Final Exam

Numerical Method, 2021/6/17

Total: 103

1. (10) e^x is approximated by the Maclaurin series truncated up to x^4:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots + \frac{x^{n+1}}{(n+1)!}e^{\xi}, \ \xi \text{ in } [0, x].$$

   a. (6) Derive the Chebyshev series of e^x and then the Chebyshev expansion (up to x^3).
   b. (4) Describe how the error of Maclaurin series and Chebyshev expansion behave for approximating e^x, and why?

2. (18) Suppose function $f(x)$ is to be approximated by Pade approximation:

$$f(x) \approx R_N(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_0 + a_1 x + \cdots + a_n x^n}{1 + b_1 x + \cdots + b_m x^m}, \quad N = n + m$$

   a. (10) Suppose $f(x)$ is approximated by an Nth-degree polynomial $p_N(x) = c_0 + c_1 x + \cdots + c_N x^N$, describe how to solve for the unknown coefficients of $P_n(x)$ and $Q_m(x)$ of Pade approximation.
   b. (3) Can we expect the derived Pade approximation to yield a rational approximation of $f(x)$ with higher significant-digit accuracy than $p_N(x) = c_0 + c_1 x + \cdots + c_N x^N$?
   c. (5) How do we introduce Chebyshev polynomial into Pade approximation? Briefly describe the formulation.

3. (10) Derive the 2 points central-difference formula for approximating $f'(x)$ and show that its error is of order $O(h^2)$.

4. (10) Based on the following table,
   a. (4) Use the central-difference approximation to approximate $f'(2.4)$ with interval size h=0.1, and
   b. (6) Get an improved estimate by using the Richardson extrapolation.

Table 5.5

i	x_i	f_i
0	2.0	0.123060
1	2.1	0.105706
2	2.2	0.089584
3	2.3	0.074764
4	2.4	0.061277
5	2.5	0.049126
6	2.6	0.038288
7	2.7	0.028722
8	2.8	0.020371
9	2.9	0.013164
10	3.0	0.007026

5. (10) Numerical integration
   c. (4) Use the data in the table to find the integral between x=1.0 and 1.8, using the trapezoidal rule with h=0.1

x	$f(x)$
1.0	1.543
1.1	1.669
1.2	1.811
1.3	1.971
1.4	2.151
1.5	2.352
1.6	2.577
1.7	2.828
1.8	3.107

   d. (6) Extrapolate from the results of 6a. to get an improved value for the integral using Romberg integration.

6. (20) The following is the general gradient search algorithm for finding the minimum of $f(x)$:

$$
\begin{aligned}
&\text{Input : Initial point } \mathrm{x}_0, i = 0\\
&\text{Repeat}\\
&\quad \text{Define a search direction } \mathrm{s}_i \text{ at } \mathrm{x}_i\\
&\quad \text{Find } \alpha_i \text{ to minimize } f(\mathrm{x}) \text{ on the direction of } \mathrm{s}_i,\\
&\qquad\qquad \text{i.e., minimize } \phi_i(\alpha) = f(\mathrm{x}_i + \alpha\,\mathrm{s}_i)\\
&\quad \mathrm{x}_{i+1} = \mathrm{x}_i + \alpha_i\,\mathrm{s}_i; \quad i = i+1\\
&\text{Until convergence criterion is satisfied}\\
&\text{Output : Approximate minimum point is } \mathrm{x}_i
\end{aligned}
$$

Suppose that the function is a quadratic polynomial

$$f(\mathrm{x}) = \frac{1}{2}\mathrm{x}^T H\,\mathrm{x} + \mathrm{b}^T\mathrm{x} + \mathrm{c}$$

   a. (3) What is the search direction at x_i used in the steepest descent method? What is the relationship between s_i and s_{i+1}?
   b. (4) What is the search direction at x_i used in the Fletcher-Reeves conjugate gradient

method? What is the relationship between s_i and s_{i+1}?

c. (6) Let $x_{i+1} = x_i + \alpha_i s_i$ be the minimum point of $f(x)$ on the search direction s_i.. Show that for the steepest descent method the gradient of f at x_{i+1} is perpendicular to the search direction s_i.

d. (7) For the steepest descent method, derive the formula for computing α_i.

7. (10) You are asked to find the minimum point of $f(x,y) = x^2 + 2y^2 + xy + 3x$ with starting point x_0=(0,0).
   a. (4) Demonstrate the steepest descent method by finding x_1.
   b. (6) Demonstrate the Fletcher-Reeves conjugate gradient method by finding x_1 and x_2. (Let $s_0 = -\nabla f(x_0)$)

8. (15) Constrained optimization
   a. (4) When you are asked to maximize $f(x)$ subject to $g(x)$=0, how do you solve the problem by using the Lagrange multiplier method?
   b. (3) What is the intuition behind the Lagrange multiplier method?
   c. (8) Find the solution of the following problem using Simplex method:

Maximize $f(x_1, x_2) = 5x_1 + 6x_2$

Subject to

$$2x_1 + 3x_2 \le 12,$$
$$3x_1 + 2x_2 \le 15,$$
$$x_1, x_2 \ge 0.$$

USEFUL INFORMATION:

Chebyshev polynomia

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x,\\
T_4(x) &= 8x^4 - 8x^2 + 1,\\
T_5(x) &= 16x^5 - 20x^3 + 5x,\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1,\\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x,\\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1,\\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x,\\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1.
\end{aligned}$$

$$\begin{aligned}
1 &= T_0,\\
x &= T_1,\\
x^2 &= \frac{1}{2}(T_0 + T_2),\\
x^3 &= \frac{1}{4}(3T_1 + T_3),\\
x^4 &= \frac{1}{8}(3T_0 + 4T_2 + T_4),\\
x^5 &= \frac{1}{16}(10T_1 + 5T_3 + T_5),\\
x^6 &= \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6),\\
x^7 &= \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7),\\
x^8 &= \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8),\\
x^9 &= \frac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9).
\end{aligned}$$