δ $h_1 = x_1 - x_2$ x_1 x_0 x_1
$h_2 = x_0 - x_1$

$x_0 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$

$\delta_1 = \frac{f(x_1) - f(x_2)}{h_1}$ $\delta_2 =$

$a = \frac{\delta_2 - \delta_1}{h_2 + h_1}$ $b = a h_2 + \delta_2$

Midterm

Numerical method, 2020/5/7

Total: 100

1. (5)
   a. (2) What are the two most important errors that are related to machine number representation and arithmetic computation?
   b. (3) If you want to test if two computed values A and B are equal in computer programs, can you do the test " if A=B, then ...."? Why? What is the correct way of doing this test?

$x_2 = x_1 - f(x_1)\frac{x_0 - x_1}{f(x_0) - f(x_1)}$

2. (15) Given an equation $f(x) = 2x^3 + x^2 - 3x - 3 = 0$,
   a. (4) Starting with $x_0 = 2.0, x_1 = 1.8$, derive x_2 using Secant method.
   b. (5) With starting value $x_0 = 2.0$, evaluate $f(2)$ and $f'(2)$ using synthetic division, and then derive x_1 using Newton's method.
   c. (6) Rewrite the polynomial into the form of $x = g(x)$ such that $x_{n+1} = g(x_n)$ converges to the exact solution with the starting value of $x_0 = 2.0$. You need to verify the convergence.

3. (10) Given a nth-degree polynomial $P_n(x)$. Show that
   a. (4) The remainder R on dividing $P_n(x)$ by $x - a$ is the value of $P_n(a)$.
   b. (6) The second remainder on dividing $Q_{n-1}(x)$ by $x - a$ is the value of derivative of $P_n(x)$ at a, i.e., $P'_n(a)$, where $P_n(x) = (x-a)Q_{n-1}(x) + R$.

$e_n = R - g(x_n) = g(R) - g(x_n)$

4. (10) Show that the Newton iteration for solving $f(x)=0$ converges quadratically.

Note that the sequence $\{x_n\}_{n=0}^{\infty}$ is called quadratically converge to r if $e_n (= x_n - r) \to 0$

in such a way that $\lim_{n\to\infty} \frac{e_n}{e_{n-1}^{\ 2}} = C_Q$, where $C_Q \neq 0$. The proof can begin with considering

the iteration of Newton method as a fixed point iteration $x=g(x)$ and doing Taylor expansion of $g(x)$ at r, with $g'(r)=0$ since $f(r)=0$.

$g'(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$ $g(r) + g'(r) + \frac{g''(r)}{2}$

5. (15) Consider the linear system $Ax=b$, where

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 5 \\ 4 & 1 & -2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -6 \\ 12 \\ -1 \end{bmatrix}$$

$\left(\frac{1}{2}(-x^2+3x+3)\right)^{\frac{1}{3}}$

$\frac{1}{3}\left(\frac{1}{2}(-x^2+3x+3)\right)^{-\frac{2}{3}} \cdot \frac{1}{2}(-2x+3)$

   a. (5) Compute the LU factorization of A.
   b. (4) Solve $Ax=b$ by using LU factorization of A.
   c. (6) Explain how the solution of 5.b can be improved by using iterative residual

correction method. You need to point out how to compute the improved solution efficiently and avoid cancellation error.

6. (10) In computing the solution of $Ax = b$, the condition number of A reveals how large the relative error in the computed solution could be for the small changes in the input, e.g, elements of b. Derive the condition number that satisfies the following inequality

$$\frac{\|x-\bar{x}\|}{\|x\|} \le \text{condition no.} \frac{\|b-\bar{b}\|}{\|b\|},$$

where x is the exact solution, i.e., $Ax = b$, $\bar{x}$ is the computed solution and $A\bar{x} = \bar{b}$. The proof can begin with $r=b-\bar{b}=Ax-A\bar{x}=A(x-\bar{x})=Ae$ and use the inequality of the matrix norm.

7. (8) Given the matrix [A:b]

$$[A:b] = \begin{bmatrix} 6 & -2 & 1 & 11 \\ 1 & 2 & -5 & -1 \\ -2 & 7 & 2 & 5 \end{bmatrix}$$

   a. (5) With starting value $x_0 = (0,0,0)$, derive the next estimate x_1 for solving Ax=b using Jacobi iteration. (You need to rewrite the system such that the iteration is convergent for any starting value.)
   b. (3) Repeat 7.a with Gauss-Seidel iteration.

8. (15) Given the following data pairs:

x	3.2	2.7	1.0	4.8	5.6
$f(x)$	22.0	17.8	14.2	38.3	51.7

   a. (7) Find the cubic Lagrangian polynomial that passes through the first 4 points.
   b. (8) Find the cubic interpolating polynomial that passes through the first 4 points using divided difference table. Is it the same as the result of (a.)? Why?

9. (12) Given N data points (x_i, Y_i), i=1, 2,.., N.
   a. (6) What do we mean by "finding a polynomial of degree n ($n < N$-1) to approximate the data in least-square sense"? Please answer the question by formulating the problem of least-square polynomial.
   b. (4) Is the resulting normal equations ill-conditioned when the degree n is high? Why?
   c. (2) What happens about the resulting polynomial if $N=n+1$? What happens if $n <$ N-1 but is high?