# Guiding Center Equation

reference :: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

cgs gaussian units

Assumption

- pitch angle is 90 degree
  so,
- $v_{\parallel} = 0$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$\vec{v} = c\frac{\vec{D}}{B^*_{\parallel}} \times \vec{b}$$

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial \vec{v_E}}{\partial t} + \nabla\frac{v_E^2}{2})$$

$$\vec{v_E} = \frac{c\vec{E}}{B} \times \vec{b}$$

$$\vec{B^*} = \vec{B} + \frac{mc}{q}(\nabla \times \vec{v_E})$$

$$B^*_{\parallel} = \vec{B^*} \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{3(\vec{m_E}\cdot\vec{\hat{x}})\vec{\hat{x}} - \vec{m_E}}{x^3}$$

$$\vec{\hat{x}} = \frac{\vec{x}}{|x|}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$\nabla f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{e_z}$$

$$\nabla \cdot \vec{A} = \frac{1}{r}(\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r\frac{\partial A_z}{\partial z})$$

$$\nabla \times \vec{A} = (\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z})\vec{e_r} + (\frac{\partial A_r}{\partial z} + \frac{\partial A_z}{\partial r})\vec{e_\theta} + \frac{1}{r}(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta})\vec{e_z}$$

Toroidal mode wave

ULF waves

$$\vec{B}_{wave} = \vec{e_\theta} B_A \sin\{\omega(t - \frac{r\theta}{v_a})\}$$

$$= \vec{e_\theta} B_A \sin\{m\omega_d(t - \frac{r\theta}{v_a})\}$$

$$= \vec{e_\theta} B_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda})\}$$

$$\vec{E}_{wave} = \vec{e_r} E_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2}\}$$

$$E_{wave} = E_A \sin\{m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2}\}$$

$$\vec{E}_{wave} = \vec{e_r} E_{wave}$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{\vec{x}})\hat{\vec{x}} - \vec{m}}{|\vec{x}|^3}$$

$$\vec{m} = m\vec{e_z}$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3mz\hat{\vec{x}} - m\vec{e_z}}{|\vec{x}|^3}$$

$$\vec{E}_0 = \vec{0}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_{wave} = \vec{E}_{wave}$$

$$\vec{B} = \vec{B}_0 + \vec{B}_{wave}$$

Apply to the fomula

$$\vec{v_E} = \frac{c\vec{E}}{B} \times \vec{b}$$

$$= \frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B}$$

$$\vec{B^*} = \vec{B} + \frac{mc}{q}(\nabla \times \vec{v_E})$$

$$= \vec{B} + \frac{mc}{q}\{\nabla \times (\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B})\}$$

$$= \vec{B} + \frac{mc}{q}\{(\vec{B} \cdot \nabla)\frac{c\vec{E}_{wave}}{|B|^2} - (\frac{c\vec{E}_{wave}}{|B|^2} \cdot \nabla)\vec{B} + (\nabla \cdot \vec{B})\frac{c\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{c\vec{E}_{wave}}{|B|^2})\vec{B}\}$$

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial\vec{v_E}}{\partial t} + \nabla\frac{v_E^2}{2})$$

$$= \vec{E}_{wave} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}\{\frac{\partial}{\partial t}(\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B}) + \frac{1}{2}\nabla|\frac{c\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2\}$$

Assumption

- $\vec{B}_{wave} << \vec{B}_0$
- $\frac{\partial B_z}{\partial z} = 0$
- $\frac{\partial B_r}{\partial z} = 0$
- $E_{wave} = E_{wave}(\theta))$

  ($r$ is const::$r = R_0$)

  so,
- $\vec{B} = \vec{B}_0$
- $\frac{\partial \vec{B}}{\partial t} = \vec{0}$
- $\vec{B} = (B_r, B_\theta, B_z) = (0, 0, B_z)$
- $\frac{\partial \vec{B}}{\partial r} = (0, 0, \xi_r)$
- $\frac{\partial \vec{B}}{\partial \theta} = (0, 0, 0)$
- $\frac{\partial \vec{B}}{\partial z} = (0, 0, 0)$
- $\vec{b} = \vec{e_z}$
- $\frac{\partial E_{wave}}{\partial r} = \frac{\partial E_{wave}}{\partial z} = 0$

$\vec{B^*} = \vec{B} + \frac{mc}{q}\{(\vec{B} \cdot \nabla)\frac{c\vec{E}_{wave}}{|B|^2} - (\frac{c\vec{E}_{wave}}{|B|^2} \cdot \nabla)\vec{B} + (\nabla \cdot \vec{B})\frac{c\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{c\vec{E}_{wave}}{|B|^2})\vec{B}\}$

$= B_z\vec{e_z} + \frac{mc}{q}\{(B_z\vec{e_z} \cdot \nabla)\frac{cE_{wave}\vec{e_r}}{B_z^2} - (\nabla \cdot \frac{cE_{wave}\vec{e_r}}{B_z^2} + \frac{cE_{wave}\vec{e_r}}{B_z^2} \cdot \nabla)B_z\vec{e_z}\}$

