the terminal to which the screens are connected is such a terminal. If therefore *earth* means the common terminal of screens in such apparatus, it has a clear significance in the light of our exposition, which is all based on the conception of capacitance. Moreover, such a use of the term would agree with ordinary laboratory usage; for such a common terminal is called "earth" in that usage; indeed, the avoidance of *earth* in our discussion of screening must have appeared artificial to many readers. Further, the usage has strong justification; for in such apparatus the sheathing of cables almost always forms part of the screens, and this sheathing is always connected to The Earth.

If, however, the observer and all his apparatus are enclosed in a complete screen through which no leads pass, this screen will always be used as the common terminal of less perfect screens within it and, so far as is possible, as one terminal of generators and detectors. There is then no need that the screen enclosing the entire apparatus should be connected to The Earth; indeed, in a high-voltage laboratory that may be positively undesirable or impossible; one man's earth may have to be another man's H.T. terminal. Accordingly, even if *earth* could always be identified with a common terminal in the sense just described, it would still be necessary to recognize that "earth" does not always mean The Earth, and, more generally, that there is no single body that can always be taken as earth.

There is another statement about "earth" made or implied in conventional text-books, especially in the description of the classical experiments of electrostatics, on which the whole subject is supposed to be based. It is that, if any charged body is connected to earth, it will be wholly discharged. According to § 13, this will be true only if "earth" means a screen surrounding the body completely or practically completely. If that is what "earth" means, earth cannot possible be a single body everywhere at the same potential; for, if there are two charged capacitors in series, the screen of one is necessarily the plate of the other. Nevertheless, the use of "earth" in this sense may have some significance in connection with the said classical experiments. For if they were conducted in the open, some of them would fail owing to the Earth's atmospheric gradient; if they are to succeed, they must be conducted within a screen, which will have to be the more complete, the more accurate the real experiments. The classical experiments are so inaccurate that the very incomplete screen formed by the partially conducting walls, ceiling and floor of the building in which they are conducted may suffice. When a charged body is discharged by touching it with the finger or with a wire connected to the water pipes, that may be because such contact provides a sufficiently conducting path between the body and the imperfect screen by which it is surrounded.

But another explanation is possible. The charge in a capacitor will be greatly reduced if its terminals are connected to an uncharged capacitor of much greater capacitance. It is possible that a charged body is discharged by the finger of the observer, not merely because he provides a conducting path to the screen, but because his body forms with all those conductors, with which the charged body has mutual capacitance, a capacitor of much greater capacitance; or, more simply, because his body is much larger than the body he discharges.

This, we suggest, is the basis of the curious statement to be found in some well known text-books (it appears to have grown commoner during the last generation) that The Earth acts as an "earth", because its capacitance is much greater than that of any body concerned in terrestrial experiments. But, though it may provide the basis, it provides no justification. For, firstly, the statement is not actually true; the capacitance of The Earth, as ordinarily calculated, is no greater than that of certain electrolytic condensers that are often used in laboratory experiments. Secondly, and more important, the capacitance of The Earth cannot possibly have any relevance to any terrestrial experiment. The facts that give significance to the conception of the capacitance of a single body (not of a pair of associated bodies) are these: (they are predicted by the field equations and confirmed by experiment.) If in a closed capacitor the distance between the plate and the screen is increased without limit, the capacitance of the capacitor tends to a lower limit; the capacitance of a single body means this lower limit towards which the capacitance of a closed capacitor tends. But since there is no accessible conductor whose distance from The Earth is large compared with The Earth's dimensions, this limit can have no relevance to experiments conducted on The Earth.

§ 15. POINT CHARGES

Point charges play a very important part in classical theory—much more important than that which they play in experimental physics. It seems therefore important to show that the propositions that we have set forth lead to the chief propositions about points charges. (These are Coulomb's law, and

$$F = e\mathscr{E} = e \,.\, dv/dq, \qquad \ldots\ldots(15.1)$$

where F is the force in the direction of the coordinate of q acting on a point charge e in a field of potential v (§ 5)).

