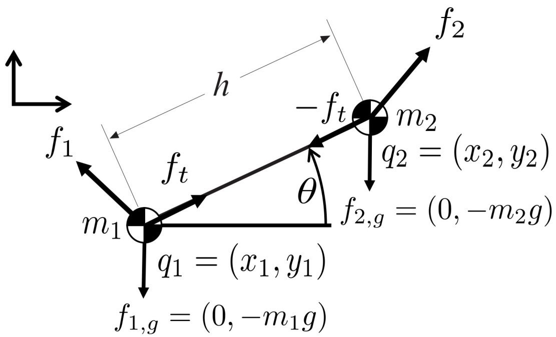
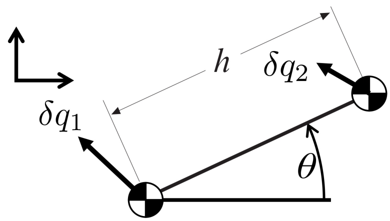
1、虚功原理 PVW Principle of Virtual Work

$$\Delta W = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f_j\right)^T \delta q_i = 0 \Rightarrow \text{系统处于平衡}$$





$$\delta W = (f_1 + f_t + f_{1,g})^T \delta q_1 + (f_2 - f_t + f_{2,g})^T \cdot \delta q_2 = 0$$

2、有关 δq

2.1 δ 可以与 d 交换计算顺序

$$\frac{d}{dt}(\delta q(t)) = \delta(\dot{q}(t)) \qquad d(\delta q(t)) = \delta(d q(t))$$

2.2 δ 可以与积分交换顺序

$$\delta \int_{t_0}^{t_f} q(t)\,dt = \int_{t_0}^{t_f} \delta q(t)\,dt$$

3、不考虑内力时，虚功原理依然成立

$$\delta W = (f_1 + f_{1,g})\delta q_1 + (f_2 + f_{2,g})\delta q_2 = 0$$

4、广义坐标：用来描述系统位形所需最少参数。

5、力旋量 wrenchs

$$F_i \triangleq \begin{bmatrix} f_i \\ q_i \times f_i \end{bmatrix} = \begin{bmatrix} f_i \\ f_i^T J q_i \end{bmatrix}$$

6、Reference:

Text reference:

Ch.1,2 of Goldstein, *Classical Mechanics*

Ch.1-3 of Cornelius, *The Variational Principles of Mechanics*

Ch.1,2 of Liberzon, *Calculus of variations and optimal control theory: a concise introduction*

Video reference:

https://www.bilibili.com/video/BV1K54y1v7Kn (full course on Goldstein's book)

See BB for a list of videos of Prof. Cornelius Lanczos' talks.

Part II、Euler-Lagrange equation

1、惯性力 inertial forces

与加速度方向相反 $-m_i\ddot{q}_i$

此时系统有加速度，不平衡，但将加速度转为惯性力后，依然可以使用PVW。

2、d'Alembert's principle 阿朗贝尔原理

$$\delta W=(f_1+f_{1,g}-m_1\ddot{q}_1)\delta q_1+(f_2+f_{2,g}-m_2\ddot{q}_2)\delta q_2=0$$

3、拉格朗日函数

广义坐标

$$L=T-V \qquad L(q_1,q_2,\cdots q_n,\dot{q}_1,\dot{q}_2\cdots\dot{q}_n)$$

动能 势能

4、Euler-Lagrange Equation $\Longleftrightarrow$ 牛顿第二定律

非保守力

$$\frac{\partial T}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=Q_j$$

(广义力，可以是力或力矩)

广义坐标q_i对应的力

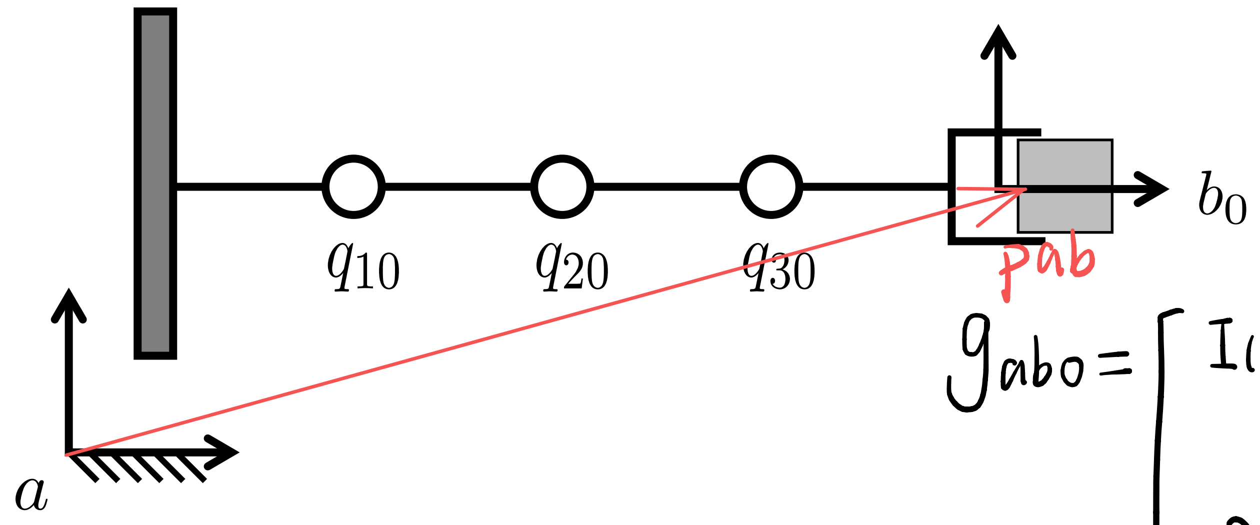
5. 推广

$h_1 = h_2 = \cdots = h_m = 0$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}_{\text{所有的惯性力、保守力}} = \underset{\substack{\downarrow \\ \text{没有被 L} \\ \text{包含的所有力}}}{Q_i} + \underbrace{\lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i}}_{\text{constrait forces}}$$

