Quiz 2 Numerical method, 2020/4/13

ID:____________Name:_______________

1. (10) Given the following system

$$Ax = \begin{bmatrix} 1 & -2 & 4 \\ 8 & -3 & 2 \\ -1 & 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

   a. (4) Solve the system by Gaussian Elimination with partial pivoting using three significant digits. (You can derive LU now for 1.b)

row1 與 row2 交換 row2 與 row3 交換

0.125

-0.125

$$\begin{bmatrix} 8 & -3 & 2 & 2 \\ 1 & -2 & 4 & 6 \\ -1 & 10 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & -1.625 & 3.75 & 5.75 \\ 0 & 9.625 & 2.25 & 4.25 \end{bmatrix} \rightarrow$$

-0.1688

$$\begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & 9.625 & 2.25 & 4.25 \\ 0 & -1.625 & 3.75 & 5.75 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & 9.625 & 2.25 & 4.25 \\ 0 & 0 & 4.1298 & 6.4674 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.125 & 1 & 0 \\ 0.125 & -0.1688 & 1 \end{bmatrix}$$

$X_1 = -0.113$

$X_2 = 0.0075$

$X_3 = 1.566$

   b. (3) Derive the LU decomposition. What is the relation between A and LU?

$A' = L * U$

$$\begin{bmatrix} 8 & -3 & 2 \\ -1 & 10 & 2 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.125 & 1 & 0 \\ 0.125 & -0.1688 & 1 \end{bmatrix} * \begin{bmatrix} 8 & -3 & 2 \\ 0 & 9.625 & 2.25 \\ 0 & 0 & 4.1298 \end{bmatrix}$$

$A' = A$(row 1 $\leftarrow\rightarrow$ row 2, row 2 $\leftarrow\rightarrow$ row 3)

   c. (3) How do you solve another system Ax=b’ with the same coefficient matrix by using LU decomposition obtained in 1.b?

$Ax = b' \rightarrow LUx = b' \rightarrow Ly = b'$ & $Ux = y$

解 $Ly = b'$ 得 y (Forward substitution)

再解 $Ux = y$ 得 x (Backward substitution)

2. (10) Given the system of equations:

$$Ax = \begin{bmatrix} 6.03 & 1.99 & 3.01 \\ 4.16 & -1.23 & 1.27 \\ -4.81 & 9.34 & 0.987 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   a. (4) Solve with a precision of four significant digits. (You can derive

LU now needed for 2.c)

沒做 row 交換

$$\begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 0 & -2.6029 & -0.8066 & 0.3101 \\ 0 & 10.9274 & 3.3881 & 1.7977 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 0 & -2.6029 & -0.8066 & 0.3101 \\ 0 & 0 & 0.0018 & 3.0996 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.6899 & 1 & 0 \\ -0.7977 & -4.1982 & 1 \end{bmatrix}$$

$X_1 = -683.3$

$X_2 = -533.7$

$X_3 = 1722$

b. (3) Is the system ill-conditioned? Why?

是 ill-conditioned，因為 LU 在 position(3,3)的值非常接近 0

c. (3) Apply two steps of iterative improvement (Residual correction) to the solution from 2.a. How do you compute the residual to avoid cancellation error?

$\bar{x} = \begin{bmatrix} -683.3 \\ -533.7 \\ 1722 \end{bmatrix}$, define : $e = x - \bar{x}$, $r = b - A\bar{x}$

$$r = b - A\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.858 \\ 0.863 \\ 1.529 \end{bmatrix} = \begin{bmatrix} 0.142 \\ 0.137 \\ -0.529 \end{bmatrix}$$

Solve: $Ae = r$, $x = \bar{x} + e$

$$e = \begin{bmatrix} 51.343 \\ 40.117 \\ -129.3792 \end{bmatrix}$$

$$x = \begin{bmatrix} -631.9 \\ -493.6 \\ 1592.6 \end{bmatrix}$$

How do you compute the residual to avoid cancellation error?

用比較高的 precision

3. (10) Given the matrix [A;b]

$$[A;b]=\begin{bmatrix} 7 & -3 & 4 & 6 \\ -3 & 2 & 6 & 2 \\ 2 & 5 & 3 & -5 \end{bmatrix}$$

a. (3) You want to solve the system with the Jacobi and Gauss-Seidel method for any starting vector. How do you arrange the system to ensure the convergence before applying the methods?

Diagonal dominant, 因此 row 2 及 row 3 交換

$$\begin{bmatrix} 7 & -3 & 4 & 6 \\ 2 & 5 & 3 & -5 \\ -3 & 2 & 6 & 2 \end{bmatrix}$$

b. (3) Following 3.a, obtain x_1 of Jacobi method using [0, 0, 0] as the starting vector x_0.

$$X_1 = \frac{1}{A_{11}} * (b_1 - A_{11} * X_2 - A_{13} * X_3) = \frac{1}{7} * 6 = \frac{6}{7}$$

$$X_2 = \frac{1}{A_{22}} * (b_2 - A_{21} * X_1 - A_{23} * X_3) = \frac{1}{5} * (-5) = -1$$

$$X_3 = \frac{1}{A_{33}} * (b_3 - A_{31} * X_1 - A_{32} * X_2) = \frac{1}{6} * 2 = \frac{1}{3}$$

c. (4) Following 3.a, obtain x_1 of Gauss-Seidel method using [0, 0, 0] as the starting vector x_0.

$$X_1 = \frac{1}{A_{11}} * (b_1 - A_{11} * X_2 - A_{13} * X_3) = \frac{1}{7} * 6 = \frac{6}{7}$$

$$X_2 = \frac{1}{A_{22}} * (b_2 - A_{21} * X_1 - A_{23} * X_3)(\text{帶入新算出的}X_1)$$

$$= \frac{1}{5} * \left(-5 - 2 * \frac{6}{7}\right) = \frac{-47}{35}$$

$$X_3 = \frac{1}{A_{33}} * (b_3 - A_{31} * X_1 - A_{32} * X_2)(\text{帶入新算出的}X_1, X_2)$$

$$= \frac{1}{6} * (2 - (-3) * \frac{6}{7} - 2 * \frac{-47}{35}) = \frac{127}{105}$$