Quiz 4 Numerical Method, 2020/5/21

ID:____________ Name:____________ $\left|\quad\right| < 0.001 \Rightarrow n = 7$

1. (12) The Maclaurin series of e^x at $x = 0$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots + \frac{x^{n+1}}{(n+1)!}e^{\xi},\ \xi \text{ in } [0, x].$$

a. (3) If you want to use Maclaurin series to approximate e^x on [-1/2, 1/2] with a precision of 0.001, how can you truncate the Maclaurin series?

$n=3 \quad \left|\frac{(\frac{1}{2})^4}{24} \times e^{\frac{1}{2}}\right| = 0.0043 > 0.001$

$n=4 \quad \left|\frac{(\frac{1}{2})^5}{120} \times e^{\frac{1}{2}}\right| = 0.00042935 < 0.001 \Rightarrow$ 算到 degree-4 $n=4$

b. (5) Let $p(x)$ be the truncated series. Describe the steps involved in finding the 5-th degree economized polynomial $p_{econ}(x)$ for $p(x)$ on $I = [-1/2, 1/2]$.

$P(X) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - (\frac{1}{720})(\frac{T_6}{32})$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{1}{720}(\frac{1}{32})(32x^6 - 48x^4 + 18x^2 - 1)$

$= 1.000043 + x + 0.499219x^2 + \frac{x^3}{6} + 0.043750x^4 + \frac{x^5}{120}$

$\frac{x^7}{7!}e^{\xi}$

$-\frac{1}{720}\frac{T_6}{32}$

c. (4) How do you measure the accuracy of $p_{econ}(x) \approx e^x$ on $I = [-1/2, 1/2]$?

The maximum value of T_6 on $[-\frac{1}{2}, \frac{1}{2}]$ is 1

$\frac{1}{720} \times \frac{T_6(x)}{32} < \frac{1}{720} \times \frac{1}{32} < 0.00005$

error $< \frac{(\frac{1}{2})^7}{7!} e^{\frac{1}{2}} + 0.00005 = 0.000052555 7$

2. (8) e^x is approximated by a Maclaurin series as shown in question 1.

a. (4) Derive the Chebyshev series of e^x and then the Chebyshev expansion (upto x^3)?

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots$

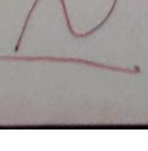
$= T_0 + T_1 + \frac{\frac{1}{2}(T_0 + T_2)}{2} + \frac{\frac{1}{4}(3T_1 + T_3)}{6} + \frac{\frac{1}{8}(3T_0 + 4T_2 + T_4)}{24} + \cdots$

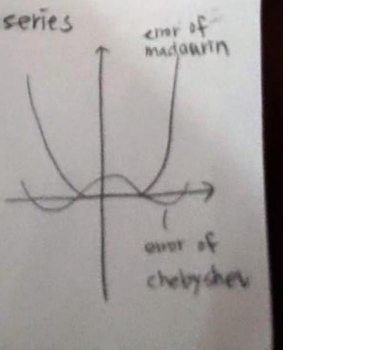
$= 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \cdots$

$= 0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3$

b. (4) Describe how the error of Maclaurin series and Chebyshev expansion behave for approximating e^x, and why?

1. Chebyshev expansion 算出的最大誤差小於 Maclaurin series
   chebyshev 的 error 在 interval 裡均勻的分布
   Maclauin 在原點附近的 error 很小，越遠 error 越大
2. 因為 chebyshev $T_i(x)$ 會在[1, -1]間振盪
   用 cos 去推導的





0. Maclaurin series 是 Local app. 在 x=0 附近 error 小 其它大 對 x=0 這點展開

3. (10) Let $f(x) = (x+1)^2$ be periodic between $-\pi$ and π. Find its Fourier series.

2 $$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)^2 dx = \frac{1}{\pi}\left(\frac{1}{3}\right)(x+1)^3 \Big|_{-\pi}^{\pi} = \frac{1}{3\pi}\left[(\pi+1)^3 - (-\pi+1)^3\right] = 8.3579$$

$$\frac{2}{3}\pi^2 + 2$$

3 $$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)^2 \cos(nx)\, dx = \frac{4}{n^2} \times (-1)^n, \quad n \geq 1$$

3 $$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)^2 \sin(nx)\, dx = \frac{4}{n} \times (-1)^{n-1}$$

2 $$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(nx) + B_n \sin(nx)\right]$$

$$= 4.17895 + \sum_{n=1}^{\infty}\left[(-1)^n \times \frac{4}{n^2}\cos(nx) + (-1)^{n-1} \times \frac{4}{n}\sin(nx)\right]$$

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