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# FAST LEGENDRE–FENCHEL TRANSFORM AND APPLICATIONS TO HAMILTON–JACOBI EQUATIONS AND CONSERVATION LAWS*

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**Abstract.** We are interested in the study of a fast algorithm introduced by Brenier computing the discrete Legendre–Fenchel transform of a real function. We present convergence results and show how the order of convergence grows with the regularity of the function to be transformed. Applications to Hamilton–Jacobi equations for front propagation problems and conservation laws are presented.

**Key words.** Legendre–Fenchel transform, Hamilton–Jacobi equation, conservation laws, Hopf's formula

**AMS subject classifications.** Primary, 49L25, 26B25, 35L65; Secondary, 65M

**1. Introduction.** The Legendre–Fenchel transform plays an important role in convex analysis and in the theory of nonlinear differential equations of first order. For example, it can be used to solve a class of Hamilton–Jacobi equations with explicit formulas.

If $u$ is a function defined on $\mathbb{R}^d$ with values in $\mathbb{R} \cup \{+\infty\}$, but $u \not\equiv +\infty$, i.e., $u$ is a *proper* function on $\mathbb{R}^d$, its Legendre–Fenchel transform is defined by

$$u^*(\mathbf{y}) := \sup_{\mathbf{x}\in\mathbb{R}^d} \{\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})\}, \qquad \mathbf{y} \in \mathbb{R}^d \tag{1.1}$$

that is a lower semicontinuous and convex function on $\mathbb{R}^d$ not necessarily proper.

Our interest here is the study of a fast algorithm introduced by Brenier [Br] to compute (1.1) when $\mathbb{R}^d$ is replaced by a bounded set. The computation of the supremum in (1.1) on an unbounded domain is, in fact, one of the two most difficult problems which occur in the discretisation of the Legendre–Fenchel transform. The other one is the computational cost that can be high if the discretisation of (1.1) is not optimal in analogy with the case of the Fourier transform.

In this paper, we consider the fast algorithm shown in [Br] ((FLT) algorithm in the sequel, for fast Legendre transform), and we investigate the convergence and the order of convergence of this one. It will be proved that the algorithm converges if $u$ satisfies the condition $\underline{(\overline{u})} = \underline{u}$, where $\underline{u}$ and $\overline{u}$ are the lower and the upper semicontinuous envelope of $u$, respectively, and that the order of the algorithm grows with the regularity of $u$. Moreover, some of the Legendre–Fenchel transform properties will be analyzed in order to determine if they still hold true for the discrete transform.

As an application, we will consider the initial value problem for the Hamilton–Jacobi equation

$$\begin{aligned} &u_t + H(Du) = 0 &&\text{in } \mathbb{R}^d \times (0,\infty), \\ &u(\mathbf{x},0) = u_0(\mathbf{x}) &&\text{on } \mathbb{R}^d. \end{aligned} \tag{1.2}$$

If it is assumed

$$\begin{aligned} &u_0 : \mathbb{R}^d \longrightarrow \mathbb{R} &&\text{uniformly Lipschitz and convex,} \\ &H : \mathbb{R}^d \longrightarrow \mathbb{R} &&\text{continuous,} \end{aligned} \tag{$H_1$}$$

then the unique viscosity solution (in the sense of Crandall and Lions [CL]) of problem (1.2) is given by

$$u(\mathbf{x},t) = \sup_{\mathbf{y}\in\mathbb{R}^d} \inf_{\mathbf{z}\in\mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \tag{1.3}$$

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while under hypotheses

$$(H_2) \qquad \begin{cases} u_0 : \mathbb{R}^d \longrightarrow \mathbb{R} & \text{uniformly Lipschitz,} \\ H : \mathbb{R}^d \longrightarrow \mathbb{R} & \text{convex,} \\ \lim_{|\mathbf{y}| \to +\infty} H(\mathbf{y}) = +\infty, & \end{cases}$$

the unique viscosity solution is

$$u(\mathbf{x}, t) = \inf_{\mathbf{z} \in \mathbb{R}^d} \sup_{\mathbf{y} \in \mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\} \tag{1.4}$$

(see [H], [BE], [L]). So using definition (1.1), it is immediate to express (1.3) and (1.4) through a combination of Legendre–Fenchel transforms and the FLT algorithm can be applied if $\mathbb{R}^d$ can be substituted with bounded sets in (1.3) and (1.4). We will show how to do that under suitable hypotheses on the Hamiltonian $H$ eventually added.

Finally, the Buckley–Leverett equation will be considered as a further test. Using the correspondence existing between Hamilton–Jacobi equations and conservation laws in one dimension, we apply the algorithm first to the Hamilton–Jacobi equation associated with the Buckley–Leverett equation and then we turn to it.

The outline of the paper is as follows. In §2 we review the FLT algorithm, we prove the convergence of this one, and we give sharp estimates to show the order of convergence. Section 3 is dedicated to the properties of the discrete Legendre–Fenchel transform, while in §4 the initial value problem (1.2) is considered and convergence results of the algorithm applied to (1.3) and (1.4) are proved. Section 5 is devoted to numerical experiments.

**2. Convergence of the FLT algorithm.** Before proving the convergence of the discrete transform, we briefly describe how the FLT algorithm works.

**2.1. Description of the FLT algorithm.** As stated in the introduction, for the numerical computation of the Legendre–Fenchel transform (1.1), the supremum in (1.1) has to be a maximum on a bounded set of $\mathbb{R}^d$. Naturally, it depends on the function $u$ if it is true or not. In particular, it is easy to see that for any compact subset $K$ of the domain $d(u^*) := \{\mathbf{y} \in \mathbb{R}^d : u^*(\mathbf{y}) < +\infty\}$ of $u^*$, there exists a compact subset $K'$ of $\mathbb{R}^d$ such that

$$u^*(\mathbf{y}) = \max_{\mathbf{x} \in K'} \{\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})\} \qquad \forall\ \mathbf{y} \in K$$

if and only if $u$ is superlinear and $d(u^*) = \cup_{\mathbf{x} \in S} (\partial_- u)(\mathbf{x})$, where

$$(\partial_- u)(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d : u(\mathbf{z}) \geq u(\mathbf{x}) + \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \quad \forall\, \mathbf{z} \in \mathbb{R}^d\}$$

is the subdifferential of $u$ at $\mathbf{x}$ and $S$ is the set of points $\mathbf{x}$ where $u$ is subdifferentiable (i.e., $(\partial_- u)(\mathbf{x}) \neq \emptyset$).

In order to have a numerical approximation of the transform, we observed previously that the Legendre–Fenchel transform can be factorized in unidimensional transforms since

$$u^*(\mathbf{y}) = \sup_{x_d} \left\{ x_d y_d + \sup_{x_{d-1}} \left\{ \cdots + \sup_{x_1} \{x_1 y_1 - u(\mathbf{x})\} \cdots \right\} \right\}. \tag{2.1}$$

Moreover, if the supremum in (2.1), with $d = 1$, has to be computed for $x$ in a bounded interval of $\mathbb{R}$, through a change of variables we can obtain a supremum in the unit interval $[0, 1]$. Therefore, it is sufficient to consider the following simplified problem without loss of generality.

Let $u$ be a function defined on $I := [0, 1]$ and let

$$u^*(y) := \max_{x\in I}\{xy - u(x)\}, \qquad y \in I, \tag{2.2}$$

and

$$h(y) := \operatorname*{Argmax}_{x\in I}\{xy - u(x)\}, \qquad y \in I, \tag{2.3}$$

be, respectively, the Legendre transform of $u$ in $I$ and the corresponding argument function from $I$ to $I$.

Hereafter we will set $I_N := \left\{\frac{1}{N}, \frac{2}{N}, \dots, 1\right\}$ with $N$ as a power of 2. A discretisation of $u^*(y)$ and $h(y)$ is immediately given for any $y \in I$ by

$$u_N^*(y) := \max_{x\in I_N}\{xy - u(x)\} \tag{2.4}$$

and

$$h_N(y) := \operatorname*{Argmax}_{x\in I_N}\{xy - u(x)\}. \tag{2.5}$$

The computation of $(u_N^*(y),\ y\in I_N)$ has an a priori cost of $0(N^2)$ operations. In order to obtain a fast algorithm it is simply necessary to observe that the problem can be decomposed into subproblems of lower degree. In fact, set $\varepsilon := \frac{1}{2N}$, we have

$$u_{2N}^*(y) = \max\{u_N^*(y), -\varepsilon y + \tilde{u}_N^*(y)\}, \qquad y \in I, \tag{2.6}$$

where

$$\tilde{u}_N^*(y) := \max_{x\in I_N}\{xy - u(x - \varepsilon)\}, \qquad y \in I, \tag{2.7}$$

and

$$h_{2N}(y) = \begin{cases} h_N(y) & \text{if } u_{2N}^*(y) = u_N^*(y), \\ \tilde{h}_N(y) & \text{if } u_{2N}^*(y) = -\varepsilon y + \tilde{u}_N^*(y), \end{cases}$$

where

$$\tilde{h}_N(y) := \operatorname*{Argmax}_{x\in I_N}\{xy - u(x - \varepsilon)\}.$$

Therefore, assuming we know the two vectors $(u_N^*(y),\ y\in I_N)$ and $(\tilde{u}_N^*(y),\ y\in I_N)$, in order to compute the discrete transform with $2N$ points $(u_{2N}^*(y),\ y\in I_{2N})$ it is sufficient to compute the two vectors $(u_N^*(y),\ y\in I_N - \varepsilon)$ and $(\tilde{u}_N^*(y),\ y\in I_N - \varepsilon)$. Finally, because $h_N$ and $\tilde{h}_N$ are increasing functions (see [Br]), we can write for any $y$ in $I_N - \varepsilon$

$$u_N^*(y) = \max\{xy - u(x);\ x\in I_N,\ h_N(y-\varepsilon) \le x \le h_N(y+\varepsilon)\}$$

and

$$\tilde{u}_N^*(y) = \max\{xy - u(x-\varepsilon);\ x\in I_N,\ \tilde{h}_N(y-\varepsilon) \le x \le \tilde{h}_N(y+\varepsilon)\}.$$

Consequently, the computation of $(u_N^*(y),\ y\in I_N - \varepsilon)$ and $(\tilde{u}_N^*(y),\ y\in I_N - \varepsilon)$ from the known vectors $(u_N^*(y),\ y\in I_N)$ and $(\tilde{u}_N^*(y),\ y\in I_N)$ has a cost of $0(N)$ operations while the total cost of a transform with $N$ points becomes $0(N\log N)$.

When we deal with more than one dimension ($d > 1$), the analogues of definitions (2.2), (2.3), (2.4), and (2.5) are immediately obtained by substitution of $I$ and $I_N$ with $I^d$ and $I_N^d$, respectively. The algorithm is as before, but since they have to be computed $d$ Legendre–Fenchel transforms, the computational cost becomes $0(N^d \log N)$.

**2.2. Convergence of the discrete Legendre–Fenchel transform.** For the sake of generality, we will investigate the convergence and the order of convergence of the discrete Legendre–Fenchel transform directly in dimension $d$. Setting $\overline{u}$ and $\underline{u}$ the upper and the lower semicontinuous envelope of $u$, respectively, we have Theorem 2.1.

