Heap Sort

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Why sorting

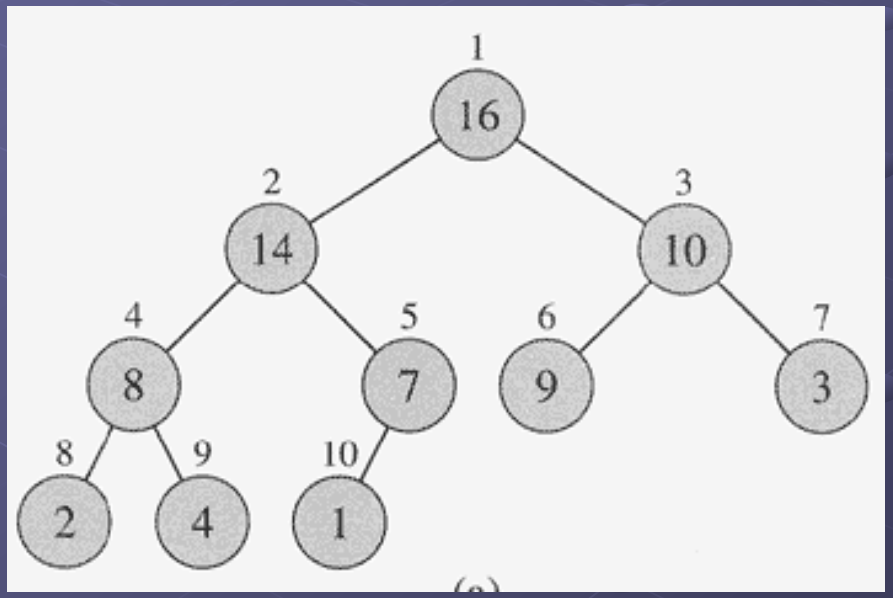
- Sometimes the need to sort information is inherent in an application.
- Algorithms often use sorting as a key subroutine.
- There is a wide variety of sorting algorithms, and they use rich set of techniques.
- Sorting problem has a nontrivial lower bound.
- Many engineering issues come to the fore when implementing sorting algorithms.

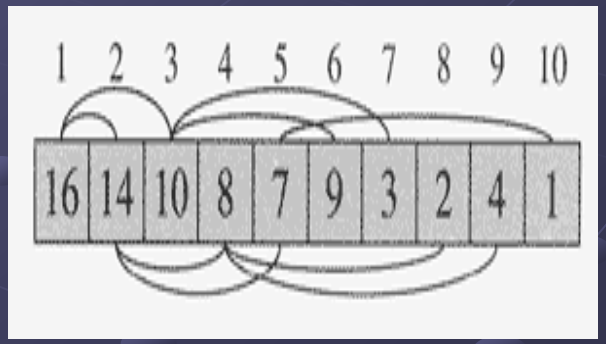
Heapsort

- Heapsort combines the better attributes of sorting algorithms:
  - Like merge sort, it's running time is O(*n lg n*).
  - Like insertion sort, it sorts in place.
- Heapsort introduce another algorithm design technique:
  - The use of a data structure to manage information during the execution of the algorithm.

Heaps[1]

- **Heap:**
  - **An array object that can be viewed as a nearly complete binary tree.**





Heaps[2]

- Max-heap (Commonly used in heapsort)
  - The *max-heap property* is that for every node *i* other than the root,

    $$A[\text{PARENT}(i)] \geq A[i]$$

  - The largest element in a max-heap is stored at the root.
- Min-heap
  - The *min-heap property* is that for every node *i* other than the root,

    $$A[\text{PARENT}(i)] \leq A[i]$$

  - The smallest element in a min-heap is stored at the root.
  - Commonly used in priority queues.

