Optical Faraday rotation

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Three calculations of optical Faraday rotation are presented in which a linearly polarized field is incident on a medium of harmonic oscillators in the presence of a longitudinal magnetic field. The rotation of the plane of polarization of the field is evaluated using classical oscillators and the Lorentz force equation, quantum oscillators and the Heisenberg equations of motion, and quantum oscillators and a Schrödinger equation approach. It is shown that a simple argument, based on the assumption that a circularly polarized field drives either $\Delta m=1$ or $\Delta m=-1$ transitions on absorption (m is the magnetic quantum number), leads to an incorrect result for the Verdet constant. © *2010 American Association of Physics Teachers.*

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I. INTRODUCTION

Faraday optical rotation refers to the rotation of the plane of polarization of an optical field as it propagates in a dielectric medium in which a static magnetic field is applied along the direction of propagation of the field.[1] Sommerfeld has given a simple derivation of the Faraday rotation, modeling the atoms in the medium as classical oscillators.[2]

Sommerfeld explained the effect as arising from a difference in the index of refraction for the left (n_+) and right (n_-) circularly polarized components of the field.[2] He used the Lorentz force to calculate the response of the oscillators to the applied fields and then obtained the polarization of the medium. From the polarization, he extracted the indices of refraction associated with the left and right circularly polarized components of the optical field and found that the difference in indices is given approximately by

$$n_- - n_+ = \frac{\rho e^3}{n m_0^2 \epsilon_0} \frac{B_0 \omega}{(\omega_0^2 - \omega^2)^2}, \tag{1}$$

where ρ is the oscillator density, ω_0 is the natural frequency, $-e$ is the charge, m_0 is the mass of an oscillator, B_0 is the magnitude of the external magnetic field, ω is the frequency of the optical field, ϵ_0 is the vacuum permittivity, and

$$n = \left(1 + \frac{\rho e^2/m_0\epsilon_0}{\omega_0^2 - \omega^2}\right)^{1/2} \tag{2}$$

is the index of refraction of the medium of oscillators in the absence of the magnetic field.

For a sample of length L, the difference in indices leads to a rotation angle

$$\theta = \frac{\omega}{c}\left(\frac{n_- - n_+}{2}\right)L = \frac{\rho e^3 B_0 L}{2 n m_0^2 c \epsilon_0} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2} \tag{3a}$$

$$= V L B_0/\mu_0, \tag{3b}$$

where μ_0 is the vacuum permeability and

$$V = \frac{\omega}{2c}\left(\frac{n_- - n_+}{B_0/\mu_0}\right) = \frac{\rho e^3 \mu_0}{2 n m_0^2 c \epsilon_0} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2} \tag{4a}$$

$$= \frac{e\mu_0}{2m_0 c}\omega \frac{dn}{d\omega} \tag{4b}$$

is the Verdet constant. It is seen that the Verdet constant is proportional to the dispersion of the medium.

Although the Faraday effect is mentioned in many optics textbooks and studied experimentally in advanced undergraduate laboratories,[3] discussions of the Faraday effect in textbooks and laboratory manuals are usually qualitative in nature. Often, there is no detailed derivation of the Faraday rotation angle. An exception is the book by Rossi,[4] which contains a derivation that is similar to Sommerfeld's. Quantum-mechanical derivations of Faraday rotation are not readily available in standard optics texts. There is a vast literature on the Faraday effect, but many of these articles are specialized, including applications to specific molecules,[5] nonlinear effects,[6] and the role of optical pumping.[7] It is useful to present a quantum calculation for a medium composed of harmonic oscillators, when a weak linearly polarized field propagates in the medium in the presence of a longitudinal magnetic field. The results should mirror that of the classical calculation; paradoxically, the predictions based on a simple quantum model seem to be at odds with the classical results.

The quantum calculation can be formulated for the level scheme of Fig. 1. Shown are the two lowest lying electronic state manifolds of a three-dimensional harmonic oscillator with natural frequency ω_0. The ground state has electronic quantum number $n=0$ and angular momentum $L=0$, and the first excited state has $n=1$ and $L=1$. If the incident field is x-polarized, we can view the field as an equal superposition of left and right circularly polarized fields or, equivalently, as an equal superposition of σ_+ and σ_- radiation, where the $\sigma_\pm$ radiation induces $\Delta m=\pm 1$ transitions on absorption and m is the magnetic quantum number. Because the only effect of the magnetic field is to shift the excited state $m=\pm 1$ sublevels, it can be deduced from Eq. (2) that the indices of refraction associated with the σ_+ and σ_- radiation are

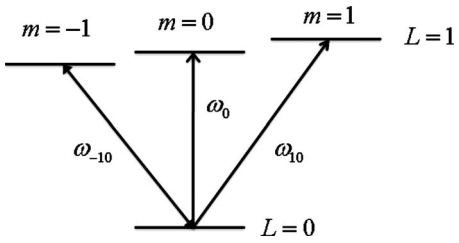

Fig. 1. Energy level scheme of an oscillator. The ground state has $n=0$ and $L=0$, while the first excited state has $n=1$ and $L=1$. A longitudinal magnetic field splits the magnetic sublevels of the $L=1$ manifold.

$$n_{\pm} = \left(1 + \frac{\rho e^2/m_0\epsilon_0}{\omega_{\pm}^2 - \omega^2}\right)^{1/2}, \tag{5}$$

where

$$\omega_{\pm} = \omega_0 \pm \frac{eB_0}{2m_0} \tag{6}$$

are the frequencies of the $\Delta m = \pm 1$ transitions from the ground to excited state, including the Zeeman shift of the levels. From Eqs. (5) and (6) it follows that

$$n_- - n_+ \approx \frac{\rho e^3}{nm_0^2\epsilon_0} \frac{\omega_0 B_0}{(\omega_0^2 - \omega^2)^2}, \tag{7}$$

which differs significantly from Eq. (1) if $\omega_0 \gg \omega$. Clearly something is wrong with this approach.

