The Experimental Basis of Electromagnetism - Part IV: Magnetic Materials

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where $\partial\Phi/\partial t$ represents the rate of change in the absence of motion, and $\mathbf{v}$ is the velocity of the element dl of the material in the circuit relative to the field $\mathbf{B}$: that is to say, the complete law must include *both* "change of linkage" and "flux-cutting". The Lorenz method for the absolute determination of the ohm constitutes an accurate verification of (5.2), which, therefore, we also regard as an established experimental law. In (5.2) the proportionality factor k of (5.1) has been made -1, which is of course the usual choice, and serves the very important practical purpose of defining the unit of E by reference to the units of Φ and $\mathbf{B}$. It would be tedious to trace in detail the reason for the negative sign, but it may be worth noting that it is determined solely by conventions laid down in connection with other laws, e.g. that fixing the sense in which circulations are linked positively, and that prescribing that in any circuit an E.M.F. and the current it drives shall have the same sign.

The next problem is to decide by what the field of $\mathbf{B}$ is determined. Here we meet the conception of "magnetic materials", and some difficulties rather different in kind from those we have discussed so far. These are dealt with in the following paper.

The Experimental Basis of Electromagnetism— Part IV: Magnetic Materials

By L. HARTSHORN
National Physical Laboratory, Teddington

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ABSTRACT. This paper concludes the inquiry which was undertaken by N. R. Campbell and the author with the object of elucidating the connection between the concepts and principles of electromagnetism and the experimental operations actually performed in the laboratory that constitute their factual basis. In Part III the vector $\mathbf{B}$ was established as measurable everywhere, even within solid bodies. The vector $\mathbf{H}$ and the scalar $\mu = \mathbf{B}/\mathbf{H}$ are now shown to be measurable in special circumstances by means of the magnetometer and permeameter, but in general their values depend on a hypothesis, which is stated. The significant facts concerning the magnetic properties of real materials are briefly reviewed, and the two-fold aspect of the science is emphasized: the self-consistent mathematical theories based on postulates, on the one hand, and the complex of experimental laws on the other. It is important that the relation between the two shall be clearly established, and not merely assumed as self-evident.

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§ 6. NON-MAGNETIC MEDIA*

In Part III of this inquiry we reached the stage at which $\mathbf{B}$ was established as a vector magnitude measurable independently of any other magnitude and characteristic of the states of certain systems, which we called magnets, and of position in and around them. We now proceed to consider what determines the distribution of $\mathbf{B}$, and for simplicity we shall restrict ourselves in the first instance to systems in which all the materials are non-magnetic. Since $\mathbf{B}$ is everywhere solenoidal, it can be determined experimentally at every point of the system. We can also readily show that $\mathbf{B}_0$ at any point (we shall use the suffix 0

* Section headings continue in sequence from Part III.

to indicate the limitation to non-magnetic media) is a linear function of the currents in the circuits of the system, and falls to zero when they all become zero. More generally, we observe that there is a very close connection between $\mathbf{B}_0$ and the magnitude $\mathbf{H}_0$, which was defined in Part I by the equation

$$\mathbf{H}_0 = S \, . \, I \, . \oint \frac{\mathbf{dl} \times \mathbf{r}_1}{r^2} = S \, . \, \mathbf{G} \, . \, I, \qquad \ldots\ldots(6.1)$$

and which therefore satisfies the equation

$$\oint \mathbf{H}_0 \, dl = 4\pi S n I. \qquad \ldots\ldots(6.2)$$

Separate measurements of $\mathbf{B}_0$ and $\mathbf{H}_0$ enable us to establish with high accuracy the experimental law

$$\mathbf{B}_0 = \mu_0 \mathbf{H}_0. \qquad \ldots\ldots(6.3)$$

In (6.3) μ_0 is a scale-factor without associated derived magnitude, since it has the same value for all non-magnetic materials. It plays the same part in (6.3) as does S in (6.1) and (6.2) and k in (5.1). By assigning convenient arbitrary values to each of these scale-factors, three units are defined in terms of other units. For example, in the M.K.S. system we assign to S the value $1/4\pi$, thereby defining the "ampere-turn per metre" as the unit of $\mathbf{H}_0$; we then assign to μ_0 the value $4\pi \times 10^{-7}$, thus defining the unit of B by reference to the ampere-turn and metre, and then give k the value -1, thereby defining the volt by reference to the ampere-turn, metre, and second. The ohm is of course defined by reference to the volt and ampere by making the scale-factor in Ohm's law unity. A different choice of scale factors gives a different system of units, but all are on the same footing.

This use of the experimental law (6.3) to obtain a unit of $\mathbf{B}$ is clearly of the greatest practical importance, for apart from some such law, the independent method of measuring $\mathbf{B}$ would demand frequent reference to one or more fixed positions near some standard magnet in order to re-calibrate fluxmeters every time a change of search coil became necessary. In virtue of this law, every "air-cored" circuit that is accurately measurable in its linear dimensions and current becomes a standard magnet, i.e. a standard field of $\mathbf{B}$.

There is some evidence that a law of the form (6.3) remains true, but with some other constant replacing μ_0, if the medium surrounding the coils and filling the whole space in which $\mathbf{B}$ is distributed is uniform but not air-equivalent. Experiments of this kind are, however, only practicable in fluids at ordinary temperatures, and these are so similar magnetically to air that the differences in the measured quantities for different media are not usually much greater than the experimental error. The question of the existence of a derived magnitude characteristic of feebly magnetic materials is therefore best considered in relation to experiments of a different kind, which will be discussed later.

§7. MAGNETIC MATERIALS: PERMEABILITY

We pass now to a consideration of the systems for which (6.3) is not even approximately true, i.e. systems which include materials that are markedly not equivalent to air in their effect on fluxmeters and compass needles, the strongly magnetic solids. The evidence which showed $\mathbf{H}_0$ as defined by (6.1) to be a significant property of non-magnetic systems now no longer applies, but it is convenient to denote by $\mathbf{H}_I$ the value of $S.\mathbf{G}.I$ of (6.1) for such a system. Very simple experiments with a fluxmeter show that $\mathbf{B}$, far from being completely determined by $\mathbf{H}_I$ in accordance with (6.3), is frequently large when $\mathbf{H}_I$ is zero. For a given system it is, however, associated with $\mathbf{H}_I$ in the sense that they invariably change together, though $\Delta B/\Delta H_I$ is usually much greater than μ_0, and not obviously characteristic of any part of the system. Useful results have however been obtained by adopting the hypothesis that the vector $\mathbf{B}$ is always associated with a vector $\mathbf{H}$, which in the special case of a non-magnetic medium reduces to $\mathbf{H}_0$, and which is such that the ratio $\mathbf{B}/\mathbf{H}=\mu$ is a scalar magnitude characteristic of the material at the point to which $\mathbf{B}$ and $\mathbf{H}$ refer. The fact that values of μ characteristic of various materials have been obtained provides some justification for the hypothesis. We must now examine the operation by which these values of μ and $\mathbf{H}$ are obtained, for according to our principles it is to these experiments that we must look for the meaning of these quantities.