Heaps[3]

- **Basic procedures:**
  - **MAX-HEAPIFY: Maintaining the max-heap property, runs in $O(lg\ n)$ time.**
  - **BUILD-MAX-HEAP: Produce a max-heap from an unordered input array, runs in linear time.**
  - **HEAPSORT: Sorts an array in place, runs in $O(n\ lg\ n)$ time.**
  - **MAX-HEAP-INSERT, HEAP-EXTRACT-MAX, HEAP-INCREASE_KEY, HEAP-MAXIMUM: Allow the heap to be used as a priority queue, run in $O(lg\ n)$ time.**

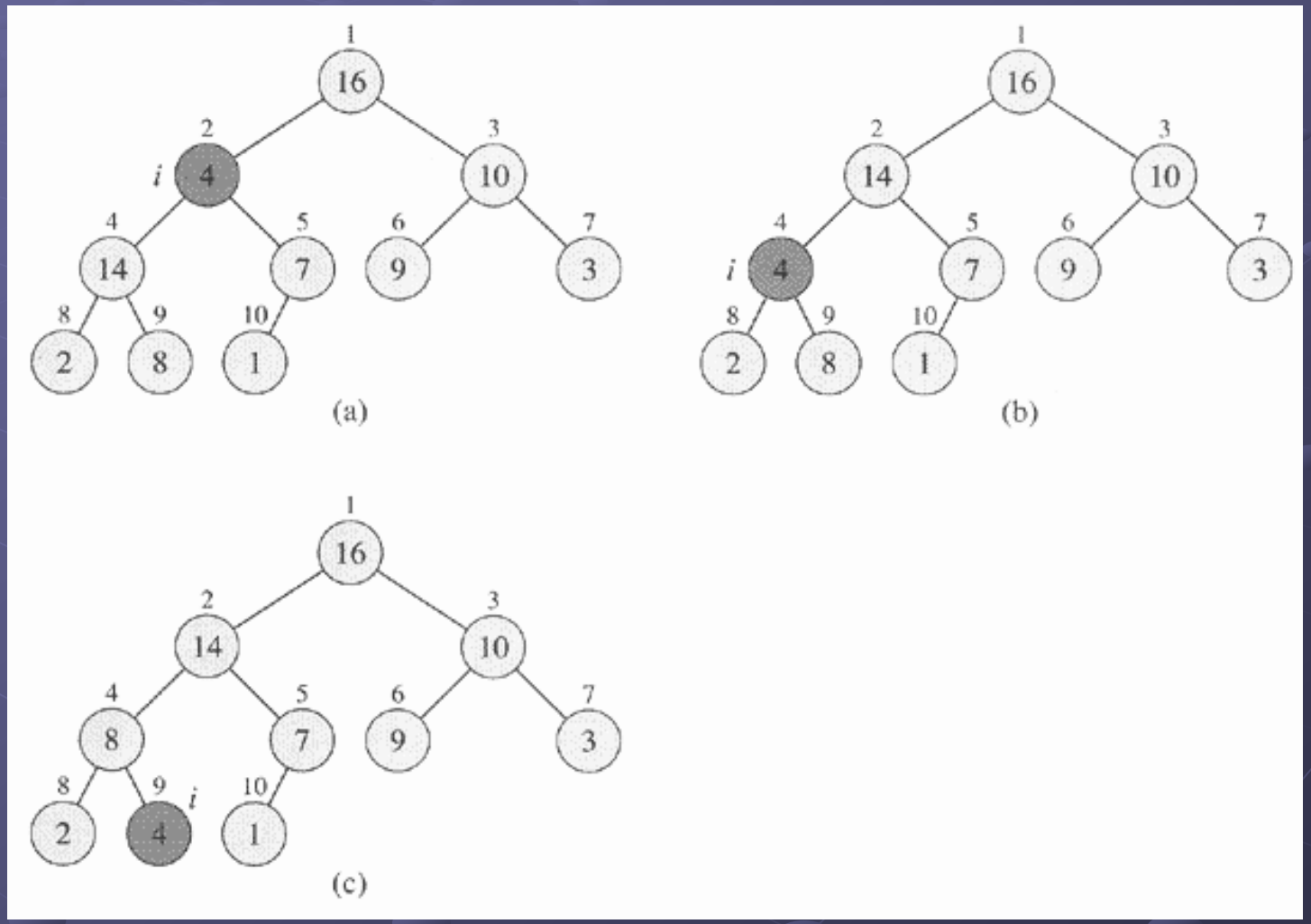
Maintaining the Heap Property[1]

☞MAX-HEAPIFY

- **Let the value at *A*[*i*] "float down" in the max-heap so that the subtree rooted at index *i* become a max-heap.**

```
MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
 2  r = RIGHT(i)
 3  if l ≤ A.heap-size and A[l] > A[i]
 4       largest = l
 5  else largest = i
 6  if r ≤ A.heap-size and A[r] > A[largest]
 7       largest = r
 8  if largest ≠ i
 9       exchange A[i] with A[largest]
10       MAX-HEAPIFY(A, largest)
```

