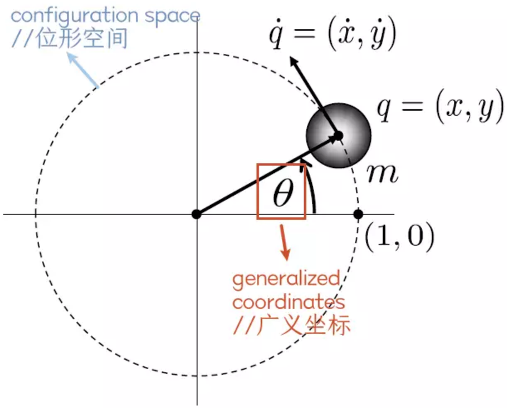
Lecture 03 / 3D Mechanics

Basic concepts / 基本概念

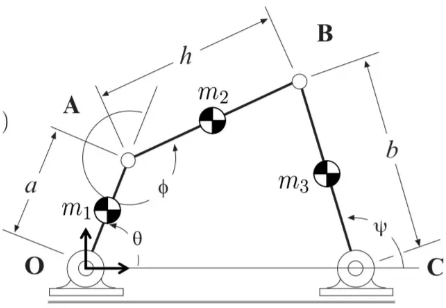
Configuration space, Generalized coordinates / 构型空间与广义坐标



//在表示圆形轨迹时使用笛卡尔坐标的问题：多解与无解点（导数求速度时会出现）

in this case，the configuration space $\mathcal{C} = \{\theta \mid \theta \in [0, 2\pi)\}$

e.g. 四连杆质心位置计算



$$\boxed{\phi = \arctan\left(\frac{b\sin\psi - a\sin\theta}{g + b\cos\psi - a\cos\theta}\right) - \theta}$$

$$\boxed{\psi(\theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2 + B^2}}\right)}$$

Kinetic energy / 动能

$$T = \frac{1}{2} m \dot{q}^T \dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}(m\dot{q}) = f \quad \left(\text{Given } \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \ \frac{\partial T}{\partial \dot{q}} = \begin{bmatrix} \partial T/\partial \dot{x} \\ \partial T/\partial \dot{y} \\ \partial T/\partial \dot{z} \end{bmatrix}\right)$$

//速度(speed)是标量

//多个质点： $T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{q}_i^{\ T} \dot{q}_i$

注意：刚体质点之间有作用力，不可以简单求导计算单个质点受力

e.g. 四连杆动能计算

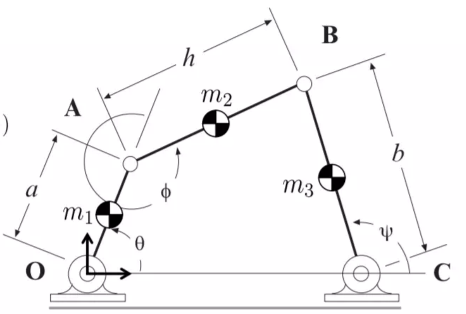
1.求质心坐标（上个例题）

2.用位置坐标对时间求导 + 利用速度间关系得到质心处的速度

3.求系统动能

Assume center of mass of the links are located at the mid-point of the links, compute the kinetic energy of the 4-bar linkage as a function of θ and $\dot{\theta}$ (assume $\phi(\theta), \psi(\theta), \dot{\phi}(\theta, \dot{\theta}), \dot{\phi}(\theta, \dot{\theta})$ are given)

$$\boxed{\frac{\dot{\psi}}{\dot{\theta}} = \frac{(\mathbf{B}-\mathbf{A})^T J \mathbf{B}}{(\mathbf{B}-\mathbf{A})^T J (\mathbf{B}-\mathbf{C})}, \quad \frac{\dot{\phi}}{\dot{\theta}} = \frac{-(\mathbf{B}-\mathbf{C})^T J \mathbf{B}}{(\mathbf{B}-\mathbf{C})^T J (\mathbf{B}-\mathbf{A})}} \quad J = R(\frac{\pi}{2})$$



Work function, Potential energy / 功函数与势能

功函数：

If there exists a *work function* $U(q)$ such that:

$$f = \frac{\partial U(q)}{\partial q}$$

we have:

$$\int_{t_0}^{t_f} dW = \int_{t_0}^{t_f} \frac{\partial U}{\partial q} dq = U(q(t_f)) - U(q(t_0))$$

