Guiding Center Equation

reference:: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

cgs gaussian units

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$ec{v} = c rac{ec{D}}{B_{\parallel}^*} imes ec{b}$$

$$ec{D}=ec{E}-rac{\mu}{q}
ablaec{B}-rac{m}{q}(rac{\partialec{v_E}}{\partial t}+
ablarac{v_E^2}{2})$$

$$ec{v_E} = rac{cec{E}}{B} imes ec{b}$$

$$ec{B^*} = ec{B} + rac{mc}{a} (
abla imes ec{v_E})$$

$$B_{\parallel}^* = \vec{B^*} \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$ec{B}(ec{x})=rac{\mu_0}{4\pi}rac{3(ec{m_E}\cdotec{\hat{x}})ec{\hat{x}}-ec{m_E}}{r^3}$$

$$ec{\hat{x}} = rac{ec{x}}{|x|}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$abla f = rac{\partial f}{\partial r} ec{e_r} + rac{1}{r} rac{\partial f}{\partial heta} ec{e_ heta} + rac{\partial f}{\partial z} ec{e_z}$$

$$abla \cdot ec{A} = rac{1}{r}(rac{\partial (rA_r)}{\partial r} + rac{\partial A_ heta}{\partial heta} + rrac{\partial A_z}{\partial z})$$

$$abla imes ec{A} = (rac{1}{r}rac{\partial A_z}{\partial heta} - rac{\partial A_ heta}{\partial z})ec{e_r} + (rac{\partial A_r}{\partial z} + rac{\partial A_z}{\partial r})ec{e_ heta} + rac{1}{r}(rac{\partial (rA_ heta)}{\partial r} - rac{\partial A_r}{\partial heta})ec{e_z}$$

Toroidal mode wave

ULF waves

$$ec{B}_{wave} = ec{e_{ heta}} B_A sin(\omega(t-rac{r heta}{v_a}))$$

$$=ec{e_{ heta}}B_{A}sin(m\omega_{d}(t-rac{r heta}{v_{c}}))$$

$$=ec{e_{ heta}}B_{A}sin(m2\pi(rac{t}{T}-rac{r heta}{\lambda}))$$

$$ec{E}_{wave} = ec{e_r} E_A sin(m2\pi(rac{t}{T} - rac{r heta}{\lambda}) + rac{\pi}{2})$$

$$E_{wave} = E_{A} sin(m2\pi(rac{t}{T}-rac{r heta}{\lambda})+rac{\pi}{2})$$

$$ec{E}_{wave} = ec{e_r} E_{wave}$$

$$ec{B_0}=rac{\mu_0}{4\pi}rac{3(ec{m}\cdotec{\hat{x}})ec{\hat{x}}-ec{m}}{|ec{x}|^3}$$

$$\vec{m} = m\vec{e_z}$$

$$ec{B_0}=rac{\mu_0}{4\pi}rac{3mz\hat{ec{x}}-mec{e_z}}{|ec{x}|^3}$$

$$\vec{E_0} = \vec{0}$$

$$ec{E} = ec{E_0} + ec{E}_{wave} = ec{E}_{wave}$$

$$ec{B} = ec{B_0} + ec{B}_{wave}$$

Apply to the fomula

$$ec{v_E} = rac{ec{cE}}{B} imes ec{b}$$

$$=rac{cec{E}_{wave}}{|B|^2} imesec{B}$$

$$ec{B^*} = ec{B} + rac{mc}{q} (
abla imes ec{v_E})$$

$$=ec{B}+rac{mc}{q}(
abla imes(rac{cec{E}_{wave}}{|B|^2} imesec{B}))$$

$$=ec{B}+rac{mc}{q}((ec{B}\cdot
abla)rac{cec{E}_{wave}}{|B|^2}-(rac{cec{E}_{wave}}{|B|^2}\cdot
abla)ec{B}+(
abla\cdotec{B})rac{cec{E}_{wave}}{|B|^2}-(
abla\cdotec{E}_{wave})ec{B})$$

$$ec{D}=ec{E}-rac{\mu}{q}
ablaec{B}-rac{m}{q}(rac{\partialec{v_E}}{\partial t}+
ablarac{v_E^2}{2})$$

$$= \vec{E}_{wave} - \tfrac{\mu}{q} \nabla \vec{B} - \tfrac{m}{q} \big(\tfrac{\partial}{\partial t} \big(\tfrac{c\vec{E}_{wave}}{|B|^2} \times \vec{B} \big) + \tfrac{1}{2} \nabla | \tfrac{c\vec{E}_{wave}}{|B|^2} \times \vec{B} |^2 \big)$$

