

Guiding Center Equation

reference :: <https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682>

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$\vec{v} = \frac{\vec{D}}{B_{\parallel}^*} \times \vec{b}$$

$$\vec{D} = \vec{E} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} \left(\frac{\partial \vec{v}_E}{\partial t} + \nabla \frac{v_E^2}{2} \right)$$

$$\vec{v}_E = \frac{\vec{E}}{B} \times \vec{b}$$

$$\vec{B}^* = \vec{B} + \frac{m}{q} (\nabla \times \vec{v}_E)$$

$$B_{\parallel}^* = \vec{B}^* \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_E \cdot \hat{\vec{x}})\hat{\vec{x}} - \vec{m}_E}{x^3}$$

$$\hat{\vec{x}} = \frac{\vec{x}}{|\vec{x}|}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$\nabla f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{e}_z$$

$$\nabla \cdot \vec{A} = \frac{1}{r} \left(\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r \frac{\partial A_z}{\partial z} \right)$$

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \vec{e}_r + \left(\frac{\partial A_r}{\partial z} + \frac{\partial A_z}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \vec{e}_z$$

Toroidal mode wave

ULF waves

$$\vec{B}_{wave} = \vec{e}_\theta B_A \sin(\omega(t - \frac{r\theta}{v_a}))$$

$$= \vec{e}_\theta B_A \sin(m\omega_d(t - \frac{r\theta}{v_a}))$$

$$= \vec{e}_\theta B_A \sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}))$$

$$\vec{E}_{wave} = \vec{e}_r E_A \sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2})$$

$$E_{wave} = E_A \sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2})$$

$$\vec{E}_{wave} = \vec{e}_r E_{wave}$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \vec{\hat{x}})\vec{\hat{x}} - \vec{m}}{|\vec{x}|^3}$$

$$\vec{m} = m\vec{e}_z$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3mz\vec{\hat{x}} - m\vec{e}_z}{|\vec{x}|^3}$$

$$\vec{E}_0 = \vec{0}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_{wave} = \vec{E}_{wave}$$

$$\vec{B} = \vec{B}_0 + \vec{B}_{wave}$$

Apply to the formula

$$\vec{v}_E = \frac{\vec{E}}{B} \times \vec{b}$$

$$= \frac{\vec{E}_{wave}}{|\vec{B}|^2} \times \vec{B}$$

$$\vec{B}^* = \vec{B} + \frac{m}{q}(\nabla \times \vec{v}_E)$$

$$= \vec{B} + \frac{m}{q}(\nabla \times (\frac{\vec{E}_{wave}}{|\vec{B}|^2} \times \vec{B}))$$

$$= \vec{B} + \frac{m}{q}((\nabla \cdot \vec{B})\frac{\vec{E}_{wave}}{|\vec{B}|^2} - (\nabla \cdot \frac{\vec{E}_{wave}}{|\vec{B}|^2})\vec{B})$$

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial\vec{v}_E}{\partial t} + \nabla\frac{v_E^2}{2})$$

$$= \vec{E}_{wave} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial}{\partial t}(\frac{\vec{E}_{wave}}{|\vec{B}|^2} \times \vec{B}) + \frac{1}{2}\nabla|\frac{\vec{E}_{wave}}{|\vec{B}|^2} \times \vec{B}|^2)$$

Assumption

- $\vec{B}_{wave} \ll \vec{B}_0$
- $\frac{\partial B_z}{\partial z} = 0$
- $\frac{\partial B_r}{\partial z} = 0$

so,

- $\vec{B} = \vec{B}_0$

- $\frac{\partial \vec{B}}{\partial t} = \vec{0}$
- $\vec{B} = (B_r, B_\theta, B_z) = (0, 0, B_z)$
- $\frac{\partial \vec{B}}{\partial r} = (0, 0, \xi_r)$
- $\frac{\partial \vec{B}}{\partial \theta} = (0, 0, 0)$
- $\frac{\partial \vec{B}}{\partial z} = (0, 0, 0)$
- $\vec{b} = \vec{e}_z$

