Guiding Center Equation

reference:: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

cgs gaussian units

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$ec{v} = c rac{ec{D}}{B_{\parallel}^*} imes ec{b}$$

$$ec{D}=ec{E}-rac{\mu}{q}
ablaec{B}-rac{m}{q}(rac{\partialec{v_E}}{\partial t}+
ablarac{v_E^2}{2})$$

$$ec{v_E} = rac{cec{E}}{B} imes ec{b}$$

$$ec{B^*} = ec{B} + rac{mc}{a} (
abla imes ec{v_E})$$

$$B_{\parallel}^* = \vec{B^*} \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$ec{B}(ec{x})=rac{\mu_0}{4\pi}rac{3(ec{m_E}\cdotec{\hat{x}})ec{\hat{x}}-ec{m_E}}{r^3}$$

$$ec{\hat{x}} = rac{ec{x}}{|x|}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$abla f = rac{\partial f}{\partial r} ec{e_r} + rac{1}{r} rac{\partial f}{\partial heta} ec{e_ heta} + rac{\partial f}{\partial z} ec{e_z}$$

$$abla \cdot ec{A} = rac{1}{r}(rac{\partial (rA_r)}{\partial r} + rac{\partial A_ heta}{\partial heta} + rrac{\partial A_z}{\partial z})$$

$$abla imes ec{A} = (rac{1}{r}rac{\partial A_z}{\partial heta} - rac{\partial A_ heta}{\partial z})ec{e_r} + (rac{\partial A_r}{\partial z} + rac{\partial A_z}{\partial r})ec{e_ heta} + rac{1}{r}(rac{\partial (rA_ heta)}{\partial r} - rac{\partial A_r}{\partial heta})ec{e_z}$$

Toroidal mode wave

ULF waves

$$ec{B}_{wave} = ec{e_{ heta}} B_A sin(\omega(t-rac{r heta}{v_a}))$$

$$=ec{e_{ heta}}B_{A}sin(m\omega_{d}(t-rac{r heta}{v_{c}}))$$

$$=ec{e_{ heta}}B_{A}sin(m2\pi(rac{t}{T}-rac{r heta}{\lambda}))$$

$$ec{E}_{wave} = ec{e_r} E_A sin(m2\pi(rac{t}{T} - rac{r heta}{\lambda}) + rac{\pi}{2})$$

$$E_{wave} = E_{A} sin(m2\pi(rac{t}{T}-rac{r heta}{\lambda})+rac{\pi}{2})$$

$$ec{E}_{wave} = ec{e_r} E_{wave}$$

$$ec{B_0} = rac{\mu_0}{4\pi} rac{3(ec{m} \cdot \hat{ec{x}}) \hat{ec{x}} - ec{m}}{|ec{x}|^3}$$

$$ec{m}=mec{e_z}$$

$$ec{B_0}=rac{\mu_0}{4\pi}rac{3mzec{\hat{x}}-mec{e_z}}{|ec{x}|^3}$$

$$\vec{E_0} = \vec{0}$$

$$ec{E} = ec{E}_0 + ec{E}_{wave} = ec{E}_{wave}$$

$$ec{B} = ec{B_0} + ec{B}_{wave}$$

Apply to the fomula

$$ec{v_E} = rac{ec{cE}}{B} imes ec{b}$$

$$=rac{cec{E}_{wave}}{|B|^2} imesec{B}$$

$$ec{B^*} = ec{B} + rac{mc}{a} (
abla imes ec{v_E})$$

$$=ec{B}+rac{mc}{q}(
abla imes(rac{cec{E}_{wave}}{|B|^2} imesec{B}))$$

$$=ec{B}+rac{mc}{q}((
abla\cdotec{B})rac{cec{E}_{wave}}{|B|^2}-(
abla\cdotrac{cec{E}_{wave}}{|B|^2})ec{B})$$

$$ec{D}=ec{E}-rac{\mu}{q}
ablaec{B}-rac{m}{q}(rac{\partialec{v_E}}{\partial t}+
ablarac{v_E^2}{2})$$

$$=ec{E}_{wave}-rac{\mu}{q}
ablaec{B}-rac{m}{q}ig(rac{\partial}{\partial t}ig(rac{cec{E}_{wave}}{|B|^2} imesec{B}ig)+rac{1}{2}
abla|rac{cec{E}_{wave}}{|B|^2} imesec{B}|^2ig)$$

