Midterm

Numerical method 2021/4/15

Total: 100

1. (15) Short questions
   a. (2) What are the two most important errors that are related to machine number representation and arithmetic computation?
   b. (3) If you want to test if two computed values A and B are equal in computer programs, what's wrong with the test " if A=B, then ...."? What is the correct way of doing this test?
   c. (6) To solve $f(x)=0$ numerically, you have learned Secant method, Newton's method, and Muller's method. Compare these three methods based on (i.) the starting points required, (ii.) how the function $f(x)$ is approximated and the next point is found, and (iii.) the convergence rate.
   d. (4) If $f(x)=0$ has a triple root r, we have known that Newton's method converges linearly to r. How can we restore the quadratic convergence of the Newton's method?

$x_1 - \dfrac{f(x_1)}{\dfrac{f(x_1)-f(x_0)}{x_1-x_0}}$

2. (15) Given an equation $f(x)=2x^3+x^2-3x-3=0$,
   a. (4) Starting with $x_0=2.0, x_1=1.8$, derive x_2 using Secant method.
   b. (5) With starting value $x_0=2.0$, evaluate $f(2)$ and $f'(2)$ using synthetic division, and then derive x_1 using Newton's method.
   c. (6) Rewrite the polynomial into the form of $x=g(x)$ such that $x_{n+1}=g(x_n)$ converges to the exact solution with the starting value of $x_0=2.0$. You need to verify the convergence.

$x = \dfrac{2x^3 - x^2 - 3}{3}$

3. (10) Note that the sequence $\{x_n\}_{n=0}^{\infty}$ is called linearly converge to r if $e_n(=x_n-r)\to 0$ in such a way that $\lim_{n\to\infty}\dfrac{e_n}{e_{n-1}}=C_L$, where $0<|C_L|<1$. Show that the fixed point iteration $x_{n+1}=g(x_n)$ converges linearly to the fixed point r if $|g'(x)|<K<1$ for all x in the interval around r. You can begin the proof by expanding $g(x)$ at r up to constant term (plus the remainder term).

$3x+3-x^2$

$\dfrac{5}{2}$

$\sqrt[3]{2.5}$

4. (15) Consider the linear system $Ax=b$, where

$$A=\begin{bmatrix}4 & -3 & 0\\ 2 & 2 & 3\\ 6 & 1 & -6\end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix}-7\\ -2\\ 6\end{bmatrix}$$

   a. (5) Solve $Ax=b$ using partial pivoting and compute the LU factorization.
   b. (2) What is the relation between $A'=LU$ and A?
   c. (3) How do you solve $Ax=b$ by using $A'=LU$. (Just list the steps)
   d. (5) Explain how the solution of 4.a can be improved by using iterative residual correction method. You need to point out how to compute the improved solution efficiently and avoid cancellation error.

$g(x) = g(r) + g'(r)(x-r) + \dfrac{g''(\xi)}{2}(x-r)^2$

$x_i \; g(x) - g(r) \; e_n$

5. (10) Given the augmented matrix [A:b]

$$[A:b] = \begin{bmatrix} 6 & -2 & 1 & 11 \\ 1 & 2 & -5 & -1 \\ -2 & 7 & 2 & 5 \end{bmatrix}$$

a. (5) With starting value $x_0 = (0,0,0)$, derive the next estimate x_1 for solving Ax=b using Jacobi iteration. (You need to rewrite the system such that the iteration is convergent for any starting value.)

b. (5) Repeat 5.a with Gauss-Seidel iteration.

6. (10) In computing the solution of $Ax = b$, the condition number of A reveals how large the relative error in the computed solution could be for the small changes in the input, e.g, elements of b. Derive the condition number that satisfies the following inequality

$$\frac{\|x - \bar{x}\|}{\|x\|} \leq \text{condition no.} \frac{\|b - \bar{b}\|}{\|b\|},$$

$\|e\| \le$

where x is the exact solution, i.e., $Ax = b$, $\bar{x}$ is the computed solution to $A\bar{x} = \bar{b}$. The proof can begin with r=b- $\bar{b}$ =Ax-$A\bar{x}$ =A(x- $\bar{x}$)=Ae and use the inequality of the matrix norm.

7. (15) Given the following divided difference table for 5 data points (x_i, f_i), i=0,1,…, 4:

Table 3.2

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, \ldots, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$	$f[x_i, \ldots, x_{i+4}]$
3.2	22.0	8.400	2.856	−0.528	0.256
2.7	17.8	2.118	2.012	0.0865	
1.0	14.2	6.342	2.263		
4.8	38.3	16.750			
5.6	51.7				

$\frac{\|r\|}{\|A\|} \le \|e\| \le \|A^{-1}\|\|r\|$

$\frac{\|b\|}{\|A\|} \le \|x\| \le \|A^{-1}\|\|$

a. (3) Compute $f[x_0, x_2]$=2.856.

b. (2) Find the quadratic interpolating polynomial that passes through the first 3 points using the above divided difference table.

c. (3) Find the cubic interpolating polynomial $P_3(x)$ that passes through the first 4 points using the above divided difference table.

d. (2) Can you say that the underlying function of the data points behaves nearly like a cubic polynomial? Why?

e. (3) Find the cubic Lagrangian polynomial $P_3(x)$ that passes through the first 4 points. Is it the same as the result of 7.c? Why?

f. (2) Although the underlying function of the data points is unknown, give an error estimation of $P_3(x)$. (Just list the form)

8. (10) We have learned that all polynomials of degree n that interpolate n+1 points are identical.

a. (3) Give your conceptual argument about this statement.

b. (7) Prove this statement by contradiction.