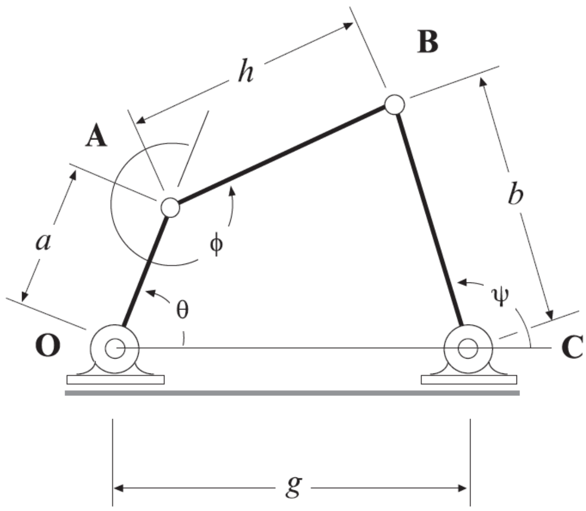
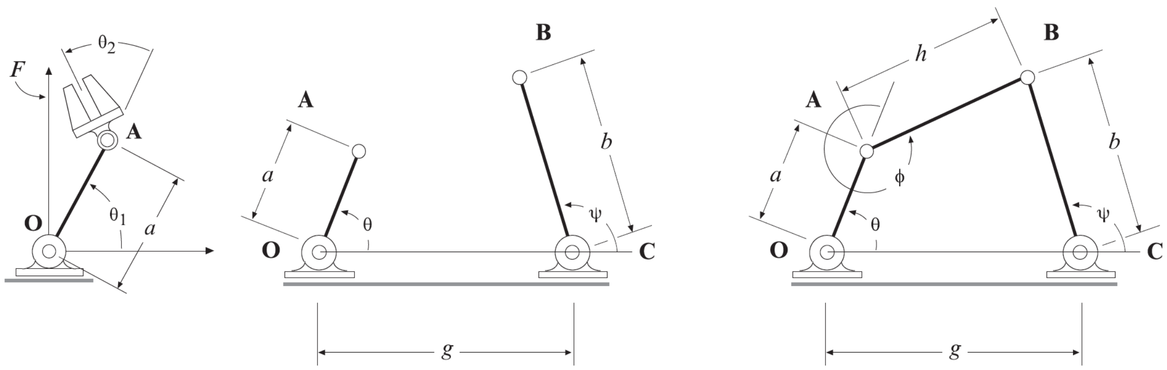
运动学
力学：如何实现运动
材料力学：考虑真实变形而非理想刚体情况
振动

力学
0.　回顾四连杆几何关系推导过程




- Forward kinematics $\theta \mapsto \psi$ (output link angle)

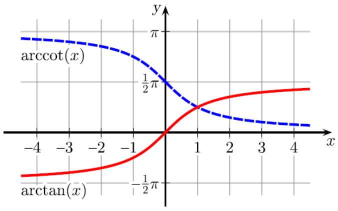


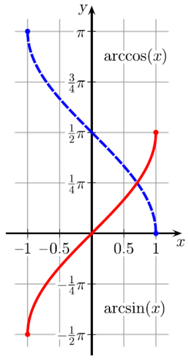

$$\mathbf{A} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} g + b\cos\psi \\ b\sin\psi \end{pmatrix} \xrightarrow{\text{Plug in}} (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) - h^2 = 0$$

- Forward kinematics $\theta \mapsto \psi$ (output link angle)

$$\boxed{\psi(\theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2+B^2}}\right)}$$

$$\begin{aligned} A(\theta) &= 2ab\cos\theta - 2gb \\ B(\theta) &= 2ab\sin\theta \\ C(\theta) &= g^2 + b^2 + a^2 - h^2 - 2ag\cos\theta \end{aligned}$$

