Quiz 2 Numerical method

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1. (10) Given the following system

$$Ax = \begin{bmatrix} 1 & -2 & 4 \\ 8 & -3 & 2 \\ -1 & 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

a. (4) Solve the system by Gaussian Elimination with partial pivoting using three significant digits. (You can derive LU now for 1.b)

$$\begin{bmatrix} 1 & -2 & 4 & 6 \\ 8 & -3 & 2 & 2 \\ -1 & 10 & 2 & 4 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 8 & -3 & 2 & 2 \\ 1 & -2 & 4 & 6 \\ -1 & 10 & 2 & 4 \end{bmatrix} \xrightarrow[0.125R_1+R_3]{-0.125R_1+R_2} \begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & -1.625 & 3.75 & 5.75 \\ 0 & 9.625 & 2.25 & 4.25 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & 9.625 & 2.25 & 4.25 \\ 0 & -1.625 & 3.75 & 5.75 \end{bmatrix}$$

$$\xrightarrow{0.1688R_2+R_3} \begin{bmatrix} 8 & -3 & 2 & 2 \\ 0 & 9.625 & 2.25 & 4.25 \\ 0 & 0 & 4.1298 & 6.4674 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -0.125 & 1 & 0 \\ 0.125 & -0.1688 & 1 \end{bmatrix}$$

$x_1 = -0.113$

$x_2 = 0.07548 = 0.0755$

$x_3 = 1.566$

b. (3) Derive the LU decomposition. What is the relation between A and LU?

$$A' = \begin{bmatrix} 8 & -3 & 2 \\ -1 & 10 & 2 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.125 & 1 & 0 \\ 0.125 & -0.1688 & 1 \end{bmatrix} \begin{bmatrix} 8 & -3 & 2 \\ 0 & 9.625 & 2.25 \\ 0 & 0 & 4.1298 \end{bmatrix} = LU$$

$A' = A$ (row1 $\leftrightarrow$ row2, row2 $\leftrightarrow$ row3)

c. (3) How do you solve another system Ax=b' with the same coefficient matrix A by using LU decomposition obtained in 1.b?

$A \xrightarrow{R_{12}} A_1 \xrightarrow{R_{23}} A'$ $\quad A'x = b'' \Rightarrow LUx = b'' \Rightarrow Ux = y$ and $Ly = b''$

$b' \xrightarrow{R_{12}} b_1' \xrightarrow{R_{23}} b''$ $\quad$ solve $Ly = b''$ to get y

solve $Ux = y$ to get x

2. (10) Given the system of equations:

$$Ax = \begin{bmatrix} 6.03 & 1.99 & 3.01 \\ 4.16 & -1.23 & 1.27 \\ -4.81 & 9.34 & 0.987 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a. (4) Solve the system by Gaussian Elimination with scaled partial pivoting using three significant digits. (You can derive LU now needed for 2.c)

$$\begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 4.16 & -1.23 & 1.27 & 1 \\ -4.81 & 9.34 & 0.987 & 1 \end{bmatrix} \xrightarrow[\frac{4.81}{6.03}R_1+R_3]{-\frac{4.16}{6.03}\times R_1+R_2} \begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 0 & -2.603 & -0.807 & 0.310 \\ 0 & 10.927 & 3.388 & 1.798 \end{bmatrix}$$

$S = [6.03, 4.16, 9.34]^T$

$$R_2 = \left[\frac{1.99}{6.03}, \frac{-2.603}{4.16}, \frac{10.927}{9.34}\right]^T = [0.330, -0.626, 1.170]$$

$\Rightarrow$ row 2 $\leftrightarrow$ row3

$$R_1 = \left[\frac{6.03}{6.03}, \frac{4.16}{4.16}, \frac{-4.81}{9.34}\right]^T$$

$= [1, 1, -0.5149]^T$

No row interchange is needed.