Equations of motion of robotic arm

SDM283



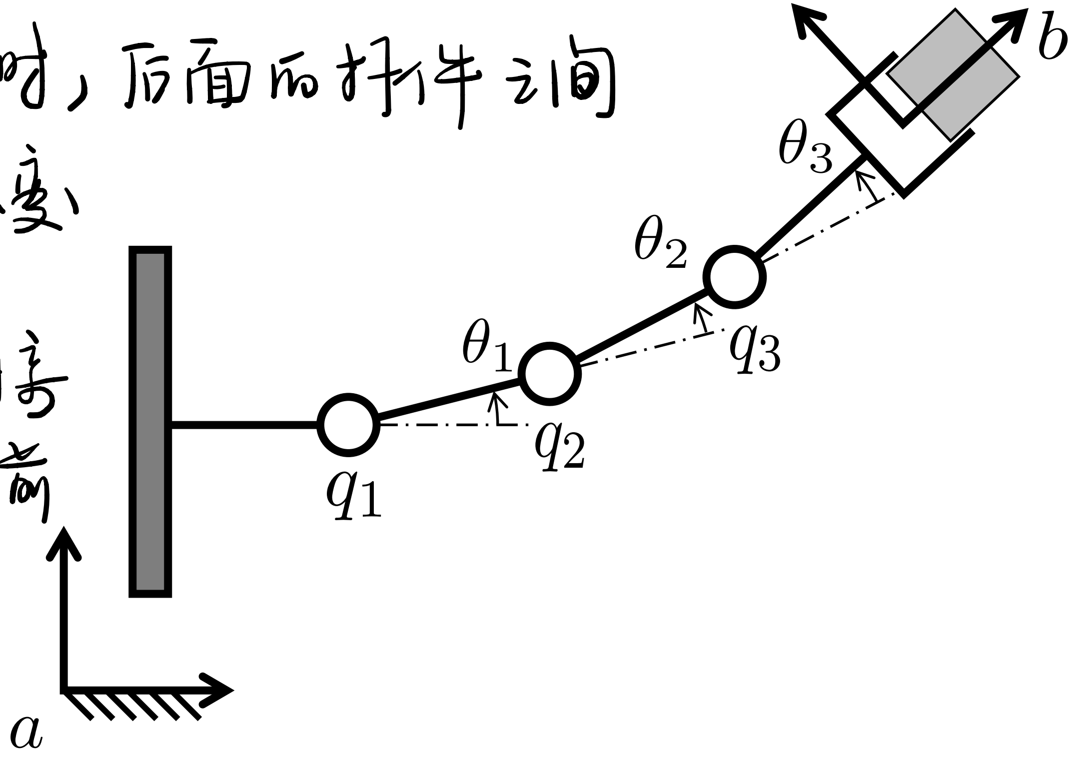
$$g_{ab0} = \begin{bmatrix} I_{(3)} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

Initial configuration

逐级迭代计算

转动第一个关节时，后面的杆件之间保持相对位置不变

角度θ为：关节之后连接的杆件在关节旋转前后的角度变化。（逆时针为正）



Current configuration

This tutorial teaches you how to compute:

