Guiding Center Equation

reference:: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

cgs gaussian units

Assumption

- pitch angle is 90 degree so,
- $v_\parallel=0$

$$rac{\partial f}{\partial t} + ec{v} \cdot rac{\partial f}{\partial ec{x}} = 0$$

$$ec{v} = c rac{ec{D}}{B_{\scriptscriptstyle \parallel}^*} imes ec{b}$$

$$ec{D}=ec{E}-rac{\mu}{q}
ablaec{B}-rac{m}{q}(rac{\partialec{v_E}}{\partial t}+
ablarac{v_E^2}{2})$$

$$ec{v_E} = rac{cec{E}}{B} imes ec{b}$$

$$ec{B^*} = ec{B} + rac{mc}{a} (
abla imes ec{v_E})$$

$$B_{\parallel}^{*}=\vec{B^{*}}\cdot\vec{b}$$

$$B=|\vec{B}|$$

$$ec{B}(ec{x})=rac{\mu_0}{4\pi}rac{3(ec{m_E}\cdotec{\hat{x}})ec{\hat{x}}-ec{m_E}}{x^3}$$

$$ec{\hat{x}} = rac{ec{x}}{|x|}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$abla f = rac{\partial f}{\partial r} ec{e_r} + rac{1}{r} rac{\partial f}{\partial heta} ec{e_ heta} + rac{\partial f}{\partial z} ec{e_z}$$

$$abla \cdot ec{A} = rac{1}{r}(rac{\partial (rA_r)}{\partial r} + rac{\partial A_ heta}{\partial heta} + rrac{\partial A_z}{\partial z})$$

$$abla imes ec{A} = (rac{1}{r} rac{\partial A_z}{\partial heta} - rac{\partial A_{ heta}}{\partial z}) ec{e_r} + (rac{\partial A_r}{\partial z} + rac{\partial A_z}{\partial r}) ec{e_{ heta}} + rac{1}{r} (rac{\partial (rA_{ heta})}{\partial r} - rac{\partial A_r}{\partial heta}) ec{e_z}$$

Toroidal mode wave

ULF waves
$$\begin{split} \vec{B}_{wave} &= \vec{e_{\theta}} \, B_A \sin\{\omega(t - \frac{r\theta}{v_a})\} \\ &= \vec{e_{\theta}} \, B_A \sin\{m\omega_d(t - \frac{r\theta}{v_a})\} \\ &= \vec{e_{\theta}} \, B_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda})\} \\ \vec{E}_{wave} &= \vec{e_r} \, E_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2}\} \\ E_{wave} &= E_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2}\} \\ \vec{E}_{wave} &= \vec{e_r} \, E_{wave} \\ \vec{B_0} &= \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \vec{\hat{x}}) \vec{\hat{x}} - \vec{m}}{|\vec{x}|^3} \end{split}$$

$$\vec{m}=mec{e_z}$$

$$ec{B_0}=rac{\mu_0}{4\pi}rac{3mzec{\hat{x}}-mec{e_z}}{|ec{x}|^3}$$

$$\vec{E_0} = \vec{0}$$

$$ec{E} = ec{E_0} + ec{E}_{wave} = ec{E}_{wave}$$

$$\vec{B} = \vec{B_0} + \vec{B}_{wave}$$

Apply to the fomula

$$\begin{split} & \vec{v_E} = \frac{c\vec{E}}{B} \times \vec{b} \\ & = \frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B} \\ & \vec{B^*} = \vec{B} + \frac{mc}{q} (\nabla \times \vec{v_E}) \\ & = \vec{B} + \frac{mc}{q} \{ \nabla \times (\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B}) \} \\ & = \vec{B} + \frac{mc}{q} \{ (\vec{B} \cdot \nabla) \frac{c\vec{E}_{wave}}{|B|^2} - (\frac{c\vec{E}_{wave}}{|B|^2} \cdot \nabla) \vec{B} + (\nabla \cdot \vec{B}) \frac{c\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{c\vec{E}_{wave}}{|B|^2}) \vec{B} \} \\ & \vec{D} = \vec{E} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} (\frac{\partial \vec{v_E}}{\partial t} + \nabla \frac{\vec{v_E}}{2}) \\ & = \vec{E}_{wave} - \frac{\mu}{a} \nabla \vec{B} - \frac{m}{a} \{ \frac{\partial}{\partial t} (\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B}) + \frac{1}{2} \nabla |\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B} |^2 \} \end{split}$$

Assumption

•
$$\vec{B}_{wave} << \vec{B}_0$$

•
$$\frac{\partial B_z}{\partial z} = 0$$

$$\begin{array}{l} \bullet \ \frac{\partial B_z}{\partial z} = 0 \\ \bullet \ \frac{\partial B_r}{\partial z} = 0 \end{array}$$

•
$$E_{wave} = E_{wave}(\theta)$$
)
(r is const:: $r = R_0$)

•
$$\vec{B} = \vec{B_0}$$

•
$$\frac{\partial \vec{B}}{\partial t} = \vec{0}$$

•
$$\vec{B}=(B_r,B_{ heta},B_z)=(0,0,B_z)$$

•
$$\frac{\partial \vec{B}}{\partial r} = (0, 0, \xi_r)$$

• $\frac{\partial \vec{B}}{\partial \theta} = (0, 0, 0)$
• $\frac{\partial \vec{B}}{\partial z} = (0, 0, 0)$