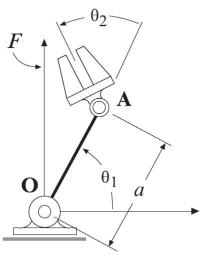


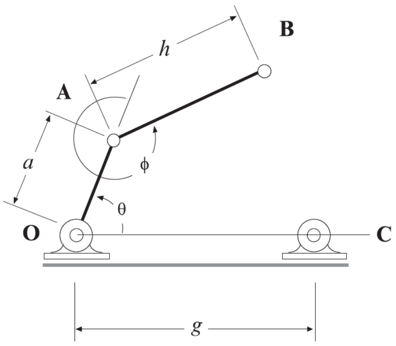



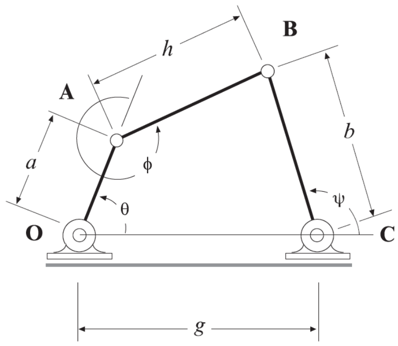

Question: solvable?

$$A^2 + B^2 - C^2 \geq 0$$

- Forward kinematics $\theta \mapsto \phi$ (coupler link angle)




$$\mathbf{A} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} a\cos\theta + h\cos(\theta+\phi) \\ a\sin\theta + h\sin(\theta+\psi) \end{pmatrix}$$

$$\begin{aligned} a\cos\theta + h\cos(\theta+\phi) &= g + b\cos\psi \\ a\sin\theta + h\sin(\theta+\phi) &= b\sin\psi \end{aligned}$$

$$\boxed{\phi = \arctan\left(\frac{b\sin\psi - a\sin\theta}{g + b\cos\psi - a\cos\theta}\right) - \theta}$$

- Input-output velocity ratio

$$\left(\dot\theta\begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} + \dot\phi\begin{pmatrix} \mathbf{J} & -\mathbf{JA} \\ \mathbf{0}^T & 0 \end{pmatrix}\right)\begin{pmatrix} \mathbf{B} \\ 1 \end{pmatrix} = \dot\psi\begin{pmatrix} \mathbf{J} & -\mathbf{JC} \\ \mathbf{0}^T & 0 \end{pmatrix}\begin{pmatrix} \mathbf{B} \\ 1 \end{pmatrix}$$

$$\Downarrow$$

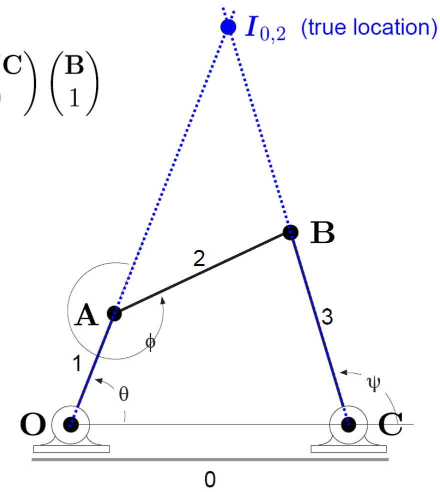
$$\dot\theta\mathbf{JB} + \dot\phi\mathbf{J}(\mathbf{B}-\mathbf{A}) = \dot\psi\mathbf{J}(\mathbf{B}-\mathbf{C})$$

$$\Downarrow$$

$$\begin{aligned} \dot\theta(\mathbf{B}-\mathbf{A})^T\mathbf{JB} &= \dot\psi(\mathbf{B}-\mathbf{A})^T\mathbf{J}(\mathbf{B}-\mathbf{C}) \\ \dot\theta(\mathbf{B}-\mathbf{C})^T\mathbf{JB} &= -\dot\phi(\mathbf{B}-\mathbf{C})^T\mathbf{J}(\mathbf{B}-\mathbf{A}) \end{aligned}$$

$$\Downarrow$$

$$\boxed{\frac{\dot\psi}{\dot\theta} = \frac{(\mathbf{B}-\mathbf{A})^T\mathbf{JB}}{(\mathbf{B}-\mathbf{A})^T\mathbf{J}(\mathbf{B}-\mathbf{C})}, \quad \frac{\dot\phi}{\dot\theta} = \frac{-(\mathbf{B}-\mathbf{C})^T\mathbf{JB}}{(\mathbf{B}-\mathbf{C})^T\mathbf{J}(\mathbf{B}-\mathbf{A})}}$$



忘记这个是怎么推导的了↑

1. Basic Concept

某质点 q $q=(x,y)$

其运动轨迹 $x^2+y^2=1$

其瞬时速度 $\dot{q}=(\dot{x},\dot{y})$

速度方程 $\frac{d}{dt}(x^2+y^2)=2x\dot{x}+2y\dot{y}=0$ $x$*径向* $\dot{x}$*切向 两者垂直，点乘结果等于*$0$