1. transformation

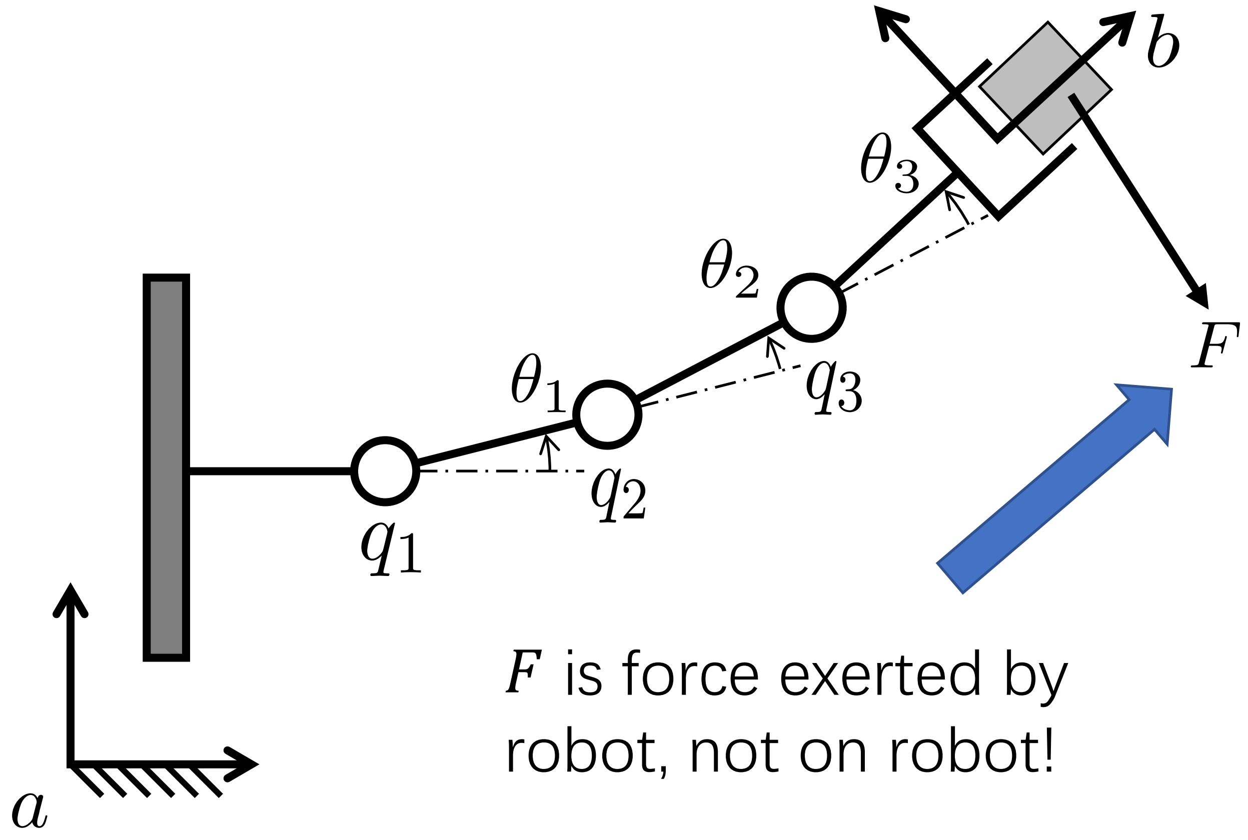
   $$g_{ab}(\theta_1, \theta_2, \theta_3)$$

2. velocity

   $$V_{ab}(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$$

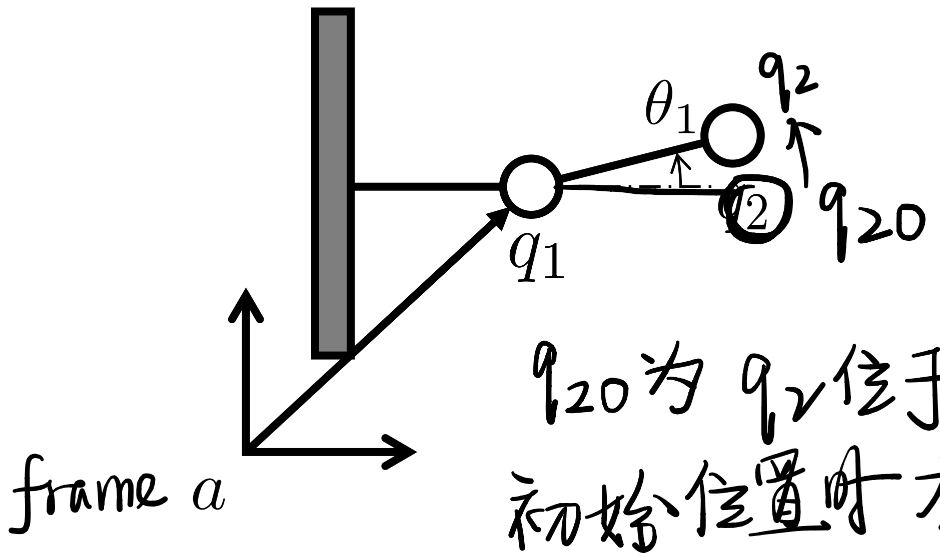
3. Torque needed to balance a wrench F

   $$\tau = (\tau_1, \tau_2, \tau_3)$$



F is force exerted by robot, not on robot!

Step 1 how to move a point



q_{20}为 q_2位于
初始位置时在
frame a下的坐标向量

$$q_1 = q_{10}$$

$$\boxed{q_2^{(t)} = \underbrace{R(q_{10}, \theta_1)}\, q_{20}}$$

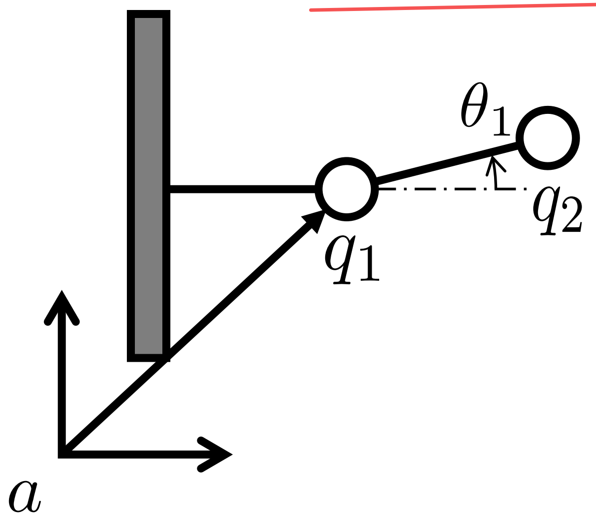
Rotation about q_{10} for θ_1
(or rather rotation about
an axis perpendicular to
the screen and passing q_1)

Step 1 how to move a point

关节最初位置在frame a下面坐标向量

某个关节的 unit twist

$$\hat{\xi}_1 = \begin{bmatrix} \hat{z} & \boxed{q_{10}} \times z \\ 0 & 0 \end{bmatrix}$$



Unit velocity of joint q_1

$$R(q_{10}, \theta_1) = e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} e^{\hat{z}\theta_1} & (I - e^{\hat{z}\theta_1})q_{10} \\ 0 & 1 \end{bmatrix}$$

z-axis is perpendicular to the screen.