Examination of the methods actually used for measuring μ shows that the one to which all others are referred consists in the use of the familiar uniformly wound long solenoid (or an equivalent toroid) and fluxmeter—the permeameter, for short. For simplicity we shall consider only the ideal form, a circuit in the form of an infinitely long solenoid, uniformly wound with N turns per unit length, and enclosing a search coil of effective area A_s, which forms part of a fluxmeter. The material to be measured takes the form of an infinitely long cylindrical core of cross-sectional area A_m; it is placed in the solenoid so that it is embraced by the search coil and so that its axis is parallel to that of the solenoid. Measurements are made by changing the steady current through the solenoid from a value I to a value $I+\Delta I$ and measuring the corresponding change of flux linkage of the search coil $\Delta\phi$ by means of the fluxmeter. The ideal is of course never completely attained, but since permeameters are only recognized as satisfactory when the inevitable departure from the ideal is concealed by the experimental error, it is unnecessary to consider such departures here.

It follows from (6.1) that H_I is zero outside the solenoid for all values of I, and inside it is everywhere $4\pi SNI$ and parallel to the axis; the solenoid is essentially a device for producing a uniform field of H_I. For simplicity we will adopt M.K.S. units so that this field is given by NI. Experiment shows that so long as the whole coil system is free from magnetic material, and the core is absent,

$$\Delta\phi=\mu_0 A_s N\Delta I. \qquad \ldots\ldots(7.1)$$

Indeed, it was mainly on evidence of this kind that (6.3) was based. When the magnetic core is introduced and the measurement repeated, $\Delta\phi$ is often greatly increased. Observations usually suggest a law of the form

$$\Delta\phi=\{\mu A_m+\mu_0(A_s-A_m)\}N\Delta I, \qquad \ldots\ldots(7.2)$$

where μ is much greater than μ_0, and depends on the nature of the core. For a class of material which is severely limited, but which includes some of technical importance ("constant-permeability alloys"), μ is found to be characteristic of the material and independent of A_m, NI and ΔI over a useful range of values. The law (7.2) shows that in a permeameter such materials have an effect which is exactly equivalent to an increase in $\mathbf{B}$, throughout the space which they occupy, from the original value $\mu_0 H_I$ to μH_I. This μ is the magnitude which is called the permeability of the material, and by analogy μ_0 is often called the permeability of free space. (It is convenient in practice to record the relative permeability $\mu/\mu_0 = \mu_r$ since this ratio has the same value in all the ordinary systems of units.) We conclude that the general vector **H**, whatever else it may be, is identical with H_I at all points in an ideal permeameter.

§ 8. THE MAGNETIC CIRCUIT

We must now consider how H is affected when there is a departure from the conditions of the permeameter, and for simplicity we shall, in the first instance, restrict ourselves to materials of constant μ. Let the material take the form of a solid ring which completely fills both the uniform toroidal winding and search coil: μ can be measured as B/H_I. Let now a narrow gap be cut in the material. Its effect is well known: the ratio $\Delta\phi/\Delta H_I$ is greatly reduced, but so long as the length of the gap l_0 is very small compared with the total length of the toroid l (both measured circumferentially) it remains independent of the position of the search coil, and we therefore conclude that **B** has the same value in the gap where the permeability is μ_0 and in the material where it is μ. If this value is B_2, then **H** must have the value B_2/μ in the material and B_2/μ_0 in the gap, and neither of these values is H_I. Thus **H** no longer satisfies (6.1): it is natural to see if it still satisfies (6.2). For this simple system (6.2) becomes

$$\frac{B_2}{\mu}(l-l_0)+\frac{B_2}{\mu_0}l_0=lNI. \qquad \ldots\ldots(8.1)$$

This law for the toroid with a small gap has been established experimentally, but it must be admitted that the experimental error is usually rather large.

Slightly more general formulae of this kind find useful application in electrical practice, and may therefore be considered as experimentally established. They are, however, limited to systems that can be described as magnetic circuits, i.e. systems in which the field of B is confined to a closed path of measurable length and cross section, the flux-linkage being constant for all cross sections, but **B** varying with change of cross section along the path. The circuit may be built up of various materials so that μ may change discontinuously from point to point along it, but μ must be uniform over any section that is everywhere perpendicular to **B**. In the simplest case, the cross section of the circuit is everywhere small enough to justify the neglect of the variation of **B** over it; then at any point where the cross-sectional area is A, B is given by ϕ/A and H by $\phi/\mu A$, and both are directed along the circuit. If the circuit can be divided into portions of length l_1, l_2, $l_3 \ldots l_p$, each of uniform permeability throughout, the values being $\mu_1, \mu_2 \ldots \mu_p$, and the cross sections of the several portions $A_1, A_2 \ldots . A_p$, then application of (6.2) yields the formula

$$\phi\{\Sigma_p l_p/(A_p\mu_p)\}=\Sigma_r n_r I_r, \qquad \ldots\ldots(8.2)$$

where I_r denotes the current in the rth current circuit linked with the magnetic circuit and n_r is the "number of turns" in this linkage. By analogy with Kirchhoff's second law, $\Sigma_r n_r I_r$ is called the magnetomotive force of the system, and the quantity in brackets the reluctance of the magnetic circuit, which is given as the sum of the reluctances of the several portions which are in series-connection in the circuit. Systems of large cross section can often be dealt with by applying (8.2) to elementary tubes of flux of small cross section into which the circuit can be divided. The details need not concern us; **H** is seen as a useful conception in making calculations of this kind, and they are certainly important in practice; but it will be noted that **H** is only used as an auxiliary variable which does not appear in the result: we must therefore look further for evidence of its significance in terms of experimental operations.

