Quiz 3 Numerical Method, 2020/4/27

ID:____________ Name:_______________

1. (15) Given 5 data points

x	$f(x)$
-1	8
3	0
2	-1
-2	15
4	3

a. (3) Derive the Lagrangian form of the interpolating polynomial using the first 3 points.

$$P(X) = \frac{(X-X_1)(X-X_2)}{(X_0-X_1)(X_0-X_2)}f(0) + \frac{(X-X_0)(X-X_2)}{(X_1-X_0)(X_1-X_2)}f(1) + \frac{(X-X_0)(X-X_1)}{(X_2-X_0)(X_2-X_1)}f(2)$$
$$= \frac{(X-3)(X-2)}{(-1-3)(-1-2)} * 8 + \frac{(X+1)(X-2)}{(3+1)(3-2)} * 0 + \frac{(X+1)(X-3)}{(2+1)(2-3)} * (-1)$$
$$= (X-3)(X-1)$$

b. (5) Construct the divided-difference table from these data

x	f(x)	Divide difference			
		1^{st}	2^{nd}	3^{rd}	4^{th}
-1	8	-2	1	0	0
3	0	1	1	0	
2	-1	-4	1		
-2	15	-2			
4	3				

c. (4) Use divided difference table to interpolate $f(2.6)$

i. (2) Using the first 3 points.

$$P_{0,2}(X) = 8 + (-2)(X+1) + 1*(X+1)(X-3) + 0*(X+1)(X-3)(X-2)$$
$$P_{0,2}(2.6) = 8 - 2*3.6 + 3.6*(-0.4) = -0.64$$

ii. (2) Using best set of 3 points. Which points should be used?

用[2 3 4]，因為與 2.6 最接近

x	f(x)	Divide difference			
		1^{st}	2^{nd}	3^{rd}	4^{th}
2	-1	1	1		
3	0	3			
4	3				

$$P_{0,2}(X) = -1 + 1*(X-2) + 1*(X-2)(X-3)$$
$$P_{0,2}(2.6) = -1 + 0.6 - 0.24 = -0.64$$

d. (3) Is this table of data come from a polynomial? If so, why? And what is its degree?

1. Yes, it come from a polynomial
2. 在$f[X_i, X_{i+1}, X_{i+2}]$時，就算出了一樣的值
3. Degree = 2

2. (8) Show that the degree n interpolating polynomial for $n+1$ points is unique.

假設有兩個不同的方程式$P_n(x)$,$Q_n(x)$是 n 次方多項式 ，$P_n(x)$,$Q_n(x)$都經過

n+1 個不同的點

$D(x) = P_n(x) - Q_n(x)$, 是兩多項式的差，且$D(x)$最高為 n 次方

因為P_n,Q_n經過 n+1 個不同的點，所以$D(x)$在這 n+1 個點的值皆為 0，表示$D(x)$有 n+1 個不同的根

除非$D(x)$=0，否則以上敘述不可能發生

與假設有矛盾

因此$P_n(x) = Q_n(x)$

(可以參考課本第 162 頁)

Suppose there are two different polynomials of degree n that agree at $n+1$ distinct points. Call these $P_n(x)$ and $Q_n(x)$, and write their difference:

$$D(x) = P_n(x) - Q_n(x),$$

where $D(x)$ is a polynomial of at most degree n. But because P and Q match at the $n+1$ points, their difference $D(x)$ is equal to zero at all $n+1$ of these x-values; that is, $D(x)$ is a polynomial of degree n at most but has $n+1$ distinct zeros. However, this is impossible unless $D(x)$ is identically zero. Hence $P_n(x)$ and $Q_n(x)$ are not different—they must be the same polynomial.

3. (7) The nth-degree Bezier curve determined by n+1 points is given by

$$P(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i p_i,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

For cubic Bezier curve, $P(u)=[x(u), y(u)]^T$ is represented by the following form:

$$x(u) = (1-u)^3 x_0 + 3(1-u)^2 u x_1 + 3(1-u)u^2 x_2 + u^3 x_3,$$

$$y(u) = (1-u)^3 y_0 + 3(1-u)^2 u y_1 + 3(1-u)u^2 y_2 + u^3 y_3.$$

a. (4) Prove that the slops at the ends of a cubic Bezier curve are the same as the slope between the two endpoints.

$$x'(u) = -3(1-u)^2 x_0 + 3[(1-u)^2 - 2(1-u)u]x_1 + 3[2(1-u)u - u^2]x_2 + 3u^2 x_3$$

$$y'(u) = -3(1-u)^2 y_0 + 3[(1-u)^2 - 2(1-u)u]y_1 + 3[2(1-u)u - u^2]y_2 + 3u^2 y_3$$

$$slops\ at\ the\ ends = \frac{y'(u)}{x'(u)} \text{，} u = 1\ or\ 0 \text{ 代入}$$

$$when\ u = 1 \text{，} slope = \frac{y'(1)}{x'(1)} = \frac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$when\ u = 0 \text{，} slope = \frac{y'(0)}{x'(0)} = \frac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \frac{y_1 - y_0}{x_1 - x_0}$$

b. (3) Argue that the cubic Bezier curve is contained in the convex hull determined by the four points.

$$B(u) = \sum \alpha_i(u) * P_i \text{，}$$

$$\sum_{i=0}^{3} \alpha_i(u) = (1-u)^3 + 3(1-u)^2 u + 3(1-u)u^2 + u^3 = 1$$

cubic Bezier curve 中的每一點是四個端點的線性組合，且對於每一個 u，線性組合之係數和為 1，因此 cubic Bezier curve 必定包含於四個端點所決定的 convex hull