Maintaining the Heap Property[2]



Maintaining the Heap Property[3]

- **The running time of MAX-HEAPIFY**
  - **Worst case: The children's subtree each have size at most 2*n*/3 when the last row of the tree is exactly half full.**

$T(n) \le T(2n/3) + \Theta(1)$ (Case 2 of Master theorem)

$\therefore\ T(n) = O(lg\ n)$

Building a Heap[1]

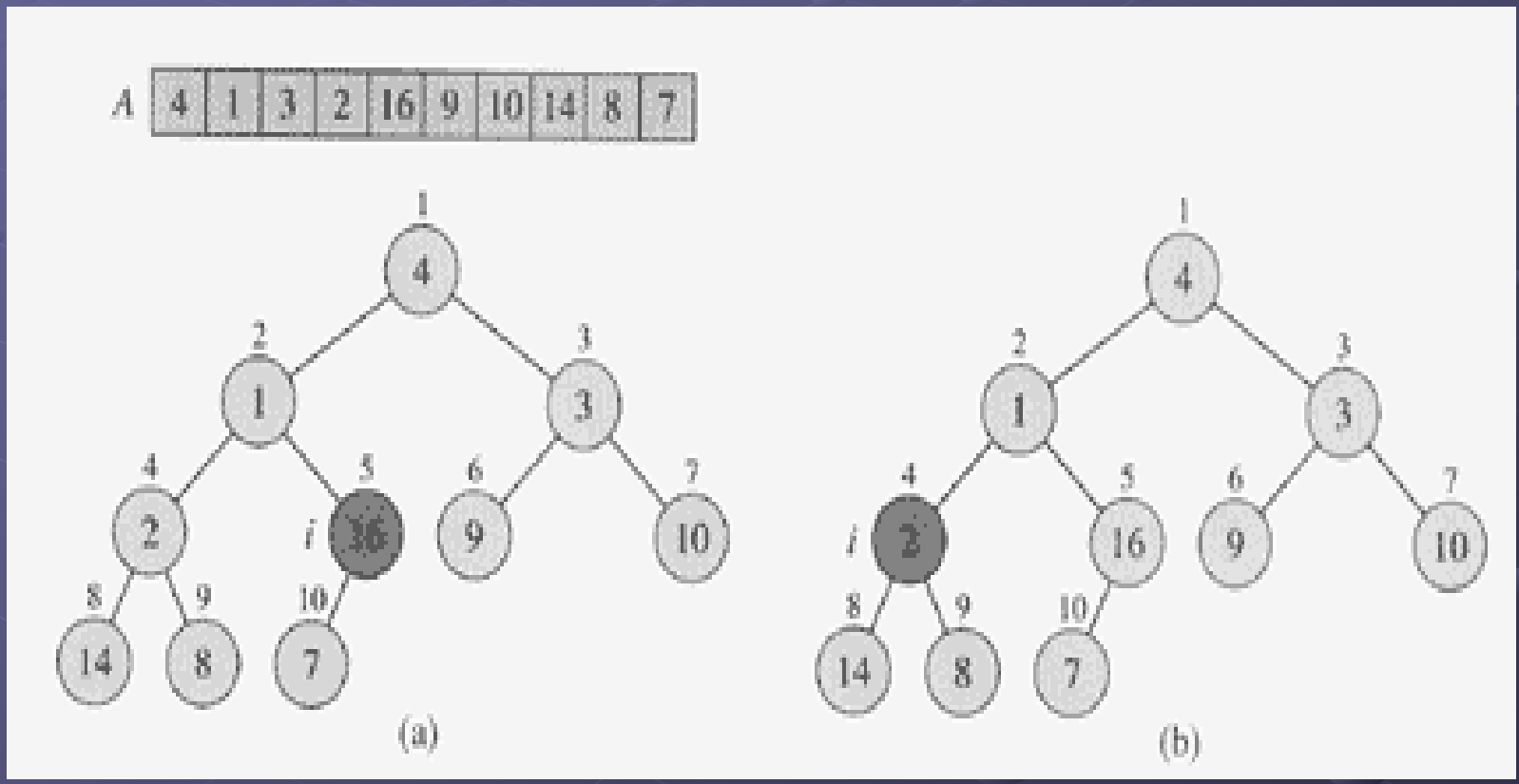
- **Algorithm**

```
BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

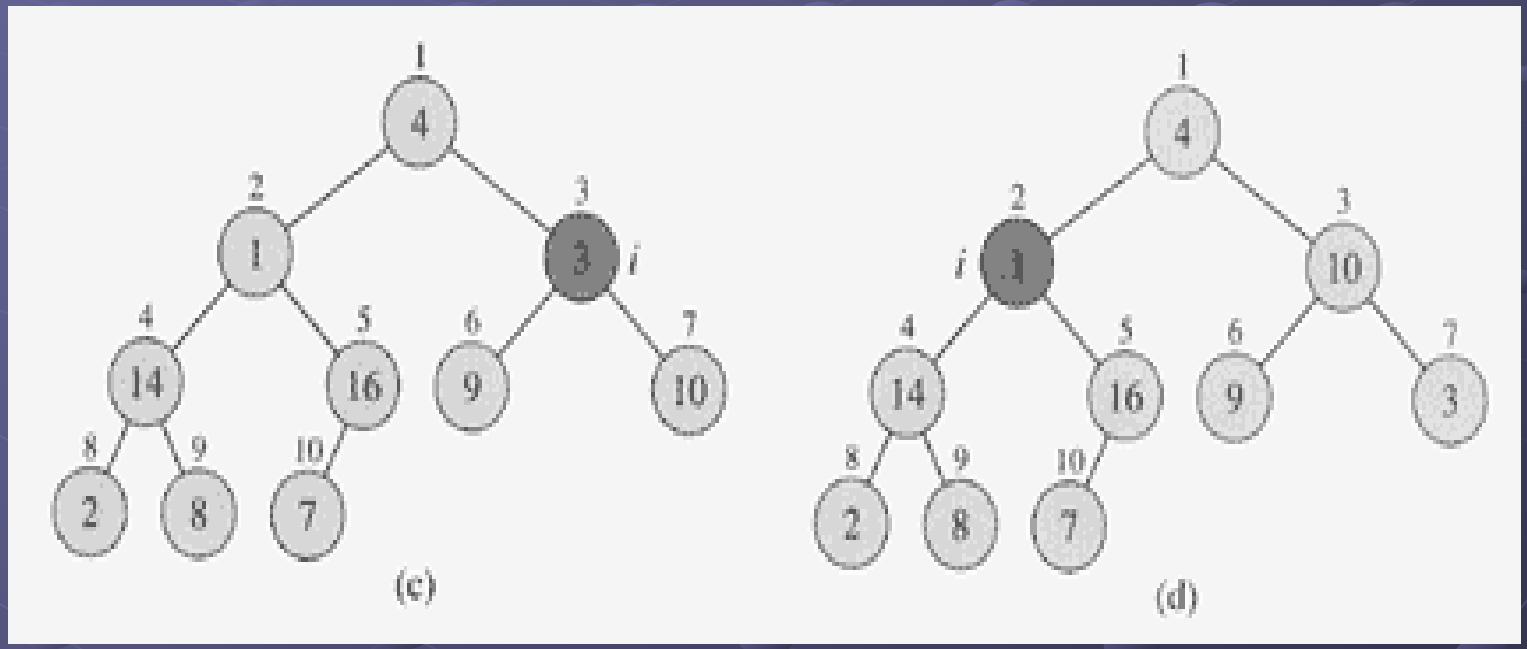
- **Correctness**
