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Quiz

Q 1. Is the function $f:[0,2\pi] \longrightarrow \mathbb{R},\ x \rightarrow x$

a) Injective

b) Surjective

c) Bijective

answer : a

Q2. Is the function $f:[0,\pi] \rightarrow [-1,1],\ x \rightarrow \cos(x)$

a) Injective

b) Surjective

c) Bijective

Answer : c

Q3. $g: \mathbb{Z} \rightarrow \mathbb{Z},\ x \rightarrow 2x$

a) Injective

b) Surjective

c) Bijective

Answer : c

Q4 $g: \mathbb{R} \rightarrow \mathbb{R},\ x \rightarrow x^2$

a) Injective

b) Surjective

c) Bijective

Answer : b

def Composition of Functions $\circ$

Let $f: A \to B$ and $g: B \to C$ be functions where A, B and C are arbitrary maps

then $g \circ f: A \to C$, $a \to g(f(a))$

is called the composition of the functions f and g.

E.g. Let $f: \mathbb{R} \to \mathbb{R}$, $x \to 2x+1$ and $g: \mathbb{R} \to \mathbb{R}$, $x \to x^2$.

Then $g \circ f: \mathbb{R} \to \mathbb{R}$, $x \to \underbrace{(2x+1)^2}$

$$\underbrace{x \to 2x+1}_{f} \underbrace{\to (2x+1)^2}_{g}$$

Remark: Neural Networks are the compositions of linear and activation functions!

"commutative"

<u>Question</u> Does $f \circ g = g \circ f$ for every function f and g?

Answer: No, for example take our previous example functions f and g

$g \circ f: \mathbb{R} \to \mathbb{R}$, $x \to (2x+1)^2$

$f \circ g: \mathbb{R} \to \mathbb{R}$, $x \to 2x^2+1$

def Inverse function

Suppose $f: A \to B$ is function

Then $f^{-1}: B \to A$ is called its inverse

satisfying $(f^{-1} \circ f)(x) = x$ for every $x \in A$ (Left inverse)

and $(f \circ f^{-1})(y) = y$ for every $y \in B$. (Right inverse)

Note: f has to be bijective!

def Cardinality

Let A be a finite set with n elements. Then the cardinality of A is n.

Denoted $|A| = n$

def Finite set

A set A is finite if there exists a bijection (or bijective function)

$$A \to \{1, 2, \dots, n\}$$

for some $n \in \mathbb{N} \cup \{0\}$

($\emptyset$ is also finite)

Q_1 Suppose A and B are disjoint i.e.

$$A \cap B = \emptyset$$

and m and n are their respective cardinalities. What is the cardinality of $A \cup B$?

Answer: $|A \cup B| = m + n$

Q_2 What if A and B are <u>not</u> disjoint?

Theorem Let A and B be finite sets. Then

$$|A \times B| = |A| \cdot |B|$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A^B| = |A|^{|B|}$$ where A^B denotes the set of functions $f : A \to B$

$$|P(A)| = 2^{|A|}$$

def Countably infinite

A set A is countably infinite if there exists a bijection $A \to \mathbb{N}$

e.g. $\mathbb{Z}$ is countably infinite

$f : \mathbb{Z} \to \mathbb{N}$, $z \to f(z)$ is a bijection

$$f(z) = \begin{cases} 2z-1 & \text{if } n \geq 0 \\ -2z-1 & \text{if } n < 0 \end{cases}$$

def Uncountably infinite

A set A is called uncountably infinite if there exists a bijection $A \to \mathbb{R}$

e.g. $]-\frac{1}{2}\pi, \frac{1}{2}\pi[$ is uncountably infinite

"interval"

$f :]-\frac{1}{2}, \frac{1}{2}\pi[\to \mathbb{R}$, $x \to \tan(x)$ is bijective def A (Binary) relation R on two sets A and B is a subset of $A \times B$.

def Let R be a relation on A i.e. R is a subset of $A \times A$. Then the relation R is called <u>reflexive</u> if for every $a \in R$ we have that $(a, a) \in R$. The latter is commonly denoted as aRa
"a is related to a"

def Let R again be a relation on A.
Then R is called <u>transitive</u> if:

- For every $a, b, c \in A$ we have that
$$aRb \text{ and } bRc \Rightarrow aRc.$$

def Let R be any relation on sets A and B.
Then R is called <u>symmetric</u> if:
For every $a \in A$ and every $b \in B$ we have that:
$$aRb \Rightarrow bRa$$

def Let R again be any relation on sets A and B
Then R is called <u>anti-symmetric</u> if
For every $a \in A$ and every $b \in A$ we have
$$(aRb \text{ and } bRa) \Rightarrow a = b$$

e.g. **Reflexive**

The $\leq$ sign is a reflexive relation on $\mathbb{N}$ since for every $n \in \mathbb{N}$ we have $n \leq n$

e.g. **Transitive**

The $\leq$ sign is a transitive relation on $\mathbb{N}$ since for every $a, b, c \in \mathbb{N}$

if $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

e.g. **Counter Symmetric**

The $\leq$ sign is not symmetric relation on $\mathbb{N}$ since for every $n, m \in \mathbb{N}$

we cannot conclude

if $m \leq n \Rightarrow n \leq m$

take $m = 1$ and $n = 2$ then $1 \leq 2$

but $2 \leq 1$ because $2 > 1$.

e.g. **Anti - Symmetric**

The relation $\leq$ on $\mathbb{N}$ is anti - symmetric since for every $m, n \in \mathbb{N}$ we have that

if $m \leq n$ and $n \leq m \Rightarrow m = n$