等价关系：

① $R(q_{i0},\theta_i) = e^{\hat{V}_i t} = e^{\hat{\xi}_i\theta_i} = \begin{bmatrix} e^{\hat{\omega}_i t} & (I-e^{\hat{\omega}_i t})q_{i0} \\ 0 & 1\end{bmatrix} = \begin{bmatrix} e^{\hat{z}\theta} & (I-e^{\hat{z}\theta})q_{i0}\\ 0 & 1\end{bmatrix}$

$\hat{V}_i = \begin{bmatrix}\hat{\omega} & v\\ 0 & 0\end{bmatrix} \xRightarrow[\text{角速度单位化}]{} \hat{\xi}_i = \begin{bmatrix} \left(\frac{\omega}{\|\omega\|}\right)^\wedge & q_{i0}\times\frac{\omega}{\|\omega\|} \\ 0 & 0\end{bmatrix}$

$\hat{V}_i t = \hat{\xi}_i\cdot\|\omega\|\cdot t = \hat{\xi}_i\theta_i$

② $R_i = (R_x \text{ or } R_y \text{ or } R_z) = e^{\hat{\omega}_i t} \xrightarrow{\text{角速度单位化}} = e^{\hat{z}\theta_i}$ ⇒ matlab中可以使用expm( )来计算e的矩阵次方

$\begin{bmatrix} 1 & 0 & 0\\ 0 & c(\omega t) & -s(\omega t)\\ 0 & s(\omega t) & c(\omega t)\end{bmatrix}$ $\begin{bmatrix} c(\omega t) & 0 & s(\omega t)\\ 0 & 1 & 0\\ -s(\omega t) & 0 & c(\omega t)\end{bmatrix}$ $\begin{bmatrix} c(\omega t) & -s(\omega t) & 0\\ s(\omega t) & c(\omega t) & 0\\ 0 & 0 & 1\end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0\\ 0 & c(x\theta) & -s(x\theta)\\ 0 & s(x\theta) & c(x\theta)\end{bmatrix}$ $\begin{bmatrix} c(y\theta) & 0 & s(y\theta)\\ 0 & 1 & 0\\ -s(y\theta) & 0 & c(y\theta)\end{bmatrix}$ $\begin{bmatrix} c(z\theta) & -s(z\theta) & 0\\ s(z\theta) & c(z\theta) & 0\\ 0 & 0 & 1\end{bmatrix}$

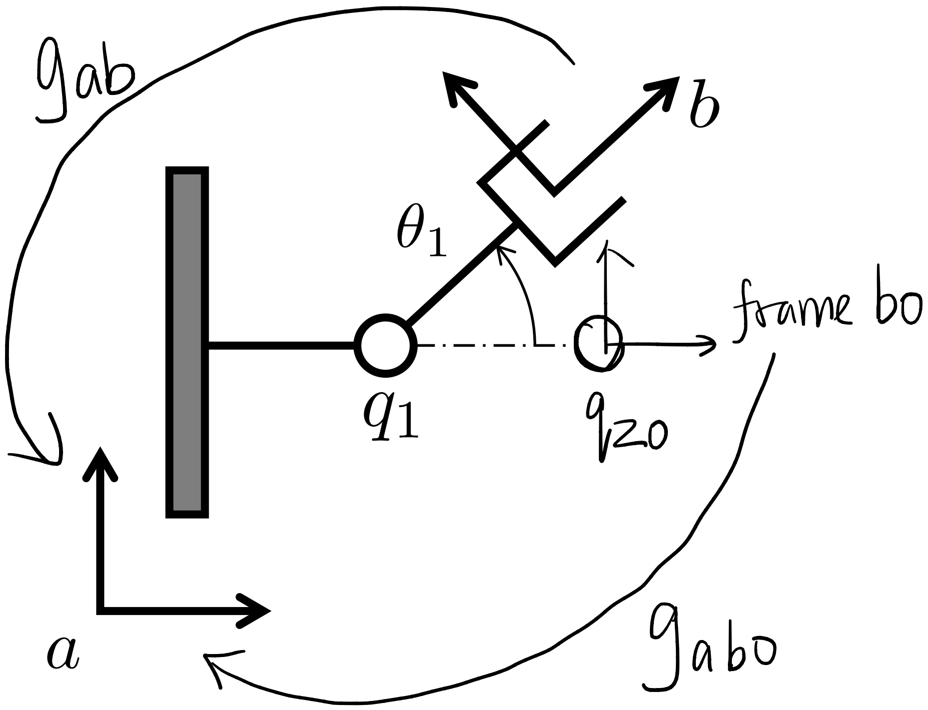
但其函数算法复杂，计算效率差

反对称阵可以用 $\boxed{\text{Rodrigues Formula}}$ 化简：

$e^{\hat{\omega} t} = I + \frac{\hat{\omega}}{\|\omega\|}\cdot\sin(\|\omega\|t) + \frac{\hat{\omega}^2}{\|\omega\|^2}\cdot(1-\cos(\|\omega\|t))$

