Guiding Center Equation

reference :: https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2010JA015682

CGS

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$$

$$\vec{v} = \frac{\vec{D}}{B_{\parallel}^{*}} \times \vec{b}$$

$$\vec{D} = \vec{E} - \frac{\mu}{q} \nabla \vec{B} - \frac{m}{q} \left(\frac{\partial \vec{v_E}}{\partial t} + \nabla \frac{v_E^2}{2} \right)$$

$$\vec{v_E} = \frac{c}{B} \vec{E} \times \vec{b}$$

$$\vec{B^*} = \vec{B} + \frac{m}{q} (\nabla \times \vec{v_E})$$

$$B_{\parallel}^{*} = \vec{B^*} \cdot \vec{b}$$

$$B = |\vec{B}|$$

$$\vec{B}(\vec{m}, \vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \vec{\hat{r}})\vec{\hat{r}} - \vec{m}}{r^3}$$

$$\vec{\hat{r}} = \frac{\vec{r}}{r}$$

Below We use cylindrical coordinates

$$\vec{x} = (r, \theta, z)$$

$$\nabla f = \frac{\partial f}{\partial r} \vec{e_r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e_\theta} + \frac{\partial f}{\partial z} \vec{e_z}$$

$$\nabla \cdot \vec{A} = \frac{1}{r} \left(\frac{\partial (r A_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + r \frac{\partial A_z}{\partial z} \right)$$

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \vec{e_r} + \left(\frac{\partial A_r}{\partial z} + \frac{\partial A_z}{\partial r} \right) \vec{e_\theta} + \frac{1}{r} \left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \vec{e_z}$$

Toroidal mode wave

ULF waves

$$\vec{B}_{wave} = \vec{e_\theta} B_A sin(\omega(t - \frac{r\theta}{v}))$$

$$= \vec{e_\theta} B_A sin(m\omega_d(t - \frac{r\theta}{v}))$$

$$= \vec{e_\theta} B_A sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}))$$

$$\vec{E}_{wave} = \vec{e_r} E_A sin(m2\pi(\frac{t}{T} - \frac{r\theta}{\lambda}) + \frac{\pi}{2})$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{\vec{x}})\hat{\vec{x}} - \vec{m}}{|\vec{x}|^3}$$

$$\vec{m} = m\vec{e_z}$$

$$\vec{B}_0 = \frac{\mu_0}{4\pi} \frac{3mz\hat{\vec{x}} - m\vec{e_z}}{|\vec{x}|^3}$$

$$\vec{E}_0 = \vec{0}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_{wave} = \vec{E}_{wave}$$

$$\vec{B} = \vec{B}_0 + \vec{B}_{wave}$$

Apply to the fomula

$$\vec{v_E} = \frac{\vec{E}}{B} \times \vec{b}$$

$$= \frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}$$

$$\vec{B}* = \vec{B} + \frac{m}{q}(\nabla \times \vec{v_E})$$

$$= \vec{B} + \frac{m}{q}(\nabla \times (\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}))$$

$$= \vec{B} + \frac{m}{q}((\nabla \cdot \vec{B})\frac{\vec{E}_{wave}}{|B|^2} - (\nabla \cdot \frac{\vec{E}_{wave}}{|B|^2})\vec{B})$$

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial\vec{v_E}}{\partial t} + \nabla\frac{v_E^2}{2})$$

$$= \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial}{\partial t}(\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}) + \frac{1}{2}\nabla|\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2)$$

Assumption $\vec{B}_{wave} << \vec{B}_0$

$$\vec{B} = \vec{B}_0$$

$\vec{b} = \vec{e_z}$ (only in the case of no spatial differentiation)

$$\vec{D} = \vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}(\frac{\partial\vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2}\nabla|\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}|^2)$$

$$\vec{v} = \frac{\vec{D}}{B_{\parallel}^*} \times \vec{b}$$

$$\vec{v} = \frac{\vec{E} - \frac{\mu}{q}\nabla\vec{B} - \frac{m}{q}\left(\frac{\partial\vec{E}_{wave}}{\partial t} \times \frac{\vec{B}}{|B|^2} + \frac{1}{2}\nabla\left|\frac{\vec{E}_{wave}}{|B|^2} \times \vec{B}\right|^2\right) + +}{(\vec{B} + \frac{m}{q}((\nabla\cdot\vec{B})\frac{\vec{E}_{wave}}{|B|^2} - (\nabla\cdot\frac{\vec{E}_{wave}}{|B|^2})\vec{B})\cdot\vec{e_z}} \times \vec{e_z}$$