The purpose of this article is to explain the breakdown in the reasoning that led to Eq. (7) and to provide a classical and two simple quantum calculations of Faraday rotation. I begin by reviewing the classical calculation based on the Lorentz force equation (essentially equivalent to that given by Sommerfeld) and then present the analogous quantum calculation using a Heisenberg operator approach. Finally, I use a Schrödinger picture approach that helps to isolate the role of the $\Delta m = \pm 1$ transitions. Such an approach allows us to uncover the fallacy in the reasoning that led to Eq. (7). Finally, I point out an interesting feature related to the choice of the atom-field interaction Hamiltonian.

II. CLASSICAL CALCULATION

The medium is assumed to be composed of a uniform density of harmonic oscillators, each having charge $-e$. The force on an oscillator is given by

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - m_0\omega_0^2\mathbf{r}, \tag{8}$$

where $\mathbf{E}$ is the electric field vector in the medium,

$$\mathbf{E}(Z,t) = \tfrac{1}{2}[E_x(Z)\hat{\mathbf{x}} + E_y(Z)\hat{\mathbf{y}}]\exp[i(nkZ - \omega t)] + \text{c.c.}, \tag{9}$$

$\mathbf{B}$ is the magnetic induction,

$$\mathbf{B} = B_0\hat{\mathbf{z}}, \tag{10}$$

$\omega = kc$, n is the index of refraction of the medium given by Eq. (2), $\mathbf{r}$ is the displacement of an oscillator from its equilibrium position produced by the applied fields, and $\mathbf{v} = \dot{\mathbf{r}}$. The field incident on the medium is polarized in the $\hat{\mathbf{x}}$ direction but acquires a y component as it propagates through the medium. The displacement $\mathbf{r}(t)$ can be written as

$$\mathbf{r}(t) = x_+(t)\hat{\mathbf{x}} + y_+(t)\hat{\mathbf{y}} + \text{c.c.}, \tag{11}$$

where $x_+(t)$ and $y_+(t)$ are positive frequency components of the displacement, varying as $\exp(-i\omega t)$. The field amplitudes $E_x(Z)$ and $E_y(Z)$ are assumed to vary slowly with the optical wavelength λ, and it is assumed that the dipole approximation is valid, $r \ll \lambda$, allowing us to evaluate both the amplitude and phase of the optical field at the equilibrium position of the oscillator.

The classical equations of motion for $x_+(t)$ and $y_+(t)$ of an oscillator that is located at position Z and subjected to the Lorentz force (8) are

$$\ddot{x}_+ + 2\gamma\dot{x}_+ + \omega_0^2 x_+ = -\frac{eE_x(Z)\exp[i(nkZ - \omega t)]}{2m_0} - \frac{eB_0\dot{y}_+}{m_0}, \tag{12a}$$

$$\ddot{y}_+ + 2\gamma\dot{y}_+ + \omega_0^2 y_+ = -\frac{eE_y(Z)\exp[i(nkZ - \omega t)]}{2m_0} + \frac{eB_0\dot{x}_+}{m_0}, \tag{12b}$$

where γ is a radiative decay rate. These equations are identical to those obtained by Sommerfeld,[2] except that I have included radiative decay, which allows for a steady-state solution. It is assumed that the magnitude of the electric field changes negligibly as a function of Z in the medium due to the fact that the field is detuned by an amount $\delta = \omega_0 - \omega \gg \gamma, eB_0/m_0$. Although the magnitude of the field remains constant, the plane of polarization rotates as the field propagates in the medium.

Sommerfeld solved Eqs. (12) by introducing the circular components $x_+ \pm iy_+$; to obtain the rate of change of the Faraday rotation angle, $d\theta/dZ$, it is sufficient to calculate the rotation angle $d\theta$ in a distance dZ for which the rotation angle is much less than unity.[8] In this limit we can take $E_x(Z) \approx E_x(0) \equiv E_0$ in the interval dZ. Moreover, we can neglect the influence of the y component of the field on the dynamics of the x coordinate and take as the equations of motion

$$\ddot{x}_+ + 2\gamma\dot{x}_+ + \omega_0^2 x_+ = -\frac{eE_0 e^{i(nkZ - \omega t)}}{2m_0}, \tag{13a}$$

$$\ddot{y}_+ + 2\gamma\dot{y}_+ + \omega_0^2 y_+ = -\frac{eE_y(Z)e^{i(nkZ - \omega t)}}{2m_0} + \frac{eB_0\dot{x}_+}{m_0}. \tag{13b}$$

The steady-state solution of these equations (that is, the solution after all transients are finished) is

$$x_+(Z,t) = -\frac{eE_0 e^{i(nkZ - \omega t)}}{2m_0} \frac{1}{\omega_0^2 - \omega^2}, \tag{14a}$$

$$y_+(Z,t) = -\frac{eE_y(Z)e^{i(nkZ - \omega t)}}{2m_0} \frac{1}{\omega_0^2 - \omega^2} + \frac{e^2 B_0 E_0 e^{i(nkZ - \omega t)}}{2m_0^2} \frac{(i\omega)}{(\omega_0^2 - \omega^2)^2}, \tag{14b}$$

where the limit $\gamma \to 0$ has been taken under the assumption that δ, ω_0, $\omega \gg \gamma$.