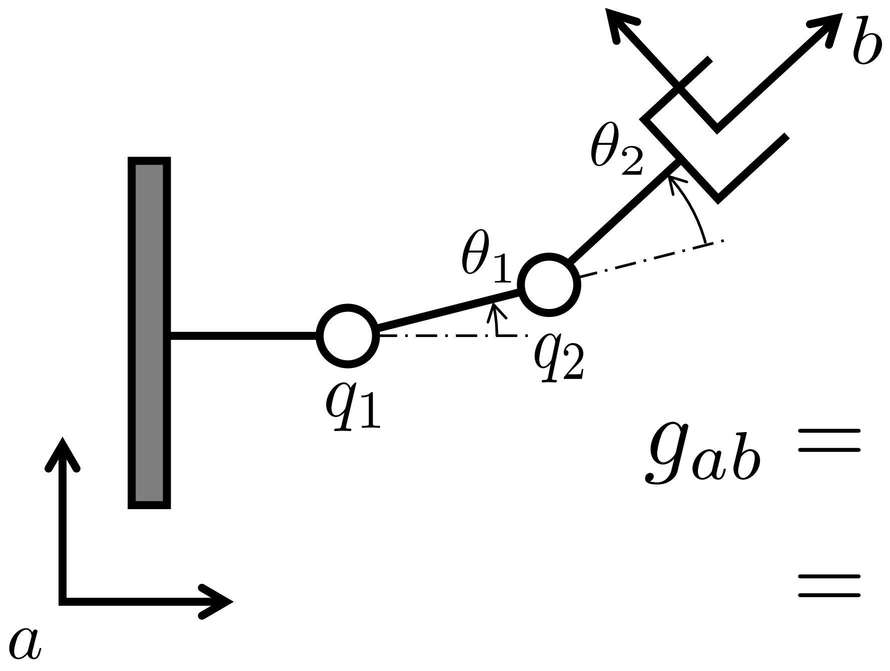
$e^{\hat{z}\theta} = I + \hat{z}\sin\theta + (\hat{z})^2\cdot(1-\cos\theta)$

Step 2 how to move a frame



$$g_{ab} = R(q_{10}, \theta_1) g_{ab_0}$$

Step 3 how to move two link robot end



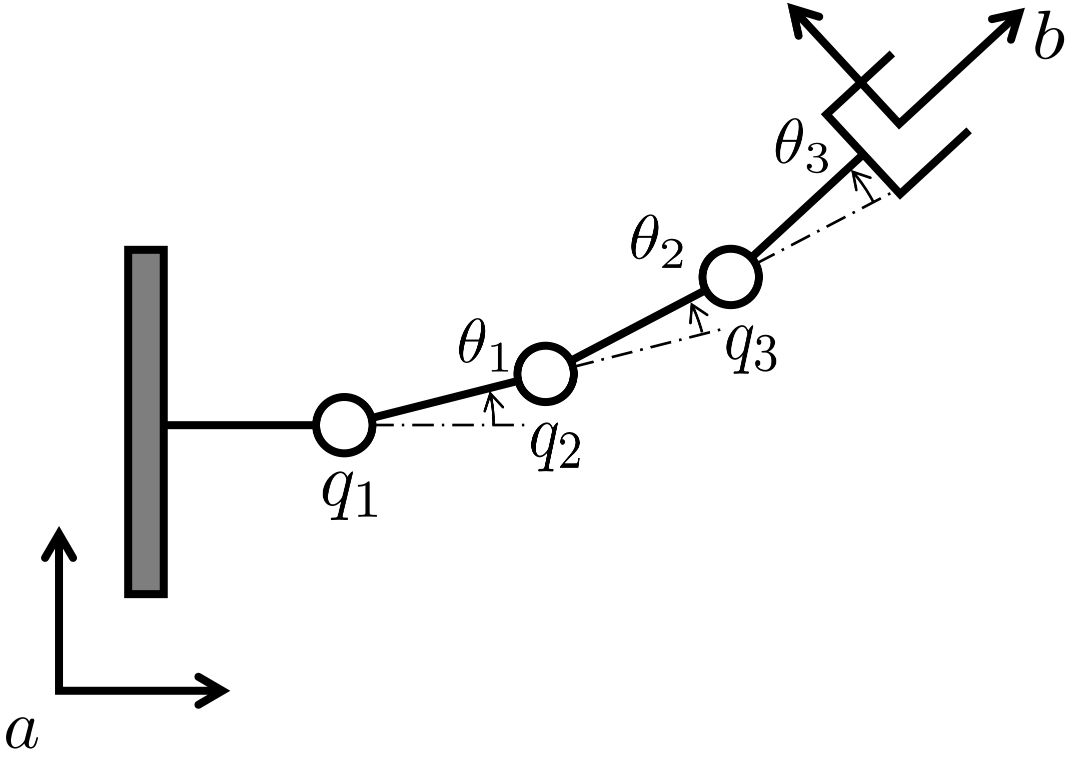
$$
\begin{aligned}
g_{ab} &= R(q_2, \theta_2) R(q_{10}, \theta_1) g_{ab_0} \\
&= R(R(q_{10}, \theta_1) q_{20}, \theta_2) R(q_{10}, \theta_1) g_{ab_0} \\
&= R(q_{10}, \theta_1) R(q_{20}, \theta_2) g_{ab_0}
\end{aligned}
$$

Can you prove the last equality?