•
$$\frac{\partial \vec{B}}{\partial \theta} = (0,0,0)$$

•
$$\frac{\partial \vec{B}}{\partial z} = (0,0,0)$$

•
$$\vec{b} = \vec{e_z}$$

•
$$\frac{\partial E_{wave}}{\partial r} = \frac{\partial E_{wave}}{\partial z} = 0$$

$$ec{B^*} = ec{B} + rac{mc}{a} \{ (ec{B} \cdot
abla) rac{cec{E}_{wave}}{|B|^2} - (rac{cec{E}_{wave}}{|B|^2} \cdot
abla) ec{B} + (
abla \cdot ec{B}) rac{cec{E}_{wave}}{|B|^2} - (
abla \cdot ec{E}_{wave}) ec{B} \}$$

$$=B_zec{e_z}+rac{mc}{g}\{(B_zec{e_z}\cdot
abla)rac{cE_{wave}ec{e_r}}{B^2}-(
abla\cdotrac{cE_{wave}ec{e_r}}{B^2}+rac{cE_{wave}ec{e_r}}{B^2}\cdot
abla)B_zec{e_z}\}$$

$$=B_z\vec{e_z}+rac{mc}{q}\{(rac{cE_{wave}\vec{e_r}}{B_z^2}\cdot
abla)B_z\vec{e_z}\}$$

$$=B_zec{e_z}+rac{mc}{q}(rac{cE_{wave}}{B_z^2}m{\xi}_rec{e_z})$$

$$B_\parallel^* = B^* \cdot ec{b} = B_z + rac{mc}{q} (rac{cE_{wave}}{B_z^2} \xi_r)$$

$$ec{D} = ec{E}_{wave} - rac{\mu}{q}
abla ec{B} - rac{m}{q} (c rac{\partial ec{E}_{wave}}{\partial t} imes rac{ec{B}}{|B|^2} + rac{1}{2}
abla |c rac{ec{E}_{wave}}{|B|^2} imes ec{B}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{a}\xi_rec{e_r}-rac{m}{a}(crac{\partial(ec{e_r}E_{wave})}{\partial t} imesrac{ec{e_z}}{B_z}+rac{c^2}{2}
abla|rac{ec{e_r}E_{wave}}{B^2} imes B_zec{e_z}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(rac{\partial(ec{e_r}E_{wave})}{\partial t} imesrac{ec{e_z}}{B_z}+rac{c}{2}
abla|rac{ec{e_r}E_{wave}}{B_z^2} imes B_zec{e_z}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+rac{c}{2}
abla|-rac{E_{wave}}{B_z^2}B_zec{e_ heta}|^2)$$

$$= \vec{e_r} E_{wave} - \frac{\mu}{q} \xi_r \vec{e_r} - \frac{mc}{q} \{ -\frac{\partial E_{wave}}{\partial t} \frac{1}{B_z} \vec{e_{ heta}} + \frac{c}{2} \nabla ((\frac{E_{wave}}{B_z^2} B_z)^2) \}$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}\{-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+crac{E_{wave}}{B_z}
abla(rac{E_{wave}}{B_z})\}$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+crac{E_{wave}}{B_z}rac{-E_{wave}
abla B_z+B_z
abla E_{wave}}{B_z^2})$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}-crac{E_{wave}}{B_z}rac{E_{wave}}{B_z^2}+crac{E_{wave}}{B_z}rac{
abla E_{wave}}{B_z})$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}-crac{E_{wave}^2\xi_r}{B_z^3}ec{e_r}+crac{E_{wave}}{B_z^2}rac{\partial E_{wave}}{R_0\partial heta}ec{e_ heta}) \ (E_{wave}=E_{wave}(heta))$$

$$\hat{E} = (E_{wave} - rac{\mu}{q} \xi_r + rac{mc^2}{q} rac{E_{wave}^2 \xi_r}{B_z^3}) ec{e_r} + rac{m}{q} (rac{\partial E_{wave}}{\partial t} rac{c}{B_z} - c^2 rac{E_{wave}}{B_z^2} rac{\partial E_{wave}}{R_0 \partial heta}) ec{e_{ heta}}$$

$$ec{v} = rac{ec{D}}{B_{\parallel}^*} imes ec{b}$$

$$\vec{v} = \{B_z + \frac{mc}{q}(\frac{cE_{wave}}{B_z^2}\xi_r)\}^{-1}\{-(E_{wave} - \frac{\mu}{q}\xi_r + \frac{mc^2}{q}\frac{E_{wave}^2\xi_r}{B_z^3})\vec{e_\theta} + \frac{m}{q}(\frac{\partial E_{wave}}{\partial t}\frac{c}{B_z} - c^2\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{R_0\partial\theta})\vec{e_r}\}$$

in 1D,

$$ec{v} = -\{B_z + rac{mc}{q}(rac{cE_{wave}}{B_z^2}\xi_r)\}^{-1}(E_{wave} - rac{\mu}{q}\xi_r + rac{mc^2}{q}rac{E_{wave}^2\xi_r}{B_z^3})ec{e_ heta}$$

$$E_{wave} = E_A \sin\{m2\pi(rac{t}{T} - rac{R_0 heta}{\lambda}) + rac{\pi}{2}\}$$

S0,

$$\vec{v} = - \big\{ B_z + \frac{mc}{q} \big(\frac{cE_A \sin\{m2\pi(\frac{t}{T} - \frac{R_0\theta}{\lambda}) + \frac{\pi}{2}\}}{B_z^2} \xi_r \big) \big\}^{-1} \big(E_A \sin\{m2\pi(\frac{t}{T} - \frac{R_0\theta}{\lambda}) + \frac{\pi}{2}\} - \frac{\mu}{q} \xi_r + \frac{mc^2}{q} \frac{(E_A \sin\{m2\pi(\frac{t}{T} - \frac{R_0\theta}{\lambda}) + \frac{\pi}{2}\})^2 \xi_r}{B_z^3} \big) \vec{e_\theta}$$