势能：

Putting the kinetic energy and work function together:

$$\int_{t_0}^{t_f} m\ddot{q}^T dq = \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{1}{2} m\dot{q}^T \dot{q}\right) dt = \int_{t_0}^{t_f} f^T dq = \int_{t_0}^{t_f} \left(\frac{\partial U}{\partial q}\right)^T dq$$

$$\Downarrow$$

$$T(t_f) - T(t_0) = U(t_f) - U(t_0)$$

$$\text{or}$$

$$T(t_f) - U(t_f) = T(t_0) - U(t_0)$$

We usually define *potential energy* $V(q) = -U(q)$

$$\Downarrow$$

$$E(t) \triangleq T(t) + V(t) = \text{constant}$$

The total energy E remains constant. $f = -\partial V / \partial q$ is therefore called a *conservative force*.

//保守力做功与路径无关；典型的非保守力：摩擦力

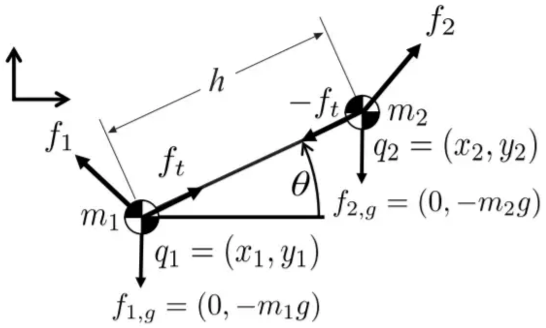
//弹簧模型：仿真形变计算基础

Principle of virtual work / 虚功原理

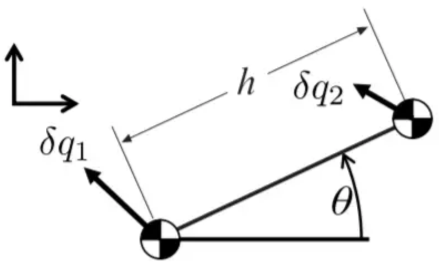
Virtual displacements, Variation

静力平衡：

$$f_1 + f_t + f_{1,g} = 0$$
$$f_2 - f_t + f_{2,g} = 0$$



假设有沿切向的微小位移（实际上并没有发生）：



$$\delta W \triangleq (f_1 + f_t + f_{1,g})^T \delta q_1 + (f_2 - f_t + f_{2,g})^T \delta q_2 = 0$$

虚功原理(Principle of Virtual Work, or PVW):

向任何方向的微小位移都不做功 = 系统平衡

虚位移(Virtual displacements): $\delta q1$, $\delta q2$ ...

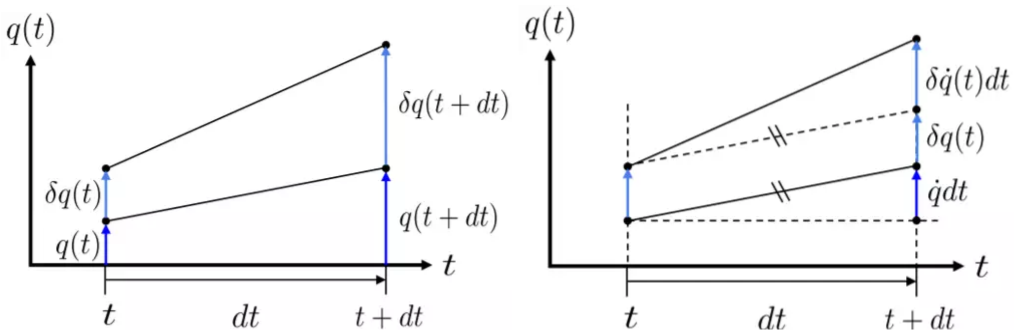
// δq_1 与 dq_1 ：相似，但是考虑虚位移时系统时间停止

$$(q_1 - q_2)^T (\dot{q}_1 - \dot{q}_2) = 0$$
$$(q_1 - q_2)^T (\delta q_1 - \delta q_2) = 0$$