§ 9. THE *B-H* HYPOTHESIS

The existence of two vectors **B** and **H**, each having a determinate value for points inside solid bodies, cannot be considered as completely established by experiment. Examination of the methods usually employed for determining **B** and **H** suggest that they depend on a hypothesis that we shall call the *B–H* hypothesis, and shall state in a general form as follows:

Any point in a system of current circuits and material bodies is characterized by the following magnitudes:

(*a*) The vector **B** determined by measurements with a fluxmeter and the equation

$$\operatorname{div} \mathbf{B} = 0. \qquad \ldots\ldots(9.1)$$

(*b*) A scalar μ and a vector **H** which are uniquely determined by the equations

$$\mathbf{B} = \mu \mathbf{H} \qquad \ldots\ldots(9.2)$$

and

$$\oint \mathbf{H}\, dl = \Sigma_r n_r I_r, \qquad \ldots\ldots(9.3)$$

where I_r is the steady current circulating in the rth circuit, and n_r is the number of times which the path around which the integral is taken is linked in the positive sense with the rth circuit.

It is to be noted that the experimental laws that are well established, viz. those for non-magnetic media, for the permeameter and for the toroid with a narrow gap are merely the results obtained when this hypothesis is applied to specially simple cases. This hypothesis therefore constitutes the definition of the general vector **H**.

The question what limitations are necessary and sufficient to ensure that the above three equations determine a unique distribution of the three magnitudes **B**, μ, **H** is of purely mathematical interest and therefore outside the scope of this paper.

§ 10. THE MEASUREMENT OF H

We have seen that **B** is (in principle) measurable everywhere. Can the same be said of **H**? In the special cases of (*a*) the system that contains no magnetic material and (*b*) the permeameter, **H** becomes equal to $\mathbf{H}_I$ and is therefore measurable in terms of current. In the more general case **H**, unlike $\mathbf{H}_I$, is not solenoidal and therefore cannot be obtained from the values of current. If the permeability is known **H** can of course be calculated as $\mathbf{B}/\mu$, but unless this ratio has some other significance nothing is gained by giving it a special name: indeed, **H** becomes no more than an auxiliary mathematical variable.

A possible method of measuring **H** directly is suggested by the experiments which originally led to the definition of $\mathbf{H}_0$ by (6.1), viz. experiments concerning the torque T on a small compass needle in air. Within the experimental error it is given by

$$T = SI\mathbf{G} \times m = \mathbf{H}_0 \times m, \qquad \ldots\ldots(10.1)$$

where m (the moment) is a derived magnitude characteristic of the needle. The suggestion is that we can measure **H** at any point by observing the torque on a compass needle of known moment at that point, in virtue of a law

$$T = \mathbf{m} \times \mathbf{H}. \qquad \ldots\ldots(10.2)$$

Measurements of this kind are certainly made in practice, but they are necessarily severely limited in scope, for compass needles cannot be inserted into solid bodies. In actual practice **H** can only be measured by this method at points in air, and for most magnets the range of **H** over which m in (10.2) proves to be a constant for the particular magnet is very small. Nevertheless the method can be used for the measurement of **H** over a very wide range if m in (10.2) is treated, not as a constant, but as a function of **H** characteristic of the particular magnet. Special magnets have been made for which this function is single-valued over a very great range of **H**, so that the value of m in (10.2) corresponding to any observed value of T can be determined by an auxiliary measurement in a standard solenoid. In practice the instrument is calibrated in this way to indicate **H** directly.

Thus **H** can be determined by such operations at points in an air space, even one included in a system of strongly magnetic bodies, and provided the small suspended magnet is not allowed to approach too closely to the surface of a strongly magnetic body the values are always found to be equal to B/μ_0, and to be consistent with the B–H hypothesis.

It is interesting to note that these instruments for measuring **H** are usually called fluxmeters, a term normally associated with **B** rather than **H**. This has arisen because they are always used in air-spaces where $B = \mu_0 H$, and are generally calibrated in the system of units for which $\mu_0 = 1$. The use of this system often leads to the notion that in any air-space B and H are identical. The facts are, that they are different in the sense that the operations by which they are measured are different, but they are "the same magnitude" in the sense in which this term was used in Part I.

The well-known methods of measuring **H** at points near the surface of solid bodies by means of very narrow search coils closely fitting the surface, or by means of the magnetic potentiometer, are equivalent to measurements of B at points in air near the surface by means of a fluxmeter, and therefore call for no special comment.

Our general conclusion is that **H** is directly measurable in air-spaces, but since it is not in general solenoidal such measurements alone are not sufficient to determine values for points inside magnetic materials. Such values depend on the B–H hypothesis.

§ 11. MAGNETIZATION CURVES: HYSTERESIS

We have so far assumed that the materials under discussion all have a definite permeability μ, which is independent of **H** or **B**. It is, however, well known that most strongly magnetic materials are so complicated in their behaviour that no single value of μ will give any adequate account of it. The permeameter is still the most generally satisfactory instrument for dealing with such materials, the law (7.2) being applied in the form

$$\Delta\phi = A_{\mathrm{m}}\Delta B + \mu_0(A_{\mathrm{s}} - A_{\mathrm{m}})\Delta H, \qquad \ldots\ldots(11.1)$$

and values of ΔB corresponding to the values ΔH being deduced from the observations of A, ϕ and ΔI. Summations of the values of these increments (i.e. $\Sigma\Delta B$ and $\Sigma\Delta H$) in a manner too familiar to be detailed here enable us to assign a value of **B** to each value of **H**, provided one such pair of values is known. A conventional procedure is usually followed in deriving these values, e.g. there is a preliminary "demagnetizing" treatment which general experience suggests will justify the assumption $\mathbf{B}=0$ when $\mathbf{H}=0$, and then the values of **H** follow a regular sequence between certain limits. It is an experimental fact that when such a procedure is followed, the relation between B and H can often be represented by a definite "magnetization curve" or a "hysteresis loop" that is characteristic of the material in the permeameter, that is to say, independent of the size and shape of the sample within the limits imposed by the permeameter, and dependent only on the composition and physical condition of the material. Thus the properties of strongly magnetic materials are represented, not by a constant μ, but by a characteristic function.

The details of the procedure need not concern us, but we must note that no law that purports to represent the magnetic behaviour of a material by a definite permeability μ is generally applicable to real magnetic materials. Such simple laws are applicable only to the relatively few "constant-permeability" materials. Nevertheless engineers often find it convenient to apply the simple formulae to their materials when they are used within a narrow range of magnetic conditions, and in such cases they adopt the simple expedient of interpreting μ as $\Delta B/\Delta H$ for changes of B and H which have been found empirically to represent the working conditions. Thus we meet the terms incremental permeability, initial permeability, reversible permeability, differential permeability, representing $\Delta B/\Delta H$ in accordance with various conventions. Such quantities cannot however be expected to have any wider significance.