Assumption

•
$$\vec{B}_{wave} << \vec{B}_0$$

•
$$\frac{\partial B_z}{\partial z} = 0$$

•
$$\frac{\partial \tilde{B}_r}{\partial z} = 0$$

•
$$\vec{B} = \vec{B_0}$$

•
$$\frac{\partial \vec{B}}{\partial t} = \vec{0}$$

•
$$\vec{B} = (B_r, B_\theta, B_z) = (0, 0, B_z)$$

•
$$\frac{\partial \vec{B}}{\partial r} = (0,0,\xi_r)$$

•
$$\frac{\partial \vec{B}}{\partial \theta} = (0,0,0)$$

•
$$\frac{\partial \vec{B}}{\partial z} = (0,0,0)$$

•
$$\vec{b} = \vec{e}$$

$$B^* = \vec{B} + rac{mc}{a}((
abla \cdot \vec{B}) rac{c\vec{E}_{wave}}{|B|^2} - (
abla \cdot rac{c\vec{E}_{wave}}{|B|^2})\vec{B})$$

$$=B_zec{e_z}+rac{mc}{q}(-(
abla\cdotrac{cec{E}_{wave}}{B_z^2})B_zec{e_z})$$

$$=B_zec{e_z}+rac{mc}{q}(-(rac{\partial}{\partial r}rac{cE_{wave}}{B_z^2})B_zec{e_z})$$

$$=B_zec{e_z}+rac{mc}{q}(-c(rac{1}{B_z^2}rac{\partial E_{wave}}{\partial r}-E_{wave}rac{2B_z\xi_r}{B_z^4})B_zec{e_z})$$

$$=B_zec{e_z}+rac{mc}{q}(c(-rac{1}{B_z}rac{\partial E_{wave}}{\partial r}+E_{wave}rac{2\xi_r}{B_z^2})ec{e_z})$$

$$=(B_z+rac{mc^2}{q}(-rac{1}{B_z}rac{\partial E_{wave}}{\partial r}+E_{wave}rac{2\xi_r}{B_z^2}))ec{e_z}$$

$$B_{\parallel}^* = B_z + rac{mc^2}{g}(-rac{1}{B_z}rac{\partial E_{wave}}{\partial r} + E_{wave}rac{2\xi_r}{B^2})$$

$$ec{D} = ec{E}_{wave} - rac{\mu}{q}
abla ec{B} - rac{m}{q} (c rac{\partial ec{E}_{wave}}{\partial t} imes rac{ec{B}}{|B|^2} + rac{1}{2}
abla |c rac{ec{E}_{wave}}{|B|^2} imes ec{B}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(rac{\partial(ec{e_r}E_{wave})}{\partial t} imesrac{ec{e_z}}{B_z}+rac{c}{2}
abla|rac{ec{e_r}E_{wave}}{B_z^2} imes B_zec{e_z}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+rac{c}{2}
abla|-rac{E_{wave}}{B_z^2}B_zec{e_ heta}|^2)$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+rac{c}{2}
abla((rac{E_{wave}}{B_z^2}B_z)^2))$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+crac{E_{wave}}{B_z}
abla(rac{E_{wave}}{B_z}))$$

$$=ec{e_r}E_{wave}-rac{\mu}{q}\xi_rec{e_r}-rac{mc}{q}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+crac{E_{wave}}{B_z}rac{E_{wave}
abla B_z^2+B_z
abla E_{wave}}{B_z^2})$$

$$=ec{e_r}E_{wave}-rac{\mu}{a}\xi_rec{e_r}-rac{mc}{a}(-rac{\partial E_{wave}}{\partial t}rac{1}{B_z}ec{e_ heta}+crac{E_{wave}}{B_z}rac{E_{wave}\xi_rec{e_r}}{B^2}+crac{E_{wave}}{B_z}rac{
abla E_{wave}}{B_z})$$

$$=\vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + c\frac{E_{wave}^2\xi_r}{B_z^3}\vec{e_r} + c\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{\partial \theta}\vec{e_\theta} + c\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{\partial r}\vec{e_r})$$

$$(E_{wave} = E_{wave}(\theta))$$

$$=(E_{wave}-rac{\mu}{q}\xi_r-rac{mc^2}{q}rac{E_{wave}^2\xi_r}{B_z^2}-crac{E_{wave}}{B_z^2}rac{\partial E_{wave}}{\partial r})ec{e_r}+rac{m}{q}(rac{\partial E_{wave}}{\partial t}rac{c}{B_z}-c^2rac{E_{wave}}{B_z^2}rac{\partial E_{wave}}{\partial heta})ec{e_ heta}$$

$$ec{v} = rac{ec{D}}{B_{\parallel}^*} imes ec{b}$$

$$\vec{v} = (B_z + \frac{mc^2}{q}(-\frac{1}{B_z}\frac{\partial E_{wave}}{\partial r} + E_{wave}\frac{2\xi_r}{B_z^2}))^{-1}(-(E_{wave} - \frac{\mu}{q}\xi_r - \frac{mc^2}{q}\frac{E_{wave}^2\xi_r}{B_z^2} - c\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{\partial r})\vec{e_\theta} + \frac{m}{q}(\frac{\partial E_{wave}}{\partial t}\frac{c}{B_z} - c^2\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{\partial \theta})\vec{e_r})$$