

1 §Ω and Z

In this section, we want to answer why Z is a more efficient technology than the good 'ol Ω.

$$P(\mu_i) = e^{-\beta E_i} \left(\frac{Z_{N-1}(\beta)}{Z_N(\beta)} \right)$$

$$P(\mu_1, \mu_2, \mu_3, \dots) = \frac{e^{-\beta E_1} e^{-\beta E_2} e^{-\beta E_3} \dots}{Z(\beta)} = \frac{e^{-\beta E}}{Z(\beta)}$$

Let's zoom into the system to see what's really going in there. As usual, in thermal equilibrium, the probability of a microstate only depends on its energy.

$$P(\mu_i) = \frac{1}{\Omega_1(E_{\mu_i})}$$

$$P(\mu_1, \mu_2, \mu_3, \dots) = \frac{1}{\Omega_1(E_1)} \frac{1}{\Omega_1(E_2)} \frac{1}{\Omega_1(E_3)} \dots \quad (1)$$

$$= \frac{1}{\Omega_N(E)} \quad (2)$$

Notice that these equations already represent thermal equilibrium, since they are characterized only by energy. Thermal equilibrium is our one and only approximation¹, hence, one shouldn't expect to do more approximation in order to arrive at Z.

There are three ways to exploit the relation between (1) and (2), but first I need to introduce several notations:

Equation (1) to (2) may look like a tiny step in L^AT_EX, but it is actually a giant step. Rewrite Ω₁ as e^{S₁}, hence the relation becomes

$$\sum_i S_1(E_i) = S_N(E) = S_N\left(\sum_i E_i\right) \quad (3)$$

1. principle of least assumption² For (3) to be true, S_N(E) must not depend on the spread of the E_i distribution, where mathematically you can freely rearrange your energy distribution as you wish

$$S_1(E_1) + S_1(E_2) = S_1(E_1 + E_2) + S_1(0)$$

From here we know that S₁ must be a linear function

$$S_1(x) = S_1(0) + \beta x$$

where β = ∂S₁ = constant.

Interpretations:

¹this includes the assumption that β is constant in the range E ± ΔE

²a more mediocre and tedious derivation is given in the appendix

- Hey, the entropy is automatically maximized, too (which means, it is redundant to further assume that entropy is maximized).

- We can now write the probability as

$$P(\mu_i) = e^{-S_1(0) - \beta E_i}$$

- any visual reasons on why the entropy increases linearly?

2. vary N Back to one particle, we are now authorized to use this relation

$$\begin{aligned} p(\mu_i) &= \frac{\Omega_{N-1}(E - E_{\mu_i})}{\Omega_N(E)} \\ &= \exp(\log \Omega_{N-1}(E - E_{\mu_i}) - \log \Omega_N(E)) \\ &= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp\left(-\frac{\partial}{\partial E} \log \Omega_{N-1} E_i - \frac{1}{2} \partial^2 \log \Omega_{N-1} E_i^2 - \dots\right) \\ &= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp\left(-\beta E_i - \frac{1}{2} \frac{\partial \beta}{\partial E} E_i^2 - \dots\right) \\ &= \frac{\exp(-\beta E_i)}{Z_1(\beta, \mathcal{O}(E_i^2))} \end{aligned}$$

There are two completely different arguments to arrive at Boltzmann distribution:

- the multiplicity is maximized
- the state only depends on energy

Another way to look at thermal equilibrium: we randomly sample a small element of our system, and yet it will still look exactly the same as its parent system (redundancy of information, rescaling, whatever).

Possible mathematical relations:

$$Z_{1R}(\beta, E_i) = \frac{\Omega_N(E)}{\Omega_{N-1}(E)} \frac{1}{\exp(+\frac{\beta}{2C_V T} E_i^2 - \dots)}$$

Now we begin to see why we need log. All of our arguments

$$\log Z_1 = \frac{\partial}{\partial N} \log \Omega_N(E)$$

The difference between the two description is that the former gives us the real-life particle description, why the latter only tells us about the smoothed out representation of the system.

In the Z description, we are instead viewing the system as a field ("N is not fixed", but we don't care).

2 APPENDIX

2.1 1. principle of least assumption: not even close to a harmonic oscillator

there are two styles to do the taylor expansion:

1. expand around average energy ϵ

$$\sum_i S_1(\epsilon) + \hat{\partial} S_1 \sum_i (E_i - \epsilon) + \hat{\partial}^2 S_1 \sum_i (E_i - \epsilon)^2 + \dots = S_N(E)$$

$$NS_1(\epsilon) + \hat{\partial} S_1(\epsilon) \cdot 0 + \hat{\partial}^2 S_1(\epsilon) \sigma_\epsilon + \dots = S_N(E)$$

$$NS_1(\epsilon) + \beta \cdot 0 + \frac{1}{2} \frac{\partial \beta}{\partial \epsilon} \sigma_\epsilon + \dots = S_N(E)$$

Unfortunately it is no longer obvious whether the higher order terms will die or not. in general it could be such that, for example $\frac{\partial \beta}{\partial \epsilon} \sigma_\epsilon = f(E)$.

2. expand around $E = 0$

$$\sum_i S_1(0) + \hat{\partial} S_1(0) \sum_i E_i + \hat{\partial}^2 S_1(0) \sum_i E_i^2 + \dots = S_N(E)$$

Now, here comes the crucial step in deriving Z , our **second approximation: we truncate our series by claiming that** (this would later means that the specific heat depends only on the variance of the energy!). This shouldn't be true for derivative with respect to microscopic variable $\frac{\partial \beta}{\partial \epsilon} = 0$.