// δq_1 、 δq_2 的取值可以是满足以上关系任意一组值，我们可以令 $\delta q_1 = dq_1$ ！

虚功(Virtual work): δW

虚位移对速度的影响：



给计算带来的好处：类似偏导，可以提出来

$$\delta q(t+dt) = \delta q(t) + \delta \dot{q}(t)dt \ \Rightarrow \begin{array}{l} \dfrac{d}{dt}(\delta q(t)) = \delta \dot{q}(t) \\ d\delta q(t) = \delta d q(t) \end{array}$$

$$\delta \int_{t_0}^{t_f} q(t)dt = \int_{t_0}^{t_f} \delta q(t)dt$$

使用虚功原理的计算：

1. 移除约束力

Recall we also have:

$$(q_1 - q_2)^T(\delta q_1 - \delta q_2) = 0$$

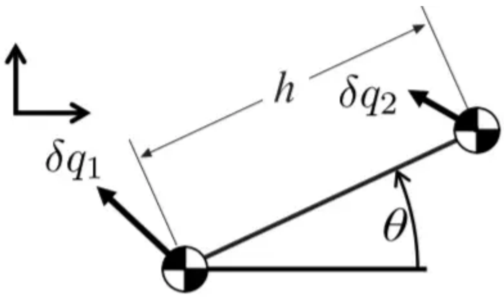
Since f_t is parallel to the vector $q_1 - q_2$, we have:

$$f_t^T(\delta q_1 - \delta q_2) = 0$$

$$\delta W = (f_1 + f_{1,g})^T \delta q_1 + (f_2 + f_{2,g})^T \delta q_2 + f_t^T(\delta q_1 - \delta q_2) = 0$$

$$= \ \delta W = (f_1 + f_{1,g})^T \delta q_1 + (f_2 + f_{2,g})^T \delta q_2 = 0$$

2. 虚位移之间的关系



$$\delta q_2 = \delta q_1 + hv\delta\theta, \quad v = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\delta W = (f_1 + f_{1,g} + f_2 + f_{2,g})^T \delta q_1 + h(f_2 + f_{2,g})^T v\delta\theta = 0$$

Since $\delta q_1, \delta\theta$ are linear independent, we have:

$$f_1 + f_{1,g} + f_2 + f_{2,g} = 0$$

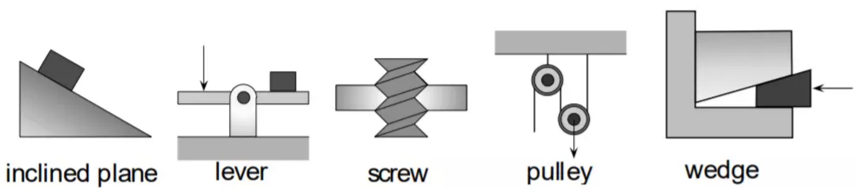
$$(f_2 + f_{2,g})^T v = 0$$

Mechanical advantage

$$f_{\text{in}}^T \delta q_{\text{in}} - f_{\text{out}}^T \delta q_{\text{out}} = 0$$

$$f_{\text{in}}^T dq_{\text{in}}/dt - f_{\text{out}}^T dq_{\text{out}}/dt = f_{\text{in}}^T \dot{q}_{\text{in}} - f_{\text{out}}^T \dot{q}_{\text{out}} = 0$$

$$\boxed{\frac{f_{\text{out}}}{f_{\text{in}}} = \frac{v_{\text{in}}}{v_{\text{out}}}}$$



Rigid body

$$\boxed{\sum_{i=1}^{n} f_i^T \delta q_i = 0} \quad \text{where } \delta q_i = \hat{\omega}_{ab} q_i + v_{ab}$$

$$\sum_{i=1}^{n}(f_i^T \hat{\omega}_{ab} q_i) + v_{ab}^T (\sum_{i=1}^{n} f_i) = \omega_{ab}^T \sum_{i=1}^{n}(q_i \times f_i) + v_{ab}^T (\sum_{i=1}^{n} f_i) = 0$$

$$\sum_{i=1}^{n} f_i = 0, \quad \sum_{i=1}^{n}(q_i \times f_i) = 0$$

$$\text{Define } \boxed{F_i \triangleq \begin{bmatrix} f_i \\ q_i \times f_i \end{bmatrix} \triangleq \begin{bmatrix} f_i \\ f_i^T J q_i \end{bmatrix}}, \text{ called wrenches}$$

// J和叉乘都用于表示法向向量，J更适合矩阵运算

$$\sum_{i=1}^{n} F_i = 0$$

Differentiation / 微分

微分与逼近（Approximation）

常数逼近

Definition. A function $f(x)$ is approximated by a constant a near x_0, denoted $f(x) \sim_1 a$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|\Delta x| < \delta \implies |f(x) - a| = |f(x) - a1| < \epsilon 1 = \epsilon.$$

常数逼近与函数连续性

Theorem. Constant approx $\iff$ continuous, and $a = f(x_0)$.

