Quiz 3

Numerical Method, 2021/5/27

ID:____________ Name:_______________

1. (10) The nth-degree Bezier curve determined by n+1 control points is given by

$$P(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i p_i,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

and u is in [0, 1]. For cubic Bezier curve, $P(u)=[x(u), y(u)]^T$ is represented by four control points $p_i(u)=[x_i, y_i]^T$, i=0, 1, 2, 3, with the following form:

$$x(u) = (1-u)^3 x_0 + 3(1-u)^2 u x_1 + 3(1-u)u^2 x_2 + u^3 x_3,$$
$$y(u) = (1-u)^3 y_0 + 3(1-u)^2 u y_1 + 3(1-u)u^2 y_2 + u^3 y_3.$$

a. (4) Prove that the slops at the ends of a cubic Bezier curve are the same as the slope between the two end control points.

b. (3) Argue that the cubic Bezier curve is contained in the convex hull determined by the four control points.

c. (3) Argue that moving a control point will change the shape of whole curve.

2. (12) The Maclaurin series of e^x at $x = 0$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots + \frac{x^{n+1}}{(n+1)!}e^{\xi},\ \xi \text{ in } [0, x].$$

a. (3) If you want to use Maclaurin series to approximate e^x on [-1/2, 1/2] with a precision of 0.001, how do you truncate the Maclaurin series to meet the precision?

b. (5) Let $p(x)$ be the truncated series obtained in 1.a. Describe the steps involved in finding the 5-th degree economized polynomial $p_{econ}(x)$ for $p(x)$ on $I = [-1/2, 1/2]$.

c. (4) How do you measure the accuracy of $p_{econ}(x) \approx e^x$ on $I = [-1/2, 1/2]$?

3. (8) e^x is approximated by the truncated Maclaurin series as shown in 1.a

   a. (4) Derive the Chebyshev series of e^x and then the Chebyshev expansion (upto x^3)?

   b. (4) Describe how the error of Maclaurin series and Chebyshev expansion behave for approximating e^x, and why?

USEFUL INFORMATION:

Chebyshev polynomial

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1, \\
T_5(x) &= 16x^5 - 20x^3 + 5x, \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x, \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1, \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x, \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1.
\end{aligned}
$$

$$
\begin{aligned}
1 &= T_0, \\
x &= T_1, \\
x^2 &= \frac{1}{2}(T_0 + T_2), \\
x^3 &= \frac{1}{4}(3T_1 + T_3), \\
x^4 &= \frac{1}{8}(3T_0 + 4T_2 + T_4), \\
x^5 &= \frac{1}{16}(10T_1 + 5T_3 + T_5), \\
x^6 &= \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6), \\
x^7 &= \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7), \\
x^8 &= \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8), \\
x^9 &= \frac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9).
\end{aligned}
$$

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0; \tag{4.15}$$

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \begin{cases} 0, & n \neq 0, \\ 2\pi, & n = 0; \end{cases} \tag{4.16}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0; \tag{4.17}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m; \end{cases} \tag{4.18}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases} \tag{4.19}$$