Let us consider then the force between two charged conductors, which should lead us to Coulomb's Law. For reasons already given we must include in our system an earth, i.e. a third conductor surrounding the two small ones (figure 4*a*); but in order to simplify the problem we shall assume that the third conductor is so large that the distance r between 1 and 2 is very small compared with their distances from 3, but large compared with the linear dimensions of 1 and 2. Unless the charges on 1 and 2 are equal and opposite, there must be a charge equal to their difference on the inner surface of 3; there may also be charges on its outer surface, but they do not concern us. We have from (11.1) with an obvious notation

$$W_V = \tfrac{1}{2}(C_{12}V_{12}^2 + C_{23}V_{23}^2 + C_{13}V_{31}^2) \qquad \ldots\ldots(15.2)$$

with

$$Q_1 = C_{12}V_{12} + C_{13}V_{13}, \qquad \ldots\ldots(15.3)$$

$$Q_2 = C_{12}V_{21} + C_{23}V_{23}, \qquad \ldots\ldots(15.4)$$

$$V_{23} = V_{21} + V_{13}. \qquad \ldots\ldots(15.5)$$

Substituting for the Vs in (15.2) from (15.3, 4, 5), we find after some reduction

$$2W_Q(C_{12}C_{23}+C_{23}C_{13}+C_{13}C_{12})=Q_1^2(C_{12}+C_{23})+Q_2^2(C_{12}+C_{13})+2Q_1Q_2C_{12}. \qquad \ldots\ldots(15.6)$$

In case (a), if the bodies 1 and 2 are small compared with their distances from the screen and from each other, C_{12} is small compared with each of C_{13}, C_{23}. It follows then from (15.6) that, if F_r is the force tending to increase the distance r between 1, 2, then

$$F_r=-dW_Q/dr=-\left\{\tfrac{1}{2}Q_1^2\frac{d}{dr}\left(\frac{1}{C_{13}}\right)+\tfrac{1}{2}Q_2^2\frac{d}{dr}\left(\frac{1}{C_{23}}\right)+Q_1Q_2\frac{d}{dr}\left(\frac{C_{12}}{C_{13}C_{23}}\right)\right\}. \qquad \ldots\ldots(15.7)$$

Now, according to a well known result (see e.g. Jeans, *Electricity and Magnetism*, § 116), if a, b are the radii of spheres to which the bodies 1, 2 approximate, and if $ab \ll r^2$

$$C_{13}=\kappa a; \quad C_{23}=\kappa b; \quad C_{12}=\kappa ab/r, \qquad \ldots\ldots(15.8)$$

where κ is the permittivity of the medium surrounding the bodies. Hence the terms of (15.7) in Q_1^2 and Q_2^2 vanish, and we have

$$F_r=-Q_1Q_2\,.\,d/dr(\kappa ab/r\,.\,1/\kappa^2ab) \qquad \ldots\ldots(15.9)$$

$$=+Q_1Q_2/\kappa r^2, \qquad \ldots\ldots(15.10)$$

which is Coulomb's Law. Thus, although the law is not strictly true for small charged conductors, as is obvious from (15.7), it is an approximation that becomes true in the limiting case in which the charged bodies become infinitely small.

Consider now the force on a single small charged body 2 in the field between two large bodies 1 and 3 (figure 4b). This is another case of the general problem of three charged bodies, and (15.6) again holds; but now C_{12}, C_{23} are both smal compared with C_{13}, as in (say) the oil-drop experiment. So we have

$$F_q=-dW_Q/dq=-\tfrac{1}{2}Q_1^2\frac{d}{dq}\left(\frac{1}{C_{13}}\right)-\tfrac{1}{2}Q_2^2\frac{d}{dq}\left(\frac{1}{C_{12}+C_{23}}\right)$$

$$-Q_1Q_2\frac{d}{dq}\left(\frac{1}{C_{13}}\cdot\frac{C_{12}}{C_{12}+C_{23}}\right), \qquad \ldots\ldots(15.12)$$

where q determines the position of the small conductor relative to the large ones, whose relative position is constant.

The term in Q_1^2 represents a force, important in some circumstances, tending to bring 2, even if it is uncharged, into the region where the field due to the charges on 1 and 3 is strongest. Since e in (15.1) is our Q_2, there is no corresponding term in (15.1) independent of Q_2. The term in Q_2^2 represents (in conventional language) the "image force" between Q_2 and the charges that it "induces" in the large conductors. If v is independent of Q_2, as the form of (15.1) suggests, there is again no corresponding term in (15.1); but one can be introduced by including in v a part due to the "induced charges"; a part due to the charge Q_2 on 2 must not be introduced, for such a part would not give rise to a force on 2. Accordingly (15.1) in its natural interpretation is at best an approximation valid only when the terms of (15.7) in Q_1^2 and Q_2^2 are negligible. That in Q_1^2 is negligible, if the field due to the charges on 1 and 3 is sufficiently uniform and the dimensions of 2 sufficiently small. The term in Q_2^2 is negligible, if Q_2/Q_1 is sufficiently small and if also 2 is sufficiently far from 1 and 3; for, when it is far from 1 and 3, dC_{12}/dq and dC_{23}/dq tend to be equal and opposite.