To calculate the Faraday rotation, we must relate the fields to the polarization in the medium. The positive frequency component of the polarization is given by

$$\mathbf{P}_+(Z,t) = -\rho e\mathbf{r}_+(t) = -\rho e[x_+(Z,t)\hat{\mathbf{x}} + y_+(Z,t)\hat{\mathbf{y}}], \tag{15}$$

and the wave equation in the medium can be written as

$$\frac{\partial^2 \mathbf{E}_+}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_+}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathbf{P}_+}{\partial t^2}, \tag{16}$$

where I have set $\mathbf{B}=\mu_0\mathbf{H}$ and $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$. Using Eqs. (16), (15), and (14), we find that the wave equation for the x component of the field (9) is

$$(n^2-1)E_0 = \frac{\rho e^2 E_0}{\epsilon_0 m_0}\frac{1}{\omega_0^2-\omega^2}, \tag{17}$$

which reproduces Eq. (2) for the index of refraction in the absence of the applied magnetic field. If we use a similar approach for the y component of the field and cancel terms using Eq. (17), we find

$$2ink\frac{\partial E_y(Z)}{\partial Z} \approx \frac{i\rho e^3 k^2 B_0 E_0}{m_0^2\epsilon_0}\frac{\omega}{(\omega_0^2-\omega^2)^2}, \tag{18}$$

where the assumption that $E_y(Z)$ varies slowly in an optical wavelength was used, that is, the $\partial^2 E_y(Z)/\partial Z^2$ term was neglected. With the boundary condition $E_y(0)=0$, the solution of Eq. (18) for $E_y(dZ)$ is

$$E_y(dZ) \approx \frac{\rho e^3 k B_0 E_0}{2nm_0^2\epsilon_0}\frac{\omega}{(\omega_0^2-\omega^2)^2}dZ, \tag{19}$$

which implies a differential change in the rotation angle for the field polarization given by

$$\frac{d\theta}{dZ} = \frac{E_y(dZ)}{E_0 dZ} = \frac{\rho e^3 B_0}{2nm_0^2 c\epsilon_0}\frac{\omega^2}{(\omega_0^2-\omega^2)^2}. \tag{20}$$

For a medium of length L, we find a rotation angle

$$\theta = \frac{\rho e^3 B_0 L}{2nm_0^2 c\epsilon_0}\frac{\omega^2}{(\omega_0^2-\omega^2)^2}, \tag{21}$$

in agreement with Eq. (3).

III. HEISENBERG EQUATIONS OF MOTION

The Hamiltonian for an oscillator having charge $-e$ in the presence of the applied fields is

$$H = \frac{[\mathbf{p}+e\mathbf{A}(\mathbf{r},Z,t)]^2}{2m_0} + \frac{1}{2}m_0\omega_0^2 r^2, \tag{22}$$

where the vector potential,

$$\mathbf{A}(Z,t) = \mathbf{A}_E(Z,t) + \mathbf{A}_B(\mathbf{r}), \tag{23a}$$

$$\mathbf{A}_E(Z,t) = \left\{\frac{1}{2i\omega}[E_x(Z)\hat{\mathbf{x}} + E_y(Z)\hat{\mathbf{y}}]\exp[i(nkZ-\omega t)] + \text{c.c.}\right\}, \tag{23b}$$

$$\mathbf{A}_B(\mathbf{r}) = -\frac{B_0}{2}[y\hat{\mathbf{x}} - x\hat{\mathbf{y}}], \tag{23c}$$

is written in the Coulomb gauge. Interactions with the vacuum field have been neglected, and thus the equations to be derived should be compared with Eq. (12) in the limit that $\gamma\to 0$. The Heisenberg equations of motion for the positive frequency components of the oscillator, obtained using $\dot{\mathbf{p}} = [\mathbf{p},H]/(i\hbar)$ and $\dot{\mathbf{r}} = [\mathbf{r},H]/(i\hbar)$, are

$$\dot{p}_{x+} = -m_0\omega_0^2 x_+ - \frac{eB_0}{2m_0}p_{y+} - \frac{e^2B_0^2}{4m_0}x_+ - \frac{e^2B_0E_y(Z)\exp[i(nkZ-\omega t)]}{4im_0\omega}, \tag{24a}$$

$$\dot{p}_{y+} = -m_0\omega_0^2 y_+ + \frac{eB_0}{2m_0}p_{x+} - \frac{e^2B_0^2}{4m_0}y_+ + \frac{e^2B_0E_y(Z)\exp[i(nkZ-\omega t)]}{4im_0\omega}, \tag{24b}$$