$= B_z\vec{e_z} + \frac{mc}{q}\{(\frac{cE_{wave}\vec{e_r}}{B_z^2} \cdot \nabla)B_z\vec{e_z}\}$

$= B_z\vec{e_z} + \frac{mc}{q}(\frac{cE_{wave}}{B_z^2}\xi_r\vec{e_z})$

$B^*_\parallel = B^* \cdot \vec{b} = B_z + \frac{mc}{q}(\frac{cE_{wave}}{B_z^2}\xi_r)$

$\vec{D} = \vec{E}_{wave} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(c\frac{\partial\vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2}\nabla|c\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2)$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{m}{q}(c\frac{\partial(\vec{e_r}\,E_{wave})}{\partial t} \times \frac{\vec{e_z}}{B_z} + \frac{c^2}{2}\nabla|\frac{\vec{e_r}\,E_{wave}}{B_z^2} \times B_z\vec{e_z}|^2)$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(\frac{\partial(\vec{e_r}\,E_{wave})}{\partial t} \times \frac{\vec{e_z}}{B_z} + \frac{c}{2}\nabla|\frac{\vec{e_r}\,E_{wave}}{B_z^2} \times B_z\vec{e_z}|^2)$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + \frac{c}{2}\nabla| - \frac{E_{wave}}{B_z^2}B_z\vec{e_\theta}|^2)$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}\{-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + \frac{c}{2}\nabla((\frac{E_{wave}}{B_z^2}B_z)^2)\}$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}\{-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + c\frac{E_{wave}}{B_z}\nabla(\frac{E_{wave}}{B_z})\}$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} + c\frac{E_{wave}}{B_z}\frac{-E_{wave}\nabla B_z + B_z\nabla E_{wave}}{B_z^2})$

$= \vec{e_r}E_{wave} - \frac{\mu}{q}\xi_r\vec{e_r} - \frac{mc}{q}(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} - c\frac{E_{wave}}{B_z}\frac{E_{wave}\xi_r\vec{e_r}}{B_z^2} + c\frac{E_{wave}}{B_z}\frac{\nabla E_{wave}}{B_z})$

$$= \vec{e_r} E_{wave} - \frac{\mu}{q}\xi_r \vec{e_r} - \frac{mc}{q}\left(-\frac{\partial E_{wave}}{\partial t}\frac{1}{B_z}\vec{e_\theta} - c\frac{E_{wave}^2 \xi_r}{B_z^3}\vec{e_r} + c\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{R_0 \partial\theta}\vec{e_\theta}\right)$$

($E_{wave} = E_{wave}(\theta)$)

$$= \left(E_{wave} - \frac{\mu}{q}\xi_r + \frac{mc^2}{q}\frac{E_{wave}^2 \xi_r}{B_z^3}\right)\vec{e_r} + \frac{m}{q}\left(\frac{\partial E_{wave}}{\partial t}\frac{c}{B_z} - c^2\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{R_0 \partial\theta}\right)\vec{e_\theta}$$

$$\vec{v} = \frac{\vec{D}}{B_\parallel^*} \times \vec{b}$$

$$\vec{v} = \left\{B_z + \frac{mc}{q}\left(\frac{cE_{wave}}{B_z^2}\xi_r\right)\right\}^{-1}\left\{-\left(E_{wave} - \frac{\mu}{q}\xi_r + \frac{mc^2}{q}\frac{E_{wave}^2 \xi_r}{B_z^3}\right)\vec{e_\theta} + \frac{m}{q}\left(\frac{\partial E_{wave}}{\partial t}\frac{c}{B_z} - c^2\frac{E_{wave}}{B_z^2}\frac{\partial E_{wave}}{R_0 \partial\theta}\right)\vec{e_r}\right\}$$

in 1D,

$$\vec{v} = -\left\{B_z + \frac{mc}{q}\left(\frac{cE_{wave}}{B_z^2}\xi_r\right)\right\}^{-1}\left(E_{wave} - \frac{\mu}{q}\xi_r + \frac{mc^2}{q}\frac{E_{wave}^2 \xi_r}{B_z^3}\right)\vec{e_\theta}$$

$$E_{wave} = E_A \sin\left\{m2\pi\left(\frac{t}{T} - \frac{R_0\theta}{\lambda}\right) + \frac{\pi}{2}\right\}$$

so,

$$\vec{v} = -\left\{B_z + \frac{mc}{q}\left(\frac{cE_A \sin\{m2\pi(\frac{t}{T} - \frac{R_0\theta}{\lambda}) + \frac{\pi}{2}\}}{B_z^2}\xi_r\right)\right\}^{-1}\left(E_A \sin\left\{m2\pi\left(\frac{t}{T} - \frac{R_0\theta}{\lambda}\right) + \frac{\pi}{2}\right\} - \frac{\mu}{q}\xi_r + \frac{mc^2}{q}\frac{(E_A \sin\{m2\pi(\frac{t}{T} - \frac{R_0\theta}{\lambda}) + \frac{\pi}{2}\})^2 \xi_r}{B_z^3}\right)\vec{e_\theta}$$