Motion of a Charged Particle in a Slowly Increasing Magnetic Field

L. J. F. BROER

Laboratorium voor Aero- en Hydrodynamica, Technische Hogeschool, Delft, Netherlands

We consider a charged particle moving in the xy plane under the influence of a homogeneous magnetic field $B=\omega(m/e)$ in the z direction. On introducing the complex coordinate $u=x+iy$, the equation of motion of the particle is

$$d^2u/dt^2+i\omega du/dt+(i/2)d\omega/dt\cdot u=0. \tag{1}$$

This equation can be shown to have a first integral of the form

$$2\mu-\omega|u|^2=\mu_0,$$

where $\mu=\mathrm{Im}\, u du^*/dt$ is the moment of momentum with respect to the origin and μ_0 is a constant of integration.

When ω is constant, the general solution of (1) is

$$u=C+A\exp(-i\omega t). \tag{2}$$

For a slowly varying field, which means that $\omega^{-2}d\omega/dt \ll 1$, we have two linearly independent solutions. For an interval of time that is short in terms of the rate of change of the field, these solutions can be taken as a fast motion, which is a gyration around the origin, tending toward $A\exp(-i\omega t)$, when $d\omega/dt$ goes to zero, and a slow motion of the center of gyration, tending toward a constant value. However, this separation in fast and slow motions is exact only in the limit of vanishing $d\omega/dt$. Otherwise an initially fast solution may aquire a slow component in the long run, and vice versa.

We now consider a slowly increasing field with the asymptotic values ω_1 and ω_2 for $t\to-\infty$ and ∞, respectively. We assume that the solution tends to $A_1\exp(-i\omega_1 t)$ for $t\to-\infty$. It is well known that in first approximation the solution tends toward $A_1(\omega_1/\omega_2)^{\frac{1}{2}}\times\exp(-i\omega_2 t)$ for $t\to\infty$ and that μ also is constant during the transition. The exact solution, however, shows a shift of the center, tending therefore toward $C_2+A_2\exp(-i\omega_2 t)$. From (2) we obtain $\omega_1|A_1| =\omega_2|A_2|^2-\omega_2|C_2|^2$. The shift, therefore, is related to an increase of the moment. An approximate expression for the shift is

$$C_2=A_2\omega_2^{-\frac{1}{2}\prime}\int_{-\infty}^{\infty}\frac{dt}{dt}\omega^{\frac{1}{2}}\cdot\exp\left(-i\int\omega dt'\right)dt. \tag{3}$$

As all derivatives of ω vanish for $|t|\to\infty$, this expression is at most of exponential order, as can be shown by repeated integration by parts. An exact solution of (1) can be given for $\omega=\omega_0(1+\Delta\tanh\gamma t)$. In this solution the shift indeed turns out to be of the order of $\exp(-\gamma/\omega_0)$.

Approximate solutions, valid for a short interval, for both fast and slow solutions can be developed by substituting $u=a\exp(-i\int fdt)$ in (1), considering both a and f to be real slowly varying functions of t. For the fast solutions μ is no longer a constant. In the second approximation it is found that

$$\mu=\omega_1A_1{}^2(1-\omega^{-\frac{3}{2}}d^2/dt^2\omega^{-\frac{1}{2}}, \tag{4}$$

Further results are contained in a forthcoming paper.[1]

Finally, (1) can be reduced to the wave equation by letting the frame of reference rotate with half the Larmor frequency. Putting $u=\varphi\exp[-(i/2)\int\omega dt]$, we obtain

$$d^2\varphi/dt^2+(\omega^2/4)\varphi=0. \tag{5}$$

Fast and slow solutions now correspond to wave solutions of (5), traveling in forward and backward directions. The problem of calculating the shift then is equivalent to the calculation of the reflection in a gradient of refractive index. It turns out that the method used to obtain (4) from (1) is equivalent to an application of the WKB method to (5), whereas a result corresponding to (3) can be obtained by treating (5) by a method due to Bremmer.[2]

[1] L. J. F. Broer and L. van Wijngaarden, Appl. Sci. Research **B8**, 159 (1960).

[2] H. Bremmer, Handel. Ned. Natuur- en Geneesk. Congr. **27**, 88 (1939).

DISCUSSION

Session Reporter: W. B. RIESENFELD

O. Laporte, *University of Michigan, Ann Arbor, Michigan:* It seems possible to decompose your particle motion into two coupled pendular motions, with changes of pendulum length. This is quite a classical problem. One can also use a perturbation approach.

L. J. F. Broer: Yes, this is a type of perturbation calculation, but the asymptotic nature of the approximation is a bit trickier. Perturbation methods would apply only to a relatively short interval of time.

D. Bershader, *Stanford University, Stanford, California:* The strictly geometrical approach to the refractivity problem yields the result that the curvature is proportional to the transverse component of the refractive index gradient. I would like to know whether there is an analogous relationship in your orbit calculation, or whether you have to take a strictly physical optical point of view.

L. J. F. Broer: There is a mathematical analogy between the equation of motion and a wave equation for inhomogeneous media. This is a one-dimensional wave equation, however; therefore, there is no analogy to curvature of rays.

C. Agostinelli, *Università di Torino, Turin, Italy:* Have you considered the relativistic case for an arbitrary field?

L. J. F. Broer: No, but there are lots of results for this sort of problem; however, the point of view generally has been that of people constructing magnetic lenses for beta-ray spectrographs. This problem is also involved in Störmer's calculation of charged particle orbits in the earth's magnetic dipole field, and their relation to the aurora borealis. This is a much more difficult problem because of the presence of inhomogeneous fields.