$$\dot{x}_+ = \frac{p_{x+}}{m_0} + \frac{eE_x(Z)\exp[i(nkZ-\omega t)]}{2im_0\omega} - \frac{eB_0}{2m_0}y_+, \tag{24c}$$

$$\dot{y}_+ = \frac{p_{y+}}{m_0} + \frac{eE_y(Z)\exp[i(nkZ-\omega t)]}{2im_0\omega} + \frac{eB_0}{2m_0}x_+, \tag{24d}$$

from which it follows that

$$\ddot{x}_+ = \frac{\dot{p}_{x+}}{m_0} - \frac{eE_x(Z)\exp[i(nkZ-\omega t)]}{2m_0} - \frac{eB_0}{2m_0}\dot{y}_+ \tag{25a}$$

$$= -\omega_0^2 x_+ - \frac{eE_x(Z)\exp[i(nkZ-\omega t)]}{2m_0} - \frac{eB_0}{m_0}\dot{y}_+, \tag{25b}$$

$$\ddot{y}_+ = \frac{\dot{p}_{y+}}{m_0} - \frac{eE_y(Z)\exp[i(nkZ-\omega t)]}{2m_0} + \frac{eB_0}{2m_0}\dot{x}_+ \tag{25c}$$

$$= -\omega_0^2 y_+ - \frac{eE_y(Z)\exp[i(nkZ-\omega t)]}{2m_0} + \frac{eB_0\dot{x}_+}{m_0}. \tag{25d}$$

These equations agree with Eq. (12) in the limit $\gamma\to 0$. If I had used Hamilton's equations for the classical calculation instead of the Lorentz force equation, the classical and quantum calculations would have been identical.

The calculation in this section illustrates the power of using the Heisenberg equations of motion. We obtained the equations of motion for the relevant operators without having to resort to a solution of Schrödinger's equation. Life was simple here, owing to the fact that I considered the quantum system to be an (intrinsically linear) oscillator. Had I tried to calculate the response of an atom to the applied fields, I would not be able to write closed form differential equations of the type (25) because the atom acts as a nonlinear object.[9]

IV. SCHRÖDINGER PICTURE CALCULATION

The two lowest lying electronic state manifolds of the oscillator are shown in Fig. 1. The ground state has angular momentum $L=0$ and the first excited state $L=1$. Under the assumption that the external fields are weak, these are the only levels that need be considered.[10] Moreover, because the external fields are weak, we can drop terms that vary as E_0^2 or B_0^2 in the Hamiltonian (22). The approximate Hamiltonian for the oscillator in this limit is

$$H=\frac{p^2}{2m_0}+\frac{1}{2}m_0\omega_0^2r^2+\left\{\frac{e[p_xE_x(Z)+p_yE_y(Z)]}{2im_0\omega}\times\exp[i(nkZ-\omega t)]+\text{adj.}\right\}-\frac{eB_0}{2m_0}[p_xy-p_yx]-\frac{e^2B_0}{2m_0}\times\left\{\frac{[yE_x(Z)-xE_y(Z)]}{2im_0\omega}\exp[i(nkZ-\omega t)]+\text{adj.}\right\} \tag{26a}$$

$$=H_0+\left\{\frac{eE_x(Z)(p_x-eyB_0/2)}{2im_0\omega}\exp[i(nkZ-\omega t)]+\text{adj.}\right\}+\left\{\frac{eE_y(Z)(p_y+exB_0/2)}{2im_0\omega}\exp[i(nkZ-\omega t)]+\text{adj.}\right\}+\frac{eL_zB_0}{2m_0}, \tag{26b}$$

where $L_z=(xp_y-yp_x)$ is the z component of angular momentum,

$$H_0=\frac{p^2}{2m_0}+\frac{1}{2}m_0\omega_0^2r^2 \tag{27}$$

is the Hamiltonian in the absence of any applied fields, and "adj." is an abbreviation for "adjoint." Note that I had to keep terms that vary as E_xB_0 or E_yB_0, originating from the $2\mathbf{A}_E\cdot\mathbf{A}_B$ cross term in the A^2 term in the Hamiltonian [see Eqs. (23)]. This term contributes to the Faraday rotation in lowest order, and its inclusion is the price we must pay for using the $e\mathbf{A}\cdot\mathbf{p}/m_0$ rather than the $e\mathbf{r}\cdot\mathbf{E}$ interaction Hamiltonian for the optical field. I will return to this point in Sec. V.

