Guiding Center Equation

reference :: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

CGS

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$\vec{v} = \frac{\vec{D}}{B_{\parallel}^{*}} \times \vec{b}$$

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}\left(\frac{\partial \vec{v_E}}{\partial t} + \nabla\frac{v_E^2}{2}\right)$$

$$\vec{v_E} = \frac{c}{B}\vec{E} \times \vec{b}$$

$$\vec{B^*} = \vec{B} + \frac{m}{q}(\nabla \times \vec{v_E})$$

$$B_{\parallel}^{*} = \vec{B^*} \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$\vec{B}(\vec{m}, \vec{r}) = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\vec{\hat{r}})\vec{\hat{r}} - \vec{m}}{r^3}$$

$$\vec{\hat{r}} = \frac{\vec{r}}{r}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$\nabla f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{e_z}$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\left(\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r\frac{\partial A_z}{\partial z}\right)$$

$$\nabla \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{e_r} + \left(\frac{\partial A_r}{\partial z} + \frac{\partial A_z}{\partial r}\right)\vec{e_\theta} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{e_z}$$

Toroidal mode wave

ULF waves

$$\vec{B}_{wave} = \vec{e_\theta} B_A sin(\omega(t - \frac{r\theta}{v_a}))$$

$$= \vec{e_\theta} B_A sin(m\omega_d(t - \frac{r\theta}{v_a}))$$

$= \vec{e_\theta} B_A sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}))$

$\vec{E}_{wave} = \vec{e_r} E_A sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2})$

$E_{wave} = E_A sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2})$

$\vec{E}_{wave} = \vec{e_r} E_{wave}$

$\vec{B_0} = \frac{\mu_0}{4\pi} \frac{3(\vec{m}\cdot\vec{\hat{x}})\vec{\hat{x}} - \vec{m}}{|\vec{x}|^3}$

$\vec{m} = m\vec{e_z}$

$\vec{B_0} = \frac{\mu_0}{4\pi} \frac{3mz\vec{\hat{x}} - m\vec{e_z}}{|\vec{x}|^3}$

$\vec{E_0} = \vec{0}$

$\vec{E} = \vec{E_0} + \vec{E}_{wave} = \vec{E}_{wave}$

$\vec{B} = \vec{B_0} + \vec{B}_{wave}$

Apply to the fomula

$\vec{v_E} = \frac{\vec{E}}{B} \times \vec{b}$

$= \frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}$

$\vec{B^*} = \vec{B} + \frac{m}{q}(\nabla \times \vec{v_E})$

$= \vec{B} + \frac{m}{q}(\nabla \times (\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}))$

$= \vec{B} + \frac{m}{q}((\nabla \cdot \vec{B})\frac{\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{\vec{E}_{wave}}{|B|^2})\vec{B})$

$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial \vec{v_E}}{\partial t} + \nabla\frac{v_E^2}{2})$

$= \vec{E}_{wave} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial}{\partial t}(\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}) + \frac{1}{2}\nabla|\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2)$

Assumption

- $\vec{B}_{wave} << \vec{B_0}$
- $\frac{\partial B_z}{\partial z} = 0$
  so,
- $\vec{B} = \vec{B_0}$
- $\frac{\partial \vec{B}}{\partial t} = \vec{0}$

- $\vec{B} = (B_r, B_\theta, B_z) = (0, 0, B_z)$
- $\frac{\partial \vec{B}}{\partial \vec{x}} = (\xi_r, 0, 0)$
- $\vec{b} = \vec{e_z}$

$$\vec{D} = \vec{E}_{wave} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} \left(\frac{\partial \vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2} \nabla \left| \frac{\vec{E}_{wave}}{|B|^2} \times \vec{B} \right|^2 \right)$$

$$\vec{D} = \vec{e_r} E_{wave} - \frac{\mu}{q} \xi_r \vec{e_r} - \frac{m}{q} \left(\frac{\partial (\vec{e_r} E_{wave})}{\partial t} \times \frac{\vec{e_z}}{B_z} + \frac{1}{2} \nabla \left| \frac{\vec{e_r} E_{wave}}{|B|^2} \times \vec{B} \right|^2 \right)$$

$$\vec{D} = \vec{e_r} E_{wave} - \frac{\mu}{q} \xi_r \vec{e_r} - \frac{m}{q} \left(- \frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} e_\theta + \frac{1}{2} \nabla \left| \frac{\vec{e_r} E_{wave}}{|B|^2} \vec{B} + \frac{\vec{e_r} E_{wave}}{|B|^2} \times \vec{B} \right|^2 \right)$$

$$\vec{v} = \frac{\vec{D}}{B_\parallel^*} \times \vec{b}$$

$$\vec{v} = \frac{\vec{E} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} \left(\frac{\partial \vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2} \nabla \left| \frac{\vec{E}_{wave}}{|B|^2} \times \vec{B} \right|^2 \right) + +}{\left(\vec{B} + \frac{m}{q} \left((\nabla \cdot \vec{B}) \frac{\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{\vec{E}_{wave}}{|B|^2}) \vec{B} \right) \cdot \vec{e_z} \right.} \times \vec{e_z}$$