Exercises for Chapter 5

A. Prove the following statements with contrapositive proof. (In each case, think about how a direct proof would work. In most cases contrapositive is easier.)

1. Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

4. Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

5. Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.

6. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

7. Suppose $a, b \in \mathbb{Z}$. If both ab and $a + b$ are even, then both a and b are even.

8. Suppose $x \in \mathbb{R}$. If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

9. Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

10. Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

11. Suppose $x, y \in \mathbb{Z}$. If $x^2(y + 3)$ is even, then x is even or y is odd.

12. Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

13. Suppose $x \in \mathbb{R}$. If $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.

B. Prove the following statements using either direct or contrapositive proof.

14. If $a, b \in \mathbb{Z}$ and a and b have the same parity, then $3a + 7$ and $7b - 4$ do not.

15. Suppose $x \in \mathbb{Z}$. If $x^3 - 1$ is even, then x is odd.

16. Suppose $x, y \in \mathbb{Z}$. If $x + y$ is even, then x and y have the same parity.

17. If n is odd, then $8 \mid (n^2 - 1)$.

18. If $a, b \in \mathbb{Z}$, then $(a + b)^3 \equiv a^3 + b^3 \pmod 3$.

19. Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod n$ and $a \equiv c \pmod n$, then $c \equiv b \pmod n$.

20. If $a \in \mathbb{Z}$ and $a \equiv 1 \pmod 5$, then $a^2 \equiv 1 \pmod 5$.

21. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod n$, then $a^3 \equiv b^3 \pmod n$.

22. Let $a \in \mathbb{Z}$, $n \in \mathbb{N}$. If a has remainder r when divided by n, then $a \equiv r \pmod n$.

23. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod n$, then $a^2 \equiv ab \pmod n$.

24. If $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $ac \equiv bd \pmod n$.

25. Let $n \in \mathbb{N}$. If $2^n - 1$ is prime, then n is prime.

26. If $n = 2^k - 1$ for $k \in \mathbb{N}$, then every entry in Row n of Pascal's Triangle is odd.

27. If $a \equiv 0 \pmod 4$ or $a \equiv 1 \pmod 4$, then $\binom{a}{2}$ is even.

28. If $n \in \mathbb{Z}$, then $4 \nmid (n^2 - 3)$.

29. If integers a and b are not both zero, then $\gcd(a, b) = \gcd(a - b, b)$.

30. If $a \equiv b \pmod n$, then $\gcd(a, n) = \gcd(b, n)$.

31. Suppose the division algorithm applied to a and b yields $a = qb + r$. Prove $\gcd(a, b) = \gcd(r, b)$.

32. If $a \equiv b \pmod n$, then a and b have the same remainder when divided by n.