In this section, I will calculate $\langle x\rangle$ and $\langle y\rangle$ from Schrödinger's equation. As in Sec. II, I assume that the field is x-polarized initially and propagates for a small distance in which the Faraday rotation angle is small, allowing one to replace $E_x(Z)$ by E_0 and to neglect the $E_y(Z)B_0$ term in Eq. (26b). As such the effective Hamiltonian is

$$H=H_0+\frac{e[E_0(p_x-eyB_0/2)+p_yE_y(Z)]}{2im_0\omega}\exp[i(nkZ-\omega t)] \tag{28a}$$

$$-\frac{e[E_0^*(p_x-eyB_0/2)+p_yE_y^*(Z)]}{2im_0\omega}\exp[-i(nkZ-\omega t)]. \tag{28b}$$

Because the external fields are weak, a perturbative approach can be used.[10]

In a perturbation theory limit, we can work with state amplitudes rather than density matrix elements. The state vector for the oscillator can be written as

$$|\psi(t)\rangle=a_0(t)|L=0\rangle+\sum_{m=-1,1}a_m(t)|L=1,m\rangle, \tag{29}$$

where $a_0(t)$ is the ground state amplitude and $a_m(t)$ ($m=\pm1$) is the excited state amplitude for sublevel m (the $m=0$ excited state sublevel is not excited because the optical electric field has no z component). In terms of these state amplitudes, the expectation values $\langle x\rangle$ and $\langle y\rangle$ for the state vector (29) are given by

$$\langle x(t)\rangle=\sum_{m=-1,1}x_{0m}a_m(t)a_0^*(t)+\text{c.c.}, \tag{30a}$$

$$\langle y(t)\rangle=\sum_{m=-1,1}y_{0m}a_m(t)a_0^*(t)+\text{c.c.}, \tag{30b}$$

where

$$x_{0m}=\langle L=0|x|L=1,m\rangle,\quad y_{0m}=\langle L=0|y|L=1,m\rangle \tag{31}$$

are matrix elements between the $|L=1,m\rangle$ excited state and the ground state.

To calculate the state amplitudes, we use the time-dependent Schrödinger equation with the Hamiltonian (28b) and the fact that the energy levels in the absence of the optical field are given by

$$E_{L=0}=\hbar\omega_0/2, \tag{32a}$$

$$E_{L=1,m}=3\hbar\omega_0/2+me\hbar B_0/2m_0. \tag{32b}$$

If the oscillator is prepared in its ground state at $t=0$, then, in lowest order perturbation theory, the ground state amplitude is given by

$$a_0(t)=\exp(-i\omega_0t/2), \tag{33}$$

and the time evolution of the excited state amplitudes is governed by

$$\dot{a}_m=-i\omega_ma_m-\gamma a_m-\frac{e}{2m_0\hbar\omega}[E_0(p_x-eyB_0/2)+p_yE_y(Z)]_{m0}\exp[i(nkZ-\omega t)]a_0(t)+\frac{e}{2m_0\hbar\omega}\times[E_0^*(p_x-eyB_0/2)+p_yE_y^*(Z)]_{m0}\times\exp[-i(nkZ-\omega t)]a_0(t), \tag{34}$$

where

$$\omega_m=\frac{E_{L=1,m}}{\hbar}=\frac{3\omega_0}{2}+\frac{emB_0}{2m_0}, \tag{35}$$

and I have included radiative damping to allow for a steady-state solution.

The steady-state solution of Eq. (34), with $a_0(t)$ given by Eq. (33), is

$$a_m(t)=-\frac{e[E_0(p_x-eyB_0/2)+p_yE_y(Z)]_{m0}}{2m_0\hbar\omega}\frac{\exp[i(nkZ-\omega t)]}{\gamma+i(\omega_{m0}-\omega)}\times e^{-i\omega_0t/2}+\frac{e[E_0^*(p_x-eyB_0/2)+p_yE_y^*(Z)]_{m0}}{2m_0\hbar\omega}\times\frac{\exp[-i(nkZ-\omega t)]}{\gamma+i(\omega_{m0}+\omega)}e^{-i\omega_0t/2}, \tag{36}$$

where

$$\omega_{m0}=\omega_m-\omega_0/2=\omega_0+\frac{emB_0}{2m_0} \tag{37}$$

is an optical transition frequency. If we keep terms to zeroth order in B_0 for $\langle x_+\rangle$ and first order in B_0 for $\langle y_+\rangle$ and neglect the back action of field component $E_y(Z)$ on x_+, we can use

Eqs. (30), (36), and (33) to show that the positive frequency components [those varying as $\exp(-i\omega t)$] of the displacement are given by

$$\langle x_+(Z,t)\rangle = -\sum_{m=-1,1}\frac{eE_0\exp[i(nkZ-\omega t)]}{2m_0\hbar\omega}\times\left\{\frac{x_{0m}(p_x)_{m0}}{\gamma+i(\omega_{m0}-\omega)}-\frac{[x_{0m}(p_x)_{m0}]^*}{\gamma-i(\omega_{m0}+\omega)}\right\}, \tag{38a}$$

$$\langle y_+(Z,t)\rangle = -\sum_{m=-1,1}\frac{e\exp[i(nkZ-\omega t)]}{2m_0\hbar\omega}\times\left\{\frac{y_{0m}[p_yE_y(Z)+p_xE_0-eyE_0B_0/2]_{m0}}{\gamma+i(\omega_{m0}-\omega)}-\frac{\{y_{0m}[p_yE_y^*(Z)+p_xE_0^*-eyE_0^*B_0/2]_{m0}\}^*}{\gamma-i(\omega_{m0}+\omega)}\right\}. \tag{38b}$$

