1 §Ω and Z

In this section, we want to answer why Z is a more efficient technology than the good 'ol Ω.

$$P(\mu_i) = e^{-\beta E_i} \left(\frac{Z_{N-1}(\beta)}{Z_N(\beta)} \right)$$

$$P(\mu_1, \mu_2, \mu_3, \ldots) = \frac{e^{-\beta E_1} e^{-\beta E_2} e^{-\beta E_3} \cdots}{Z(\beta)} = \frac{e^{-\beta E}}{Z(\beta)}$$

Let's zoom into the system to see what's really going in there. As usual, in thermal equilibrium, the probability of a microstate only depends on its energy.

$$P(\mu_i) = \frac{1}{\Omega_1(E_{\mu_i})}$$

$$P(\mu_1, \mu_2, \mu_3, \ldots) = \frac{1}{\Omega_1(E_1)} \frac{1}{\Omega_1(E_2)} \frac{1}{\Omega_1(E_3)} \cdots \tag{1}$$

$$= \frac{1}{\Omega_N(E)} \tag{2}$$

Notice that these equations already represent thermal equilibrium, since they are characterized only by energy. Thermal equilibrium is our one and only approximation[1], hence, one shouldn't expect to do more approximation in order to arrive at Z.

There are three ways to exploit the relation between (1) and (2), but first I need to introduce several notations:

Equation (1) to (2) may look like a tiny step in LaTeX, but it is actually a giant step. Rewrite Ω_1 as e^{S_1}, hence the relation becomes

$$\sum_i S_1(E_i) = S_N(E) = S_N(\sum_i E_i) \tag{3}$$

1. principle of least assumption [2]   For (3) to be true, $S_N(E)$ must not depend on the spread of the E_i distribution, where mathematically you can freely rearrange your energy distribution as you wish

$$S_1(E_1) + S_1(E_2) = S_1(E_1 + E_2) + S_1(0)$$

From here we know that S_1 must be a linear function

$$S_1(x) = S_1(0) + \beta x$$

where $\beta = \partial S_1 = $ constant.

Interpretations:

---

[1] this includes the assumption that β is constant in the range $E \pm \Delta E$

[2] a more mediocre and tedious derivation is given in the appendix

- Hey, the entropy is automatically maximized, too (which means, it is redundant to further assume that entropy is maximized).

- We can now write the probability as
$$P(\mu_i) = e^{-S_1(0)-\beta E_i}$$

- any visual reasons on why the entropy increases linearly?

2. vary N Back to one particle, we are now authorized to use this relation

$$
\begin{aligned}
p(\mu_i) &= \frac{\Omega_{N-1}(E - E_{\mu_i})}{\Omega_N(E)} \\
&= \exp(\log \Omega_{N-1}(E - E_{\mu_i}) - \log \Omega_N(E)) \\
&= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp(-\frac{\partial}{\partial E} \log \Omega_{N-1} E_i - \frac{1}{2}\partial^2 \log \Omega_{N-1} E_i^2 - \cdots) \\
&= \frac{\Omega_{N-1}(E)}{\Omega_N(E)} \exp(-\beta E_i - \frac{1}{2}\frac{\partial \beta}{\partial E} E_i^2 - \cdots) \\
&= \frac{\exp(-\beta E_i)}{Z_1(\beta, \mathcal{O}(E_i^2))}
\end{aligned}
$$

There are two completely different arguments to arrive at Boltzmann distribution:

- the multiplicity is maximized

- the state only depends on energy

Another way to look at thermal equilibrium: we randomly sample a small element of our system, and yet it will still look exactly the same as its parent system (redundancy of information, rescaling, whatever).

Possible mathematical relations:

$$Z_{1R}(\beta, E_i) = \frac{\Omega_N(E)}{\Omega_{N-1}(E)} \frac{1}{\exp(+\frac{\beta}{2C_v T} E_i^2 - \cdots)}$$

Now we begin to see why we need log. All of our arguments

$$\log Z_1 = \frac{\partial}{\partial N} \log \Omega_N(E)$$

The difference between the two description is that the former gives us the real-life particle description, why the latter only tells us about the smoothed out representation of the system.

In the Z description, we are instead viewing the system as a field ("N is not fixed", but we don't care).

2 APPENDIX

2.1 1. principle of least assumption: not even close to a harmonic oscillator

there are two styles to do the taylor expansion:

1. expand around average energy ϵ

$$\sum_i S_1(\epsilon) + \hat{\partial} S_1 \sum_i (E_i - \epsilon) + \hat{\partial}^2 S_1 \sum_i (E_i - \epsilon)^2 + \cdots = S_N(E)$$

$$NS_1(\epsilon) + \hat{\partial} S_1(\epsilon) \cdot 0 + \hat{\partial}^2 S_1(\epsilon)\sigma_\epsilon + \cdots = S_N(E)$$

$$NS_1(\epsilon) + \beta \cdot 0 + \frac{1}{2}\frac{\partial\beta}{\partial\epsilon}\sigma_\epsilon + \cdots = S_N(E)$$

Unfortunately it is no longer obvious whether the higher order terms will die or not. in general it could be such that, for example $\frac{\partial\beta}{\partial\epsilon}\sigma_\epsilon = f(E)$.

2. expand around $E = 0$

$$\sum_i S_1(0) + \hat{\partial} S_1(0) \sum_i E_i + \hat{\partial}^2 S_1(0) \sum_i E_i^2 + \cdots = S_N(E)$$

Now, here comes the crucial step in deriving Z, our **second approximation: we truncate our series by claiming that** (this would later means that the specific heat depends only on the variance of the energy!). This shouldn't be true for derivative with respect to microscopic variable $\frac{\partial\beta}{\partial\epsilon} = 0$.