Quiz 1 ID

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1. (6) The function $f(x) = x^2 - 0.8x + 0.15$ has two zeros at 0.3 and 0.5. Starting with [0, 0.49],

   a. (2) Apply bisection method to get next interval.

$x_0 = 0 \quad f(x_0) = 0.15$

$x_1 = 0.49 \quad f(x_1) = -0.0019$

$x_2 = \frac{x_0 + x_1}{2} = 0.245 \quad f(x_2) = 0.014$

$\because f(x_1)f(x_2) < 0$

$\therefore \underline{[0.245, 0.49]}$ #

   b. (4) Apply the secant method to get next interval.

$x_0 = 0$

$x_1 = 0.49$

$|f(x_0)| > |f(x_1)| \Rightarrow$ no swap

$x_2 = x_1 - f(x_1) \cdot \frac{x_0 - x_1}{f(x_0) - f(x_1)}$

$= 0.49 + 0.0019 \frac{-0.49}{0.1519} = 0.48387$

① next interval = $\underline{[0.48387, 0.49]}$ #

or ② $\because f(x_0)f(x_2) < 0$

$\therefore$ next interval $= \underline{[0, 0.48387]}$ #

2. (12) Given a degree 3 polynomial $f(x) = 2x^3 + x^2 - 3x - 3$,

   a. (4) One of the real roots of $P(x)$ is near 1.5. Perform Newton's method to get the next approximate x_1 using initial point x_0=1.5. You need to use synthetic division to do function evaluation for $f(1.5)$ and $f'(1.5)$.

2	1	-3	-3	1.5
	3	6	4.5	
2	4	3	1.5	

$f(1.5) = \underline{1.5}$ #

2	4	3	1.5
	3	10.5	
2	7	13.5	

$f'(1.5) = \underline{13.5}$ #

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1.5 - \frac{1.5}{13.5}$

$= \underline{1.3\overline{8}}$ #

   b. (3) For Problem 2(a), to get $f'(1.5)$ we apply synthetic division to the reduced polynomial resulted from dividing f(x) by (x-1.5). Explain why?

$f(x) = (x-1.5)Q(x) + R$

$\because$ order of $(x-1.5) = 1 \quad \therefore R$ is constant

$\Rightarrow f'(x) = (x-1.5)Q'(x) + Q(x)$

$f'(1.5) = Q(1.5)$

$Q(x) = (x-1.5)Q_1(x) + R_1$ Similarly, R_1 is constant.

$\therefore Q(1.5) = R_1$ □

c. (5) What is the rate of convergence for Newton method? Briefly describe why.

Quadratically, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$

$$g'(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$$

By Taylor expansion of $g(x)$ at root r,

$$g(x) = g(r) + g'(r)(x-r) + \frac{g''(\xi)}{2}(x-r)^2$$

$\xi \in [x_n, r]$

$\because f(r)=0 \therefore g'(r)=0$

$$\Rightarrow g(x_n) = g(r) + \frac{g''(\xi)}{2}(x_n - r)^2$$

$$\begin{aligned} e_{n+1} &= x_{n+1} - r \\ &= g(x_n) - g(r) \\ &= \frac{g''(\xi)}{2} e_n^2 \end{aligned}$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \left|\frac{g''(\xi)}{2}\right| = \left|\frac{g''(r)}{2}\right| \neq 0$$

3. (12) Given $f(x) = x^2 - 2x - 3$, with roots -1 and 3,

a. (4) Convert $f(x)=0$ to all possible forms for fixed-point iteration.

$x^2 - 2x - 3 = 0 \Rightarrow$

① $2x = x^2 - 3$ $\quad x = \frac{x^2-3}{2}$

② $x(x-2) = 3$ $\quad x = \frac{3}{x-2}$

③ $x^2 = 2x+3$ $\quad x = \sqrt{2x+3}$

b. (3) Choose one fixed-point iteration $x=g(x)$ that is **convergent** to a root, and compute the first approximate x_1 using the initial point x_0 near to the root.

Choose $x = \sqrt{2x+3}$

Let $x_0 = 3.1$ $\quad x_1 = \sqrt{2x_0+3} = \sqrt{9.2} = 3.03315$

c. (5)

✓ (1) Is $x_{n+1}=g(x_n)$ in 3.b linearly convergent?

Yes

✓ (1) What is Aitken acceleration of this fixed-point iteration for 3.b?

✓ (1) What is the convergent rate of Aitken acceleration for $x_{n+1}=g(x_n)$?

Linearly

✓ (2) How faster it is compared to $x_{n+1}=g(x_n)$? Try to be specific.

c.-2

$\Delta x_n = x_{n+1} - x_n$

$\Delta^k x_n = \Delta^{k-1}(\Delta x_n)$

$$\xi_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n} = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}$$

c.-4

N': the number of iterations required for convergence by Aitken's iteration.

N: " by fixed-point iteration

$$\lim_{n\to\infty} \frac{N'}{N} = \frac{1}{2}$$