The matrix elements for the 3D oscillator are

$$x_{0m}=\frac{1}{2}\sqrt{\frac{\hbar}{m_0\omega_0}}(\delta_{m,1}-\delta_{m,-1}), \tag{39a}$$

$$y_{0m}=y_{m0}^*=\frac{i}{2}\sqrt{\frac{\hbar}{m_0\omega_0}}(\delta_{m,1}+\delta_{m,-1}), \tag{39b}$$

$$(p_x)_{m0}=\frac{i}{2}\sqrt{\hbar m_0\omega_0}(\delta_{m,1}-\delta_{m,-1}), \tag{39c}$$

$$(p_y)_{m0}=\frac{1}{2}\sqrt{\hbar m_0\omega_0}(\delta_{m,1}+\delta_{m,-1}), \tag{39d}$$

where $\delta_{i,j}$ is the Kronecker delta. The matrix elements depend on ω_0 and not ω_{m0} because they are taken with respect to the unperturbed basis. We combine Eqs. (38) and (39) and find

$$\langle x_+(Z,t)\rangle = -\sum_{m=-1,1}\frac{eE_0e^{i(nkZ-\omega t)}}{4m_0(\omega_{m0}^2-\omega^2)}\approx-\frac{eE_0e^{i(nkZ-\omega t)}}{2m_0(\omega_0^2-\omega^2)}, \tag{40a}$$

$$\langle y_+(Z,t)\rangle = -\sum_{m=-1,1}\frac{eE_y(Z)e^{i(nkZ-\omega t)}}{4m_0(\omega_{m0}^2-\omega^2)}-\sum_{m=-1,1}\frac{ieE_0e^{i(nkZ-\omega t)}\omega_{m0}}{4m_0\omega(\omega_{m0}^2-\omega^2)}\times\left[(\delta_{m,1}-\delta_{m,-1})+\frac{eB_0}{2m_0\omega_0}\right], \tag{40b}$$

where I have taken the $\gamma\to0$ limit. Using Eq. (37), we can show that

$$\sum_{m=-1,1}\frac{\omega_{m0}}{(\omega_{m0}^2-\omega^2)}\left[(\delta_{m,1}-\delta_{m,-1})+\frac{eB_0}{2m_0\omega_0}\right]\approx\left[\frac{\omega_{10}}{\omega_{10}^2-\omega^2}-\frac{\omega_{-10}}{\omega_{-10}^2-\omega^2}+\frac{eB_0}{m_0(\omega_0^2-\omega^2)}\right]=\left[\frac{\omega_0+\frac{eB_0}{2m_0}}{\left(\omega_0+\frac{eB_0}{2m_0}\right)^2}-\frac{\omega_0-\frac{eB_0}{2m_0}}{\left(\omega_0-\frac{eB_0}{2m_0}\right)^2}+\frac{eB_0}{m_0(\omega_0^2-\omega^2)}\right]\approx-\frac{2eB_0\omega^2}{m_0(\omega_0^2-\omega^2)^2}. \tag{41}$$

As a consequence, Eq. (40b) reduces to

$$\langle y_+(Z,t)\rangle\approx\left\{-\frac{eE_y(Z)}{2m_0(\omega_0^2-\omega^2)}+\frac{e^2B_0E_0}{2m_0^2}\frac{(i\omega)}{(\omega_0^2-\omega^2)^2}\right\}e^{i(nkZ-\omega t)}. \tag{42}$$

Equations (40) and (42) agree with Eq. (14).

The Schrödinger approach has led to a result that is consistent with the classical result. Moreover, it can help us to understand the breakdown of the reasoning given in Sec. I. To understand what went wrong, consider a σ_+ circularly polarized field with the electric field vector,

$$\mathbf{E}(Z,t,+)=\frac{E_0}{2}\left[\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}\right]e^{i(n_+kZ-\omega t)}+\text{c.c.}, \tag{43}$$

with E_0 real, that drives transitions between the $L=0$ ground state and the $L=1$ excited state. Using the Hamiltonian (see Sec. V)

$$H=H_0+e\mathbf{r}\cdot\mathbf{E}(Z,t,+)+\frac{eL_zB_0}{2m_0}, \tag{44}$$

with the state vector (29) and $a_0(t)\approx\exp(-i\omega_0t/2)$, I find that the excited state amplitudes evolve as

$$\dot{a}_m=-i\omega_ma_m-\gamma a_m-i\frac{E_0e^{i(n_+kZ-\omega t)}}{2\hbar\sqrt{2}}[x+iy]_{m0}\times\exp(-i\omega_0t/2) \tag{45a}$$

$$-i\frac{E_0e^{-i(n_+kZ-\omega t)}}{2\sqrt{2}}[x-iy]_{m0}\exp(-i\omega_0t/2), \tag{45b}$$

where I have included radiative decay. The needed matrix elements are given by Eq. (39), and it follows that

$$\dot{a}_m=-i\omega_ma_m-\gamma a_m-ieE_0\sqrt{\frac{1}{8\hbar m_0\omega_0}}\times[e^{i(n_+kZ-\omega t)}\delta_{m,1}-e^{-i(n_+kZ-\omega t)}\delta_{m,-1}]\exp(-i\omega_0t/2). \tag{46}$$

From Eq. (46) it is clear that for the Hamiltonian (44), the only effect of the magnetic field is to shift the energy of the $m=\pm1$ excited state sublevels. Magnetic-field-induced changes in the index of refraction associated with the σ_+ and σ_- components of the electric field result solely from these magnetic-field-induced level shifts. It seems that arguments

of the type presented in Sec. I lead to an inconsistent value for the Verdet constant. However, this line of reasoning is valid only if σ_+ radiation drives only $\Delta m=1$ and σ_- radiation only $\Delta m=-1$ transitions on absorption.

