Def Number sets

| Notation | Name | Set notation |
|---|---|---|
| $\mathbb{Z}$ | "Integers" | $\{\ldots,-2,-1,0,1,2,\ldots\}$ |
| $\mathbb{N}$ | "Natural numbers" | $\{1,2,3,\ldots\}$ |
| $\mathbb{Q}$ | "Rational numbers" | $\{\frac{a}{b} : a,b \in \mathbb{Z}$ and $b \neq 0$ s.t. $\gcd(a,b)=1\}$ |
| $\mathbb{R}$ | "Real numbers" | "Includes all elements of $\mathbb{Q}$ and $\pi$, $e$ etc" |
| $\mathbb{C}$ | "Complex numbers" | $\{a+bi : a,b \in \mathbb{R}$ where $i^2=-1\}$ |

With irrational numbers we mean $\mathbb{R} \setminus \mathbb{Q}$ i.e. the real numbers without the rational numbers.

e.g. $\sqrt{2}$, $\pi$

N.B. "if and only if" is often abbreviated to <u>iff</u>.

Def Any integer $n \in \mathbb{Z}$ is divisible by some $m \in \mathbb{Z}$ iff $n = mk$ for some $k \in \mathbb{Z}$.

We denote $n$ is divisible by $m$ as

$m \mid n$

e.g. $2 \mid 8$ namely $8 = 2 \cdot 4$

$3 \mid 9$ namely $9 = 3 \cdot 3$

etc.

Def  Proof by contrapositive
Sometimes its easier to prove the state
$\neg q \Rightarrow \neg p$ than $p \Rightarrow q$. Even though
they are logically equivalent $\square$

e.g. lets first prove that $n|3$ iff $n^2|3$.
i.e. $n$ is divisible by 3 only if $n^2$ is
divisible by 3.

<u>Proof</u>
Suppose $n$ is divisible by 3 i.e
$n = 3k$ for some $k \in \mathbb{Z}$.
Then $n^2 = 9k^2 = 3(3k^2)$
Now notice since $3k^2$ is again some
integer we know that $3|n^2$ if $n|3$.

Suppose $n$ is not divisible by 3 i.e.
$n = 3k+1$ or $n = 3k+2$ for some $k \in \mathbb{Z}$.
Then $n^2 = (3k+1)^2$ or $n^2 = (3k+2)^2$
$n^2 = 9k^2+6k+1$ or $n^2 = 9k^2+12k+4$
$n^2 = 3(3k^2+2)+1$ or $n^2 = 3(3k^2+4k+1)+1$
Now notice that $3k^2+2$ and $3k^2+4k+1$
are again some integer. Therefore
if $3 \nmid n$ then $3 \nmid n^2$. Now using
the contrapositive we also know that
if $3|n^2$ then $3|n$.

combining these results we conclude that $3 \mid n$ iff $3 \mid n^2$ $\square$

<u>def</u> Proof By Contradiction

We assume the negation of the statement we want to prove true. Then show that this leads to a contradiction i.e a statement we know to be false. Contradiction we denote as $\lightning$

e.g. prove that $\sqrt{3}$ is irrational

<u>Proof</u>

Assume $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ s.t. $b \neq 0$ and $a, b$ share no common factors.

observe then that

$$\sqrt{3} = \frac{a}{b}$$

$$3 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

$$3b^2 = a^2$$

Now notice that $b^2$ is some integer. therefore $3 \mid a^2$ then using the previous example we now also know that $3 \mid a$ i.e $a = 3k$ for some $k \in \mathbb{Z}$.

revisiting $3 = \frac{a^2}{b^2}$ and subst $a = 3k$ we get

$$3 = \frac{a^2}{b^2}$$

$$3 = \frac{9k^2}{b^2}$$

$$b^2 = 3k^2$$

In other words $3 \mid b^2$ and therefore $3 \mid b$. $\lightning$

This is a contradiction because we assumed a and b had no common factors but if $\sqrt{3}$ is rational then they do. Therefore we conclude using contradiction that $\sqrt{3}$ is irrational $\square$

Def "Weak" Induction Principal

Let $P(n)$ be a statement where $n \in \mathbb{N}$ i.e. $n = 1, 2, \dots$. Then $P(n)$ is true for all $n \in \mathbb{N}$ if

- $P(1)$ is true (Base Case)
- Assume $P(K)$ is true for some $K \in \mathbb{N}$
- Then prove that $P(K)$ is true implies that $P(K+1)$ is true.

e.g. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

Proof

Base Case: $n = 1$ then $1 = \frac{1(1+1)}{2}$ ✓

Induction Hypothesis: Assume $\sum_{i=1}^{K} i = \frac{K(K+1)}{2}$ for some $K \in \mathbb{N}$.

Induction step

Have: $\sum_{i=1}^{K} i = \frac{K(K+1)}{2}$

Want: $\sum_{i=1}^{K} i = \frac{K(K+1)}{2} \Rightarrow \sum_{i=1}^{K+1} i = \frac{(K+1)(K+2)}{2}$

Now notice that $\sum_{i=1}^{K+1} i = K+1 + \sum_{i=1}^{K} i$

Now subst. $\sum_{i=1}^{K} i$ for $\frac{K(K+1)}{2}$ then

$$\sum_{i=1}^{K+1} i = K+1 + \frac{K(K+1)}{2} = \frac{2K+2+K(K+1)}{2}$$

$$= \frac{2K+2+K^2+K}{2} = \frac{K^2+3K+2}{2} = \frac{(K+1)(K+2)}{2}$$

Therefore $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ if $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Thus we conclude by principal of induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$ $\square$

e.g. Prove that $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ for any $n \in \mathbb{N}$ and any $r \in \mathbb{R}$

<u>proof</u>

Base Case: $r^0 = 1$ and $\frac{1-r^1}{1-r} = 1$

hence $r^0 = \frac{1-r^1}{1-r}$. ✓

Induction Hypothesis

Assume $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$ for some $k \in \mathbb{N}$.

Induction step:

Have: $\sum_{i=0}^{k} r^k = \frac{1-r^{k+1}}{1-r}$

Want: $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r} \Rightarrow \sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$

Now notice that $\sum_{i=0}^{k+1} r^i = r^{k+1} + \sum_{i=0}^{k} r^i$

Now subst. $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$. Then we get:

$$\sum_{i=0}^{k+1} r^i = r^{k+1} + \frac{1-r^{k+1}}{1-r}$$

$$= \frac{r^{k+1}(1-r)}{1-r} + \frac{1-r^{k+1}}{1-r}$$

$$= \frac{r^{k+1}-r^{k+2}}{1-r} + \frac{1-r^{k+1}}{1-r}$$

$$= \frac{1-r^{k+2}}{1-r}$$

Thus $\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$. $\square$

Therefore by principal of induction we have shown that $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ for any $n \in \mathbb{N}$ and any $r \in \mathbb{R}$ $\square$