1 Ω and Z

In this section, we want to answer why Z is a more efficient technology than the good 'ol Ω.

$$P(\mu_i) = e^{-\beta E_i} \left(\frac{Z_{N-1}(\beta)}{Z_N(\beta)} \right)$$

$$P(\mu_1, \mu_2, \mu_3, \ldots) = \frac{e^{-\beta E_1} e^{-\beta E_2} e^{-\beta E_3} \cdots}{Z(\beta)} = \frac{e^{-\beta E}}{Z(\beta)}$$

Let's zoom into the system to see what's really going in there. As usual, in thermal equilibrium, the probability of a microstate only depends on its energy.

$$P(\mu_i) = \frac{1}{\Omega_1(E_{\mu_i})}$$

$$P(\mu_1, \mu_2, \mu_3, \ldots) = \frac{1}{\Omega_1(E_1)} \frac{1}{\Omega_1(E_2)} \frac{1}{\Omega_1(E_3)} \cdots \tag{1}$$

$$= \frac{1}{\Omega_N(E)} \tag{2}$$

Notice that these equations already represent thermal equilibrium, since they are characterized only by energy and both the element and the whole system are described by the SAME characteristic function, Ω. Thermal equilibrium is our one and only approximation, hence, one shouldn't expect to do more approximation in order to arrive at Z. Boltzmann relation means your system is renormalizable.

There are three ways to exploit the relation between (1) and (2):

1. not even close to a harmonic oscillator Equation (1) to (2) may look like a tiny step in LaTeX, but it is actually a giant step. Rewrite Ω_1 as e^{S_1}, hence the relation becomes

$$\sum_i S_1(E_i) = S_N(E) = S_N(\sum_i E_i)$$

which means S_N must not depend on the spread of the E_i distribution, to see this mathematically, let's taylor-expand around ϵ

$$\sum_i S_1(\epsilon) + \hat{\partial} S_1 \sum_i (E_i - \epsilon) + \hat{\partial}^2 S_1 \sum_i (E_i - \epsilon)^2 + \cdots = S_N(E)$$

$$N S_1(\epsilon) + \hat{\partial} S_1(\epsilon) \cdot 0 + \hat{\partial}^2 S_1(\epsilon) \sigma_E + \cdots = S_N(E)$$

or, $\hat{\partial} S_1(\epsilon) = \beta$ where β is a constant.

Hey, the entropy is automatically maximized, too (which means, it is redundant to further assume that entropy is maximized).

2. vary N Back to one particle, we are now authorized to use this relation

$$p(\mu_i) = \frac{\Omega_{N-1}(E - E_{\mu_i})}{\Omega_N(E)}$$

$$= \exp(\log \Omega_{N-1}(E - E_{\mu_i}) - \log \Omega_N(E))$$

$$= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp(-\frac{\partial}{\partial E} \log \Omega_{N-1} E_i - \frac{1}{2}\partial^2 \log \Omega_{N-1} E_i^2 - \cdots)$$

$$= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp(-\beta E_i - \frac{1}{2}\frac{\partial\beta}{\partial E} E_i^2 - \cdots)$$

$$= \frac{\exp(-\beta E_i)}{Z(\beta, \mathcal{O}(E_i^2))}$$

where β is just a label (until this point, you can safely assume that the). There are two completely different arguments: one, the multiplicty is maximized, two, the state only depends on energy

Another way to look at thermal equilibrium: we randomly sample a small element of our system, and yet it will still look exactly the same as its parent system (redundancy of information, rescaling, whatever).

Possible mathematical relations:

$$Z_{1R}(\beta, E_i) = \frac{\Omega_N(E)}{\Omega_{N-1}(E)} \frac{1}{\exp(+\frac{\beta}{2C_vT} E_i^2 - \cdots)}$$

Now we begin to see why we need log. All of our arguments

$$\log Z_1 = \frac{\partial}{\partial N} \log \Omega_N(E)$$

The difference between the two description is that the former gives us the real-life particle description, why the latter only tells us about the smoothed out representation of the system.

In the Z description, we are instead viewing the system as a field ("N is not fixed", but we don't care).

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