THEOREM 2.1. *If $u$ satisfies the condition $\underline{(\overline{u})} = \underline{u}$, then $u_N^*$ converges pointwise to $u^*$ on $I^d$ when $N$ goes to infinity. Moreover, if $u$ is continuous, the convergence is uniform on $I^d$.*

*Proof.* We immediately observe that for any $\mathbf{y} \in I^d$ and independently of the properties of $u$, $u_N^*(\mathbf{y})$ is an increasing sequence upper bounded by $u^*(\mathbf{y})$ and then it converges. In order to prove that $u_N^*$ converges to $u^*$, let $\mathbf{j}(N) := (j_1, \dots, j_d) \in \mathbb{N}^d$ with $1 \leq j_i \leq N$, and for any $\mathbf{x} \in I^d$ let $\mathbf{x}_{\mathbf{j}(N)} := \mathbf{j}(N)/N$ be the node of the uniform grid $I_N^d$ such that

$$|\mathbf{x} - \mathbf{x}_{\mathbf{j}(N)}| \leq \sqrt{d}\, N^{-1}.$$

Since $u$ satisfies $\underline{(\overline{u})} = \underline{u}$ and $u^* = (\underline{u})^*$ for any function $u$, we have

$$\sup_{\mathbf{x} \in I^d}\{\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})\} = \sup_{\mathbf{x} \in I^d}\{\mathbf{x} \cdot \mathbf{y} - \overline{u}(\mathbf{x})\} \qquad \forall\ \mathbf{y} \in I^d.$$

Given any $\varepsilon > 0$ there exists $\mathbf{z} \in I^d$ such that

$$\sup_{\mathbf{x} \in I^d}\{\mathbf{x} \cdot \mathbf{y} - \overline{u}(\mathbf{x})\} \leq \mathbf{z} \cdot \mathbf{y} - \overline{u}(\mathbf{z}) + \frac{\varepsilon}{3},$$

while, for such $\varepsilon$ and $\mathbf{z}$, there exists a neighbourhood $U(\mathbf{z})$ of $\mathbf{z}$ such that

$$\overline{u}(\mathbf{x}) \leq \overline{u}(\mathbf{z}) + \frac{\varepsilon}{3} \qquad \forall\ \mathbf{x} \in U(\mathbf{z}).$$

Finally, if $N$ is great enough to have $\mathbf{z}_{\mathbf{j}(N)} \in I_N^d \cap U(\mathbf{z})$ and $d\, N^{-1} \leq \varepsilon/3$, we obtain

$$\begin{aligned} 0 \leq u^*(\mathbf{y}) - u_N^*(\mathbf{y}) &= \sup_{\mathbf{x} \in I^d}\{\mathbf{x} \cdot \mathbf{y} - \overline{u}(\mathbf{x})\} - \max_{\mathbf{x} \in I_N^d}\{\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})\} \\ &\leq \mathbf{z} \cdot \mathbf{y} - \overline{u}(\mathbf{z}) + \frac{\varepsilon}{3} - \mathbf{z}_{\mathbf{j}(N)} \cdot \mathbf{y} + u(\mathbf{z}_{\mathbf{j}(N)}) \qquad (2.8) \\ &\leq |\mathbf{y}||\mathbf{z} - \mathbf{z}_{\mathbf{j}(N)}| + \overline{u}(\mathbf{z}_{\mathbf{j}(N)}) - \overline{u}(\mathbf{z}) + \frac{\varepsilon}{3} \leq \varepsilon; \end{aligned}$$

i.e., $u_N^*(\mathbf{y})$ converges to $u^*(\mathbf{y})$.

If $u$ is continuous on $I^d$, then $u$ is uniformly continuous, $u = \overline{u}$, the supremum is achieved, and inequality (2.8) becomes uniform on $\mathbf{y} \in I^d$. $\square$

*Remark* 2.2. Obviously any upper semicontinuous function $u$ verifies $\underline{(\overline{u})} = \underline{u}$ so that the convergence is assured. On the contrary, if $u$ is lower semicontinuous, $u_N^*(\mathbf{y})$ may not converge to $u^*(\mathbf{y})$. As an example we consider the function

$$u(x) := \begin{cases} |x - k| & \text{if } x \neq k, \\ -1 & \text{if } x = k, \end{cases}$$

where $k \in (0, 1]/I_N$ for any $N$. It is easy to see that

$$u^*(y) = \begin{cases} 1 + ky & \text{if } |y| \leq 1, \\ +\infty & \text{if } |y| > 1, \end{cases}$$

while, for any $y \in [-1, 1]$,

$$\begin{aligned} u_N^*(y) &= \max_{x \in I_N}\{xy - u(x)\} \\ &= \begin{cases} \frac{1}{N}y - (\frac{1}{N} - k) & \text{if } 0 < k < \frac{1}{N}, \\ \max\left\{\frac{j+1}{N}(y-1) + k;\ \frac{j}{N}(y+1) - k\right\} & \text{if } \frac{j}{N} < k < \frac{j+1}{N}, \end{cases} \end{aligned}$$

where $j \in \{1, \dots, N-1\}$. Then $u_N^*(y)$ converges uniformly on $[-1, 1]$ to $w^*(y) = ky$ that is the Legendre–Fenchel transform in $[-1, 1]$ of $w(x) = |x - k|$.

In order to have an error estimate, it is necessary to suppose $u$ more than continuous. In particular, the order of convergence grows with the regularity of $u$ as shown in the following proposition.

THEOREM 2.3. *If $u$ is Lipschitz continuous on $I^d$, then*

(i) $\|u_{2N}^* - u_N^*\|_{L^\infty(I^d)} \le (\frac{1}{2}\sqrt{d}L)N^{-1}$,

(ii) $\|u^* - u_N^*\|_{L^\infty(I^d)} \le c_1 N^{-1}$,

*where $L$ is the Lipschitz constant of $u$ and $c_1 = c_1(L, \sqrt{d})$. Moreover, if $u$ is a $C^2$ function on a neighbourhood of $I^d$, then*

(iii) $\|u^* - u_N^*\|_{L^\infty(I^d)} \le c_2 N^{-2}$,

*where $c_2 = c_2(\|D^2 u\|_{L^\infty(I^d)}, \sqrt{d})$.*

*Proof.* First let us prove (i) in one dimension. For all $y \in I$ and for all $x \in I_N$, we have

$$u_N^*(y) = h_N(y)y - u(h_N(y)) \ge xy - u\left(x - \frac{1}{2N}\right) - \frac{L}{2N}$$

and from (2.7) we have

$$\tilde{u}_N^*(y) \le u_N^*(y) + \frac{L}{2N}.$$

Then the estimate follows by (2.6).

For $d = 2$ it is sufficient to write the analogues of (2.6); i.e.,

$$u_{2N}^*(\mathbf{y}) = \max\left[u_N^*(\mathbf{y});\ \tilde{u}_N^{*1}(\mathbf{y}) - \varepsilon y_1;\ \tilde{u}_N^{*2}(\mathbf{y}) - \varepsilon y_2;\ \tilde{u}_N^{*12}(\mathbf{y}) - \varepsilon y_1 - \varepsilon y_2\right] \tag{2.9}$$

where

$$\tilde{u}_N^{*1}(\mathbf{y}) := \max_{\mathbf{x} \in I_N^2}\{\mathbf{x} \cdot \mathbf{y} - u(x_1 - \varepsilon, x_2)\},$$

$$\tilde{u}_N^{*2}(\mathbf{y}) := \max_{\mathbf{x} \in I_N^2}\{\mathbf{x} \cdot \mathbf{y} - u(x_1, x_2 - \varepsilon)\},$$

$$\tilde{u}_N^{*12}(\mathbf{y}) := \max_{\mathbf{x} \in I_N^2}\{\mathbf{x} \cdot \mathbf{y} - u(x_1 - \varepsilon, x_2 - \varepsilon)\},$$

and $\mathbf{y} \in I^2$. Reasoning as before, (2.9) immediately gives (i). The generalization to higher dimension is obvious.

Set $\mathbf{z} := h(\mathbf{y})$ and define $\mathbf{z}_{\mathbf{j}(N)}$ as in the proof of Theorem 2.1; we have for any $\mathbf{y} \in I^d$

$$u^*(\mathbf{y}) = \mathbf{z} \cdot \mathbf{y} - u(\mathbf{z}) \le |\mathbf{y}||\mathbf{z} - \mathbf{z}_{\mathbf{j}(N)}| + L|\mathbf{z} - \mathbf{z}_{\mathbf{j}(N)}| + u_N^*(\mathbf{y}) \le \frac{(L+\sqrt{d})\sqrt{d}}{N} + u_N^*(\mathbf{y}),$$

i.e., estimate (ii).

Finally, using the regularity of $u$ and deriving $\{\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})\}$ with respect to $x_i$, $i = 1, \dots, d$, we have that $\mathbf{y} = Du(\mathbf{z})$, where $\mathbf{z} := h(\mathbf{y})$. The Taylor expansion of $u$ at $\mathbf{z}$ gives us

$$\begin{aligned} u^*(\mathbf{y}) &= \mathbf{z} \cdot Du(\mathbf{z}) - u(\mathbf{z}) \\ &= \mathbf{z}_{\mathbf{j}(N)} \cdot Du(\mathbf{z}) - (\mathbf{z}_{\mathbf{j}(N)} - \mathbf{z}) \cdot Du(\mathbf{z}) - u(\mathbf{z}) \\ &= \mathbf{z}_{\mathbf{j}(N)} \cdot Du(\mathbf{z}) - u(\mathbf{z}_{\mathbf{j}(N)}) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial u(\xi)}{\partial x_i x_j}(\mathbf{z}_{\mathbf{j}(N)} - \mathbf{z})_i(\mathbf{z}_{\mathbf{j}(N)} - \mathbf{z})_j; \end{aligned}$$

hence,

$$u^*(\mathbf{y}) \le u_N^*(\mathbf{y}) + \|D^2u\|_{L^\infty(I^d)}\ |\mathbf{z}-\mathbf{z}_{\mathbf{j}(N)}|^2$$

and estimate (iii) is proved. □

*Remark* 2.4. In the general case when $I^d$ is replaced by an arbitrary hypercube $K \subset \mathbb{R}^d$ of size $l$, Theorems 2.1 and 2.3 still hold true. In this case $I_N^d$ is replaced by the uniform grid $K_N$ obtained while introducing a uniform grid of mesh size $lN^{-1}$ on every edge of $K$, so that $|\mathbf{x}'_N - \mathbf{x}''_N| \le l\sqrt{d}N^{-1}$ for any $\mathbf{x}'_N, \mathbf{x}''_N \in K_N$.

*Remark* 2.5. It can be proved that both estimates (ii) and (iii) in Theorem 2.3 are precise. First let us consider the Lipschitz continuous function on $\mathbb{R}$, $u(x) := |x-k|$ with $k \in (0,1]/I_N$ for any $N$, and we have

$$u^*(y) = \begin{cases} ky & \text{if } |y| \le 1, \\ +\infty & \text{if } |y| > 1, \end{cases}$$

and, for any $y \in [-1,1]$,

$$u_N^*(y) = \begin{cases} \frac{1}{N}y - (\frac{1}{N} - k) & \text{if } 0 < k < \frac{1}{N}, \\ \max\left\{\frac{j}{N}y - u\left(\frac{j}{N}\right);\ \frac{j+1}{N}y - u\left(\frac{j+1}{N}\right)\right\} & \text{if } \frac{j}{N} < k < \frac{j+1}{N}, \end{cases}$$

where $j \in \{1,\dots,N-1\}$. Then

$$u^*(y) - u_N^*(y) = (y-1)(k - \frac{1}{N}) \qquad \text{if } 0 < k < \frac{1}{N}$$

or

$$u^*(y) - u_N^*(y) = \begin{cases} (y+1)(k - \frac{j}{N}) \\ (y-1)(k - \frac{j+1}{N}) \end{cases} \qquad \text{if } \frac{j}{N} < k < \frac{j+1}{N}.$$

Anyway, $\|u^* - u_N^*\|_{L^\infty([-1,1])} = 0(N^{-1})$, so estimate (ii) is accurate.

Let us now consider the $C^2(\mathbb{R})$ function $u(x) := \frac{1}{3}|x|^3$. Its Legendre–Fenchel transform is $u^*(y) := \frac{2}{3}|y|^{3/2}$ for any $y \in \mathbb{R}$, while the associated argument function is $h(y) = \sqrt{y}$. For the discrete transform computed in $y \in I$ we have

$$u_N^*(y) = \max_{x \in I_N}\left\{xy - \frac{1}{3}|x|^3\right\} = yh_N(y) - \frac{1}{3}|h_N(y)|^3$$

where $h_N(y) = \frac{1}{N}$ if $0 \le \sqrt{y} < \frac{1}{N}$, otherwise $h_N(y)$ is equal to $\frac{j}{N}$ or to $\frac{j+1}{N}$ where $j \in \{1,\dots,N-1\}$ is such that $\frac{j}{N} \le \sqrt{y} \le \frac{j+1}{N}$. In any case we obtain that

$$\begin{aligned} u^*(y) - u_N^*(y) &= yh(y) - u(h(y)) - yh_N(y) + u(h_N(y)) \\ &= u(h_N(y)) - u(h(y)) + u'(h(y))\,(h(y) - h_N(y)) \\ &= \frac{1}{2}u''(\xi)\,(h_N(y) - h(y))^2 = 0(N^{-2}); \end{aligned}$$

i.e., $\|u^* - u_N^*\|_{L^\infty(I)} = 0(N^{-2})$, which shows that estimate (iii) is also precise.

**2.3. Convergence of the discrete subdifferentials.** It is important to not forget that whenever $u$ is a proper function on $\mathbb{R}^d$, the Legendre–Fenchel transform $u^*$ is a lower semicontinuous convex function on $\mathbb{R}^d$ and that the same is true for $u_N^*$ on $I^d$. Then classical results of convex analysis give us the convergence of $(\partial_- u_N^*)$ to $(\partial_- u^*)$.