It now becomes obvious that the determination of the distribution of **B**, **H** and μ by the application of the B–H hypothesis to a real system including strongly magnetic materials, is necessarily a very complicated matter. Instead of the constant μ first considered, we have now a function represented by a curve or a table of values, and not even a single-valued function of H in most cases. In such cases the application must be made by numerical computation, and μ must be interpreted as the value of the permeability ($=B/H$) given by the characteristic curve for the material occupying the point in question, for the particular value of **B** which satisfies equations (9.1), (9.2), (9.3). When the characteristic curve includes an appreciable hysteresis loop it gives more than one value of μ for each value of B, except at saturation, and computation is then only possible if the previous magnetic history of the system is known sufficiently well to indicate clearly which of the values is appropriate. A preliminary demagnetization (or magnetization to saturation) will often provide the necessary information, but the general problem is one of great complexity and a detailed solution for many actual systems is impracticable, although relaxation methods (Southwell 1946) would probably enable a wider range of conditions to be investigated than has so far been done. The direct evidence for the B–H hypothesis is obviously meagre, being limited to the simple cases already mentioned. It does however provide satisfactory coordination for a great amount of experimental work, and this constitutes its chief justification.

§ 12. SATURATION: INTENSITY OF MAGNETIZATION

There remains however an important experimental law that gives **H** a significance that is more than mathematical; it is the one suggested by the term saturation, mentioned in the last section. If in the permeameter the current is steadily increased, then for the large class of strongly magnetic materials of §11, provided the value of $\mathbf{H}$ exceeds some particular value which is different for different materials and can only be found by experiment, the relation between ΔB and ΔH_I satisfies the law (7.1), which can be written $\Delta B = \mu_0 \Delta H$; in other words, in this region the increments of ΔB and ΔH obey the same law as if the material were absent. Thus, beyond the particular value of $\mathbf{H}$, the value of $B - \mu_0 H$ remains constant however much $\mathbf{H}$ is increased, and within this range of values $\mathbf{H}$ has the same significance within the magnetic material as it does within a non-magnetic material or air. Moreover, for many materials the stationary value of $B - \mu_0 H$ is found to be more accurately characteristic of the material than μ.

The most obvious interpretation of this law is that the material is an open structure of some kind in space, and that the total flux-density B when the material is present always includes a component $\mu_0 H$, which is the flux-density in that portion of space whether the material is present or not, so that $\mathbf{B} - \mu_0 \mathbf{H} = \mathbf{J}$, say, represents the contribution of the structure itself, and this contribution cannot exceed some limiting value J_s which is characteristic of the structure. The whole idea necessarily implies that B, H, μ and J only represent properties of the system on a scale that is macroscopic relative to the microscopic scale of the structure itself, but we have no reason for not accepting such a limitation to the significance of our measurements. J is of course the intensity of magnetization, and J_s the saturation magnetization. The general definition of J is the equation

$$\mathbf{B} - \mu_0 \mathbf{H} = S_J \mathbf{J}. \qquad \ldots\ldots(12.1)$$

The scale factor, S_J, is frequently chosen to be 4π instead of unity, but this is purely a matter of convention. The ratio $J/\mu_0 H$ is defined as the susceptibility, κ, of the material, and we have therefore

$$\mu_r - 1 = S_J \kappa. \qquad \ldots\ldots(12.2)$$

The susceptibility κ is therefore characteristic of the material to the same extent as μ_r, but J_s is often more accurately representative of the material than either. The fact that J has this significance in terms of experiment implies that the conception of $\mathbf{H}$ within a magnetic solid, on which it partly depends, is also not without physical significance.

§13. THE MEASUREMENT OF SMALL SUSCEPTIBILITY

The susceptibility of a non-magnetic material is zero by definition. It is well known that there is a very large class of materials that are nearly but not quite non-magnetic, and susceptibility provides a very convenient measure of the magnetic behaviour of these materials. In accordance with our principles we must examine the methods by which κ is determined for such materials in actual practice. The Gouy apparatus may be taken as typical. The material to be measured takes the form of a long straight rod of cross section A, permeability μ_r and susceptibility κ, suspended vertically from the beam of a balance in a medium of permeability μ_0. Its lower end lies in a strong uniform magnetic field H_l (generally, but not necessarily horizontal); its upper end in a much weaker field H_u, which, if appreciable, must also be uniform. The field H_l can be removed and established at will, and the force F on the specimen associated with the field H_l is determined by observation of the increase of weight indicated by the balance when the field is established. The law established for the method may be written,

$$F = A(\mu - \mu_0)(H_1^2 - H_u^2) \qquad \ldots\ldots(13.1)$$

or

$$F = S_J \kappa \mu_0 A(H_1^2 - H_u^2). \qquad \ldots\ldots(13.2)$$

Thus the measured force is proportional to κ, other things being equal, which is no doubt why the method is usually regarded as one for determining susceptibility rather than permeability. The materials for which the law has been accurately established fall into two classes: the paramagnetic materials for which κ is positive and the diamagnetic materials for which it is negative. For both classes it is usually much smaller than unity, and it is for this reason that μ_r $(=1+4\pi\kappa)$ is an inconvenient magnitude for discriminating between materials in these classes.

In writing (13.1) and (13.2) we have tacitly assumed that μ and H are the magnitudes that we have already defined in terms of the permeameter and the B–H hypothesis and that the scale factor in (13.1) is unity for the units already chosen. All the available evidence is consistent with this law, and it is fairly convincing, because a slight modification of the permeameter technique, described previously, enables us to measure $(\mu - \mu_0)$ with accuracy for specimens of the same form as are required for the Gouy method. A long solenoid and a search coil of many turns are employed; a large current giving an intense field in the solenoid is established; the fluxmeter circuit, previously open, is then closed, and the specimen suddenly withdrawn from the search coil. In these circumstances the change of flux-linkage ϕ measured is given by

$$\Delta\phi = A_s(\mu - \mu_0)H_I, \qquad \ldots\ldots(13.3)$$

and since both A_s and H_I are very large $(\mu - \mu_0)$ can be accurately determined although it is very small. This is indeed another standard method for the measurement of κ, and one that is of great importance in enabling us to establish that κ, as measured by the Gouy and Faraday methods and their variants, is "the same magnitude", in the sense in which we employ that term, as $\mu_r - 1$ measured by the permeameter or determined by the B–H hypothesis.