当 x 确定时（取消一个自由度） $y=\pm\sqrt{1-x^2}$

$$\dot{y}=\mp\frac{x\dot{x}}{\sqrt{1-x^2}}$$

本处做法问题：1 正负号选哪个 2 x=1 时方程无意义

当 y 确定时情况类似

解决方式：不必要使用笛卡尔坐标系，可以使用极坐标

极坐标与笛卡尔坐标的变换方式：

$$x=cos\theta \quad y=sin\theta$$

$$\dot{x}=-sin\theta\dot{\theta} \quad \dot{y}=cos\theta\dot{\theta}$$

质点可以存在的空间：configuration space 位形空间 C $C=\{\theta|\theta\in[0,2\pi)\}$ 开区间原因：圆环上 0 与 2pi 为同一个点

与质点本身位置（x,y）无关的坐标形式叫做广义坐标 generalized coordinates

2. Kinetic energy

牛顿第二定律 $f=\ddot{q}$

定义动能 $T=\frac{1}{2}m\dot{q}^T\dot{q}$

动能函数反推牛顿方程

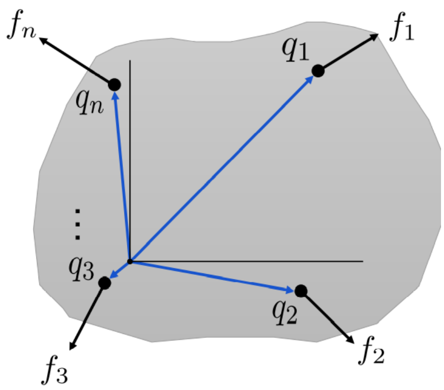
Kinetic energy becomes very useful when you have a whole system of particles (for rigid body, replace summation with integration)

$$T=\frac{1}{2}\sum_{i=1}^{n}m_i\dot{q}_i^T\dot{q}_i$$

Question: do we have the following equation?

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)=\frac{d}{dt}(m_i\dot{q}_i)=f_i, \qquad i=1,\ldots,n$$

(Hint: think about the force that keeps a particle on the circle)

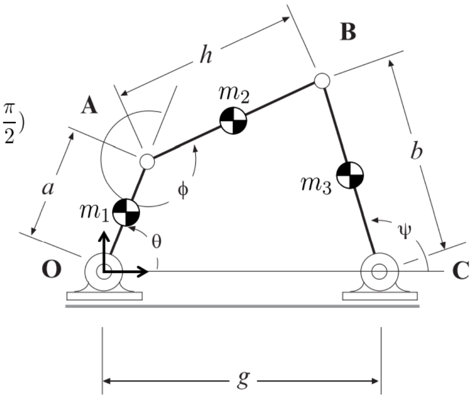


$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}(m\dot{q}) = f \qquad \left(\text{Given } \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix},\ \frac{\partial T}{\partial \dot{q}} = \begin{bmatrix} \partial T/\partial \dot{x} \\ \partial T/\partial \dot{y} \\ \partial T/\partial \dot{z} \end{bmatrix}\right)$$

问题：有内力

Assume center of mass of the links are located at the mid-point of the links, compute the kinetic energy of the 4-bar linkage as a function of $\theta$ and $\dot{\theta}$ (assume $\phi(\theta), \psi(\theta), \dot{\phi}(\theta,\dot{\theta}), \dot{\phi}(\theta,\dot{\theta})$ are given)

$$\boxed{\frac{\dot{\psi}}{\dot{\theta}} = \frac{(\mathbf{B}-\mathbf{A})^T J\mathbf{B}}{(\mathbf{B}-\mathbf{A})^T J(\mathbf{B}-\mathbf{C})}, \quad \frac{\dot{\phi}}{\dot{\theta}} = \frac{-(\mathbf{B}-\mathbf{C})^T J\mathbf{B}}{(\mathbf{B}-\mathbf{C})^T J(\mathbf{B}-\mathbf{A})}} \qquad J = R(\tfrac{\pi}{2})$$




求三联杆系统的动能

1 直接求导 2 先算出角速度之间的关系再计算

Consider the infinitesimal work done by $f$:

$$dW = f^T dq$$

If there exists a *work function* $U(q)$ such that:

$$f = \frac{\partial U(q)}{\partial q}$$

we have:

$$\int_{t_0}^{t_f} dW = \int_{t_0}^{t_f} \frac{\partial U}{\partial q} dq = U(q(t_f)) - U(q(t_0))$$