  - **Loop invariant**

    At the start of each iteration of the for loop, each node $i + 1, i + 2, \ldots, n$ is the root of a max-heap.

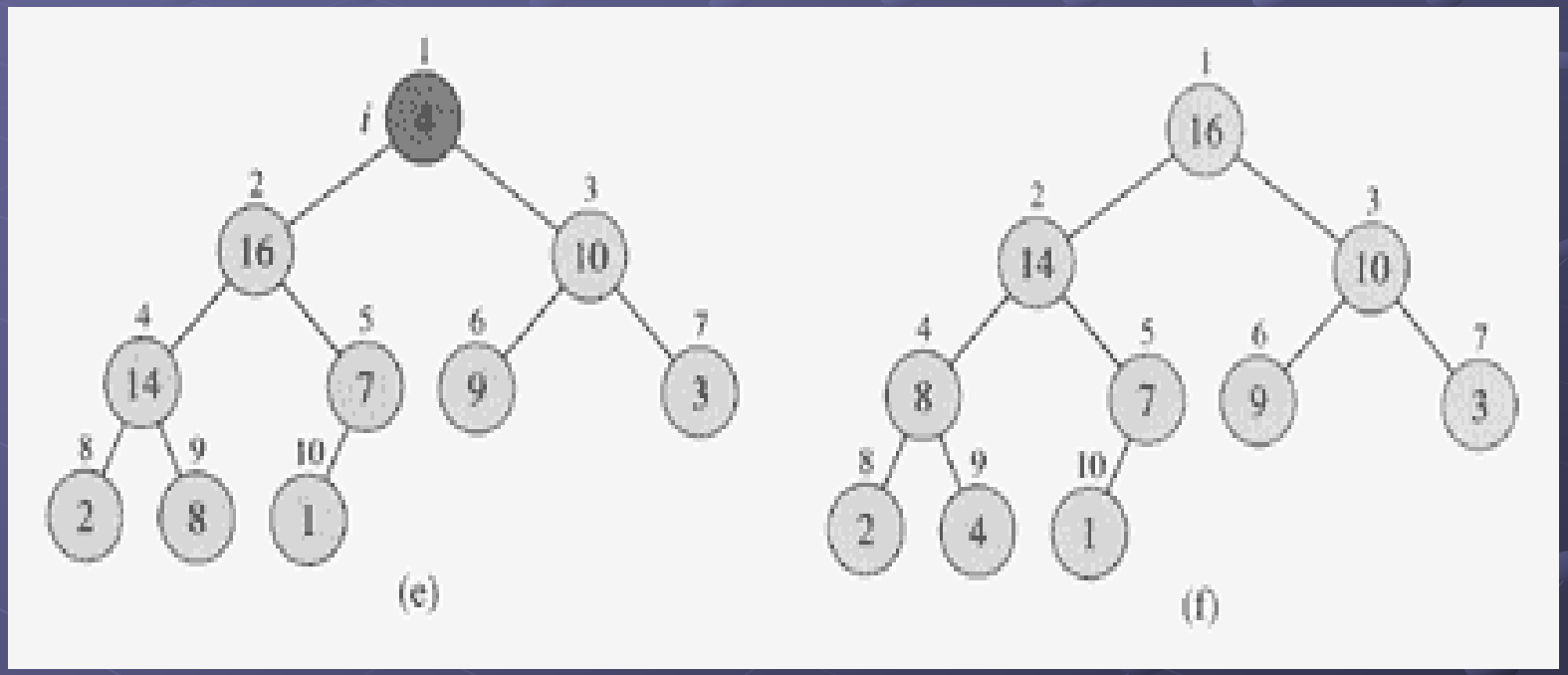
Building a Heap[2]



Building a Heap[3]



Building a Heap[4]



Building a Heap[5]

- Initialization: Prior to the first iteration of the loop, $i = \lfloor n/2 \rfloor$.
- Maintenance
  - The children of node i are both roots of max-heaps.
  - MAX-HEAPIFY(A, i) makes node i a max-heap root.
- Termination: At termination $i = 0$.