THEOREM 2.6. *If $u$ is a continuous function on $\mathbb{R}^d$ and $u^*$ is proper, given any $\mathbf{y} \in \mathring{I}^d$, $\delta > 0$ and $\varepsilon > 0$ such that $B(\mathbf{y}, \varepsilon^\delta) \subseteq \mathring{I}^d$, there exists $\overline{N}$ such that*

$$(\partial_- u_N^*)(\mathbf{y}) \subseteq \bigcup_{\mathbf{y}' \in \overline{B}(\mathbf{y}, \varepsilon^\delta)} \left( \bigcup_{\mathbf{x}' \in \partial_- u^*(\mathbf{y}')} \overline{B}(\mathbf{x}', \varepsilon^{1-\delta}) \right) \tag{2.10}$$

*and*

$$(\partial_- u^*)(\mathbf{y}) \subseteq \bigcup_{\mathbf{y}' \in \overline{B}(\mathbf{y}, \varepsilon^\delta)} \left( \bigcup_{\mathbf{x}' \in \partial_- u_N^*(\mathbf{y}')} \overline{B}(\mathbf{x}', \varepsilon^{1-\delta}) \right) \tag{2.11}$$

*whenever $N \geq \overline{N}$. If $u$ is also Lipschitz continuous on $I^d$, we can substitute $\varepsilon$ with $N^{-1}$.*

In order to prove the theorem, we need the following lemma.

LEMMA 2.7. *Let $u$ be a proper convex function on $\mathbb{R}^d$ and let $d(u) := \{\mathbf{x} \in \mathbb{R}^d : u(\mathbf{x}) < +\infty\}$ be the domain of $u$. Given any compact set $K \subseteq d(u)$, define*

$$(\partial_- u)_K(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d : u(\mathbf{z}) \geq u(\mathbf{x}) + \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \ \ \forall\, \mathbf{z} \in K\}. \tag{2.12}$$

*Then*

$$(\partial_- u)_K(\mathbf{x}) \equiv (\partial_- u)(\mathbf{x}) \qquad \forall\ \mathbf{x} \in \mathring{K}.$$

*Proof.* The inclusion $(\partial_- u)(\mathbf{x}) \subset (\partial_- u)_K(\mathbf{x})$ is obvious. On the other hand, let $\mathbf{x} \in \mathring{K}$ and $\mathbf{y} \in (\partial_- u)_K(\mathbf{x})$. Supposing there exists $\mathbf{z} \in K^c \cap d(u)$ such that

$$u(\mathbf{z}) < u(\mathbf{x}) + \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}), \tag{2.13}$$

let $t \in (0, 1)$ be such that $t\mathbf{x} + (1 - t)\mathbf{z} \in K$. Then from the convexity of $u$ and definitions (2.12) and (2.13), it follows

$$\begin{aligned} tu(\mathbf{x}) + (1-t)u(\mathbf{z}) &\geq u(t\mathbf{x} + (1-t)\mathbf{z}) \geq u(\mathbf{x}) + (1-t)\mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \\ &> tu(\mathbf{x}) + (1-t)u(\mathbf{z}). \end{aligned}$$

The last inequality gives us that (2.13) may not hold true for any $\mathbf{z} \in K^c \cap d(u)$. Hence $\mathbf{y} \in (\partial_- u)(\mathbf{x})$. □

*Proof of Theorem* 2.6. Given any $\mathbf{y} \in \mathring{I}^d$ and $\delta > 0$, let $\varepsilon > 0$ be sufficiently small such that $B(\mathbf{y}, \varepsilon^\delta) \subseteq \mathring{I}^d$ and let $\mathbf{x} \in (\partial_- u_N^*)(\mathbf{y})$. Then

$$u_N^*(\mathbf{y}) + \mathbf{x} \cdot (\mathbf{z} - \mathbf{y}) \leq u_N^*(\mathbf{z}) \qquad \forall\ \mathbf{z} \in \mathbb{R}^d.$$

Theorem 2.1 and the last inequality with $N$ sufficiently large give us

$$u^*(\mathbf{y}) + \mathbf{x} \cdot (\mathbf{z} - \mathbf{y}) \leq u^*(\mathbf{z}) + \varepsilon \qquad \forall\ \mathbf{z} \in I^d.$$

Defining

$$\tilde{u}^*(\mathbf{z}) := \begin{cases} u^*(\mathbf{z}) \text{ if } \mathbf{z} \in I^d, \\ +\infty \ \text{ if } \mathbf{z} \notin I^d, \end{cases}$$

we obtain

$$\tilde{u}^*(\mathbf{y}) + \mathbf{x} \cdot (\mathbf{z} - \mathbf{y}) \leq \tilde{u}^*(\mathbf{z}) + \varepsilon \qquad \forall\ \mathbf{z} \in \mathbb{R}^d.$$

The last equation means that $\mathbf{x}$ is an $\varepsilon$-subgradient of $\tilde{u}^*$ in $\mathbf{y}$. Since $\tilde{u}^*$ is a lower semicontinuous proper convex function, we have that there exists $\mathbf{y}' \in B(\mathbf{y}, \varepsilon^\delta)$ and $\mathbf{x}' \in (\partial_- \tilde{u}^*)(\mathbf{y}')$ such that

$$|\mathbf{x} - \mathbf{x}'| \leq \varepsilon^{1-\delta}$$

(see [ET, Chap. II, Thm. 6.2]). Moreover, from the definition of $\tilde{u}^*$ we have that $\mathbf{x}' \in (\partial_- u^*)_{I^d}(\mathbf{y}')$. Since $\mathbf{y}' \in \overset{\circ}{I}{}^d$ and $I^d \subseteq d(u^*)$ for numerical reasons, Lemma 2.7 gives us that $(\partial_- u^*)_{I^d}(\mathbf{y}') \equiv (\partial_- u^*)(\mathbf{y}')$ and (2.10) is proved.

The proof of (2.11) is analogous to that of (2.10), while if $u$ is Lipschitz continuous, Theorem 2.3 gives us the estimate. □

**3. Properties of the discrete Legendre–Fenchel transform.** In order to complete the analysis of the discrete transform, we will determine in this section if the properties of the Legendre–Fenchel transform still hold true for its numerical approximation.

It is well known that if $u$ is a *lower semicontinuous proper convex* function on $\mathbb{R}^d$, then

$$u^{**}(\mathbf{x}) = u(\mathbf{x}) \qquad \forall\ \mathbf{x} \in \mathbb{R}^d, \tag{3.1}$$

while for any function $u : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{+\infty\}$

$$u^{***}(\mathbf{y}) = u^*(\mathbf{y}) \qquad \forall\ \mathbf{y} \in \mathbb{R}^d \tag{3.2}$$

(see [ET] and [R]).

Equality (3.1) is not true, in general, for the discrete case. In fact, as an example, let us consider the lower semicontinuous proper convex function on $\mathbb{R}$

$$u(x) := \begin{cases} xk & \text{if } x \in I, \\ +\infty & \text{if } x \notin I, \end{cases}$$

with $k \in (0, 1]/I_N$ for any $N$. Then

$$\begin{aligned} u^*(y) &= \sup_{x \in \mathbb{R}} \{xy - u(x)\} = \max_{x \in I} \{xy - xk\} \\ &= \begin{cases} y - k & \text{if } y \geq k, \\ 0 & \text{if } y < k, \end{cases} \end{aligned}$$

and

$$u_N^*(y) = \begin{cases} y - k & \text{if } y \geq k, \\ \frac{1}{N}(y - k) & \text{if } y < k. \end{cases}$$

The discrete transform of $u_N^*$ computed in $x \in I_N$ is given by

$$\begin{aligned} u_{NN}^{**}(x) &= \max_{y \in I_N} \{xy - u_N^*(y)\} \\ &= \begin{cases} \frac{1}{N}(x - 1) + k & \text{if } 0 < k < \frac{1}{N}, \\ \max\{\frac{j}{N}(x - \frac{1}{N}) + \frac{K}{N}\,;\ \frac{j+1}{N}(x - 1) + k\} & \text{if } \frac{j}{N} < k < \frac{j+1}{N}; \end{cases} \end{aligned}$$

i.e.,

$$u^{**}_{NN}(x) = \begin{cases} \frac{1}{N}(x-1)+k & \text{if } 0<k<\frac{1}{N} \ \forall x \in I_N, \\ \frac{j}{N}(x-\frac{1}{N})+\frac{k}{N} & \text{if } x \le a \text{ and } \frac{j}{N}<k<\frac{j+1}{N}, \\ \frac{j+1}{N}(x-1)+k & \text{if } x > a \text{ and } \frac{j}{N}<k<\frac{j+1}{N}, \end{cases}$$

where $j \in \{1,\dots,N-1\}$ and $a := \left(\frac{j+1}{N}-k\right)N+\left(k-\frac{j}{N}\right)$. Therefore, we have obtained that $u^{**}_{NN}(x) \le u(x)$ for any $x \in I_N$ and for any $N$, and where the equality holds true only in $x = \frac{1}{N}, 1$.

More generally, let $u$ be a *proper* function on $\mathbb{R}^d$ and let $d(u)$ and $d(u^*)$ be the domains of $u$ and $u^*$, respectively, regardless of their geometry and boundedness. Then (1.1) can be written as

$$u^*(\mathbf{y}) = \sup_{\mathbf{x}\in d(u)} \{\mathbf{x}\cdot\mathbf{y} - u(\mathbf{x})\}, \qquad \mathbf{y} \in d(u^*), \tag{3.3}$$

while

$$u^{**}(\mathbf{x}) = \sup_{\mathbf{y}\in d(u^*)} \{\mathbf{x}\cdot\mathbf{y} - u^*(\mathbf{y})\}, \qquad \mathbf{x} \in d(u). \tag{3.4}$$

Let $d_N(u)$ and $d_N(u^*)$ be any arbitrary subsets of $d(u)$ and $d(u^*)$, respectively, and $c_1$ and $c_2$ be two real constants depending only on the geometry of $d(u)$ and $d(u^*)$ such that $\operatorname{dist}(\mathbf{x}, d_N(u)) \le c_1N^{-1}$ and $\operatorname{dist}(\mathbf{y}, d_N(u^*)) \le c_2N^{-1}$ uniformly on $\mathbf{x} \in d(u)$ and $\mathbf{y} \in d(u^*)$. Let us write

$$u^*_N(\mathbf{y}) = \sup_{\mathbf{x}\in d_N(u)} \{\mathbf{x}\cdot\mathbf{y} - u(\mathbf{x})\}, \qquad \mathbf{y} \in d_N(u^*), \tag{3.5}$$

$$u^{**}_{NN}(\mathbf{x}) = \sup_{\mathbf{y}\in d_N(u^*)} \{\mathbf{x}\cdot\mathbf{y} - u^*_N(\mathbf{y})\}, \qquad \mathbf{x} \in d_N(u), \tag{3.6}$$

and

$$u^{***}_{NNN}(\mathbf{y}) = \sup_{\mathbf{x}\in d_N(u)} \{\mathbf{x}\cdot\mathbf{y} - u^{**}_{NN}(\mathbf{x})\}, \qquad \mathbf{y} \in d_N(u^*). \tag{3.7}$$

THEOREM 3.1. *For the discrete transform it holds true that*

$$u^{**}_{NN}(\mathbf{x}) \le u(\mathbf{x}) \qquad \forall\ \mathbf{x} \in d_N(u) \tag{3.8}$$

*while*

$$u^{***}_{NNN}(\mathbf{y}) = u^*_N(\mathbf{y}) \qquad \forall\ \mathbf{y} \in d_N(u^*). \tag{3.9}$$

*Proof.* Obviously

$$u^*_N(\mathbf{y}) \ge \mathbf{x}\cdot\mathbf{y} - u(\mathbf{x}) \qquad \forall\ \mathbf{x} \in d_N(u) \quad \text{and} \quad \forall\ \mathbf{y} \in d_N(u^*).$$

Hence $u^{**}_{NN}(\mathbf{x}) \le u(\mathbf{x})$.