Theorem. Suppose $f(x) \sim_1 a$ and $g(x) \sim_1 b$ near x_0.

- If $a > b$, then $f(x) > g(x)$ near x_0.
- $f(x) + g(x) \sim_1 a + b$
- $f(x)g(x) \sim_1 ab$.

线性逼近（需要平滑）

Definition. Linear approximation $f(x) \sim_{\Delta x} a + b\Delta x$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|\Delta x| < \delta \implies |f(x) - a - b\Delta x| \le \epsilon |\Delta x|.$$

We say f is differentiable at x_0.

导数：逼近的计算

Definition. The derivative of a function $f(x)$ at x_0 is the coefficient b in the linear approximation $f(x) \sim_{\Delta x} a + b\Delta x$.

$$f'(x_0) = b = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Example. Find linear approximation of $f(x) = x^3$ at x_0

$$f(x) = (x_0+\Delta x)^3 = x_0^3 + 3x_0^2\Delta x + 3x_0\Delta x^2 + \Delta x^3$$

We have $a = x_0^3 = f(x_0)$, and $b = 3x_0^2$.

Leibniz rule：逼近的乘法

Suppose $f \sim_{\Delta x} a + b\Delta x$ and $g \sim_{\Delta x} c + d\Delta x$ near x_0.

Then

$$\begin{aligned} f + g &\sim_{\Delta x} (a+b\Delta x) + (c+d\Delta x) \\ &= (a+c) + (b+d)\Delta x. \end{aligned}$$

We get

$$(f+g)'(x_0) = b + d = f'(x_0) + g'(x_0).$$

Then

$$\begin{aligned} fg &\sim_{\Delta x} (a+b\Delta x)(c+d\Delta x) \\ &= ac + (ad+bc)\Delta x + bd\Delta x^2 \\ &\sim_{\Delta x} ac + (ad+bc)\Delta x. \end{aligned}$$

We get the Leibniz rule

$$(fg)'(x_0) = ad + bc = f(x_0)g'(x_0) + f'(x_0)g(x_0).$$

链式法则：逼近的复合

Suppose $y = f(x) \sim_{\Delta x} a + b\Delta x$ and $g(y) \sim_{\Delta y} c + d\Delta y$ near $y_0 = f(x_0)$.

Then (e.g.: $(1+2y)\circ(3+4x) = 1 + 2(3+4x) = 7 + 8x$)

$$\begin{aligned} g(f(x)) &\sim_{\Delta x} (c+d\Delta y)\circ(a+b\Delta x) \\ &= c + db\Delta x. \end{aligned}$$

We get the chain rule

$$(g\circ f)'(x_0) = db = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0).$$

*链式法则基于可以微分成立…

*可导一定可微，可微不一定可导

高阶导数：真的难算，只用组合即可

Theorem. If $f^{(n)}(x_0)$ exists, then

$$f(x) \sim_{\Delta x^n} f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}\Delta x^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\Delta x^n.$$

Basic high order approximations at $x_0 = 0$ ($\Delta x = x$):

$$\begin{aligned} (1+x)^p &= 1 + px + \tfrac{p(p-1)}{2!}x^2 + \cdots + \tfrac{p(p-1)\cdots(p-n+1)}{n!}x^n + o(x^n), \\ e^x &= 1 + x + \tfrac{1}{2!}x^2 + \cdots + \tfrac{1}{n!}x^n + o(x^n), \\ \log(1+x) &= x - \tfrac{1}{2}x^2 + \cdots + (-1)^{n-1}\tfrac{1}{n}x^n + o(x^n), \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots + (-1)^n\tfrac{1}{(2n)!}x^{2n} + o(x^{2n+1}), \\ \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + (-1)^n\tfrac{1}{(2n+1)!}x^{2n+1} + o(x^{2n+2}). \end{aligned}$$

This is the last time you use high order derivatives!!!

多变量逼近

*偏导数是多变量逼近的系数!