Step 4 The result for g_{ab}

从右至左理解本公式

末端先动，前面不跟随动
前端先动，后面会跟随动
} 先动末端



$$g_{ab} = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{ab_0}$$

$$\hat{\xi}_i = \begin{bmatrix} \hat{\omega}_{i0} & q_{i0} \times \omega_{i0} \\ 0 & 0 \end{bmatrix}$$

ω_{i0} denotes initial unit direction of joint i ($= z$ here)

逐级迭代计算

转动第一个关节时，后面的构件之间保持相对位置不变。

角度θ为：关节之后连接的构件在关节旋转前后的角度变化。（逆时针为正）

q_{i0} 为每个关节最初始位置在 frame a 下的坐标向量

Step 5 Compute velocity

g_{ab}^{-1} : 先穿的后脱

$$g_{ab} = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{ab_0}$$

$g_{ab}^{-1} = g_{ab_0}^{-1} \cdot e^{-\hat{\xi}_3\theta_3} \cdot e^{-\hat{\xi}_2\theta_2} \cdot e^{-\hat{\xi}_1\theta_1}$

$\dot{g}_{ab}$ 注意 $\hat{\xi}_i$ 矩阵乘法不能随便交换位置

$(e^{\hat{\xi}_1\theta_1})^{\cdot} = \hat{\xi}_1 \cdot e^{\hat{\xi}_1\theta_1} \cdot \dot{\theta}_1$

对t求导

$$\hat{V}_{ab} = \dot{g}_{ab} g_{ab}^{-1} = \hat{\xi}_1\dot{\theta}_1 + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{-\hat{\xi}_1\theta_1}\dot{\theta}_2 + (e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2})\hat{\xi}_3 (e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1})\dot{\theta}_3$$

乘法求导展开

Step 6 Recall convenient equation

$$g\hat{V}g^{-1} = Ad_g\hat{V}$$

$$Ad_gV = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} V$$

for $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ a rigid motion and $V = \begin{bmatrix} v \\ \omega \end{bmatrix}$ a twist.

Step 7 Taking the hat ∧ off

$$V_{ab} = \xi_1 \dot{\theta}_1 + \underbrace{Ad_{e^{\hat{\xi}_1 \theta_1}} \xi_2}_{\xi_2'} \dot{\theta}_2 + \underbrace{Ad_{e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2}} \xi_3}_{\xi_3'} \dot{\theta}_3$$

Twists at new location

$$V_{ab} = \underbrace{\begin{bmatrix} \xi_1 & \xi_2' & \xi_3' \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}}_{\dot{\theta}}$$

Jacobian matrix

Step 8 Principle of Virtual Work!

$$V_{ab}^T F = (J\dot{\theta})^T F = \dot{\theta}^T J^T F = \dot{\theta}^T \tau$$

$$\Downarrow$$

$$\tau = J^T F$$

Here F only accounts for robot payload, but not inertial force and gravity.

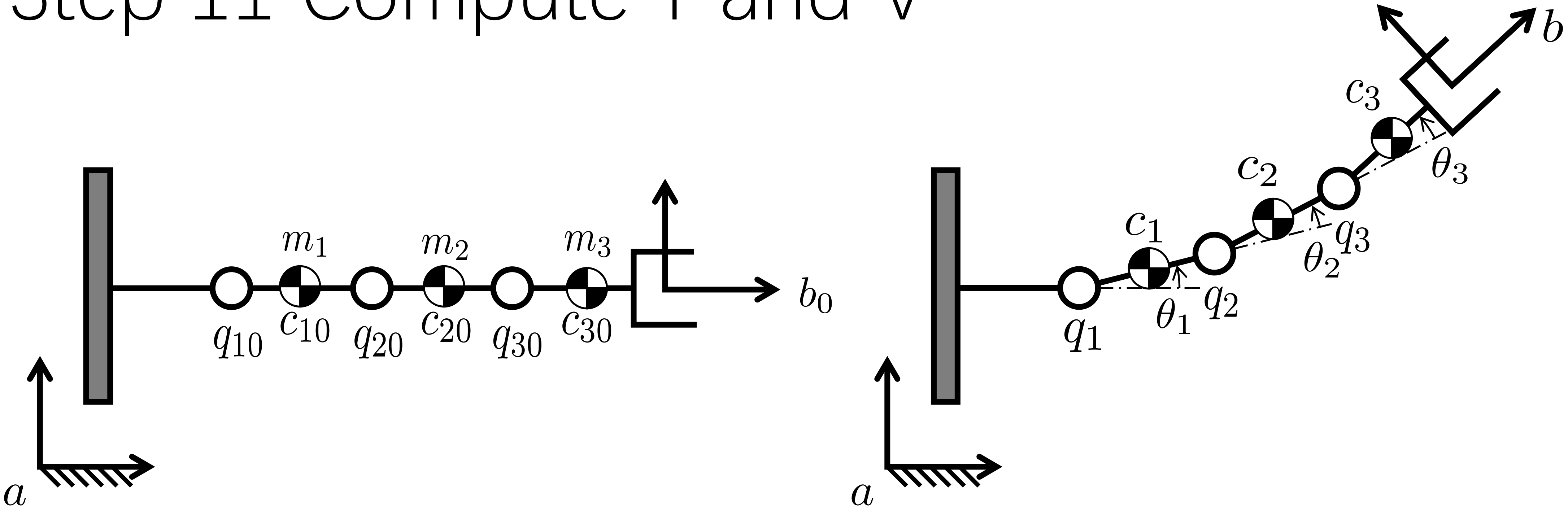
Step 9 Account for gravity

$$\tau = -\frac{\partial L}{\partial \vec{\theta}} + J^T F$$

Step 10 Account for both gravity and inertial forces

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{\theta}}}\right) - \frac{\partial L}{\partial \vec{\theta}} + J^T F$$