§14. ENERGY OF THE MAGNETIC FIELD

The law of the Gouy method calls for some consideration of the conception of the energy associated with the magnetic field. When the field is wholly due to currents in a non-magnetic medium, it follows from (4.1), (6.1) and (6.3) that the magnetic flux through the rth circuit due to the currents in the others is given by

$$\phi_r = \Sigma_s M_{rs} I_s, \qquad \ldots\ldots(14.1)$$

where

$$M_{rs} = (\mu_0/4\pi)N \qquad \ldots\ldots(14.2)$$

and N is Neumann's integral for the rth and sth circuits. Here and in what follows we insert the scale factor appropriate to M.K.S. units unless the contrary is stated.

It now follows from (5.1) that if e_r is the instantaneous E.M.F. induced in the rth circuit

$$e_r = \Sigma_s M_{rs}\, dI_s/dt \qquad \ldots\ldots(14.3)$$

and

$$\int -(\Sigma_r e_r I_r)\, dt = \tfrac{1}{2}\Sigma_r \Sigma_s M_{rs} I_r I_s = W, \qquad \ldots\ldots(14.4)$$

and if this E.M.F. has all the properties of direct E.M.F. the quantity W will be the energy that must be expended in order to start the currents, apart from that required to maintain them, and also the energy that will appear as heat or in some other form when the currents are stopped. There is strong experimental evidence for these laws taken together; M_{rs} proves to be accurately measurable in virtue of (14.3) and the values are consistent with (14.2) to a very high order of accuracy; also Ampère's law is obtained by taking the appropriate derivative of W, which may therefore be accepted as a "force function" for such systems.

If now the system includes magnetic material of constant permeability, and if M_{rs} now means the quantity defined by (14.3)—the mutual inductance, which is accurately measurable by, say, A.C. technique—then we find that (14.2) is no longer true. The experimental evidence suggests that for uniform media of constant permeability μ the law becomes

$$M_{rs} = (\mu/4\pi)N. \qquad \ldots\ldots(14.5)$$

It follows from the B–H hypothesis that for systems which include such materials we have

$$W = \tfrac{1}{2}\Sigma_r \Sigma_s M_{rs} I_r I_s = \tfrac{1}{2}\int \mu H^2\, d\tau, \qquad \ldots\ldots(14.6)$$

where $d\tau$ is an element of volume and the integration is taken throughout the whole region where the integrand is non-zero. All the experimental evidence supports the conclusion that W is a force function and that $F_q = \partial W_F/\partial q$ gives not only the forces tending to move and distort circuits, but also those tending to move magnetic material in their field. The Gouy method is one example. When we turn to the more complicated magnetic materials, we find hysteresis effects; the systems can no longer be considered as conservative and no force function can be regarded as established. A familiar argument shows that the energy that must be expended in order to change the magnetic induction from B_0 to B is given by

$$W = \int d\tau \int_{B_0}^{B} H\,dB, \qquad \ldots\ldots(14.7)$$

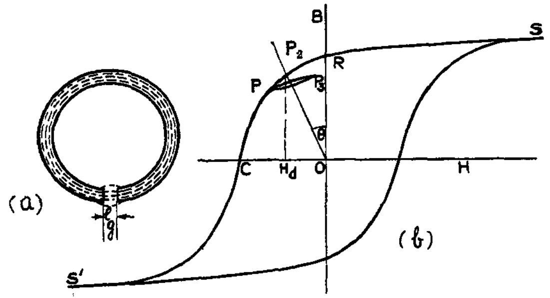
and this leads to the conclusion that during cyclic magnetization the energy lost per cycle per unit volume of material is determined by the area of the hysteresis loop. This conclusion is supported by the experimental evidence obtained by measuring the power loss by calorimeter and other methods, but the accuracy is not very high since power arising from other causes is nearly always a disturbing factor. The position is that the concept of a distribution of energy throughout materials finds little application in experiments on real magnetic material magnetic apart from those that have a constant permeability.

In order to complete our survey of the methods in general use for the measurement of magnetic properties, it will be necessary to consider the use of the magnetometer for this purpose. The term magnetometer is commonly applied to any instrument the use of which involves a measurement of the torque T exerted on a suitably suspended permanent magnet. We have already considered the use of such instruments for the measurement of H in air: the Schuster magnetometer was mentioned as an important instrument of high precision in Part I, and the "fluxmeters" mentioned in § 10 above can also be regarded as magnetometers. The historic Kew magnetometer and the magnetometers employed for measurements of μ or κ involve an additional law, that concerning the torque exerted by one magnet on another, and it is this law that we must consider. But before doing so it will be well to consider the properties of "permanent magnets", using this term in the sense in which it is used in magnetometry.

§ 15. PERMANENT MAGNETS

The historical development of magnetism was based on the conception of the permanent magnet characterized by a definite axis and constant moment. We have employed the conception so far as to refer to compass needles, for which a derived vector magnitude **m** can be established in restricted conditions; indeed, "magnetometer needles" having this property are familiar enough to all physicists, but as we have already mentioned, the experimental evidence contradicts any idea that **m** can be regarded as constant for any given magnet; in particular, it must be expected to vary with **H** and μ in its immediate vicinity. It was no doubt for this reason that the magnetometer gave place to the fluxmeter as the standard instrument in this field of work. (It is interesting to note that even as late as 1902 the fluxmeter, although its principle had long been known, was treated in the *Handbuch der Physik* as entirely subsidiary to the magnetometer.) Instruments of the "moving magnet" type are however still of practical, as well as historical, importance, and the properties of the magnets actually employed must therefore be considered.