Definition. Linear approximation $f(\vec{x}) \sim_{\Delta\vec{x}} a + b(\vec{x})$, if $\cdots$,

$$\|\Delta\vec{x}\| < \delta \implies |f(\vec{x}) - a - b(\Delta\vec{x})| \le \epsilon\|\Delta\vec{x}\|.$$

We say f is differentiable at x_0, and denote the (total) derivative

$$f'(\vec{x}_0) = b, \quad f'(\vec{x}_0)(\vec{v}) = \vec{b}\cdot\vec{v} = \nabla f(\vec{x}_0)\cdot\vec{v}.$$

Calculation:

$$a = f(x_{10}, x_{20}),\ b_1 = \frac{\partial f}{\partial x_1}(x_{10}, x_{20}),\ b_2 = \frac{\partial f}{\partial x_2}(x_{10}, x_{20}).$$

线性变换与雅可比行列式:

More generally, consider multifunction of multivariable

$$F(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x})) \colon \mathbb{R}^n \to \mathbb{R}^m$$

A linear map (multifunction) is

$$P(\vec{x}) = \vec{a} + B\Delta\vec{x}, \quad \Delta\vec{x} = \vec{x} - \vec{x}_0, \quad \vec{a} \in \mathbb{R}^m.$$

$B\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, or an $m \times n$ matrix.

Definition. A map $F(\vec{x})\colon \mathbb{R}^n \to \mathbb{R}^m$ is approximated by a linear map $\vec{a} + B\Delta\vec{x}$ near $\vec{x}_0$, and denoted $F(\vec{x}) \sim_{\Delta\vec{x}} \vec{a} + B\Delta\vec{x}$, if for ar $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\Delta\vec{x}\| < \delta \implies \|F(\vec{x}) - \vec{a} - B\Delta\vec{x}\| \le \epsilon\|\Delta\vec{x}\|.$$

We have

$$\vec{a} = F(\vec{x}_0), \quad B = F'(\vec{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}.$$

多变量的Leibniz rule和链式法则:

- Leibniz rule: $F \sim_{\|\Delta\vec{x}\|^n} P$ and $G \sim_{\|\Delta\vec{x}\|^n} Q \implies FG \sim_{\|\Delta\vec{x}\|^n} PQ$, and cutting terms of order $> n$ in PQ.
- Chain rule: $F(\vec{x}) \sim_{\|\Delta\vec{x}\|^n} P(\vec{x})$ near $\vec{x}_0$, and $G(\vec{y}) \sim_{\Delta\vec{y}^n} Q(\vec{y})$ near $\vec{y}_0 = f(\vec{x}_0) \implies G(F(\vec{x})) \sim_{\|\Delta\vec{x}\|^n} Q(P(\vec{x}))$, and cutting terms of order $> n$.

Chain rule $(G \circ F)'(\vec{x}_0) = G'(\vec{y}_0)F'(\vec{x}_0)$ means

$$\begin{pmatrix} \frac{\partial g_1(f_1,f_2,f_3)}{\partial x_1} & \frac{\partial g_1(f_1,f_2,f_3)}{\partial x_2} \\ \frac{\partial g_2(f_1,f_2,f_3)}{\partial x_1} & \frac{\partial g_2(f_1,f_2,f_3)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}.$$

线性逼近可以求逆 –> 函数可逆（对矩阵也一致

矩阵的逆映射与求导:

Example.
Derivative of inverse matrix map $F(X) = X^{-1} \colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$.

(1) Derivative $F'(I)$ at $X_0 = I$. Denote $H = \Delta X = X - I$.

$$X^{-1} = (I + H)^{-1} = I - H + H^2 - H^3 + \cdots .$$

Therefore $F'(I)(H) = -H$, $\frac{1}{2!}F''(I)(H) = H^2$, $\frac{1}{3!}F'''(I)(H) = -H^3$

(2) Derivative $F'(A)$ at $X_0 = A$. F is the composition:

$$X_A \xrightarrow{A^{-1}?} A^{-1}X_I \xrightarrow{?^{-1}} X^{-1}A_I \xrightarrow{?A^{-1}} X^{-1}.$$