$$\xrightarrow{R_{23}} \begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 0 & 10.927 & 3.388 & 1.798 \\ 0 & -2.603 & -0.807 & 0.310 \end{bmatrix} \xrightarrow{\frac{2.603}{10.927}\times R_2+R_3} \begin{bmatrix} 6.03 & 1.99 & 3.01 & 1 \\ 0 & 10.927 & 3.388 & 0.310 \\ 0 & 0 & 0.00008 & 1.872 \end{bmatrix}$$

$\downarrow$

$x_1 = -9286.055$
$x_2 = -7255.320$
$x_3 = 23400$

b. (3) Is the system ill-conditioned? Why?

Yes, the position (3,3) of LU is near to 0.
$\downarrow$
0.00008

c. (3) Apply one step of iterative improvement (Residual correction) to the solution from 2.a. How do you compute the residual to avoid cancellation error?

C.-1

Define: $e = x - \bar{x}$ $\quad r = b - A\bar{x}$

$$e = \begin{bmatrix} 8654.344 \\ 6761.859 \\ -21807.906 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} -9286.055 \\ -7255.320 \\ 23400 \end{bmatrix}$$

$$r = b - A\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.0016 \\ 12.055 \\ -2.964 \end{bmatrix} = \begin{bmatrix} -0.0016 \\ -11.055 \\ 3.964 \end{bmatrix}$$

$$x = \bar{x} + e = \begin{bmatrix} -631.711 \\ -493.461 \\ 1592.094 \end{bmatrix}$$

solve $Ae = r$, $x = \bar{x} + e$

C.-2: Use higher precision

3. (10) Given the matrix [A;b]

$$[A;b] = \begin{bmatrix} 7 & -3 & 4 & 6 \\ -3 & 2 & 6 & 2 \\ 2 & 5 & 3 & -5 \end{bmatrix}$$

a. (3) You want to solve the system with the Jacobi and Gauss-Seidel method for any starting vector. How do you arrange the system to ensure the convergence before applying the methods?

Diagonal dominant

$|A_{ii}| \geq \sum_{i \neq j} |A_{ij}|$ for all i

$\therefore$ row2 $\leftrightarrow$ row3

$$\begin{bmatrix} 7 & -3 & 4 & 6 \\ 2 & 5 & 3 & -5 \\ -3 & 2 & 6 & 2 \end{bmatrix}$$

b. (3) Following 3.a, obtain x_1 of Jacobi method using the starting vector x_0=[0, 0, 0].

$$(x_1)_1 = \frac{1}{A_{11}}(b_1 - A_{12} \times (x_0)_2 - A_{13} \times (x_0)_3) = \frac{1}{7} \times 6 = \frac{6}{7}$$

$$(x_1)_2 = \frac{1}{A_{22}}(b_2 - A_{21} \times (x_0)_1 - A_{23} \times (x_0)_3) = \frac{1}{5} \times (-5) = -1$$

$$(x_1)_3 = \frac{1}{A_{33}}(b_3 - A_{31} \times (x_0)_1 - A_{32} \times (x_0)_2) = \frac{1}{6} \times 2 = \frac{1}{3}$$

c. (4) Following 3.a, obtain x_1 of Gauss-Seidel method using the starting vector x_0=[0, 0, 0].

$$(x_1)_1 = \quad '' \quad = \frac{1}{7} \times 6 = \frac{6}{7}$$

$$(x_1)_2 = \frac{1}{A_{22}}(b_2 - A_{21} \times (x_1)_1 - A_{23} \times (x_0)_3) = \frac{1}{5}\left(-5 - 2 \times \frac{6}{7}\right) = \frac{-47}{35}$$

$$(x_1)_3 = \frac{1}{A_{33}}(b_3 - A_{31} \times (x_1)_1 - A_{32} \times (x_1)_2 = \frac{1}{6} \times \left(2 - (-3) \times \frac{6}{7} - 2 \times \frac{-47}{35}\right) = \frac{127}{105}$$