4 | Are there any exercises you want me to cover?

Exercise | Disprove the statement:
There exist odd integers a and b s.t.

$$4 \mid 3a^2 + 7b^2$$

Proof | Suppose there exist odd integers a and b s.t.

$$4 \mid 3a^2 + 7b^2$$

Then we can write that

$$4 \mid 3(2m+1)^2 + 7(2n+1)^2 \text{ for some } m,n \in \mathbb{Z}.$$

i.e. $4 \mid 3(4m^2+4m+1) + 7(4n^2+4n+1)$

$4 \mid 4(3m^2+3m)+3 + 4(7n^2+7n+1)+3$

$4 \mid 4(3m^2+3m+7n^2+7n+2)+2$

i.e. $4K = 4(3m^2+3m+7n^2+7n+2)+2$

for some $K \in \mathbb{Z}$. This is a contradiction

since $K = 3m^2+3m+7n^2+7n+1+\frac{1}{2}$

in this instance which is not a member of $\mathbb{Z}$ since $3m^2 + \dots + 1 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$ therefore their sum which equals K is not a member of $\mathbb{Z}$. Thus we conclude that there do not exist any odd integers a and b s.t. $4 \mid 3a^2 + 7b^2$ $\square$

Exercise: Prove that if n is odd then $7n-5$ is even using:

- Direct proof
- Contrapositive proof
- Contradiction proof

Direct Proof: Suppose n is odd, observe then that:

$n = 2m+1$ for some $m \in \mathbb{Z}$

$7n = 14m+7$

$7n-5 = 14m+2 = 2(7m+1)$

Now $7n-5$ is odd iff it can be written in the form $2k+1$ for some $k \in \mathbb{Z}$.

Notice $7m+1$ is some element of $\mathbb{Z}$

Hence $7n-5$ is odd $\square$

Contrapos. Proof: Suppose $7n-5$ is odd. Then $7n-5 = 2m+1$ for some $m \in \mathbb{Z}$. Now observe then that:

$7n-5 = 2m+1$

$7n = 2m+6$

$7n = 2(m+3)$

This means that $7n$ is even. Now since 7 is odd n must be even. Therefore with the help of the contrapositive we conclude that if n is odd then $7n-5$ is even $\square$

Contradict Proof: Suppose n is odd and $7n-5$ is odd.

Then $n = 2k+1$ for some $k \in \mathbb{Z}$

Now subst. this equality in $7n-5$ we get:

$7(2k+1)-5 = 14k+2 = 2(7k+2)$

In other words $7n-5$ is even which is contradictory to our assumptions. Therefore we conclude that if n is odd then $7n-5$ is even $\square$

Def   Recursive definitions

e.g.   Fibonacci sequence

we define $f_0 = 0$, $f_1 = 1$

and $f_n = f_{n-1} + f_{n-2}$ for consecutive $n \geq 2$.

using this def. we can compute the sequence:

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 5$$
$$f_6 = f_5 + f_4 = 8.$$
$$\vdots \qquad \vdots$$

Def.   "Strong" Induction Principle

Let $P(n)$ be a statement you want to prove where $n = 0, 1, \ldots$ is any integer then $P(n)$ is true for all $n = 0, 1, \ldots$ if:

- $P(0)$ is true (Base Case)
- Assume $P(0) \wedge \ldots \wedge P(k)$ is true for some integer $k \geq 1$ (Inductive Hypothesis)
- Prove that $(P(0) \wedge \ldots \wedge P(k))$ is true implies that $P(k+1)$ is true.

e.g. "Prove that for the fibonacci sequence i.e. $f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$ and $f_1 = 1$. it holds that $f_n > (\frac{3}{2})^{n-1}$ for every integer $n \geq 6$.

Proof

Base Case

Notice that $f_6 = 8$ and $f_7 = 13$

and that $(\frac{3}{2})^{6-1} = (\frac{3}{2})^5 = 7\frac{19}{32}$

and $(\frac{3}{2})^{7-1} = (\frac{3}{2})^6 = 7\frac{19}{32} \cdot (\frac{3}{2}) = 11\frac{25}{64}$

Therefore $f_n > (\frac{3}{2})^{n-1}$ for $n = 6$ and $n = 7$ thereby satisfying our base case.

Induction hypothesis

We continue by assuming that $f_6 > (\frac{3}{2})^{6-1}$ $, \ldots, f_k > (\frac{3}{2})^{k-1}$ hold for $1, \ldots, k$. for some integere $k \geq 7$. (i.e $k \geq 6+1$).

Inductive Step

Next, we want to show that given our induction hypothesis it follows that

$$f_{k+1} > (\tfrac{3}{2})^{(k+1)-1}$$

i.e. $f_{k+1} > (\frac{3}{2})^k$

Now notice that $f_{k+1} = f_k + f_{k-1} > (\frac{3}{2})^{k-1} + (\frac{3}{2})^{k-2}$

where $(\frac{3}{2})^{k-1} + (\frac{3}{2})^{k-2} = (\frac{3}{2})^{-1} \cdot (\frac{3}{2})^k + (\frac{3}{2})^{-2} \cdot (\frac{3}{2})^k$

$= (\frac{2}{3} + \frac{4}{9})(\frac{3}{2})^k > (\frac{3}{2})^k$

"Notice that $(\frac{2}{3} + \frac{4}{9}) > 1$"

Thus $f_{k+1} < (\frac{3}{2})^{k-}$ given that $f_{k-1} < (\frac{3}{2})^{k-2}$ and $f_k < (\frac{3}{2})^{k-1}$. Therefore we conclude by principal of "strong" induction that $f_n > (\frac{3}{2})^{n-1}$ for every integer $n \geq 6$ □