In the resonance or rotating-wave approximation, the last term in Eq. (46) can be dropped. In this approximation, it is correct to say that a σ_+ circularly polarized excites only $\Delta m=1$ transitions on absorption. If the last term in Eq. (46) is included, this statement no longer holds. Thus the breakdown in the reasoning in Sec. I is that a circularly polarized field drives both $\Delta m=\pm 1$ transitions if the resonance approximation is not made. Thus, there are contributions to the index of refraction of σ_+ radiation from both the $\Delta m=\pm 1$ transitions.

Equations (30) and (46) can be used to calculate n_+. The steady-state solution of Eq. (46) is

$$a_m(Z,t) = -ieE_0\sqrt{\frac{1}{8\hbar m_0\omega_0}}\left[\frac{e^{i(n_+kZ-\omega t)}\delta_{m,1}}{\gamma+i(\omega_{m0}-\omega)} - \frac{e^{-i(n_+kZ-\omega t)}\delta_{m,-1}}{\gamma+i(\omega_{m0}+\omega)}\right]\exp(-i\omega_0 t/2). \tag{47}$$

To obtain an expression for n_+ in terms of $a_m(Z,t)$, I first write the polarization as

$$\mathbf{P}(Z,t) = (P_{x_+}\hat{\mathbf{x}} + iP_{y_+}\hat{\mathbf{y}})e^{i(n_+kZ-\omega t)} + \text{c.c.} \tag{48a}$$

$$= \left[P_+(+)\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + P_+(-)\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{\sqrt{2}}\right]e^{i(n_+kZ-\omega t)} + \text{c.c.}, \tag{48b}$$

where the positive frequency component of the circular components of the polarization is given by

$$P_+(\pm) = \frac{P_{x_+}\mp iP_{y_+}}{\sqrt{2}} = -\rho e\frac{\langle x_+\rangle \mp i\langle y_+\rangle}{\sqrt{2}}. \tag{49}$$

By combining Eqs. (30), (39), (47), and (49), we can show that $P_+(-)=0$ (the σ_+-polarized incident field induces only a σ_+ component of the polarization) and

$$P_+(+) = \frac{\rho e^2 E_0}{4m_0\omega_0}\left[\frac{1}{\omega_+-\omega}+\frac{1}{\omega_-+\omega}\right], \tag{50}$$

where the limit $\gamma\to 0$ has been taken (recall that $\omega_\pm=\omega_{\pm 10}=\omega_0\pm eB_0/2m_0$). Finally, because the polarization and field amplitudes in Eqs. (48) and (43) are related by

$$P_+(+) = (n_+^2-1)\epsilon_0 E_0/2, \tag{51}$$

it follows that

$$n_+^2 - 1 = \frac{\rho e^2}{2m_0\epsilon_0\omega_0}\left[\frac{1}{\omega_+-\omega}+\frac{1}{\omega_-+\omega}\right]. \tag{52}$$

Similarly, for a σ_- circularly polarized field, we find

$$n_-^2 - 1 = \frac{\rho e^2}{2m_0\epsilon_0\omega_0}\left[\frac{1}{\omega_--\omega}+\frac{1}{\omega_++\omega}\right], \tag{53}$$

implying that

$$n_- - n_+ \approx \frac{\rho e^3}{nm_0^2\epsilon_0}\frac{\omega B_0}{(\omega_0^2-\omega^2)^2}, \tag{54}$$

in agreement with Eq. (1). Note that the index of refraction for each circularly polarized field in Eqs. (52) and (53) contain contributions from both $\Delta m=\pm 1$ transitions.

V. DISCUSSION

I have presented three calculations of Faraday rotation, a classical calculation using the Lorentz force equation, a quantum calculation using a Heisenberg operator approach, and a quantum calculation using a Schrödinger wave function approach. The three methods yield consistent results. An oversimplified picture in which one assigns an index of refraction for $\sigma_\pm$ circularly polarized radiation resulting only from $\Delta m=\pm 1$ transitions on absorption breaks down when non-rotating-wave approximation terms are included in the calculation.

Finally, I return to the question as to the correct form of the Hamiltonian in weak fields. It might be thought that using a Hamiltonian of the form

$$H_1 = H_0 + \frac{e\mathbf{p}\cdot\mathbf{A}_E(Z,t)}{m_0} + \frac{eL_zB_0}{2m_0} \tag{55}$$

with neglect of the A^2 term would lead to correct results. However, we have seen that this assumption is not the case. It is essential to include the $2\mathbf{A}_E\cdot\mathbf{A}_B$ cross term to arrive at the correct expression for the Faraday rotation for both the quantum calculation and a classical calculation based on Hamilton's equations. In contrast, if we start from the Hamiltonian

$$H_2 = H_0 + e\mathbf{r}\cdot\mathbf{E}(Z,t) + \frac{eL_zB_0}{2m_0}, \tag{56}$$

the correct Faraday rotation is obtained without any problem. This result is another example of where it is advantageous to use the $e\mathbf{r}\cdot\mathbf{E}$ rather than the $e\mathbf{p}\cdot\mathbf{A}/m_0$ form of the interaction potential for interactions of atoms or oscillators with optical fields in the dipole approximation.[11]

