Exercise: let p, q, r be any proposition. Show with the help of a truth table that

$$p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$$

i.e. that $p \Rightarrow (q \vee r)$ and $(p \Rightarrow q) \vee (p \Rightarrow r)$ are logically equivalent.

p	q	r	$(q \vee r)$	$p \Rightarrow (q \vee r)$	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Now, with the help of Boolean algebra:

Notice that $p \Rightarrow q \equiv \neg p \vee q$ and $p \Rightarrow r \equiv \neg p \vee r$

Therefore $(p \Rightarrow q) \vee (p \Rightarrow r) \equiv (\neg p \vee q) \vee (\neg p \vee r)$

by the substitution law. Furthermore due to associativity of the $\vee$ operator it follows that

$(\neg p \vee q) \vee (\neg p \vee r) \equiv \neg p \vee (q \vee \neg p) \vee r$

Then due to the commutitative property of the $\vee$ operator it follows that:

$$\neg p \vee (q \vee \neg p) \vee r \equiv \neg p \vee (\neg p \vee q) \vee r$$

Now again apply associativity of $\vee$ we get:

$$\neg p \vee (\neg p \vee q) \vee r \equiv (\neg p \vee \neg p) \vee (q \vee r) \; \Delta$$

Now notice due to distributivity of $\vee$ we can write

$$\neg p \vee \neg p \equiv \neg (p \vee p)$$

Therefore due to the idempotent laws of $\vee$

we have $\neg(p \vee p) \equiv \neg p \; \nabla$

Now combining Δ and ∇ we get:

$$\neg p \vee (q \vee r) \equiv p \Rightarrow (q \vee r)$$

$\square$

Recall that a proposition consists of an entity / subject and ~~and~~ predicate.

e.g. entity "The moon". predicate "is made of cheese"

Let a be the entity "The moon" and P be the predicate "is made of cheese" then P(a) means "The moon is made of cheese" is a proposition.

Another entity is b = "dutchmen"

then P(b) means "dutchmen are made of cheese."

Def Domain of Discourse and One-Place Predicate

A <u>one-place predicate</u> P associates single proposition with every entity of some collection of entities.

We call this collection of entities the <u>domain of discourse</u> of predicate P

Let P be an arbitrary predicate and a an entity in its domain of discourse Then P(a) is a proposition

We also say predicate P <u>maps</u> entity a to proposition P(a).

Def <u>Two-place</u> or more-place predicate

Associates two or more entities to a predicate.

e.g. Let a be the entity "Jesse"
 , b ~~be the~~ "Jane".
and P be the two-place predicate "loves"
Then P(a, b) is "$\underbrace{\text{Jesse}}_{a}$ $\underbrace{\text{loves}}_{P}$ $\underbrace{\text{Jane}}_{b}$".

e.g. Compound Proposition

let R be the one-place predicate "is red"

Then

$R(a) \wedge R(b)$ "a is red and b is red" ← Open statement

$\neg R(a)$ "a is not red"

Free variable

Def Quantifiers

universal

$\forall$ means "all"

$\exists$ means "there exists" or "some"
$\exists!$ means "there exists exactly one" } Existential

Def Bound

A free variable in an open statement that can be quantified say x is called bound.

e.g. let $L(x,y)$ represent "x loves y".

Then $\forall y (L(x,y))$ means:

"x loves everyone" ← open statement

and $\exists x (\forall y L(x,y))$ means

"There is someone that loves everyone".

↑ Proposition

e.g. let a be an entity in the d.o.d.

Then $P(a) \Rightarrow Q(a)$ is a proposition

$P(x) \Rightarrow Q(x)$ is open

$$\left.\begin{array}{l} \forall x(P(x) \Rightarrow Q(x)) \\ P(a) \wedge (\exists x\, Q(x)) \\ (\forall x\, P(x)) \Rightarrow (\exists x\, P(x)) \end{array}\right\} \text{Propositions}$$

e.g. let $H(x)$ represent "x is happy"

Then $\forall x\,(H(x))$ means

"everyone is happy"

let $O(x, y)$ represent "x owns y"

and $C(y)$ represent "y is a computer"

Then $\forall x(O(\text{"Jack"}, x) \Rightarrow C(x))$

means "everything Jack owns is a computer"

other examples can be found in the book.