Assumption

•
$$\vec{B}_{wave} << \vec{B}_0$$

•
$$\frac{\partial B_z}{\partial z} = 0$$

$$\begin{array}{l} \bullet \ \frac{\partial B_z}{\partial z} = 0 \\ \bullet \ \frac{\partial B_r}{\partial z} = 0 \end{array}$$

•
$$E_{wave} = E_{wave}(\theta)$$
)
(r is const)

SO,
•
$$\vec{B} = \vec{B_0}$$

• $\frac{\partial \vec{B}}{\partial t} = \vec{0}$
• $\vec{B} = (B_r, B_\theta, B_z) = (0, 0, B_z)$
• $\frac{\partial \vec{B}}{\partial r} = (0, 0, \xi_r)$
• $\frac{\partial \vec{B}}{\partial \theta} = (0, 0, 0)$
• $\frac{\partial \vec{B}}{\partial z} = (0, 0, 0)$
• $\vec{b} = \vec{e_z}$
• $\frac{\partial E_{wave}}{\partial r} = \frac{\partial E_{wave}}{\partial z} = 0$
 $\vec{B}^* = \vec{B} + \frac{mc}{q}((\vec{B} \cdot \nabla)\frac{c\vec{E}_{wave}\vec{e_r}}{|\vec{B}|^2} - (\frac{c\vec{E}_{wave}\vec{e_r}}{|\vec{B}|^2})$

•
$$\frac{\partial E_{wave}}{\partial r} = \frac{\partial E_{wave}}{\partial z} = 0$$

$$\begin{split} B^* &= \vec{B} + \frac{mc}{q} ((\vec{B} \cdot \nabla) \frac{c\vec{E}_{wave}}{|B|^2} - (\frac{c\vec{E}_{wave}}{|B|^2} \cdot \nabla) \vec{B} + (\nabla \cdot \vec{B}) \frac{c\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{c\vec{E}_{wave}}{|B|^2}) \vec{B}) \\ &= B_z \vec{e_z} + \frac{mc}{q} ((B_z \vec{e_z} \cdot \nabla) \frac{cE_{wave} \vec{e_r}}{B_z^2} - (\nabla \cdot \frac{cE_{wave} \vec{e_r}}{B_z^2} + \frac{cE_{wave} \vec{e_r}}{B_z^2} \cdot \nabla) B_z \vec{e_z}) \\ &= B_z \vec{e_z} + \frac{mc}{q} ((\frac{cE_{wave} \vec{e_r}}{B_z^2} \cdot \nabla) B_z \vec{e_z}) \\ &= B_z \vec{e_z} + \frac{mc}{q} (\frac{cE_{wave}}{B_z^2} \xi_r \vec{e_z}) \\ \vec{D} &= \vec{E}_{wave} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} (c \frac{\partial \vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2} \nabla |c \frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2) \end{split}$$

$$D = E_{wave} - \frac{\mu}{q} \nabla B - \frac{m}{q} \left(c \frac{\partial \omega_{uave}}{\partial t} \times \frac{\partial \omega_{$$

$$=\vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + c\frac{E_{wave}^2\xi_r}{B_z^3}\vec{e_r} + c\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{r\partial\theta}\vec{e_\theta})$$

$$(E_{wave} = E_{wave}(\theta))$$

$$= (E_{wave} - \frac{\mu}{q} \xi_r - \frac{mc^2}{q} \frac{E_{wave}^2 \xi_r}{B_z^3}) \vec{e_r} + \frac{m}{q} (\frac{\partial E_{wave}}{\partial t} \frac{c}{B_z} - c^2 \frac{E_{wave}}{B_z^2} \frac{\partial E_{wave}}{r \partial \theta}) \vec{e_\theta}$$

$$ec{v} = rac{ec{D}}{B_{\parallel}^*} imes ec{b}$$

$$\vec{v} = (B_z \vec{e_z} + \frac{mc}{q} (\frac{cE_{wave}}{B_z^2} \xi_r \vec{e_z}))^{-1} (-(E_{wave} - \frac{\mu}{q} \xi_r - \frac{mc^2}{q} \frac{E_{wave}^2 \xi_r}{B_z^3}) \vec{e_\theta} + \frac{m}{q} (\frac{\partial E_{wave}}{\partial t} \frac{c}{B_z} - c^2 \frac{E_{wave}}{B_z^2} \frac{\partial E_{wave}}{\partial t}) \vec{e_r})$$