☞ Running time: $O(n \lg n)$

- Each call to MAX-HEAPIFY cost $O(\lg n)$ time.
- There are $O(n)$ such call.
- Not asymptotic tight.

Building a Heap[6]

- Tighter analysis
  - An n-element heap has height $\lfloor lg\ n \rfloor$ and at most $\lceil n/2^{h+1} \rceil$ nodes of any height h.
  - The time required by MAX-HEAPIFY when called on a node of height h is $O(h)$.

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\begin{aligned} \sum_{h=0}^{\infty} \frac{h}{2^h} &= \frac{1/2}{(1-1/2)^2} \\ &= 2 . \end{aligned}$$

$$\begin{aligned} O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) &= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) \\ &= O(n) . \end{aligned}$$

Deriving the equation in p.159

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad for\ |x| < 1$$

Substituting $x = \frac{1}{2}$ into the above equation.

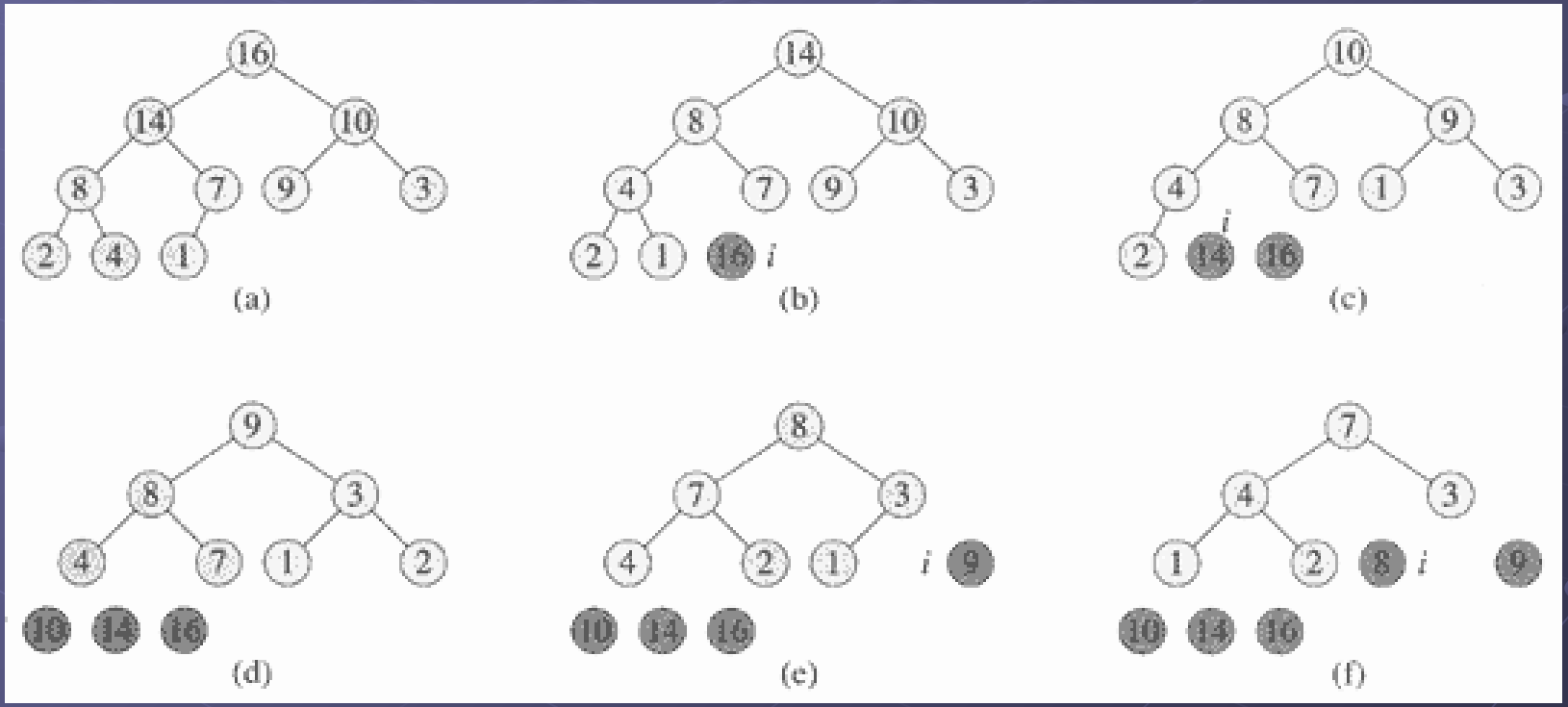
The Heapsort Algorithm[1]

- **Algorithm**

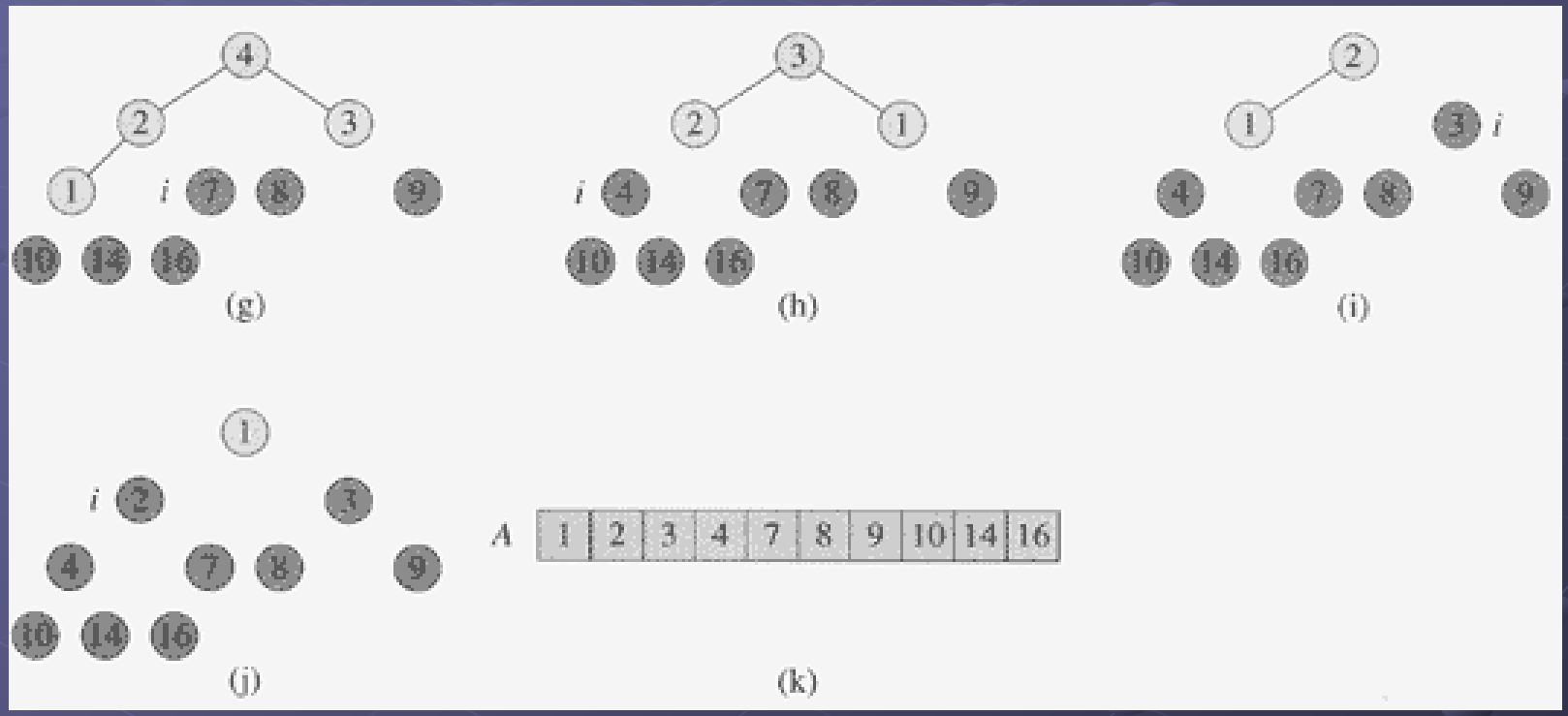
```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```