Predicate Logic Algebra

$$\neg(\forall x\ P(x)) \equiv \exists x(\neg P(x))$$

$$\neg(\exists x\ P(x)) \equiv \forall x\,(\neg P(x))$$

$$\forall x \forall y\ Q(x,y) \equiv \forall y \forall x\ Q(x,y)$$

$$\exists x \exists y\ Q(x,y) \equiv \exists y \exists x\, Q(x,y)$$

e.g. $\neg(\forall Y(R(Y) \vee Q(Y))) \equiv \exists Y \neg(R(Y) \vee Q(Y))$

$\equiv \exists Y (\neg R(Y) \wedge Q(Y))$

Deduction

argument
Premise { "$\underbrace{\text{If today is tuesday}}_{\text{'p'}}$ $\underbrace{\text{then}}_{\Rightarrow}$ $\underbrace{\text{this is in amsterdam}}_{\text{"q"}}$"
"today is tues, day" } "p"

Conclusion { $\therefore$ This is amsterdam

e.g. let p and q be propositions

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline \therefore q \end{array}$$
} Premise i.e. assume true

$$\begin{array}{l} p \Rightarrow q \equiv \text{"}\neg q \Rightarrow \neg p\text{"} \\ \neg q \\ \hline \therefore \neg p \end{array} \qquad \begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore p \Rightarrow r \end{array}$$

eg. let $P(X)$ be the statement "X is mortal"
and $Q(X)$ be the statement "Socrates is X"
Then:

$$\begin{array}{l} \forall X(P(X) \Rightarrow Q(X)) \\ P(a) \\ \hline \therefore Q(a) \end{array}$$

i.e. All humans are mortal
Socrates is human
$\therefore$ Socrates is mortal

here entity a is socrates.

e.g.

$$\begin{array}{l} (p \wedge r) \Rightarrow s \\ q \Rightarrow p \\ t \Rightarrow r \\ q \\ t \\ \hline \therefore s \end{array}$$

Same follows for predicate logic..

e.g.

$$\begin{array}{l} \forall x \, (P(x) \Rightarrow Q(x)) \\ P(a) \\ \hline \therefore Q(a) \end{array}$$

e.g. let $P(n)$ be the statement
"n is even"
and $P(n^2)$ is the statement
"n^2 is even"
prove the statement

$$\begin{array}{l} P(n) \\ \hline \therefore P(n) \Rightarrow P(n^2) \end{array}$$

i.e. "If n is even then n^2 is even".

Proof
Since n is even we can write $n = 2k$ for some integer k. Notice then that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ Now since $2k^2$ is also some integer we conclude that n^2 is even given that n is even $\square$

e.g. Prove the statement

$$\frac{\neg P(n)}{\therefore \neg P(n) \Rightarrow \neg P(n^2)}$$

i.e. "If n is odd then n^2 is odd"

Proof
Since n is odd we can write that
$n = 2k+1$ for some integer k.
Now notice that $n^2 = (2k+1)^2$
$= 4k^2 + 4k + 1 = 2(2k^2+2k)+1$
Now since $2k^2+2k$ is again some integer we conclude that n^2 is odd given that n is odd $\square$

e.g. Is the following argument valid?

$$\begin{array}{l} \neg P(n) \Rightarrow \neg P(n^2) \\ P(n) \Rightarrow P(n^2) \\ \hline \therefore P(n) \Leftrightarrow P(n^2) \end{array}$$

i.e.

"n is odd then n^2 is odd"
"n is even then n^2 is even"
∴ "n is even only if n^2 is even"

Answer: Yes, since
$\neg P(n) \Rightarrow \neg P(n^2) \equiv P(n^2) \Rightarrow P(n)$ "contrapositive"
Using this equivalence we rewrite the argument as:

$$\begin{array}{l} P(n^2) \Rightarrow P(n) \\ P(n) \Rightarrow P(n^2) \\ \hline \therefore P(n) \Leftrightarrow P(n^2) \end{array}$$

which is trivially true because this is the definition of the bi-conditional operator "$\Leftrightarrow$" $\square$