Inequality (3.8) gives us

$$u^*_N(\mathbf{y}) \le u^{***}_{NNN}(\mathbf{y}) \qquad \forall\, \mathbf{y} \in d_N(u^*).$$

At the same time $u^{**}_{NN}(\mathbf{x}) \ge \mathbf{x}\cdot\mathbf{y} - u^*_N(\mathbf{y})$ for any $\mathbf{y} \in d_N(u^*)$ and $\mathbf{x} \in d_N(u)$. So

$$u^*_N(\mathbf{y}) \ge u^{***}_{NNN}(\mathbf{y}) \qquad \forall\, \mathbf{y} \in d_N(u^*)$$

and (3.9) is proved. $\square$

Now, if $u$ is a *proper lower semicontinuous convex* function and if we extend definition (3.6) to all $\mathbf{x} \in d(u)$, we have the following estimate.

THEOREM 3.2. *If $d(u)$ is bounded, then there exists a real constant $c$ independent from $N$ such that*

$$u^{**}_{NN}(\mathbf{x}) \le u(\mathbf{x}) \le u^{**}_{NN}(\mathbf{x}) + cN^{-1} \qquad \forall\ \mathbf{x} \in d_N(u). \tag{3.10}$$

*Moreover, if $d(u^*)$ is also bounded, given any compact subset $K$ of $\overset{\circ}{d}(u)$ there exists a real constant $c'$ independent from $N$ such that*

$$-c'N^{-1} + u^{**}_{NN}(\mathbf{x}) \le u(\mathbf{x}) \le u^{**}_{NN}(\mathbf{x}) + cN^{-1} \qquad \forall\ \mathbf{x} \in K. \tag{3.11}$$

*If $u$ is uniformly Lipschitz continuous on $d(u)$, estimate* (3.11) *holds true for any* $\mathbf{x} \in d(u)$.

*Proof.* For any $\mathbf{x} \in d(u)$ and any $\mathbf{y} \in d(u^*)$ let $\mathbf{z}_\mathbf{x} \in d_N(u)$ and $\mathbf{z}_\mathbf{y} \in d_N(u^*)$ be such that $|\mathbf{x} - \mathbf{z}_\mathbf{x}| \le c_1 N^{-1}$ and $|\mathbf{y} - \mathbf{z}_\mathbf{y}| \le c_2 N^{-1}$, respectively, according to the definition of $d_N(u)$ and $d_N(u^*)$. Observe that for any $N$ and for any $\mathbf{x} \in d(u)$ there exists $\mathbf{y}_N \in d(u^*)$ such that

$$u^{**}(\mathbf{x}) < [\mathbf{x} \cdot \mathbf{y}_N - u^*(\mathbf{y}_N)] + \frac{1}{N}$$

and we have

$$\begin{aligned} u(\mathbf{x}) - u^{**}_{NN}(\mathbf{x}) &= u^{**}(\mathbf{x}) - u^{**}_{NN}(\mathbf{x}) \\ &\le [\mathbf{x} \cdot \mathbf{y}_N - u^*(\mathbf{y}_N)] - [\mathbf{x} \cdot \mathbf{z}_{\mathbf{y}_N} - u^*_N(\mathbf{z}_{\mathbf{y}_N})] + \frac{1}{N} \\ &\le |\mathbf{x}||\mathbf{y}_N - \mathbf{z}_{\mathbf{y}_N}| + [u^*_N(\mathbf{z}_{\mathbf{y}_N}) - u^*(\mathbf{z}_{\mathbf{y}_N})] + [u^*(\mathbf{z}_{\mathbf{y}_N}) - u^*(\mathbf{y}_N)] + \frac{1}{N} \\ &\le |u^*(\mathbf{z}_{\mathbf{y}_N}) - u^*(\mathbf{y}_N)| + (c_2 \operatorname{diam}(d(u)) + 1)N^{-1}, \end{aligned} \tag{3.12}$$

where $\operatorname{diam}(d(u))$ is the diameter of $d(u)$. As before, for any $N$ and for any $\mathbf{y} \in d(u^*)$ there exists $\mathbf{x}_N \in d(u)$ such that

$$u^*(\mathbf{y}) \le \mathbf{y} \cdot \mathbf{x}_N - u(\mathbf{x}_N) + \frac{1}{N}.$$

Therefore, given any $\mathbf{y}', \mathbf{y}'' \in d(u^*)$, it follows that

$$\begin{aligned} u^*(\mathbf{y}') - u^*(\mathbf{y}'') &\le [\mathbf{y}' \cdot \mathbf{x}_N - u(\mathbf{x}_N)] - [\mathbf{y}'' \cdot \mathbf{x}_N - u(\mathbf{x}_N)] + \frac{1}{N} \\ &\le |\mathbf{x}_N||\mathbf{y}' - \mathbf{y}''| + \frac{1}{N} \\ &\le \operatorname{diam}(d(u))|\mathbf{y}' - \mathbf{y}''| + \frac{1}{N}\ . \end{aligned}$$

In the same way, we obtain

$$u^*(\mathbf{y}') - u^*(\mathbf{y}'') \ge -\operatorname{diam}(d(u))|\mathbf{y}' - \mathbf{y}''| - \frac{1}{N}$$

and (3.12) gives us

$$u(\mathbf{x}) - u^{**}_{NN}(\mathbf{x}) \le cN^{-1} \qquad \forall\ \mathbf{x} \in d(u), \tag{3.13}$$

where $c = 2(c_2 \operatorname{diam}(d(u)) + 1)$.

Finally, if $\mathbf{x} \in d_N(u)$, it has been proved in the previous theorem that $u(\mathbf{x}) - u^{**}_{NN}(\mathbf{x}) \geq 0$ and from (3.13) it follows (3.10). Otherwise, given any compact subset $K$ of $\overset{\circ}{d}(u)$ and $\mathbf{x}, \mathbf{z_x} \in K$ with $\mathbf{z_x}$ as before, since $u$ is Lipschitz continuous on $K$ (see [R]), it follows that

$$\begin{aligned} u^*_N(\mathbf{y}) &\geq \mathbf{z_x} \cdot \mathbf{y} - u(\mathbf{z_x}) = (\mathbf{z_x} - \mathbf{x}) \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y} - u(\mathbf{z_x}) \\ &\geq -|\mathbf{y}||\mathbf{z_x} - \mathbf{x}| + [\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})] + u(\mathbf{x}) - u(\mathbf{z_x}) \\ &\geq [\mathbf{x} \cdot \mathbf{y} - u(\mathbf{x})] - c_1(\mathrm{diam}(d(u^*)) + L_K)N^{-1} \end{aligned}$$

for any $\mathbf{y} \in d(u^*)$ and where $L_K$ is the Lipschitz constant of $u$ on $K$. Then

$$u(\mathbf{x}) \geq u^{**}_{NN}(\mathbf{x}) - c' N^{-1}$$

so that (3.11) is proved. □

*Remark* 3.3. Even if $u$ is not lower semicontinuous and convex but the argument function $h_N : d_N(u^*) \to d_N(u)$ associated $u^*_N$ is surjective, the equality in (3.8) holds true. In fact, if $\mathbf{x} \in d_N(u)$ is an argument point corresponding to some $\tilde{\mathbf{y}} \in d_N(u^*)$, i.e., $\mathbf{x} = h_N(\tilde{\mathbf{y}})$, we have

$$\begin{aligned} u^{**}_{NN}(\mathbf{x}) &\geq \mathbf{x} \cdot \tilde{\mathbf{y}} - u^*_N(\tilde{\mathbf{y}}) \\ &= \mathbf{x} \cdot \tilde{\mathbf{y}} - (\mathbf{x} \cdot \tilde{\mathbf{y}} - u(\mathbf{x})) \\ &= u(\mathbf{x}) \end{aligned}$$

and, together with (3.8), we have $u^{**}_{NN}(\mathbf{x}) = u(\mathbf{x})$. In the continuous case, if the argument function $h$ associated with $u^*$ is surjective, we have in the same way that $u^{**} = u$, i.e., that $u$ is convex and lower semicontinuous as it has to be.

As an example of a function $u$ satisfying the equality in (3.8), let us consider the function $u(x) := x^2$. Its Legendre–Fenchel transform computed in $y \in [0, 2]$ is given by

$$u^*(y) = \sup_{x \in \mathbb{R}}\{xy - x^2\} = \max_{x \in I}\{xy - x^2\} = \frac{1}{4}y^2$$

and the argument function associated is $h(y) = \frac{1}{2}y$. So for any $y \in \{2j/N;\ j = 1, \ldots, N\} =: [0, 2]_N$ one has that $h(y) = h_N(y) \in I_N$. Then

$$u^*_N(y) = u^*(y) \qquad \forall\ y \in [0, 2]_N$$

and

$$u^{**}_{NN}(x) = u^{**}(x) \qquad \forall\ x \in I_N.$$

*Remark* 3.4. It is easy to see that if $u$ is not convex but it is Lipschitz continuous on $d(u)$, and if $d(u)$ and $d(u^*)$ are bounded, the estimate (3.11) still holds true on $d(u)$ for $u^{**}$ instead of $u$.

**4. Application to Hamilton–Jacobi equations.** Here we apply the results of the previous section to approximate the initial value problem (1.2) under hypotheses $(H_1)$ or $(H_2)$. In order to simplify the discussion, we will treat the two cases separately.

**4.1. Convex initial condition.** As mentioned in the introduction, under hypothesis $(H_1)$ the unique viscosity solution of (1.2) is given by (1.3), i.e., using the definition of Legendre transform

$$\begin{aligned} u(\mathbf{x}, t) &= \sup_{\mathbf{y} \in \mathbb{R}^d}\{\mathbf{y} \cdot \mathbf{x} - u^*_0(\mathbf{y}) - tH(\mathbf{y})\} \\ &= (u^*_0 + tH)^*(\mathbf{x}) \end{aligned} \tag{4.1}$$

for $(\mathbf{x}, t) \in \mathbb{R}^d \times (0, +\infty)$.

Since $u$ is given by a combination of two Legendre–Fenchel transforms, whenever we can substitue $\mathbb{R}^d$ with bounded sets in formula (4.1), the FLT algorithm can be applied for its numerical computation. In order to do that, let us suppose $H$ locally Lipschitz continuous on $\mathbb{R}^d$ and let $L$ and $L_H$ be, respectively, the Lipschitz constant of $u_0$ and the Lipschitz constant of $H$ on $\overline{B}(\mathbf{0}, L)$. Let $K_1$ be a hypercube containing $\overline{B(\mathbf{0}, R + L_H t)}$ where $t$ and $R$ are arbitrary real positive constants and let $K_2$ be a hypercube containing $\overline{B}(\mathbf{0}, L)$. Finally, let $K_{i,N}$, $i = 1, 2$, be the corresponding discrete hypercubes (as described in Remark 2.4).

We define

$$u_N(\mathbf{x}, t) := \max_{\mathbf{y}\in K_{2,N}} \{\mathbf{y} \cdot \mathbf{x} - \tilde{u}^*_{0,N}(\mathbf{y}) - tH(\mathbf{y})\} \tag{4.2}$$

and

$$\tilde{u}^*_{0,N}(\mathbf{y}) := \begin{cases} \max_{\mathbf{z}\in K_{1,N}} \{\mathbf{y} \cdot \mathbf{z} - u_0(\mathbf{z})\} = u^*_{0,N}(\mathbf{y}) & \text{if } \mathbf{y} \in Q_N, \\ +\infty & \text{if } \mathbf{y} \in K_{2,N} \setminus Q_N, \end{cases} \tag{4.3}$$

where

$$Q := \bigcup_{\mathbf{z}\in K_1} (\partial_- u_0)(\mathbf{z}). \tag{4.4}$$

and

$$Q_N := Q \cap K_{2,N} \tag{4.5}$$

if $Q$ is explicitly known, or

$$Q_N := \bigcup_{\mathbf{z}\in K_{1,N}} \left[ \left(\partial_- v^{**}_{NN}\right)(\mathbf{z}) \cap K_{2,N} \right] \tag{4.6}$$

in the general case. The double discrete transform $v^{**}_{NN}$ in (4.6) is performed as follows:

$$v^{**}_{NN}(\mathbf{z}) := \max_{\mathbf{y}\in K_{2,N}} \{\mathbf{z} \cdot \mathbf{y} - v^*_N(\mathbf{y})\} \qquad \forall \ \ \mathbf{z}\in K_3 \tag{4.7}$$

with

$$v^*_N(\mathbf{y}) := \max_{\mathbf{z}\in K_{3,N}} \{\mathbf{z} \cdot \mathbf{y} - u_0(\mathbf{z})\} \qquad \forall \ \ \mathbf{y}\in K_2,$$

where $K_3$ is a hypercube such that $K_1 \subseteq \overset{\circ}{K_3}$ and $K_{3,N}$ is the corresponding discrete one.