The Hamiltonians (22) and (56) must lead to the same expectation values for any physical observables because they are related by a unitary transformation.[11,12] However, the dynamics of the state evolution for each Hamiltonian can be quite different. For example, it follows from Eq. (34) that for the Hamiltonian (26b), the magnetic field shifts the excited state sublevels, the optical field couples the ground and excited states, and the combined action of the optical and magnetic fields couples the ground and excited states. In contrast, for the Hamiltonian (56), the magnetic field shifts the excited state sublevels and the optical field couples the ground and excited states, but there is no cross term involving the combined action of the optical and magnetic fields. As a consequence, it is easier to solve for the dynamics and to interpret the roles of the fields if we use the $e\mathbf{r}\cdot\mathbf{E}$ Hamiltonian in Eq. (56).

The calculation for the oscillators would be identical to that for atoms with ground state angular momentum $L=0$ and excited state angular momentum $L=1$ as long as the fields are weak. In contrast to the oscillator problem, however, there are nonlinear effects for atoms that enter as the optical field strength is increased.[6] Moreover, for ground states in which there is magnetic state degeneracy, fine or

hyperfine structure optical pumping can modify the Faraday rotation.[7] For atoms, it is not possible to obtain a closed from differential equation for the Heisenberg operator $\mathbf{r}(t)$ as it was for the oscillator. Instead, using either a Heisenberg or Schrödinger approach, we must calculate the expectation value of the dipole moment operator of the atoms.

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[1] See, for example, A. D. Buckingham and P. J. Stephens, "Magnetic optical activity," Annu. Rev. Phys. Chem. **17**, 399–432 (1966), and references therein.

[2] A. Sommerfeld, *Optics* (Academic, New York, 1964), pp. 101–106. Sommerfeld used n_- for left circularly polarized radiation and n_+ for right circularly polarized radiation. I have interchanged his notation because $n_\pm$ corresponds to $\sigma_\pm$ radiation for a field propagating in the $\hat{\boldsymbol{z}}$ direction, where $\sigma_\pm$ radiation induced $\Delta m = \pm 1$ transitions on absorption (m is the magnetic quantum number).

[3] F. J. Loeffler, "A Faraday rotation experiment for the undergraduate physics laboratory," Am. J. Phys. **51**, 661–663 (1983); F. L. Pedrotti and P. Bandettini, "Faraday rotation in the undergraduate advanced laboratory," *ibid.* **58**, 542–545 (1990).

[4] B. Rossi, *Optics* (Addison-Wesley, Reading, MA, 1957), pp. 427–430.

[5] See, for example, D. M. Bishop and S. M. Cybulski, "Magnetic optical rotation in H_2 and D_2," J. Chem. Phys. **93**, 590–599 (1990); M. Krykunov, A. Banerjee, T. Ziegler, and J. Autschbach, "Calculation of Verdet constants with time-dependent density functional theory: Implementation and results for small molecules," *ibid.* **112**, 074105-1–7 (2005).

[6] See, for example, F. Schuller, M. J. D. MacPherson, and D. N. Stacey, "Saturation and collisional effects on magnetic optical rotation," Physica C **147**, 321–331 (1987); D. Budker, D. J. Orlando, and V. Yashchuk, "Nonlinear laser spectroscopy and magneto-optics," Am. J. Phys. **67**, 584–592 (1999).

[7] F. Schuller, M. J. D. MacPherson, and D. N. Stacey, "Magneto-optical rotation in an atomic vapor," Opt. Commun. **71**, 61–64 (1989); F. Schuller, D. N. Stacey, R. B. Warrington, and K. P. Zetie, "Theory of Faraday rotation produced by atoms near a $J_g=\frac{1}{2}\rightarrow J_g=\frac{1}{2}$ transition," J. Phys. B **28**, 3783–3790 (1995). This paper also includes collisional and saturation effects.

[8] For successive intervals, we can redefine the direction of polarization as the x direction and then calculate the rotation in the next small interval. The treatment is strictly valid only for $eB_0\omega/2m_0 \ll (\omega_0^2-\omega^2)$.

[9] P. R. Berman, "Two-level approximation in atomic systems," Am. J. Phys. **42**, 992–997 (1974).

[10] Because the equations of motion for the displacement operator of a harmonic oscillator contain a source term that is linear in the external electric field, a perturbative calculation of the Heisenberg operator $\mathbf{r}(t)$ yields a result that is correct to all orders in the electric field amplitude.

[11] See, for example, J. R. Ackerhalt and P. W. Milonni, "Interaction Hamiltonian of quantum optics," J. Opt. Soc. Am. B **1**, 116–120 (1984).

[12] The Hamiltonians are related by the unitary transformation $H'=UHU^\dagger$ with $U=\exp[ie\mathbf{r}\cdot\mathbf{A}_E(Z,t)/\hbar]$, where H is given by Eq. (22), $\mathbf{A}_E(Z,t)$ is given by Eq. (23b), and $H'=H_2+e^2B_0^2(x^2+y^2)/8m_0$ is equal to H_2 [Eq. (56)], plus a term corresponding to the quadratic Zeeman shift.