The most detailed information is that provided by the technicians who design and make such magnets, and we find that the results of their work, which naturally proceeds on empirical lines, can be summarized as follows. Permanent magnets of all forms, including the needle, are most conveniently regarded as magnetic circuits composed of two elements in series, one of a material specially chosen because its characteristic hysteresis loop is of large area and suitable shape, and the other an air path which may be of any form, from a narrow gap in a circular magnet to the air return path from one end of a needle to the other. The term magnet is commonly applied to the circuit-element of strongly marked hysteresis, but the air path is an essential feature. The properties of the magnet can best be understood by considering the procedure followed in making it, and for simplicity we will consider the case of a circular magnet of small and uniform cross section, with a narrow air-gap (see Figure). The gap is first completely filled



with material of very high permeability (soft iron), and the ring, having been provided with a suitable winding, is magnetized to saturation by passing a heavy current through the winding. If now the current is varied cyclically between this saturation value and an equal but oppositely directed one, the behaviour of the material can be represented by the hysteresis loop shown in the Figure. *B* and

H will not be quite uniform over any cross section of the magnet, but we may suppose them to be the values for the material within a central tube of flux of small cross section. The curve can be determined by the methods previously discussed, and we find that by first switching on the current corresponding to S and then reducing it to zero (switching off) we bring the material into the condition represented by R. If now we remove the soft iron from the gap we find that there is a reduction of flux corresponding to the attainment of a condition represented by P. The magnet is now complete: the winding can be removed, and experiment shows that the flux $\phi = BA$ of the magnet remains constant so long as the magnet is not disturbed. The value is, however, liable to change with change of temperature, mechanical shock, and especially with the application of any field of H or any change in the reluctance of the air gap of the magnet, e.g. the introduction of any material not air-equivalent. An applied negative H brings the material down the loop to P_2; a positive H takes it along a branch curve to P_3, while cyclic variations of H take it round a succession of subsidiary loops, each successive member being slightly lower than its predecessor, until a stable condition is reached. The final state is always represented by some point on or within such a subsidiary loop, and this state may be regarded as the working condition of a magnet in practice. For a good magnet the subsidiary loop is thin and only slightly tilted with respect to the H axis, so that ϕ, although not strictly constant, is, to the accuracy that is possible in much experimental work, a single-valued function of H.

The hysteresis loop is characteristic of the material from which the magnet is made, and is the same for magnets of all sizes and shapes, although in using it one must allow for any variation of B from point to point in magnets, which, unlike the simple example quoted, do not possess an approximately uniform distribution of B.

The position of the point P may be calculated as follows. Let l denote the length of the magnet along the central tube of flux and l_g that of the gap. When the magnetizing current is reduced to zero we have by the B–H hypothesis

$$Hl + Bl_g/\mu_g = 0; \qquad \ldots\ldots(15.1)$$

thus

$$H = -(l_g/l\mu_g)B. \qquad \ldots\ldots(15.2)$$

When the gap is filled with soft iron μ_g is very large and, therefore, $H=0$, corresponding to the point R. When the iron is removed, μ_g becomes μ_0, and equation (15.2) can be represented by the line OP in the Figure, where $\tan\theta = l_g/l\mu_0$. Thus the wider the gap, the lower down the curve is P. As the gap widens the flux ϕ spreads over a larger area, i.e. the sectional area of the magnetic circuit increases as we enter the gap. The effect may be represented by dividing B in (15.2) by a leakage factor α which is greater than unity and can be determined from fluxmeter measurements. The important point is that the properties of actual magnets can be predicted by such considerations from the characteristic curves of the material and the dimensions of the magnet and its air path. The principles are therefore established as affording a description of the properties of real magnets that is consistent with all the known facts. High accuracy is impossible except in very limited conditions.

§ 16. THE MAGNETOMETER: MAGNETIC DIPOLES

A magnetometer is essentially a permanent magnet supported so as to be capable of rotation about an axis of symmetry that is perpendicular to the axis of its air path. Such magnets have already been taken for granted as familiar objects adequately described by the term "compass needle"; and we have shown that they can be used to measure H in air spaces in virtue of an experimental law,

$$T = \mathbf{m} \times \mathbf{H}, \qquad \ldots\ldots(16.1)$$

valid over a range of $\mathbf{H}$ that must be found by experiment for the particular magnet used. It can now be added that the deviations from the law that are observed in conditions that are beyond the range established for the instrument are consistent with the laws of permanent magnets described in the previous section.

The magnetometer "needle" may take the form of the ring with a narrow air gap there discussed: it is only necessary that the axis of suspension shall pass symmetrically between the walls of the gap. It is then found that the moment of the magnet $\mathbf{m}$ is given by

$$\mathbf{m} = \mathbf{B}Al_0, \qquad \ldots\ldots(16.2)$$

and that the conditions under which $\mathbf{m}$ can be treated as constant can therefore be obtained from the B–H curve for the magnet.

Alternatively, the magnet may be literally of needle form, or rather a straight bar of uniform cross section A and of length l, which is large compared with the dimensions of the cross section. Such magnets are prepared by a process analogous to that described for the circular magnet; the magnetizing winding takes the form of a long solenoid instead of a toroid and the closed magnetic circuit may consist of two such bars, each with its own solenoid, and with their ends linked by pieces of soft iron. Again we magnetize the circuit to saturation, switch off the current, and remove the soft iron links; the changes in the value of B are exactly similar to those for the ring, and can be represented in the same way by the portion RPC of the hysteresis loop, though in this case the calculation of the point P, which represents the working condition of the magnet, is not so simple. It is found, however, that magnets made from a given material can be usefully represented by a single curve RPC, and values of $\tan\theta$ characteristic of the shape of the magnet, increasing, for example, consistently as the magnet becomes shorter, other factors remaining constant. Similarly magnets of a given shape but different materials have properties which are consistent with a constant value of $\tan\theta$ applied to the "demagnetization curves" characteristic of the several materials. A magnet in its working condition is therefore regarded as characterized by the demagnetization curve for the material; and a demagnetizing factor $D = \tan\theta$, dependent on the shape of the magnet, and determining a demagnetizing field $H_\mathrm{d} = -DB$, which is identified with H defined by the B–H hypothesis. The long thin needle and the ring with the narrow gap are the two forms for which D is small and B nearly uniform throughout the material. D increases as the needle becomes shorter and as the gap in the ring becomes wider; and in both cases the distribution of B at the same time departs from uniformity, the lines of flux opening out, with a consequent diminution of B, as they pass from the material into the air path. Thus in any permanent magnet made according to the standard procedure B increases as we pass along the magnet from the air path and reaches a maximum value in a central region, which is larger the longer the magnet or the narrower the air gap. The values of B and of $\phi = BA$ for this central region can be measured accurately by slipping the search coil of a fluxmeter off the magnet, and it can be shown by experiment that long needle magnets conform to the law

$$\mathbf{m} = BAl. \qquad \ldots\ldots(16.3)$$

For these magnets, $D=0$ and $H_d=0$, so that in the working condition $B=J$, and we may also write $\mathbf{m}=\mathbf{J}Al$. The same argument applied to the ring magnet with a narrow gap leads to $\mathbf{m}=\mathbf{J}Al_0$. These two equations show clearly that we must not define J as "magnetic moment per unit volume" if we are thinking of quantities that are actually measurable, for the relevant volume in the latter case is that of the air gap and in the former that of the magnet.