THEOREM 4.1. *$u_N$ converges to $u$ uniformly on $\overline{B}(\mathbf{0}, R) \times (0, +\infty)$ if the set $Q$ is explicitly known or if $u_0$ is a $C^1$ function on $K_1$.*

Before proving the theorem, several results explaining the definitions previously introduced have to be given.

PROPOSITION 4.2. *It holds true that*

$$u(\mathbf{x}, t) = \max_{\mathbf{y}\in Q} \min_{\mathbf{z}\in K_1} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\} \tag{4.8}$$

*for any $\mathbf{x}\in \overline{B}(\mathbf{0}, R)$.*

*Proof.* Since $u_0$ is Lipschitz continuous, for any $\mathbf{z} \in \mathbb{R}^d$ we have

$$u_0(\mathbf{z}) \le |\mathbf{z}|L + u_0(\mathbf{0}).$$

Then for any $\mathbf{y} \in \overline{B}^c(0, L)$, set $\mathbf{z} := \alpha\mathbf{y}/|\mathbf{y}|$ with $\alpha \in \mathbb{R}_+$, we obtain

$$\mathbf{z} \cdot \mathbf{y} - u_0(\mathbf{z}) \geq \alpha(|\mathbf{y}| - L) - u_0(\mathbf{0}),$$

and since $\alpha$ can be arbitrarily large, the last inequality gives

$$u_0^*(\mathbf{y}) = +\infty$$

so that

$$u(\mathbf{x}, t) = \sup_{\mathbf{y} \in \overline{B}(\mathbf{0}, L)} \inf_{\mathbf{z} \in \mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\}. \tag{4.9}$$

At the same time, since $H$ is locally Lipschitz continuous, the property of the cone of dependence holds; i.e., if $u$ and $v$ are two viscosity solutions of (1.2) such that $u(\mathbf{x}, 0) = v(\mathbf{x}, 0)$ on $|\mathbf{x}| \leq R$, then

$$u(\mathbf{x}, t) = v(\mathbf{x}, t) \qquad \text{on } |\mathbf{x}| \leq R - L_H t$$

(see [CL] for the proof).

This property allows us to "preserve" in formula (4.9) only the points $\mathbf{y} \in \overline{B}(\mathbf{0}, L)$ such that the corresponding infimum is achieved and consequently attained in $\overline{B}(\mathbf{x}, L_H t)$. Moreover, it is easy to see that for any $\mathbf{y} \in \overline{B}(\mathbf{0}, L)$ the infimum is attained at $\mathbf{z} \in \overline{B}(\mathbf{x}, L_H t)$ if and only if $\mathbf{y} \in (\partial_- u_0)(\mathbf{z})$ and that it can exist $\mathbf{y} \in \overline{B}(\mathbf{0}, L)$ such that $\mathbf{y} \notin (\partial_- u_0)(\mathbf{z})$ for any $\mathbf{z} \in \overline{B}(\mathbf{x}, L_H t)$. Notice that $(\partial_- u_0)(\mathbf{z}) \subseteq \overline{B}(\mathbf{0}, L)$ for any $\mathbf{z} \in \mathbb{R}^d$.

We can conclude that

$$u(\mathbf{x}, t) = \max_{\mathbf{y} \in Q(\mathbf{x})} \min_{\mathbf{z} \in \overline{B}(\mathbf{x}, L_H t)} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\},$$

where

$$Q(\mathbf{x}) := \bigcup_{\mathbf{z} \in \overline{B}(\mathbf{x}, L_H t)} (\partial_- u_0)(\mathbf{z})$$

is a closed subset of $\overline{B}(\mathbf{0}, L)$ (see [R]).

Finally, since we want to compute the solution $u$ in $\overline{B}(\mathbf{0}, R)$, we have uniformly in $\mathbf{x} \in \overline{B}(\mathbf{0}, R)$

$$u(\mathbf{x}, t) = \max_{\mathbf{y} \in Q} \min_{\mathbf{z} \in K_1} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\}$$

with $Q$ and $K_1$ given by (4.4). $\square$

The previous proposition shows that it is possible to substitute $\mathbb{R}^d$ with bounded, closed subsets in formula (4.1). However, in order to apply the FLT algorithm to (4.8), the last formula has to be read as follows:

$$u(\mathbf{x}, t) = \max_{\mathbf{y} \in K_2} \{\mathbf{y} \cdot \mathbf{x} - \tilde{u}_0^*(\mathbf{y}) - tH(\mathbf{y})\} \qquad \forall\ \mathbf{x} \in \overline{B}(\mathbf{0}, R), \tag{4.10}$$

where

$$\tilde{u}_0^*(\mathbf{y}) := \begin{cases} \max_{\mathbf{z} \in K_1} \{\mathbf{y} \cdot \mathbf{z} - u_0(\mathbf{z})\} = u_0^*(\mathbf{y}) & \text{if } \mathbf{y} \in Q, \\ +\infty & \text{if } \mathbf{y} \in K_2 \setminus Q. \end{cases} \tag{4.11}$$

If the set $Q$ is explicitly known, (4.2)–(4.3) and (4.5) immediately give us the numerical approximation of the viscosity solution (4.10)–(4.11), while if $Q$ is unknown, it has to be computed numerically. In view of doing this, let us define

$$v^*(\mathbf{y}) := \max_{\mathbf{z}\in K_3}\{\mathbf{z}\cdot\mathbf{y} - u_0(\mathbf{z})\} \qquad \forall \ \ \mathbf{y}\in K_2$$

and

$$v^{**}(\mathbf{z}) := \max_{\mathbf{y}\in K_2}\{\mathbf{z}\cdot\mathbf{y} - v^*(\mathbf{y})\} \qquad \forall \ \ \mathbf{z}\in K_3.$$

We observe that $v^*(\mathbf{y}) \equiv u_0^*(\mathbf{y})$ for any $\mathbf{y} \in \cup_{\mathbf{z}\in K_3}(\partial_- u_0)(\mathbf{z}) \supseteq Q$ and $v^*(\mathbf{y}) \le u_0^*(\mathbf{y})$ for any $\mathbf{y} \in K_2 \setminus \cup_{\mathbf{z}\in K_3}(\partial_- u_0)(\mathbf{z})$, while $v^{**}(\mathbf{z}) = u_0(\mathbf{z})$ for any $\mathbf{z} \in K_3$.

If $h : K_3 \to K_2$ is the argument function associated with $v^{**}$, we have the following characterization of $Q$.

PROPOSITION 4.3. *It holds true that*

$$Q \equiv \{h(\mathbf{z});\ \mathbf{z}\in K_1\}. \tag{4.12}$$

*Proof.* If $\mathbf{y} \in Q$, there exists $\mathbf{z} \in K_1$ such that $\mathbf{y} \in (\partial_- u_0)(\mathbf{z})$. Then

$$v^*(\mathbf{y}) = \mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z})$$

and

$$v^*(\mathbf{y}) + \mathbf{z}\cdot(\mathbf{x} - \mathbf{y}) = \mathbf{z}\cdot\mathbf{x} - u_0(\mathbf{z}) \le v^*(\mathbf{x}) \qquad \forall \ \ \mathbf{x}\in K_2.$$

The last inequality means that $\mathbf{z} \in (\partial_- v^*)_{K_2}(\mathbf{z})$ and it gives

$$\mathbf{z}\cdot\mathbf{x} - v^*(\mathbf{x}) \le \mathbf{z}\cdot\mathbf{y} - v^*(\mathbf{y}) \qquad \forall \ \ \mathbf{x}\in K_2;$$

i.e., $\mathbf{y} \in \{h(\mathbf{z});\ \mathbf{z}\in K_1\}$.

Conversely, if $\mathbf{y} \in \{h(\mathbf{z});\ \mathbf{z}\in K_1\}$, there exists $\mathbf{z} \in K_1$ such that

$$v^{**}(\mathbf{z}) = u_0(\mathbf{z}) = \mathbf{y}\cdot\mathbf{z} - v^*(\mathbf{y});$$

hence

$$v^*(\mathbf{y}) = \mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z}) \ge \mathbf{y}\cdot\mathbf{x} - u_0(\mathbf{x}) \qquad \forall \ \ \mathbf{x}\in K_3,$$

i.e., $\mathbf{y} \in (\partial_- u_0)_{K_3}(\mathbf{z})$. Since $\mathbf{z} \in K_1 \subseteq \overset{\circ}{K_3}$, Lemma 2.7 gives us $(\partial_- u_0)_{K_3}(\mathbf{z}) \equiv (\partial_- u_0)(\mathbf{z})$ and $\mathbf{y} \in Q$. $\square$

From (4.12) it follows a natural approximation of $Q$; that is,

$$\tilde{Q}_N := \{h_N(\mathbf{z});\ \mathbf{z}\in K_{1,N}\}, \tag{4.13}$$

where $h_N$ is the argument function associated with $v_{NN}^{**}$ given by (4.7). The set $\tilde{Q}_N$ can be computed numerically through the computation of $v_N^*$ and $v_{NN}^{**}$. Moreover, extending the definition of $v_{NN}^{**}$ to all $\mathbf{z} \in \mathbb{R}^d$, we obtain the equality of (4.13) with the initial definition (4.6).

PROPOSITION 4.4. *It holds true that*

$$Q_N \equiv \tilde{Q}_N.$$

*Proof.* If $\mathbf{y} \in \tilde{Q}_N$, there exists $\mathbf{z} \in K_{1,N}$ such that $v_{NN}^{**}(\mathbf{z}) = \mathbf{y} \cdot \mathbf{z} - v_N^*(\mathbf{y})$ and

$$v_{NN}^{**}(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) = \mathbf{y} \cdot \mathbf{x} - v_N^*(\mathbf{y}) \leq v_{NN}^{**}(\mathbf{x}) \qquad \forall \quad \mathbf{x} \in \mathbb{R}^d.$$

So we have obtained

$$\mathbf{y} \in (\partial_- v_{NN}^{**})(\mathbf{z}) \cap K_{2,N}. \tag{4.14}$$

Conversely, if $\mathbf{y} \in (\partial_- v_{NN}^{**})(\mathbf{z}) \cap K_{2,N}$ with $\mathbf{z} \in K_{1,N}$, then

$$v_{NN}^{**}(\mathbf{x}) \geq v_{NN}^{**}(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) \qquad \forall \quad \mathbf{x} \in \mathbb{R}^d$$

and

$$v_{NNN}^{***}(\mathbf{y}) = \mathbf{y} \cdot \mathbf{z} - v_{NN}^{**}(\mathbf{z}). \tag{4.15}$$

From Theorem 3.1, since $\mathbf{y} \in K_{2,N}$, we have $v_{NNN}^{***}(\mathbf{y}) = v_N^*(\mathbf{y})$ and (4.15) becomes

$$v_{NN}^{**}(\mathbf{z}) = \mathbf{y} \cdot \mathbf{z} - v_N^*(\mathbf{y});$$

i.e., $\mathbf{y} \in \tilde{Q}_N$. □

Finally, setting $d_H(A, B) := \inf\{r > 0 : \ A \subseteq V_r(B) \text{ and } \ B \subseteq V_r(A)\}$, the Hausdorff distance between compact sets of $\mathbb{R}^d$, where $V_r(A) := \{\mathbf{x} \in \mathbb{R}^d \ : \ \mathrm{dist}(\mathbf{x}, A) \leq r\}$, we have the following proposition.