- **Running time: $O(n \lg n)$.**

The Heapsort Algorithm[2]



The Heapsort Algorithm[3]



Priority Queues[1]

- **Priority queue:**
  - A data structure for maintaining a set S of elements, each with an associated value called a *key*.
- **Max-priority queue**
  - INSERT(S, x)
  - MAXIMUM(S)
  - EXTRACT-MAX(S)
  - INCREASE-KEY(S, x, k)
- **Min-priority queue**

Priority Queues[2]

- **HEAP-MAXIMUM(*A*)** **Θ(1)**

```
1  return A[1]
```

- **HEAP-EXTRACT-MAX(*A*)** **O(*lg n*)**

```
1  if  heap-size < 1
2     then error "heap underflow"
3  max = A[1]
4  A[1] =A[A.heap-size]
5  A.heap-size = A.heap-size - 1
6  MAX-HEAPIFY (A, 1)
7  return max
```

Priority Queues[3]

- **HEAP-INCREASE-KEY (*A*, *i*, *key*) O(*lg n*)**

```
1  if  key < A[i]
2    then error "new key is smaller than current key"
3  A[i] = key
4  while i > 1 and A[PARENT(i)] < A[i]
5    exchange A[i] with A[PARENT(i)]
6    i = PARENT(i)
```

- **MAX-HEAP-INSERT(*A*, *key*) O(*lg n*)**

```
1  A.heap-size = A.heap-size + 1
2  A[A.heap-size] = -∞
3  HEAP-INCREASE-KEY (A, A.heap-size, key)
```

Priority Queues[4]