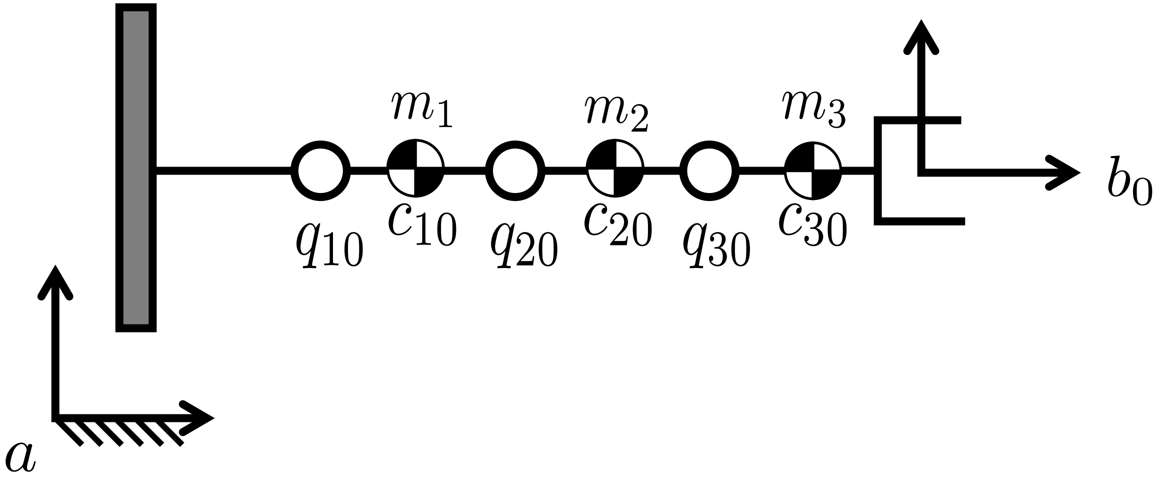
Step 11 Compute T and V

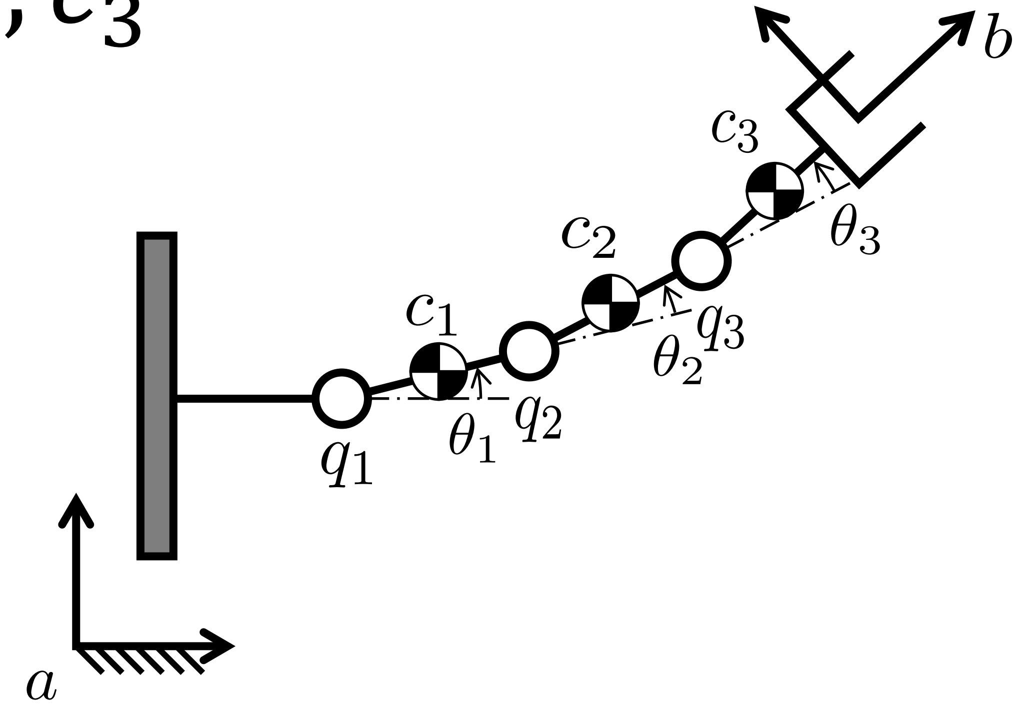


$$V = -m_1 g c_{1,y} - m_2 g c_{2,y} - m_3 g c_{3,y}$$

$$T = \frac{1}{2} m_1 \dot{c}_1^T \dot{c}_1 + \frac{1}{2} m_2 \dot{c}_2^T \dot{c}_2 + \frac{1}{2} m_3 \dot{c}_3^T \dot{c}_3$$

