

Exercises in Proof by Induction

Here's a summary of what we mean by a "proof by induction":

The Induction Principle: Let $P(n)$ be a statement which depends on $n = 1, 2, 3, \dots$. Then $P(n)$ is true for all n if:

- $P(1)$ is true (the *base case*).
- Prove that $P(k)$ is true implies that $P(k+1)$ is true. This is sometimes broken into two steps, but they go together: Assume that $P(k)$ is true, then show that with this assumption, $P(k+1)$ must be true.

Exercises

1. Prove each using induction:

- | | |
|---|--|
| (a) $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ | (e) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ |
| (b) $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ | (f) $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$ |
| (c) $\sum_{i=1}^n 2^{i-1} = 2^n - 1$ | (g) $\sum_{i=1}^n (2i-1) = n^2$ |
| (d) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ | (h) $n! > 2^n$ for $n \geq 4$. |
| | (i) $2^{n+1} > n^2$ for all positive integers. |

2. This exercise refers to the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

The sequence is defined recursively by $f_1 = 1$, $f_2 = 1$, then $f_{n+1} = f_n + f_{n-1}$ for each $n > 2$. As before, prove each of the following using induction. You might want to investigate each with several examples before you start.

- (a) $f_1 + f_2 + \dots + f_n = f_{n+2} - 1$
- (b) $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$
- (c) $f_1 + f_3 + f_5 + \dots + f_{2n-1} = f_{2n}$