1 Optimization 2025 - Problems for Tutorial 1

The focus of this tutorial is modeling linear programs.

1.1 Basic LP modeling

1. Vanderbei, Exercise 1.1. Draw also the set of feasible solutions.

2. The Superb Soft Drink Company is producing three products: Alice's Morning Juice, Bertram's Orange Soda, and Carol's Lemonade. All products are sold to distributors in large quantities and priced by volume.

   - To produce one liter of Alice's Morning Juice, the company needs 0.5 liter of orange juice, 0.2 liters of lemon juice, 6 grams of sugar, and some water.
   - To produce one liter of Bertram's Orange Soda, the company needs 0.1 liters of orange juice, 30 grams of sugar, and some water.
   - To produce one liter of Carol's Lemonade, the company needs 0.6 liters of lemon juice, 50 grams of sugar, and some water.

   Alice's Morning Juice is sold at 2 dollars per liter, Bertram's Orange Soda at 1 dollar per liter, and Carol's Lemonade at 3 dollars per liter. The company can purchase sugar at 1 cent per gram, orange juice at 100 cents per liter and lemon juice at 400 cents per liter. The price of water is negligible. The company can produce and ship a combined total of at most 10000 liters of the three products in a day. Write a linear program that expresses the problem of finding the appropriate amount of each of the products to produce daily so as to maximize profit. You do not have to solve the linear program.

3. Vanderbei, Exercise 1.2. You do not have to solve the linear program.

1.2 Advanced LP modeling

1. At the first lecture, a transformation of general linear programs to linear programs in standard form was described. This reduction will in general increase the number of variables and constraints by a factor of two each, which is undesirable in practice. Give an alternative transformation that only adds *one* variable and *one* constraint to the program when transforming it to standard form.

   Hint: The case of variables might be the easiest to think about first. You can think about an unconstrained variable as a surplus over a baseline. For constraints, enforcing an equality can be viewed as disallowing both surplus and slack in inequalities. Consider bundling the restrictons on surplus, say, into a single inequality.

2. You are given a convex polygon P in $\mathbb{R}^2$ by an *unordered* list of its corner points $\{p_1, p_2, \dots, p_m\}$ with $p_j = (a_j, b_j)$. Construct a linear program with the following properties:

   - If the origin $(0,0)$ is in the interior of P, then the program is feasible, and the optimal value of its objective function is strictly positive.

- If the origin $(0,0)$ is not in the interior of P, then the program is either infeasible, or it is feasible and the optimal value of the objective function is at most 0.

Here, recall that a point is said to be in the interior of P if it is in P but not on the boundary of P, that is, it is neither a corner of P nor on an edge of P.

Hint: You may also find it useful to know that a point is in the interior of P if and only if it is a weighted average of the corner points of P, with all weights being strictly positive.

3. Given two lists of points $p_1, p_2, \ldots, p_k \in \mathbb{R}^n$ (the pink points) and $g_1, g_2, \ldots, g_\ell \in \mathbb{R}^n$ (the green points), we want to construct a simple device that distinguishes the pink points from the green points. A type of device that has been used in artificial intelligence research is the perceptron. A perceptron is given by a list of real-valued weights $w_0, w_1, \ldots, w_n$ and classifies an input point $z \in \mathbb{R}^n$ as green if $\sum_{i=1}^n w_i z_i \leq w_0$ and as pink if $\sum_{i=1}^n w_i z_i > w_0$.

   Devise a linear program whose optimal solution value is strictly positive if and only if there exists a perceptron correctly classifying the two lists of points.

4. A different way of classifying points is to use a ball. A ball in $\mathbb{R}^n$ is a set $B(y,r) = \{z \in \mathbb{R}^n \mid \text{dist}(y,z) \leq r\}$, where $y \in \mathbb{R}^n$ is called the center of the ball, $r \in \mathbb{R}$ the radius and $\text{dist}(y,z) = \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$ is the usual Euclidean distance between points y and z. We say ball $B(y,r)$ classifies a point z as green if $z \in B(y,r)$ and as pink if $z \notin B(y,r)$.

   - Show that there exists $r > 0$ such that $B(y,r)$ correctly classifies two lists of points $p_1, p_2, \ldots, p_k \in \mathbb{R}^n$ and $g_1, g_2, \ldots, g_\ell \in \mathbb{R}^n$ if and only if $\text{dist}(y, g_i) < \text{dist}(y, p_j)$ for all i and j.
   - Devise a linear program whose optimal solution value is strictly positive if and only if there exists a ball correctly classifying the two lists of points.

5. You are given a set of inequalities $(\sum_{j=1}^n a_{ij} x_j \leq b_i)_{i \in \{1,\ldots,m\}}$. You are promised that the set of inequalities has a non-empty set of feasible solutions in $\mathbb{R}^n$ and that this set of feasible solutions is in fact a bounded polyhedron. Your task is to develop a method for determining the largest ball that will fit inside this polyhedron.

   - Show that a ball $B(y,r)$ with center y and radius r lies inside (i.e., is a subset of) the polyhedron if and only if
     (a) For all $i \in \{1, \ldots, m\}$ it holds that $\sum_{j=1}^n a_{ij} y_j \leq b_i$, and
     (b) For all i, the distance between the point y and the hyperplane obtained from the i-th inequality, $\{x \in \mathbb{R}^n \mid \forall i : \sum_j a_{ij} x_j = b_i\}$, is at least r.
   - Show how to use a linear programming solver to efficiently determine the center and radius of the largest ball (i.e., the one with largest radius) that fits inside the polyhedron. You can ignore numerical issues, i.e., you may assume that you have arbitrary precision arithmetic on real numbers at your disposal, including a $\sqrt{\cdot}$ operation.