PROPOSITION 4.5. *If $u_0$ is a $C^1$ function on $K_1$, the set $Q_N$ converges to $Q$ with respect to the Hausdorff metric.*

*Proof.* From Theorem 3.2 it follows that $v_{NN}^{**}$ converges uniformly to $u_0$ on $K_3$. Hence, given any $\varepsilon > 0$, there exists an index $\overline{N}$ such that

$$\left(\partial_- v_{NN}^{**}\right)(\mathbf{z}) \subseteq \overline{B}(Du_0(\mathbf{z}), \varepsilon/2) \qquad \forall \ \mathbf{z} \in K_1 \quad \forall \ N \geq \overline{N}$$

(see [R, Thm. 24.5]). Therefore,

$$Q_N \subseteq V_{\varepsilon/2}(Q) \qquad \forall \ N \geq \overline{N}.$$

Since $u_0$ is $C^1$, there exists $\delta > 0$ such that $|Du_0(\mathbf{z}') - Du_0(\mathbf{z}'')| < \varepsilon/2$ whenever $\mathbf{z}', \ \mathbf{z}'' \in K_1$ and $|\mathbf{z}' - \mathbf{z}''| < \delta$. At the same time, for any $\mathbf{z} \in K_1$ there exists $\mathbf{z}_N \in K_{1,N}$ such that $|\mathbf{z} - \mathbf{z}_N| < l\sqrt{d}N^{-1}$, where $l$ is the size of $K_1$ (see Remark 2.4). Choosing $N \geq \max\{\overline{N}, l\sqrt{d}\delta^{-1}\}$ and giving any $\mathbf{y} \in \{Du_0(\mathbf{z}), \mathbf{z} \in K_1\} = Q$ and $\mathbf{y}_N \in \left(\partial_- v_{NN}^{**}\right)(\mathbf{z}_N) \cap K_{2,N} \subseteq \overline{B}(Du_0(\mathbf{z}_N), \varepsilon/2)$, we have

$$|\mathbf{y} - \mathbf{y}_N| \leq |Du_0(\mathbf{z}) - Du_0(\mathbf{z}_N)| + |Du_0(\mathbf{z}_N) - \mathbf{y}_N| < \varepsilon;$$

i.e.,

$$Q \subseteq V_\varepsilon(Q_N)$$

and the proposition is proved. □

The previous propositions allow us to define the numerical approximation of the viscosity solution (4.10)–(4.11) even when the set $Q$ is unknown. The approximation will be given as before through formulas (4.2)–(4.3) with $Q_N$ given by (4.6). Moreover, the convergence of $Q_N$ towards $Q$ gives us the tool we need to prove Theorem 4.1 in the general case.

*Proof of Theorem* 4.1. Setting

$$v_N(\mathbf{x},t) := \max_{\mathbf{y}\in Q}\ \min_{\mathbf{z}\in K_{1,N}} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\},$$

we obtain

$$\begin{aligned} |u(\mathbf{x},t) - u_N(\mathbf{x},t)| &\le |u(\mathbf{x},t) - v_N(\mathbf{x},t)| + |v_N(\mathbf{x},t) - u_N(\mathbf{x},t)| \\ &\le \max_{\mathbf{y}\in Q}\left|\max_{\mathbf{z}\in K_1}\{\mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z})\} - \max_{\mathbf{z}\in K_{1,N}}\{\mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z})\}\right| \\ &\quad + |v_N(\mathbf{x},t) - u_N(\mathbf{x},t)|. \end{aligned} \tag{4.16}$$

Theorem 2.3 gives us that there exists a constant $c$ depending on $L$ and on the diameter of $K_1$ and $K_2$ such that

$$\left|\max_{\mathbf{z}\in K_1}\{\mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z})\} - \max_{\mathbf{z}\in K_{1,N}}\{\mathbf{y}\cdot\mathbf{z} - u_0(\mathbf{z})\}\right| \le cN^{-1} \tag{4.17}$$

uniformly on $\mathbf{y} \in Q$. Moreover, since $v_N$ is defined as the maximum of the continuous function on $\mathbf{y}$

$$\min_{\mathbf{z}\in K_{1,N}} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}$$

over the closed, bounded set $Q$, from Proposition 4.5 it follows that

$$|v_N(\mathbf{x},t) - u_N(\mathbf{x},t)| \le \varepsilon \tag{4.18}$$

for any $N$ sufficiently large and uniformly in $(\mathbf{x}, t)$.

Set (4.17) and (4.18) in (4.16); the convergence is proved. □

*Remark* 4.6. If the set $Q$ is known and $Q_N$ is given by (4.5), we can obtain the estimate

$$|v_N(\mathbf{x},t) - u_N(\mathbf{x},t)| \le cN^{-1}$$

with $c$ independent from $N$. Consequently

$$\|u(\cdot,t) - u_N(\cdot,t)\|_{L^\infty(\overline{B}(\mathbf{0},R))} \le cN^{-1}.$$

**4.2. Convex Hamiltonian.** If hypotheses $(H_2)$ hold true, the unique viscosity solution of (1.2) is given by (1.4), i.e.,

$$u(\mathbf{x},t) = \inf_{\mathbf{z}\in\mathbb{R}^d}\{u_0(\mathbf{z}) + (tH)^*(\mathbf{x}-\mathbf{z})\}, \tag{4.19}$$

where the difference with (4.1) is only in the interchange of the infimum and the supremum.

In this case, it seems that we cannot apply the FLT algorithm to formula (4.19) because the solution of the Hamilton–Jacobi equation is not given by a combination of two Legendre–Fenchel transforms. However, in one dimension it can be shown that the argument function associated to the infimum in (4.19) is increasing from $\mathbb{R}$ to $\mathbb{R}$. Then, following the same arguments used to construct the FLT algorithm, we can obtain an algorithm which approximates the viscosity solution even in this case. In fact, setting

$$h(x) := \sup \operatorname*{Arginf}_{z\in\mathbb{R}} \left\{u_0(z) + (tH)^*(x-z)\right\}, \tag{4.20}$$

for any $x' < x''$ and any $z < h(x')$, the Jensen's inequality gives us

$$(tH)^*(x'-z) + (tH)^*(x''-h(x')) \le (tH)^*(x''-z) + (tH)^*(x'-h(x')). \tag{4.21}$$

Since

$$u_0(h(x')) + (tH)^*(x'-h(x')) \le u_0(z) + (tH)^*(x'-z),$$

from (4.21) it follows that

$$u_0(h(x')) + (tH)^*(x''-h(x')) \le u_0(z) + (tH)^*(x''-z) \qquad \forall\ z < h(x').$$

Therefore, $h(x'') \ge h(x')$.

In order to write the algorithm, it is again necessary to have a bounded interval instead of $\mathbb{R}$ in formula (4.19) with $d = 1$. Assuming that this is true, for simplicity of notation and without loss of generality, we write

$$u(x,t) = \min_{z\in I} \left\{u_0(z) + (tH)^*(x-z)\right\} \qquad x \in I \tag{4.22}$$

and

$$(tH)^*(\xi) = \max_{y\in I}\{\xi y - tH(y)\}. \tag{4.23}$$

The discretisation of (4.22)–(4.23) will be, as usual,

$$u_N(x,t) := \min_{z\in I_N} \left\{u_0(z) + (tH)^*_N(x-z)\right\}$$

and

$$(tH)^*_N(\xi) = \max_{y\in I_N}\{\xi y - tH(y)\}$$

while

$$h_N(x) := \sup \operatorname*{Argmin}_{z\in I_N} \left\{u_0(z) + (tH)^*_N(x-z)\right\}$$

is the discrete argument function associated with $u_N$.

Setting $\varepsilon := \frac{1}{2N}$, we observe that

$$u_{2N}(x,t) = \min\left\{u'_N(x,t);\ u''_N(x,t)\right\}$$

where

$$u'_N(x,t) := \min_{z\in I_N} \left\{u_0(z) + (tH)^*_{2N}(x-z)\right\}$$

and

$$u''_N(x,t) := \min_{z\in I_N} \left\{u_0(z-\varepsilon) + (tH)^*_{2N}(x-z+\varepsilon)\right\},$$

while for the argument function it holds true that

$$h_{2N}(x) = \begin{cases} h'_N(x) \text{ if } u_{2N}(x,t) = u'_N(x,t), \\ h''_N(x) \text{ if } u_{2N}(x,t) = u''_N(x,t), \end{cases}$$

where

$$h'_N(x) := \sup \operatorname*{Argmin}_{z \in I_N} \left\{u_0(z) + (tH)^*_{2N}(x - z)\right\}$$

and

$$h''_N(x) := \sup \operatorname*{Argmin}_{z \in I_N} \left\{u_0(z - \varepsilon) + (tH)^*_{2N}(x - z + \varepsilon)\right\}.$$

Hence, for any fixed $t \in (0, +\infty)$, the two known vectors $(u'_N(x, t),\ x \in I_N)$ and $(u''_N(x, t),\ x \in I_N)$, for the computation of $(u_{2N}(x, t),\ x \in I_{2N})$ it is sufficient to compute the two vectors $(u'_N(x, t),\ x \in I_N - \varepsilon)$ and $(u''_N(x, t),\ x \in I_N - \varepsilon)$. Using the same argument as before, it follows that $h'_N$ and $h''_N$ are increasing functions of $x$. Then, for any $x$ in $I_N - \varepsilon$,

$$u'_N(x, t) = \min\{u_0(z) + (tH)^*_{2N}(x - z);\ z \in I_N, h'_N(x - \varepsilon) \le z \le h'_N(x + \varepsilon)\}$$

and

$$u''_N(x, t) = \min\{u_0(z - \varepsilon) + (tH)^*_{2N}(x - z + \varepsilon);\ z \in I_N, h''_N(x - \varepsilon) \le z \le h''_N(x + \varepsilon)\}.$$

So, starting from the known vectors $(u'_N(x, t),\ x \in I_N)$ and $(u''_N(x, t),\ x \in I_N)$, the computational cost of $(u_{2N}(x, t),\ x \in I_{2N})$ is $0(N)$ while the total cost of the vector $(u_N(x, t),\ x \in I_N)$ is $0(N \log N)$.

In more than one dimension, a difficulty arises due to the fact that formula (4.22) cannot be factorized as the Legendre–Fenchel transform. However, we can write (4.22) with $d = 2$ for simplicity as

$$u(\mathbf{x}, t) = \min_{z_2 \in I} \left\{ \min_{z_1 \in I} \{u_0(z_1, z_2) + (tH)^*(x_1 - z_1, x_2 - z_2)\} \right\} \qquad \forall\ \mathbf{x} \in I^2$$

with its numerical approximation

$$u_N(\mathbf{x}, t) = \min_{z_2 \in I_N} \left\{ \min_{z_1 \in I_N} \{u_0(z_1, z_2) + (tH)^*(x_1 - z_1, x_2 - z_2)\} \right\} \qquad \forall\ \mathbf{x} \in I_N^2.$$

With fixed $x_2$ and $z_2$ in $I_N$, the algorithm can be used to compute the minimum with respect to $z_1 \in I_N$ and for any $x_1 \in I_N$ with a cost of $0\,(N \log N)$ numerical operations. Then, for any point $(x_1, x_2) \in I_N^2$, extracting the minimum with respect to $z_2 \in I_N$ between the $N$ computed values $\{\min_{z_1 \in I_N} \{u_0(z_1, z_2) + (tH)^*(x_1 - z_1, x_2 - z_2)\}\,;\ z_2 \in I_N\}$, we obtain $\left(u_N(\mathbf{x}, t);\ \mathbf{x} \in I_N^2\right)$ with a total computational cost $0\left(N^3 \log N\right)$. Naturally, the same argument gives us an algorithm in any dimension $d$, but it will be less interesting since the computational cost grows with $d$.

Next we show how to substitute $\mathbb{R}^d$ with compact sets in formula (4.19) in order to apply the previous algorithm. First we observe that since $H$ is convex, it is locally Lipschitz continuous on its domain $d(H)$ (see [R]). Supposing that $d(H) \equiv \mathbb{R}^d$ and setting $L$ the Lipschitz constant of $u_0$ and $L_H$ the Lipschitz constant of $H$ on $\overline{B}(\mathbf{0}, L)$, we have the following proposition.

PROPOSITION 4.7. *Under hypotheses* $(H_2)$*, given any* $R > 0$*, any fixed* $t > 0$*, and any hypercube* $K_1 \supseteq \overline{B}(\mathbf{0}, R + L_H t)$*, there exists* $M > 0$ *depending on* $R$ *and on the diameter of* $K_1$ *such that*

$$u(\mathbf{x}, t) = \min_{\mathbf{z} \in K_1} \max_{\mathbf{y} \in \overline{B}(\mathbf{0}, M)} \{u_0(\mathbf{z}) + \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) - tH(\mathbf{y})\} \tag{4.24}$$

*for any* $\mathbf{x} \in \overline{B}(\mathbf{0}, R)$.