Another law established by means of the magnetometer is that concerning the torque exerted by one magnet on another. It is most conveniently expressed by the following experimental law for magnets of needle form: A permanent magnet of moment $\mathbf{m}$ is associated with a field of $\mathbf{H}$ in the space surrounding it, the value at any point P being given by

$$\mathbf{H}=S\ \mathrm{grad}\ \{(\mathbf{m}\,.\,\mathbf{r}/r^3)f(l,r)\}, \qquad \ldots\ldots(16.4)$$

where $\mathbf{r}$ is the radius vector OP, O being the centre of the axis $\mathbf{l}$ of the magnet, and f a function that is known, but need not be considered beyond saying that it tends to 1 as $l/r \rightarrow 0$. This field is superposed on any field that may exist at P in the absence of the magnet. (It would be well to note at this point that $\mathbf{H}$ in (16.1) is the value at the point in question in the absence of the magnet.)

It is scarcely necessary to trace in detail the experiments required to justify this law. Every physicist will recognize that the standard procedure associated with the Kew magnetometer will provide much of the evidence required. The well-known use of the magnetometer method for determining magnetization curves and hysteresis loops is also to the point. We must note that the law holds good for long solenoids, either with or without cores of magnetic material, as well as for permanent magnets, if we apply (16.3) to such solenoids. Results obtained in this way are consistent with those obtained with the permeameter, and this agreement also provides some support for the conception of magnetic moment as an important quantity. The theoretical conception of the magnetic dipole is derived directly from this conception and (16.4) in the limiting case when $f(l,r)=1$. The dipole is indeed defined as an entity of very small dimensions which obeys the equation

$$\mathbf{H}=S\ \mathrm{grad}\ \{(\mathbf{m}\,.\,\mathbf{r}/r^3)\}, \qquad \ldots\ldots(16.5)$$

where $\mathbf{m}$ is constant. This is exactly the same equation as was obtained for a current I circulating in a small circuit of area A surrounding the point O in a vacuous medium if $\mathbf{m}=IA$, and has the direction of the normal to the surface A (Part I, § 15). It follows that the magnetic dipole can be conceived as either a minute current whirl or a minute permanent magnet in a vacuous medium. In either case we must assume that the dipole differs from the corresponding real object not only in the smallness of its dimensions, but also in freedom from resistance in the case of the current whirl, and freedom from all changes of moment in the other case. The magnetic field of a real circuit can be shown to be the same as that of a surface distribution of such dipoles, which is the "magnetic shell" of the old treatises. It is scarcely necessary to point out that the shell has by definition properties entirely different from those of real permanent magnets.

It is well known that the law of the torque between dipoles is mathematically equivalent to a law for point poles of the same form as those for point masses and point electric charges, and this has led to the development of mathematical theories in which the point pole with this law of force is the basic postulate. Such theories are quite divorced from the experimental basis of the subject as we understand it, but the law is so frequently quoted as the fundamental experimental law of magnetism that it is impossible to ignore it completely. The most curious thing about the law is that even today, after generations of experiment, there is no general agreement about its exact form. There are two schools of thought: one stating the law in the form

$$F = Sq_1q_2/d^2, \qquad \ldots\ldots(16.6)$$

where S is a mere scale-factor or universal constant. the same for all materials. and the other in the form

$$F = Sq_1q_2/\mu d^2, \qquad \ldots\ldots(16.7)$$

where μ, the permeability, varies from one medium to another. Self-consistent theories can be built up from either law by making the assumptions necessary to secure some sort of correspondence with the known experimental facts. One law requires the recognition of "induced charges" in the medium, also subject to the law, while the other does not. Livens (1947) has recently discussed the matter in detail and has decided in favour of (16.6) on the grounds of the simplicity, scope, and generality of the resulting theory, which is essentially the original theory of Poisson, Kelvin and Maxwell. It is no part of our purpose to question his decision; we would only point out that the other law has been so widely quoted as fundamental in recent years that one can only make sense of the situation by recognizing that these are not experimental laws at all.

§ 17. CONCLUSION

We have now completed our survey of the operations which are most widely employed in experimental investigations of magnetism, and which therefore give the basic concepts of the science their meaning. The further development of the subject depends largely on mathematical theory, the form of which is dictated by mathematical operations rather than experimental operations. The theory takes the form of a self-consistent body of mathematical doctrine which, proceeding from certain postulates, is found to contain relations of the same form as the experimental laws. The theory is therefore accepted as a working mathematical model of the world of the experimenter, certain of the mathematical concepts being identified with the corresponding physical concepts. It is not to be expected that the basic postulates of the theory will correspond with the experimenter's basic concepts, and we have shown that they do not; the magnetic dipole or the point pole which forms the starting point of many theories is remote from the experimental basis of the subject and, provided the fact is recognized, there can be no objection to the procedure. We should, however, not confuse the laws of force between poles or dipoles with experimental laws; neither should we assume that because certain parts of the theoretical model correspond with the real thing that all its parts necessarily have their counterparts in the real world. It is from this point of view that our work should be considered. We believe that the actual correspondence between the experimental laws and the theoretical concepts requires a much closer scrutiny than is usually given to it. The self-consistency of a theory, although of great importance, affords no evidence whatever of its relevance to experiment; that evidence must be sought independently by actual examination of experiments. Self-consistency of the theories has always been examined with great care: consistency among the experimental concepts and the actual correspondence between those of theory and experiment are too frequently taken for granted. It is sometimes said that these things are known by "physical intuition", a quality that can be cultivated by practice but not described. To us it seems to be only a cloak for a certain mistiness of thought which is out of place in physics, and which it is our chief aim to dispel.

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DISCUSSION

Dr. N. R. Campbell. The fundamental difference between Dr. Hartshorn and myself which has prevented my sharing with him the authorship of his last paper (IV), as I have that of our earlier papers I, II, III, concerns the significance of his " general vector " **H** ; we are perfectly agreed concerning **B** and $\mathbf{H}_I$.

1. I do not think it desirable to denote by **H** the vector, measured by the torque on a compass needle, that is the subject of IV, § 10. When it is measured in an air space it is the same " magnitude " as **B** in the sense of I, § 10, this being the only sense in which that term seems to me useful. Theorists have discussed * elaborately what the vector would be if it could be measured in a space filled with appreciably magnetic matter, and have usually concluded that it would differ from **B** by a factor involving μ for that matter ; it has been suggested that its relation to **B** would differ according as the field was due to currents or to permanent magnets. None of these speculations have any experimental support ; it seems to me foolish to introduce a special symbol in order to express them. I should therefore denote the magnitudes of IV, § 10 (in suitable units), by **B** and call it flux density.