Putting the kinetic energy and work function together:

$$\int_{t_0}^{t_f} m\ddot{q}^T dq = \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{1}{2}m\dot{q}^T\dot{q}\right)dt = \int_{t_0}^{t_f} f^T dq = \int_{t_0}^{t_f}\left(\frac{\partial U}{\partial q}\right)^T dq$$

$$\Downarrow$$

$$T(t_f) - T(t_0) = U(t_f) - U(t_0)$$

or

$$T(t_f) - U(t_f) = T(t_0) - U(t_0)$$

We usually define *potential energy* $V(q) = -U(q)$

$$\Downarrow$$

$$E(t) \triangleq T(t) + V(t) = \text{constant}$$

The total energy $E$ remains constant. $f = -\partial V/\partial q$ is therefore called a *conservative force*.

之前使用动能表示牛顿定律有问题，没表达出内力。应使用动能+势能表示牛顿定律。其中势能对应的为保守力。

3. 虚功原理

   约束力

   在切向方向上进行一小段微小位移（任取），如果系统处于平衡状态（合力为 0），合力在该微小位移上做的功为 0. 如果该功为 0，也能证明力平衡.

   虚位移：1 沿切线 2 不真实，用于测试系统是否平衡 3 发生位移时，时间停止 4 性质上类似速度

The virtual displacements $\delta q_1, \delta q_2$ act like $dq_1, dq_2$, but occurs when the system freezes in time. Recall:

$$\dot{q}_1 = \frac{dq_1}{dt}, \dot{q}_2 = \frac{dq_2}{dt}$$

$$(q_1 - q_2)^T(q_1 - q_2) = h^2 \quad \Rightarrow \quad (q_1 - q_2)^T(\dot{q}_1 - \dot{q}_2) = 0$$

Similarly, we have:

$$(q_1 - q_2)^T(\delta q_1 - \delta q_2) = 0$$

$$\frac{d}{dt}(u^T v) = \dot{u}^T v + u^T \dot{v}$$

虚位移可以是关于时间的函数，不同时间下虚位移不同。

两个重要运算性质:

$$\delta q(t+dt) = \delta q(t) + \delta \dot{q}(t)dt \ \Rightarrow$$
$$\frac{d}{dt}(\delta q(t)) = \delta \dot{q}(t) \text{ or } d\delta q(t) = \delta dq(t)$$

Similarly, $\delta$ commutes with $\int dt$.

$$\delta \int_{t_0}^{t_f} q(t)dt = \int_{t_0}^{t_f} \delta q(t)dt$$

$$\begin{aligned}\delta W &\triangleq (f_1 + f_t + f_{1,g})^T \delta q_1 + (f_2 - f_t + f_{2,g})^T \delta q_2 \\ &= (f_1 + f_{1,g})^T \delta q_1 + (f_2 + f_{2,g})^T \delta q_2 + f_t^T (\delta q_1 - \delta q_2) \\ &= 0\end{aligned}$$

Recall we also have:

$$(q_1 - q_2)^T (\delta q_1 - \delta q_2) = 0$$

Since $f_t$ is parallel to the vector $q_1 - q_2$, we have:

$$f_t^T (\delta q_1 - \delta q_2) = 0$$

$$\delta W = (f_1 + f_{1,g})^T \delta q_1 + (f_2 + f_{2,g})^T \delta q_2 = 0$$

约束力不做功，对虚位移也不做功，所以使用虚功原理可以不考虑约束力。
两个虚位移线性相关

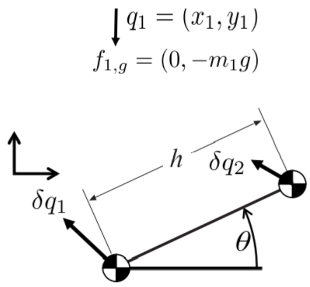
However, if we notice:

$$\delta q_2 = \delta q_1 + hv\delta\theta, \quad v = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\delta W = (f_1 + f_{1,g} + f_2 + f_{2,g})^T \delta q_1 + h(f_2 + f_{2,g})^T v \delta\theta = 0$$