$$B^* = \vec{B} + \frac{m}{q}((\nabla \cdot \vec{B}) \frac{\vec{E}_{wave}}{|\vec{B}|^2} - (\nabla \cdot \frac{\vec{E}_{wave}}{|\vec{B}|^2}) \vec{B})$$

$$= B_z \vec{e}_z + \frac{m}{q}(-(\nabla \cdot \frac{\vec{E}_{wave}}{B_z^2}) B_z \vec{e}_z)$$

$$= B_z \vec{e}_z + \frac{m}{q}(-(\frac{\partial}{\partial r} \frac{E_{wave}}{B_z^2}) B_z \vec{e}_z)$$

$$= B_z \vec{e}_z + \frac{m}{q}(E_{wave} (\frac{2B_z \xi_r}{B_z^4}) B_z \vec{e}_z)$$

$$= B_z \vec{e}_z + \frac{m}{q}(E_{wave} \frac{2\xi_r}{B_z^2}) \vec{e}_z$$

$$= (B_z + \frac{m}{q} E_{wave} \frac{2\xi_r}{B_z^2}) \vec{e}_z$$

$$B_{||}^* = B_z + \frac{m}{q} E_{wave} \frac{2\xi_r}{B_z^2}$$

$$\vec{D} = \vec{E}_{wave} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q}(\frac{\partial \vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|\vec{B}|^2} + \frac{1}{2} \nabla |\frac{\vec{E}_{wave}}{|\vec{B}|^2} \times \vec{B}|^2)$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(\frac{\partial(\vec{e}_r E_{wave})}{\partial t} \times \frac{\vec{e}_z}{B_z} + \frac{1}{2} \nabla |\frac{\vec{e}_r E_{wave}}{B_z^2} \times B_z \vec{e}_z|^2)$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{1}{2} \nabla |-\frac{E_{wave}}{B_z^2} B_z \vec{e}_\theta|^2)$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{1}{2} \nabla ((\frac{E_{wave}}{B_z^2} B_z)^2))$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{E_{wave}}{B_z} \nabla (\frac{E_{wave}}{B_z}))$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{E_{wave}}{B_z} \frac{E_{wave} \nabla B_z + B_z \nabla E_{wave}}{B_z^2})$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{E_{wave}}{B_z} \frac{E_{wave} \xi_r \vec{e}_r}{B_z^2} + \frac{E_{wave}}{B_z} \frac{\nabla E_{wave}}{B_z})$$

$$= \vec{e}_r E_{wave} - \frac{\mu}{q} \xi_r \vec{e}_r - \frac{m}{q}(-\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e}_\theta + \frac{E_{wave}^2 \xi_r}{B_z^3} \vec{e}_r + \frac{E_{wave}}{B_z^2} \frac{\partial E_{wave}}{\partial \theta} \vec{e}_\theta)$$

$$(E_{wave} = E_{wave}(\theta))$$

$$= (E_{wave} - \frac{\mu}{q} \xi_r - \frac{m}{q} \frac{E_{wave}^2 \xi_r}{B_z^3}) \vec{e}_r + \frac{m}{q}(\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} - \frac{E_{wave}}{B_z^2} \frac{\partial E_{wave}}{\partial \theta}) \vec{e}_\theta$$

$$\vec{v} = \frac{\vec{D}}{B_{||}^*} \times \vec{b}$$

$$\vec{v} = (B_z + \frac{m}{q} E_{wave} \frac{2\xi_r}{B_z^2})^{-1} (-(E_{wave} - \frac{\mu}{q} \xi_r - \frac{m}{q} \frac{E_{wave}^2 \xi_r}{B_z^3}) \vec{e}_\theta + \frac{m}{q} (\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} - \frac{E_{wave}}{B_z^2} \frac{\partial E_{wave}}{\partial \theta}) \vec{e}_r)$$