If these conditions are fulfilled (15.7) reduces to

$$F_q = -Q_1Q_2 \cdot \frac{d}{dq}\left(\frac{1}{C_{13}} \cdot \frac{C_{12}}{C_{12}+C_{23}}\right); \qquad \ldots\ldots(15.13)$$

and this is the equation that really has to be compared with (15.1). In that equation v is the potential relative to the screen 3 of the point occupied by 2, when 1 is at its actual potential V_{13}. This may be identified with the voltage that would be established between 3 and a small uncharged conductor in the position of 2, if the voltage between 1 and 3 were increased from 0 to V_{13}. The small conductor, being insulated, cannot become charged during the process; and we have from (15.4)

$$0 = C_{12}V_{21} + C_{23}V_{23} = C_{12}(v - V_{13}) + C_{23}v$$

or

$$v = V_{13} \, . \, C_{12}/(C_{12} + V_{23}). \qquad \ldots\ldots(15.14)$$

Hence (15.13) becomes

$$F_q = -Q_1Q_2 \cdot \frac{d}{dq}\left(\frac{v}{C_{13}V_{13}}\right) \qquad \ldots\ldots(15.15)$$

or from (15.3),

$$= -Q_2 \cdot \frac{d}{dq}\left(v \cdot \frac{Q_1}{Q_1 - C_{12}V_{12}}\right), \qquad \ldots\ldots(15.16)$$

(15.16) is identical with (15.1) only if $C_{12}V_{12} \ll Q_1$. This does not follow immediately from (15.3), (15.4) together with $Q_2 \ll Q_1$ and $(C_{12}, C_{23}) \ll C_{13}$; for the two terms in (15.4) may be of opposite sign. But it can hardly be doubted that there are conditions in which (15.16) approaches (15.1) as a limit as Q_2 approaches zero. We leave to the mathematician the task of deciding exactly what those conditions are; they will certainly involve the interpretation given to v in (15.1). Further, a correction, which will certainly involve terms in Q_2/Q_1 only, has to be made for the fact that, while (15.1) relates to a state in which V_{13} remains constant during the motion of Q_2, we have calculated the force on the assumption that Q_1 remains constant during that motion. It is sufficient for our purpose to note that (15.1) can be deduced by the argument that we have presented only as a highly specialized limit.

The complexity of the argument necessary to deduce the very simple classical formula (15.1) may suggest the criticism that, though our exposition of electrostatics may be ultimately sounder than that of classical theory (at least if that theory is supposed to have any relation to experiment), the classical treatment is always to be preferred in dealing with any practical problem. We reject any such suggestion. The physicist who applies the simple formula (15.1) to any real experimental system must always make due allowance for the difference between his actual system and the ideal system for which the formula is strictly valid. The simplicity of (15.1) is obtained only at the cost of complexity of interpretation in terms of measurable quantities, and our methods remove this complexity. For general problems in experimental physics we suggest that our methods are far more perspicuous physically, and no more difficult algebraically, than classical methods. This is particularly true of the problems (§§ 10–12) arising in "systems of conductors"; actual experience has convinced us that the treatment of such problems in terms of the equivalent capacitance network has great advantages.

§ 16 CHARGES ON INSULATORS

It has been assumed in the two preceding sections, explicitly or tacitly, that charged bodies are always conductors. This is not true. Of the bodies which, when introduced into a Faraday cage produce a non-zero δV_{12}, some are insulators. Such bodies cannot form a plate of a capacitor, cannot be charged or discharged by a current circuit, and have no measurable voltage to other bodies; accordingly many of the propositions of those sections have no application to them. Nevertheless they have a determinate charge, because they satisfy the criterion of § 14, namely that n equivalent bodies, introduced together into the cage, produce n times the effect of one of them. Further, if we suppose that a conductor and an insulator, having the same charge according to the Faraday cage, differ only in the distribution of the charge, all propositions concerning point charges must be equally true of both of them; for in the limit when the dimensions of the body are zero, the distribution of the charge in the body cannot matter. Accordingly (15.1) can be applied to the measurement of the charge on an oil drop, by observing the force on it in a uniform field, although the drop may be an insulator; or to the deflection of a cathode ray in an electric field, without making an assumption about the constitution of an electron, except that it is small and charged.