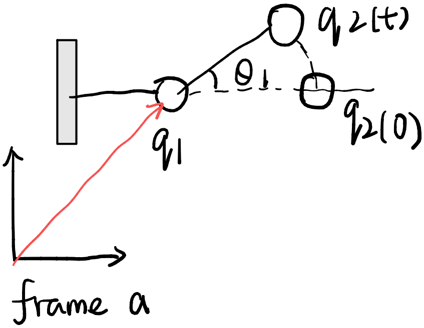
Since $\delta q_1, \delta\theta$ are linear independent, we have:

$$\begin{aligned} f_1 + f_{1,g} + f_2 + f_{2,g} &= 0 \\ (f_2 + f_{2,g})^T v &= 0 \end{aligned}$$




虚功

1、move a point $q_2(0) \longrightarrow q_2(t)$

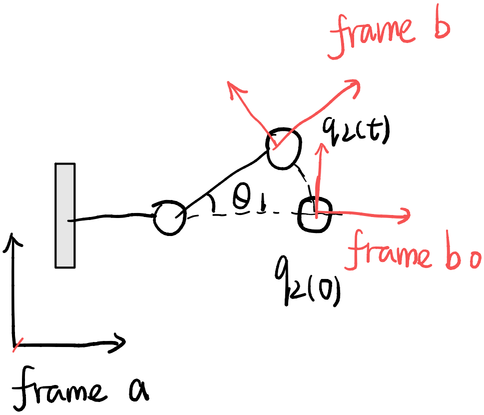



$$\boxed{q_2(t) = \boxed{R(q_1, \theta_1)}\; q_2(0)}$$

fixed axis rotation
(about axis $q_1$ for $\theta_1$)

$$R(q_1,\theta_1) = e^{\hat{V}t} = \begin{bmatrix} e^{\hat{\omega} t} & (I-e^{\hat{\omega} t})p \\ 0 & 1 \end{bmatrix} \xrightarrow[p = q_{1a}]{\hat{\omega} = \hat{z}\cdot\|w\|} \begin{bmatrix} e^{\hat{z}\theta_1} & (I-e^{\hat{z}\theta_1})q_1 \\ 0 & 1 \end{bmatrix}$$

2、move a frame




frame a → fram b0 : $g_{ab0}$

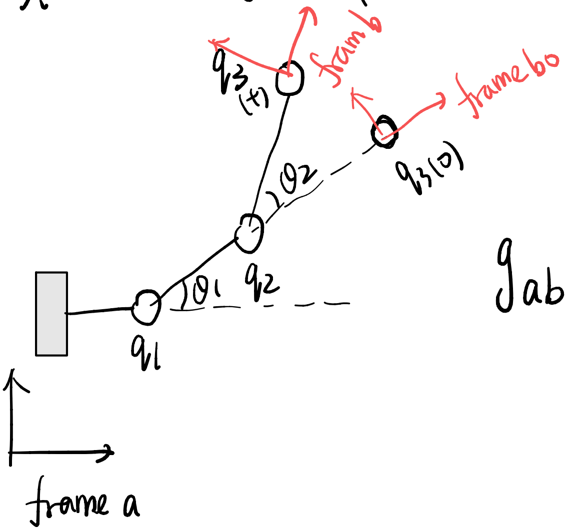
$$\begin{bmatrix} q_{2a}(0) \\ 1 \end{bmatrix} = g_{ab0} \cdot \begin{bmatrix} q_{2b0} \\ 1 \end{bmatrix}$$

frame a → frame b : $g_{ab}$

$$\begin{bmatrix} q_{2a}(t) \\ 1 \end{bmatrix} = g_{ab_1} \cdot \begin{bmatrix} q_{2b} \\ 1 \end{bmatrix}$$

$$\boxed{g_{ab} = R(q_1, \theta_1)\, g_{ab0}}$$

3. move a frame (2 link robot end)




$$g_{ab} = R(q_{10}, \theta_1)\, R(q_{20}, \theta_2) \cdot g_{ab0}$$

4、$g_{ab}$ (定轴转动情况，$\dot{p}=0$)

$$g_{ab} = \prod_{i=1}^{n} e^{\hat{\xi}_i \theta_i} \cdot g_{ab0}$$

$$\hat{\xi}_i = \begin{bmatrix} \hat{\omega}_{i0} & q_{i0} \times \omega_{i0} \\ 0 & 0 \end{bmatrix}$$

$\boxed{\omega_{i0}\text{ 为单位绕轴转动的角速度}}$

一般情况:

$$\hat{V} = \begin{bmatrix} \hat{\omega} = \dot{R}R^T & \dot{p}(t) - \hat{\omega} \cdot p(t) \\ 0 & 0 \end{bmatrix}$$