2. If this were done, **H** would always mean the auxiliary variable characteristic of the *B–H* hypothesis, which I should prefer to state thus :—It is possible to calculate the distribution of **B** about any collection of current circuits and magnetic bodies by assuming (*a*) that **B** is everywhere solenoidal, (*b*) that it is associated at each point with a vector **H**, of which it is true (*c*) that

$$\oint \mathbf{H}\, dl = \Sigma_r n_r I_r$$

and (*d*) that it is related to **B** at each point in the same way as $\mathbf{H}_I$ is related to **B** in a permeameter whose core consists of the material occupying that point.

3. The appearance of **H** in a proposition would then be an indication that the *B–H* hypothesis was involved in it. This would justify some unexpected appearances. Thus in IV, § 13, the field is usually measured by a fluxmeter (in the sense of III, § 2) ; it might seem therefore that **B** would be more appropriate than **H** in IV, (13.1). But the flux density is measured in the absence of the magnetic material. Now it is a consequence of the *B–H* hypothesis, not often stated explicitly, that, if a *small* quantity of magnetic material is introduced into an air-space in which **B** has been measured, then, though the introduction will in general change both **B** and **H** everywhere, in the neighbourhood of the material introduced, the change of **H** will be much less than the change of **B**. Accordingly, by expressing IV, (13.1) in terms of **H**, not **B**, we make sure that the unknown error, due to making no allowance for the effect of the material on the field, is as small as possible.

4. Research during the last generation, by revealing the existence of " domains ", has shown that no piece of ferromagnetic material as large as a permeameter core can be homogeneous ; it cannot be truly characterized by a single parameter or function ; the *B–H* hypothesis cannot really be true of it. Since **H** derives its significance from the *B–H* hypothesis, it cannot really be applicable to bulk material.

5. The best that can be said of the hypothesis and the magnitude dependent on it is that they are valid over the whole of some practically important range of experiment, so long as no methods of measurement are used more precise than those normally employed in such

* See e.g., H. Chipart, *C.R. Acad. Sci. Paris*, 1921, **172**, 589–591, 750–753; L. R. Wilberforce, *Proc. Phys. Soc.*, 1933, **44**, 82–86; L. Page, *Phys. Rev.*, 1933, **45**, 112–115.
These authors, though all professing to expound the same classical theory, differ notably in their conclusions.

experiment. But I submit that even this is not true. The B–H hypothesis is of no practical use unless some subsidiary simplifying hypothesis is made about the B–H relation. Thus the conception of a magnetic circuit, having reluctance and containing M.M.F., is valueless unless it is assumed that B/H is constant; that can almost always be shown to be false by the very experiments used to determine the constant value or by some slightly modified variant. Again, the assumption of constant μ, involved in attributing a definite inductance to a coil with a dust-core, is almost always accompanied by the contradictory assumption that the core material has " hysteresis loss ". Again, in Evershed's theory of permanent magnets (which theorists conspire to neglect, although it is much more important practically than propositions to which they devote reams of paper, and at least equally nearly true), the B–H relation is identified with the demagnetizing branch of the hysteresis curve; this, of course, is true only for very small changes.

6. I admire the gallantry and ingenuity of Dr. Hartshorn's attempt to base the significance of **H** on constant-permeability alloys and the extension of the hysteresis curve beyond saturation. But he will hardly convince anyone who remembers that the facts on which his defence rests were unknown until his **H**, and the " classical theory " on which it rests, had been established for at least two generations, and even now are not usually recorded in text-books. The real position, I suggest, is that the theory has been taken seriously only because it was propounded before any of the important facts about ferromagnetism were known, and because, by the time that those facts were discovered, Maxwell's reputation was so firmly based on other work that anything he said had become (and has remained) for most physicists) outside the range of dispassionate examination.

7. The analogy between magnetic materials and dielectrics, which was undoubtedly one of the foundation stones of the classical theory, is invalid in respect of the matters considered here. The concept of permittivity has survived the discovery that all dielectrics are imperfect, because their imperfection can be represented by the substitution of a complex for a real permittivity ; there is no analogous way of representing the fact that B/H_I is not single-valued.

8. If, as I maintain, the sole use of **H** is to make statements about magnetic materials that are either definitely false or unsupported by any evidence, it is particularly out of place in a theory that explains away magnetic materials as the electron theory explains away dielectrics. The essence of the modern theory of magnetic materials is that certain atoms are dipoles of zero dimensions, having magnetic moments independent of the field in which they find themselves ; their behaviour is governed by equations of the form of IV, (15.1), (15.4)—the latter expressed in quantum-theory terms—by which it is possible, in the Kew magnetometer, to measure both of the vectors **m**, **H** without measuring any other magnetic magnitude. **H** so measured turns out to be the same magnitude as **B** ; for in the presence of magnetic materials the equations cease to be true and **H** to be measurable by this method. Since the theory which explains magnetic materials cannot suppose that there is magnetic material in the field of the dipoles representing magnetic material, the vector **H** concerned in it cannot fail to be the same as **B** ; nowadays it is sheer perversity, based on the accidents of history, to represent it otherwise.

I do not say that all propositions about theoretical dipoles to be found in current literature remain true if **B** is substituted for the usual **H**. But if they do not, then there must be some error or ambiguity that requires the attention of experts.

Dr. L. Hartshorn (in reply). In view of the lamentable death of Dr. Campbell before these notes have appeared in print I think it more appropriate to outline the circumstances in which they were written than to attempt to reply in detail. The two papers III and IV were originally drafted by me as a single paper based on earlier drafts prepared by Dr. Campbell and extensive discussions with him. My expectation was that the whole would be published as a joint paper after Dr. Campbell had made whatever amendments he found necessary. This was his own idea also at first, but after a careful study of the matter he came to the conclusion that although he had little objection to anything actually stated in the paper, it showed that our views were essentially divergent. He therefore abandoned the attempt to produce an amended version on which we could be unanimous, and suggested that the form in which the papers now appear should be adopted. The difficulties raised by Dr. Campbell seem to me either unimportant or adequately covered by Part IV, but I am glad that it has been possible to state them in his own words, so that points that I may have missed are faithfully recorded.