*Proof.* We suppose initially that the Hamiltonian $H$ is superlinear. Again, using the property of cone of dependence, we obtain immediately

$$\begin{aligned} u(\mathbf{x},t) &= \inf_{\mathbf{z}\in\overline{B}(\mathbf{0},R+L_H t)} \sup_{\mathbf{y}\in\mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \\ &= \inf_{\mathbf{z}\in K_1} \sup_{\mathbf{y}\in\mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \end{aligned} \tag{4.25}$$

for any $\mathbf{x}\in\overline{B}(\mathbf{0},R)$. Moreover, for any $\lambda > 0$ there exists $M_\lambda$ such that

$$tH(\mathbf{y}) - tH(\mathbf{0}) > \lambda|\mathbf{y}| \qquad \forall\ \mathbf{y}\in\overline{B}^c(\mathbf{0},M_\lambda). \tag{4.26}$$

Choosing $\lambda := R + \operatorname{diam}(K_1)$ and setting $M$ equal to the corresponding $M_\lambda$, we show that it follows uniformly on $\mathbf{x}\in\overline{B}(\mathbf{0},R)$ and $\mathbf{z}\in K_1$ that

$$\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y}) < |\mathbf{y}|(|\mathbf{x}-\mathbf{z}| - \lambda) - tH(\mathbf{0}) \le -tH(\mathbf{0}) \qquad \forall\ \mathbf{y}\in\overline{B}^c(\mathbf{0},M);$$

i.e.,

$$\sup_{\mathbf{y}\in\mathbb{R}^d}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} = \max_{\mathbf{y}\in\overline{B}(\mathbf{0},M)}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}$$

and formula (4.25) becomes

$$u(\mathbf{x},t) = \min_{\mathbf{z}\in K_1} \max_{\mathbf{y}\in\overline{B}(\mathbf{0},M)} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}.$$

In the general case, when the Hamiltonian $H$ satisfies

$$\lim_{|\mathbf{y}|\to+\infty} H(\mathbf{y}) = +\infty,$$

we define

$$\tilde{H}(\mathbf{y}) := H(\mathbf{y}) + \left[|\mathbf{y}|^2 - L^2\right]^+ \qquad \forall\ \mathbf{y}\in\mathbb{R}^d.$$

Since $\tilde{H}$ is convex and superlinear, the unique viscosity solution $\tilde{u}$ of the initial value problem

$$\begin{cases} u_t + \tilde{H}(Du) = 0 \text{ in } \mathbb{R}^d\times(0,\infty), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{on } \mathbb{R}^d, \end{cases}$$

will be for any $\mathbf{x}\in\overline{B}(\mathbf{0},R)$

$$\begin{aligned} \tilde{u}(\mathbf{x},t) &= \inf_{\mathbf{z}\in\mathbb{R}^d} \sup_{\mathbf{y}\in\mathbb{R}^d} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \\ &= \min_{\mathbf{z}\in K_1} \max_{\mathbf{y}\in\overline{B}(\mathbf{0},M)} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}. \end{aligned}$$

Using the definition of viscosity solution (see [CL], [CEL]), it is easy to see that $\tilde{u}$ is also a viscosity solution of the original initial value problem (1.2). The unicity gives us $\tilde{u}(\mathbf{x},t) \equiv u(\mathbf{x},t)$ and the proposition is proved. □

In formula (4.24), choosing a hypercube $K_2 \supseteq \overline{B}(\mathbf{0},M)$, we have

$$u(\mathbf{x},t) = \min_{\mathbf{z}\in K_1} \max_{\mathbf{y}\in K_2} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \qquad \forall\ \mathbf{x}\in\overline{B}(\mathbf{0},R).$$

Setting

$$u_N(\mathbf{x},t) := \min_{\mathbf{z}\in K_{1,N}} \max_{\mathbf{y}\in K_{2,N}} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\},$$

we obtain the following convergence result.

THEOREM 4.8. *Under hypotheses* $(H_2)$, $u_N(\mathbf{x},t)$ *converges to* $u(\mathbf{x},t)$ *uniformly on* $\overline{B}(\mathbf{0},R)\times(0,+\infty)$. *Moreover,*

$$\|u(\cdot,t) - u_N(\cdot,t)\|_{L^\infty(\overline{B}(\mathbf{0},R))} \le cN^{-1},$$

*where c is a constant independent from N.*

*Proof.* Set

$$v_N(\mathbf{x},t) := \min_{\mathbf{z}\in K_{1,N}} \max_{\mathbf{y}\in K_2} \{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\},$$

we have

$$\begin{aligned} |u(\mathbf{x},t) - u_N(\mathbf{x},t)| &\le |u(\mathbf{x},t) - v_N(\mathbf{x},t)| + |v_N(\mathbf{x},t) - u_N(\mathbf{x},t)| \\ &\le \max_{\mathbf{z}\in K_{1,N}} \left[ \max_{\mathbf{y}\in K_2}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} - \max_{\mathbf{y}\in K_{2,N}}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} \right] \\ &\quad + |u(\mathbf{x},t) - v_N(\mathbf{x},t)|. \end{aligned} \tag{4.27}$$

Since $H$ is locally Lipschitz continuous, using Theorem 2.3 we have that there exists a constant $c$ independent from $N$ such that

$$|\max_{\mathbf{y}\in K_2}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\} - \max_{\mathbf{y}\in K_{2,N}}\{\mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}| \le cN^{-1} \tag{4.28}$$

uniformly on $\mathbf{x}\in\overline{B}(\mathbf{0},R)$ and $\mathbf{z}\in K_1$.

Observe that the function

$$\max_{\mathbf{y}\in K_2}\{u_0(\mathbf{z}) + \mathbf{y}\cdot(\mathbf{x}-\mathbf{z}) - tH(\mathbf{y})\}$$

is Lipschitz continuous with respect to $\mathbf{z}$ uniformly in $(\mathbf{x},t)$ and since for any $\mathbf{z}\in K_1$ there exists $\mathbf{z}_N\in K_{1,N}$ such that $|\mathbf{z}-\mathbf{z}_N| \le l\sqrt{d}N^{-1}$ where $l$ is the size of $K_1$ (see Remark 2.4), we can easily obtain

$$|u(\mathbf{x},t) - v_N(\mathbf{x},t)| \le cN^{-1} \tag{4.29}$$

uniformly on $(\mathbf{x},t)\in\overline{B}(\mathbf{0},R)\times(0,+\infty)$.

Setting (4.28) and (4.29) into (4.27), we prove the convergence and the estimate. □

**5. Numerical tests.** The first equation we have considered to investigate the behaviour of the FLT algorithm is the Buckley–Leverett equation

$$u_t + \frac{Q}{\phi} f(u)_x = 0, \qquad (x,t)\in\mathbb{R}\times(0,+\infty), \tag{5.1}$$

describing the flow of oil and water through sand, where $u$ is the water saturation in the sand, $f(u)$ is the flux function of the flowing stream, $Q$ is the total flow, and $\phi$ is the porosity. In particular, here we have considered the model flux function

$$f(u) := \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2},$$

we have set $Q/\phi := 1$, and we have chosen the initial distribution as follows:

$$u_0(x) := \begin{cases} 0.1/(x+0.1)^{-1}, & x \geq 0, \\ 1, & x < 0 \end{cases} \tag{5.2}$$

(see [CP]).

Equation (5.1) is a conservation law and our interest in it is due to the well-known connection between scalar conservation laws and Hamilton–Jacobi equations in one dimension. More precisely, if $u$ is an entropy solution (in Kruzkov's sense, see [K]) of the initial valued problem

$$\begin{cases} u_t + f(u)_x = 0 \text{ in } \mathbb{R} \times (0, +\infty), \\ u(x, 0) = u_0(x) \text{ on } \mathbb{R}, \end{cases} \tag{5.3}$$

then $v_\alpha := \int_\alpha^x u(\xi, t)\, d\xi$ is a viscosity solution of

$$\begin{cases} v_t + f(v_x) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \\ v(x, 0) = v_{\alpha 0}(x) \text{ on } \mathbb{R}, \end{cases} \tag{5.4}$$

where $v_{\alpha 0}(x) := \int_\alpha^x u_0(\xi)\, d\xi$ and $\alpha \in \mathbb{R}$. Therefore, if hypotheses $(H_1)$ or $(H_2)$ hold for problem (5.4), we can apply the FLT algorithm to compute $v_\alpha$. Finally, the solution of (5.3) can be obtained, for example, by centered finite differences or observing that $h(y) = \frac{d}{dy} g^*(y)$ whenever the Legendre transform $g^*$ of any function $g$ is regular and where $h$ is the argument function associated with $g^*$.

In the case of problems (5.1)–(5.2), if we substitute $f(u)$ with $\tilde{f}(u) := -f(u)$ and $u_0$ with $\tilde{u}_0(x) := u_0(-x)$, we obtain a Cauchy problem for a conservation law with initial data bounded and increasing. Then, taking $\alpha = 0$, the initial data for the corresponding problem (5.4), $v_0(x) := \int_0^x \tilde{u}_0(\xi)\, d\xi$, is a $C^1$ Lipschitz continuous and convex function. Since $\tilde{f}$ is locally Lipschitz continuous, hypothesis $(H_1)$ holds and the FLT algorithm can be applied to (5.4). After the solution $\tilde{u}$ of the corresponding modified conservation law has been computed, the solution of the original problem (5.1)–(5.2) will be given by $u(x, t) = \tilde{u}(-x, t)$.

Figure 5.1 shows the solution $u$ of (5.1)–(5.2) computed at time $T = 0.5$ and with $N = 64, 128, 256, 1024$ nodes in the interval of computation. The graphs are exactly that of the argument function associated with the second transform in $(v_0^* + t\tilde{f})^*(-x)$. In this way we have obtained a sharp shock while a certain amount of diffusion would be present if the solution had been computed with centered finite differences. Figure 5.2 shows the evolution of $u$ and the formation of the shock.

As a further test we have considered the equation describing the propagation of a flame front which moves in the direction normal to itself with constant speed and when the front at time $t = 0$ is, respectively, a closed region in $\mathbb{R}^d$ with $d = 2, 3$ and a surface (see [Ba], [FGL], [OS], [S]).

In the first case the equation is given by

$$u_t - c|Du| = 0 \qquad \text{in } \mathbb{R}^d \times (0, +\infty) \tag{5.5}$$

with initial Lipschitz condition

$$u_0(\mathbf{x}) := [1 - d(\mathbf{x}, \overline{\Omega})]^+ + d(\mathbf{x}, \Omega^c) \qquad \text{on } \mathbb{R}^d \tag{5.6}$$

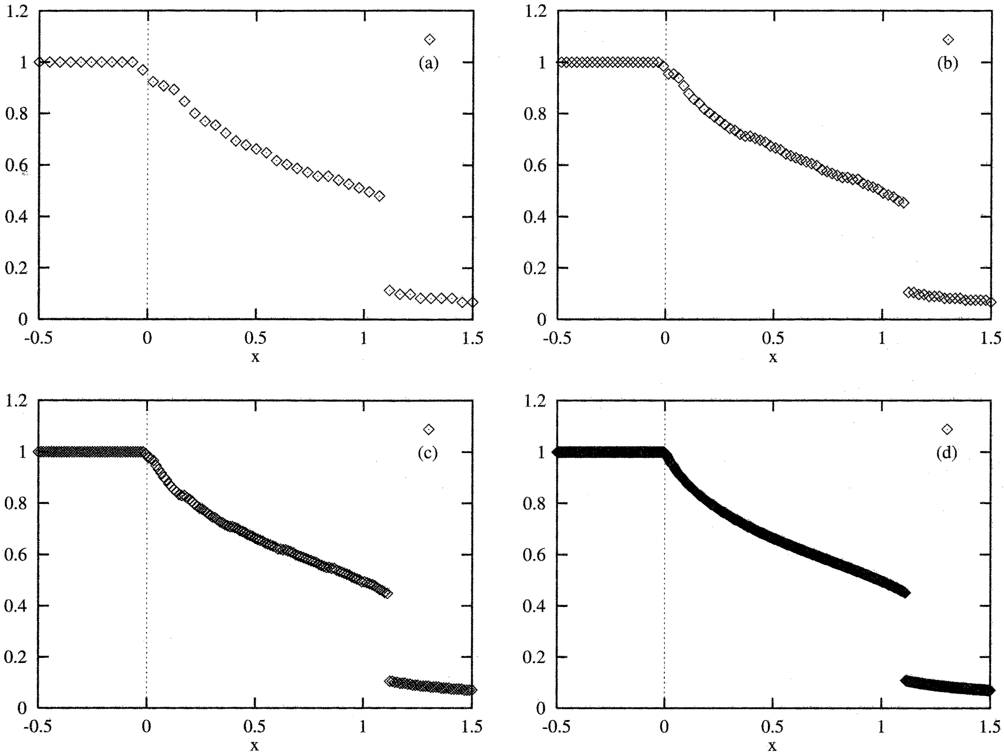


FIG. 5.1. *Buckley–Leverett equation. Solution computed at times* $T = 0.5$ *with* (a) $N = 64$, (b) $N = 128$, (c) $N = 256$, *and* (d) $N = 1024$.