But the distribution of charge on an insulator enters into other problems. They can be solved in a manner that we shall not discuss in detail by means of a proposition about the relation between the field v (§ 6) about an insulator and the distribution of charge in and over it. The conventional proposition is that if σ is the surface density of the charge and ρ the volume density, then v must satisfy the differential equations

$$4\pi\sigma = -\left(\frac{dv}{dn_1} + \frac{dv}{dn_2}\right), \qquad \ldots\ldots(16.1)$$

where n_1, n_2 are the normals to the surface on opposite sides of it

$$4\pi\rho = -\left\{\frac{d}{dx}\left(\kappa\frac{dv}{dx}\right) + \frac{d}{dy}\left(\kappa\frac{dv}{dy}\right) + \frac{d}{dz}\left(\kappa\frac{dv}{dz}\right)\right\}. \qquad \ldots\ldots(16.2)$$

The direct evidence for these propositions is very slight. Some evidence in favour of (16.1) can be obtained by means of "proof planes", i.e. small conductors applied to the surface of the insulator and then transferred to a Faraday cage, if it is assumed that the proof plane takes up the surface density of the surface to which it is applied. (It may be noted incidentally that, by means of proof planes, evidence can be obtained for the presence of the charges $-Q_1'$ and $+Q_1'$ on the inner and outer surfaces of the intermediate plate of the two capacitors in series discussed in § 5.) Volume charge can be detected only by breaking the body and transferring the fragment to the cage. Actually volume charges in solids are at present of little experimental importance. The force between two bodies, one at least of which is an insulator, arising from their charges, can be obtained by means of a proposition that might have been stated earlier. It can be shown by purely mathematical reasoning that (11.1), (11.2) are equivalent to

$$W = \frac{1}{8\pi}\iiint \kappa\left\{\left(\frac{dv}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} dxdydz. \quad \ldots\ldots (16.3)$$

It is assumed that this formula for the energy holds even when some or all of the charges reside on insulators. The results deduced from this assumption, together with (16.1), (16.2), apparently agree with such experiments as can be made.

Finally, it might appear that, if the dielectric forming part of a capacitor were charged, the rule for calculating capacitance given in §4 would no longer hold. Experiment, in accordance with deduction from (16.1), (16.2), shows that such charges, though they modify the values of v, do not modify the capacitance.

§ 17. INDUCED CHARGE

In the conventional exposition of electrostatics a magnitude called *induced charge* is introduced, and considerable space is devoted to a discussion how far "induced" and "real" charges have the same properties. The expression *induced charge* is often useful in drawing attention to the fact that, when an insulated conductor is introduced into an electric field, the distribution of the charge on it, as measured by ϕ_e, changes, while its total charge, measured by $\int I dt$, does not. But it appears to us that attempts to give the expression definite quantitative significance introduce confusion rather than clarity.

REFERENCES.

CAMPBELL, N. R. and HARTSHORN, L., 1946, *Proc. Phys Soc.*, **58**, 634.
HARTSHORN, L., 1940, *Radio-Frequency Measurements by Bridge and Resonance Methods* (London : Chapman and Hall).
ROSA and DORSEY, 1907, *Bull. Bur. Stand.*, **3**, 43

Measurement of Absolute Humidity in Extremely Dry Air

BY A. W. BREWER, B. CWILONG AND G. M. B. DOBSON,
Oxford

MS. received 29 *May* 1947

ABSTRACT. To meet an urgent demand for a hygrometer capable of use at all heights in the atmosphere, the dew-point hygrometer, which is shown to have many advantages, has been developed primarily for use in aircraft to measure dew-points, or rather frost-points, down to $-90°$ C.

It is necessary that the instrument should operate at the lowest possible frost-points as it has been discovered that the air in the stratosphere is very dry Laboratory studies of the deposition of water and ice from the vapour at low temperature are described.

Below $-90°$ C. it is not possible to operate a frost-point hygrometer because the deposit is in the form of an invisible glassy layer, but the instrument gives correct results at temperatures close to this limit.

Details are given of the construction of different forms of hand-operated hygrometers, and work is now going on to develop a fully automatic frost-point hygrometer.