Step 12 Compute $\dot{c}_1, \dot{c}_2, \dot{c}_3$





$$\begin{cases} c_1 = e^{\hat{\xi}_1\theta_1} c_{10} \\ c_2 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} c_{20} \\ c_3 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} c_{30} \end{cases} \Rightarrow \begin{cases} \dot{c}_1 = \dot{\theta}_1 \hat{\xi}_1 c_1 \\ \dot{c}_2 = (\dot{\theta}_1 \hat{\xi}_1 + \dot{\theta}_2 \hat{\xi}'_2) c_2 \\ \dot{c}_3 = (\dot{\theta}_1 \hat{\xi}_1 + \dot{\theta}_2 \hat{\xi}'_2 + \dot{\theta}_3 \hat{\xi}'_3) c_3 \end{cases}$$

Step 13 Compute $\frac{\partial L}{\partial \vec{\theta}} = \frac{\partial (T-V)}{\partial \vec{\theta}}$

$$\frac{\partial}{\partial \theta_1}\left(\frac{1}{2} m_1 \dot{c}_1^T \dot{c}_1\right) = m_1 \dot{c}_1^T \frac{\partial \dot{c}_1}{\partial \theta_1} = m_1 \dot{c}_1^T \dot{\theta}_1 \hat{\xi}_1 \frac{\partial c_1}{\partial \theta_1} = m_1 \dot{c}_1^T \dot{\theta}_1 \hat{\xi}_1 \hat{\xi}_1 c_1$$

$$\frac{\partial}{\partial \theta_2}\left(\frac{1}{2} m_1 \dot{c}_1^T \dot{c}_1\right) = 0$$

$$\frac{\partial}{\partial \theta_3}\left(\frac{1}{2} m_1 \dot{c}_1^T \dot{c}_1\right) = 0$$

Step 13 Compute $\frac{\partial L}{\partial \vec{\theta}} = \frac{\partial (T - V)}{\partial \vec{\theta}}$

$$\begin{aligned}
\frac{\partial}{\partial \theta_1}\left(\frac{1}{2} m_2 \dot{c}_2^T \dot{c}_2\right) &= m_2 \dot{c}_2^T \frac{\partial}{\partial \theta_1}\left((\dot{\theta}_1 \hat{\xi}_1 + \dot{\theta}_2 \hat{\xi}_2')c_2\right) \\
&= m_2 \dot{c}_2^T \dot{\theta}_2 \left(\frac{\partial \hat{\xi}_2'}{\partial \theta_1} c_2 + \hat{\xi}_2' \frac{\partial c_2}{\partial \theta_1}\right)
\end{aligned}$$

$$\frac{\partial \hat{\xi}_2'}{\partial \theta_1} = \hat{\xi}_1 \hat{\xi}_2' - \hat{\xi}_2' \hat{\xi}_1, \qquad \frac{\partial c_2}{\partial \theta_1} = \hat{\xi}_1 c_2$$

Step 13 Compute $\frac{\partial L}{\partial \vec{\theta}} = \frac{\partial (T-V)}{\partial \vec{\theta}}$

$$\begin{aligned}
\frac{\partial}{\partial \theta_2}\left(\frac{1}{2}m_2 \dot{c}_2^T \dot{c}_2\right) &= m_2 \dot{c}_2^T \frac{\partial}{\partial \theta_2}\left((\dot{\theta}_1 \hat{\xi}_1 + \dot{\theta}_2 \hat{\xi}_2')c_2\right) \\
&= m_2 \dot{c}_2^T \dot{\theta}_2 \left(\frac{\partial \hat{\xi}_2'}{\partial \theta_2} c_2 + \hat{\xi}_2' \frac{\partial c_2}{\partial \theta_2}\right)
\end{aligned}$$

$$\frac{\partial \hat{\xi}_2'}{\partial \theta_2} = 0, \qquad \frac{\partial c_2}{\partial \theta_2} = \hat{\xi}_2' c_2$$

Step 13 Compute $\frac{\partial L}{\partial \vec{\theta}} = \frac{\partial (T - V)}{\partial \vec{\theta}}$

$$\frac{\partial}{\partial \theta_3}\left(\frac{1}{2} m_2 \dot{c}_2^T \dot{c}_2\right) = 0$$

Now do the same for $\frac{1}{2} m_3 \dot{c}_3^T \dot{c}_3$ and V!

Step 14 Compute $\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial L}{\partial \dot{\vec{\theta}}}\right) = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial T}{\partial \dot{\vec{\theta}}}\right)$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}_1}\left(\frac{1}{2}m_1\dot{c}_1^T\dot{c}_1\right)\right) = \frac{d}{dt}\left(m_1\dot{c}_1^T\hat{\xi}_1 c_1\right) = m_1\ddot{c}_1^T\hat{\xi}_1 c_1 + m_1\dot{c}_1^T\hat{\xi}_1\dot{c}_1$$

$$\dot{c}_1 = \dot{\theta}_1\hat{\xi}_1 c_1 \Rightarrow \ddot{c}_1 = \ddot{\theta}_1\hat{\xi}_1 c_1 + \dot{\theta}_1\hat{\xi}_1\dot{c}_1 = \ddot{\theta}_1\hat{\xi}_1 c_1 + \dot{\theta}_1^2\hat{\xi}_1^2 c_1$$

Now do the same for the rest, and get your Euler-Lagrange equation in the end!