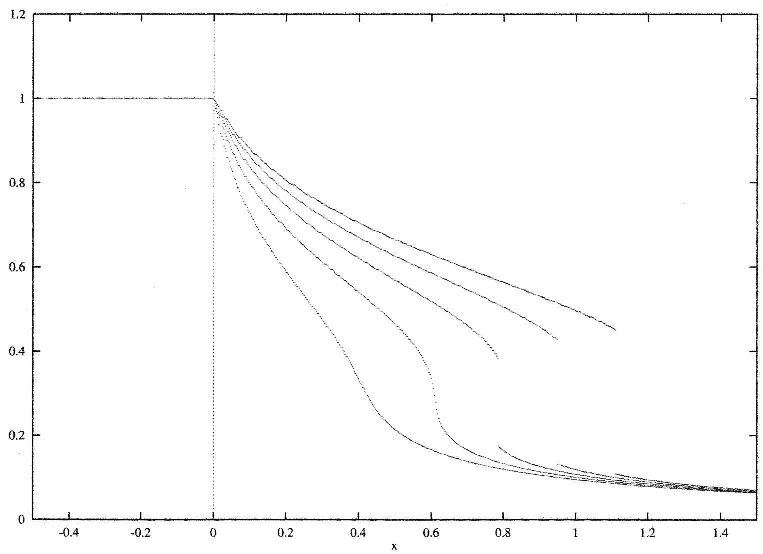


FIG. 5.2. *Buckley–Leverett equation. Solution computed at time* $T = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ *with* $N = 1024$.

where $c$ is the constant speed of propagation and $\overline{\Omega}$ the burnt region at time $t = 0$. In particular, if $\Omega$ is equal to the unit ball $B(0, 1)$ in $\mathbb{R}^3$, the initial data (5.6) become

$$u_0(\mathbf{x}) = \begin{cases} 2 - |\mathbf{x}| & \text{if } \mathbf{x} \in \overline{B}(0,1), \\ [2 - |\mathbf{x}|]^+ & \text{if } \mathbf{x} \notin \overline{B}(0,1). \end{cases} \tag{5.7}$$

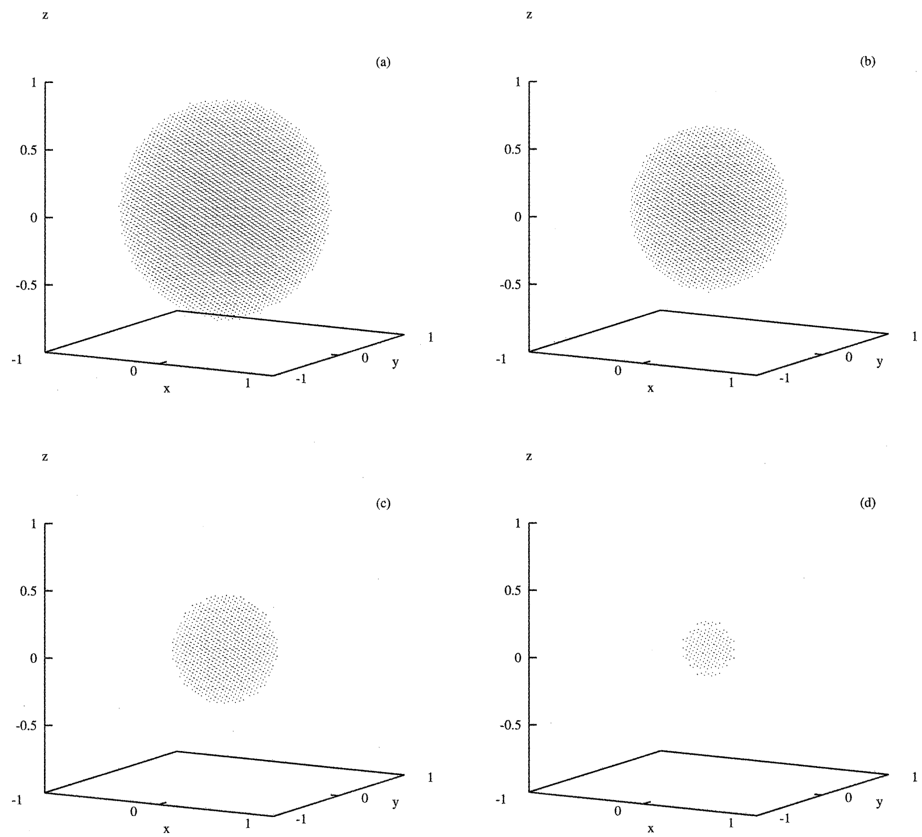


FIG. 5.3. *Collapsing sphere computed with* $N = 64$ *nodes per side of the domain of computation at times* (a) $T = 0.2$, (b) $T = 0.4$, (c) $T = 0.6$, *and* (d) $T = 0.8$.

It is easy to see that the solution of problems (5.5)–(5.7) does not change if the data are substituted with

$$u_0(\mathbf{x}) := 2 - |\mathbf{x}|. \tag{5.8}$$

In this way, we obtain a Cauchy problem of type (1.2) with a concave initial condition. Moreover, since the Hamiltonian $H$ in (5.5) depends only on $|Du|$, the solution $u$ of (5.5)–(5.8) is given by $u(\mathbf{x}, t) = -\tilde{u}(\mathbf{x}, t)$ where $\tilde{u}$ is the solution of $u_t + c|Du| = 0$ with a convex initial condition $\tilde{u}_0(\mathbf{x}) := |\mathbf{x}| - 2$. Again, hypotheses $(H_1)$ hold for the last problem and the FLT algorithm can be applied. Setting the speed of propagation $c := -1$, we obtain a collapsing sphere whose evolution is shown in Figure 5.3.

It is interesting to observe that the initial condition (5.6) is a Lipschitz continuous function and that the Hamiltonian $H$ in (5.5) is concave. Hence, if the initial burnt region $\Omega$ is such that $u_0$ in (5.6) is not convex or concave as before, we can apply the algorithm to equation $u_t + c|Du| = 0$ with initial condition $\tilde{u}_0(\mathbf{x}) := -u_0(\mathbf{x})$ anyway since hypotheses $(H_2)$ hold. If $\tilde{u}$ is the solution of the last problem, the solution of the original initial value problem will be $u(\mathbf{x}, t) = -\tilde{u}(\mathbf{x}, t)$. Using this observation, we have computed the evolution of a star moving at speed $c = 1$ (see Figure 5.4).

Finally, when the initial front is given by a surface $y = u_0(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, the corresponding equation is

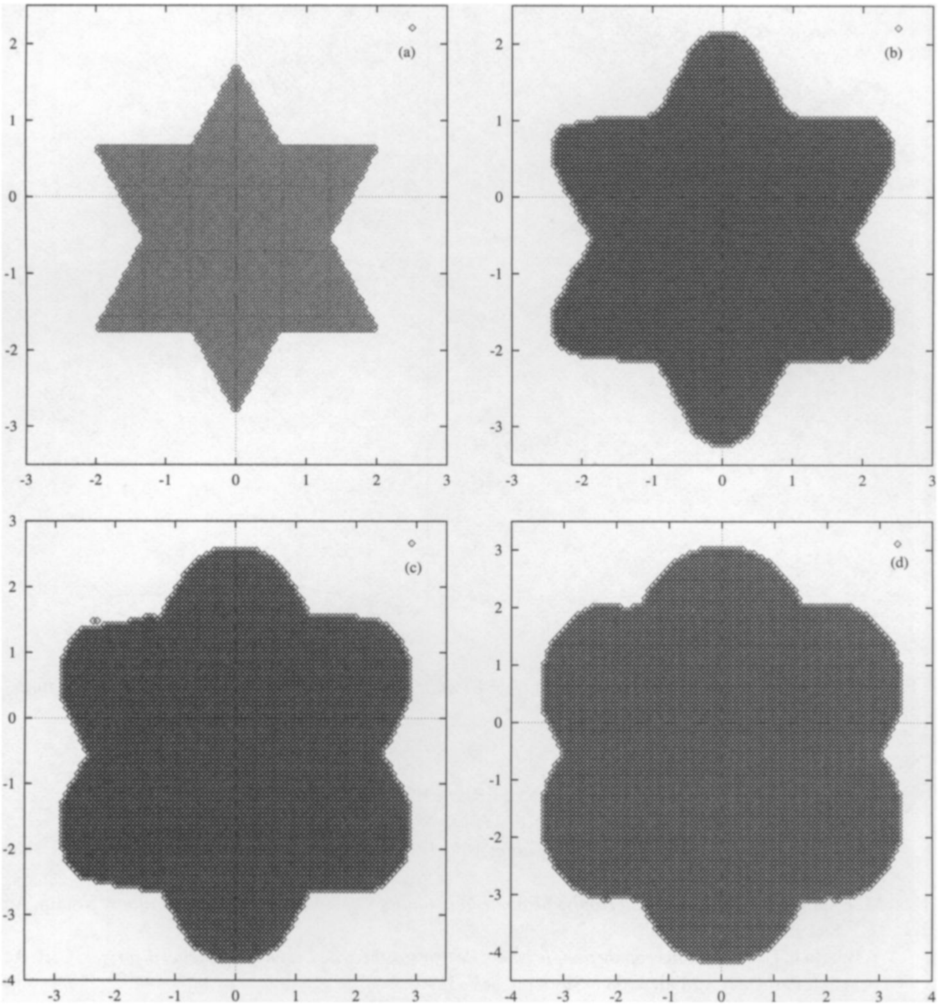

FIG. 5.4. *Expanding star computed with* $N = 128$ *nodes per side of the domain of computation at times* (a) $T = 0$, (b) $T = 0.5$, (c) $T = 1$, *and* (d) $T = 1.5$.

$$u_t - c\sqrt{1 + |Du|^2} = 0 \qquad \text{in } \mathbb{R}^2 \times (0, +\infty)$$

with initial data

$$u(\mathbf{x}, 0) := u_0(\mathbf{x}) \qquad \text{on } \mathbb{R}^2.$$

Here we have chosen

$$u_0(\mathbf{x}) := \begin{cases} |\mathbf{x}|^2 & \text{if } \mathbf{x} \in \overline{B}(0, 1), \\ 2|\mathbf{x}| - 1 & \text{if } \mathbf{x} \notin \overline{B}(0, 1), \end{cases}$$

and since hypotheses $(H_1)$ hold true, the algorithm can be applied directly on it. In Figure 5.5 we show the front at various times. The speed of propagation is $c = 1$.

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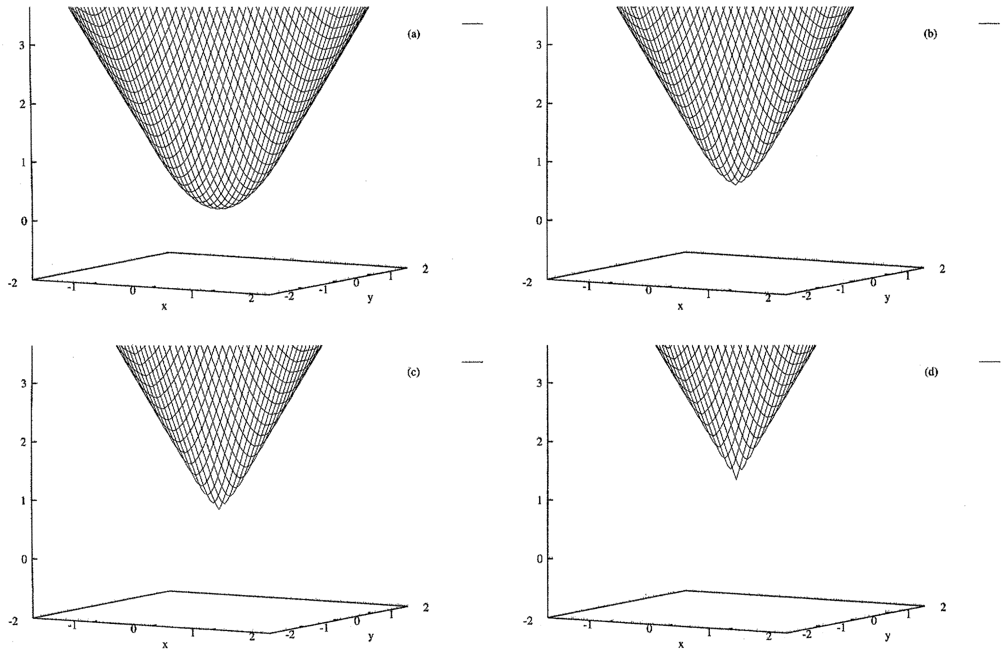


FIG. 5.5. *Propagating surface computed with* $N = 64$ *nodes per side of the domain of computation at times* (a) $T = 0.1$, (b) $T = 0.5$, (c) $T = 0.7$, *and* (d) $T = 1$.

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