The maps $X \to A^{-1}X$ and $X \to XA^{-1}$ are linear. Therefore

$$(A^{-1}?)'(A)(H) = A^{-1}H, \quad (?A^{-1})'(I)(H) = HA^{-1}.$$

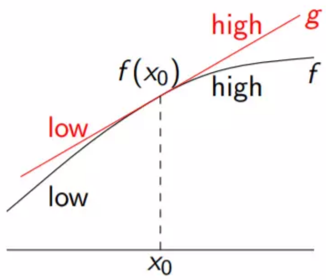
Then

$$F'(A)(H) = [(?A^{-1})'(I) \circ F'(I) \circ (A^{-1}?)'(A)](H) = -A^{-1}HA^{-1}.$$

Local Extreme

→ 线性逼近（Linear Approximation）：导数判别法背后的原理

Suppose $f(x) \sim_{\Delta x} p(x) = a + b\Delta x$. If $b > 0$, then
high on right and low on left $\implies$ high on right and low on left
$\implies$ not local extreme.



Similarly, $b < 0 \implies$ not local extreme.

- **Theorem.** If f is two-sided differentiable at x_0, then x_0 is a local extreme $\implies f'(x_0) = 0$.
- 一定要两边都能逼近：论证中需要此条件

→ 二次逼近（Quadratic Approximation）

Suppose $f(x) \sim_{\Delta x^2} p(x) = a + c\Delta x^2$ (this means $f'(x_0) = 0$).

$c > 0 \implies p(x) > a = p(x_0)$ for $\Delta x \neq 0$ (local minimum of p)
$\implies f(x) > f(x_0)$ for $\Delta x \neq 0$ (local minimum of f).

Theorem. Suppose $f(x) \sim_{\Delta x^2} p(x) = a + c\Delta x^2$ near x_0.

- If $c > 0$, then x_0 is a local minimum.
- If $c < 0$, then x_0 is a local maximum.

Remark. By Taylor expansion, we have $f'(x_0) = b = 0$ and $f''(x_0) = 2c > 0 \implies$ local minimum.

Theorem. Suppose $f(x) \sim_{\Delta x^n} p(x) = a + c\Delta x^n$ near x_0.

- If n is odd and $c \neq 0$, then x_0 is not a local extreme.
- If n is even and $c > 0$, then x_0 is a local minimum.
- If n is even and $c < 0$, then x_0 is a local maximum.

- n次逼近中低阶项变成常数，所以只考虑高阶项

→ 双变量与二次逼近

- 通过二次型正定/负定判别确定c的正负

Suppose $f(\vec{x}) \sim_{\|\Delta\vec{x}\|^2} p(\vec{x}) = a + c(\Delta\vec{x})$, where c is a quadratic function

$$c(v_1, v_2) = c_{11}v_1^2 + c_{22}v_2^2 + 2c_{12}v_1v_2.$$

Theorem.
$c(\vec{v}) > 0$ for $\vec{v} \neq \vec{0}$ (positive definite) $\implies$ local minimum.
$c(\vec{v}) < 0$ for $\vec{v} \neq \vec{0}$ (negative definite) $\implies$ local maximum.

Remark. In terms of partial derivatives,

$$b_1 = f_{x_1}(\vec{x_0}) = 0,\ b_2 = f_{x_2}(\vec{x_0}) = 0,$$

$$c_{11} = \tfrac{1}{2}f_{x_1^2}(\vec{x_0}),\ c_{22} = \tfrac{1}{2}f_{x_2^2}(\vec{x_0}),\ c_{12} = \tfrac{1}{2}f_{x_1x_2}(\vec{x_0}).$$

Remark. Positive definite $\iff$ $c_{11} > 0$ and $c_{11}c_{22} - c_{12}^2 > 0$

- $\iff$ $f_{x_1^2} > 0$ and $f_{x_1^2}f_{x_2^2} - f_{x_1x_2}^2 > 0.$

→ 引入约束域

- 线性逼近转换题目

Problem. Find local extreme $\vec{x_0} = (x_0, y_0)$ of $f(x,y) = xy$ on the circle $g(x,y) = x^2 + y^2 = 1$.

We have $g(\vec{x_0}) = 1$, and need linear approximation for all data:

$$\begin{aligned} f(\vec{x}) \sim p_f(\vec{x}) &= f(\vec{x_0}) + f_x(\vec{x_0})\Delta x + f_y(\vec{x_0})\Delta y \\ &= f(\vec{x_0}) + y_0\Delta x + x_0\Delta y, \\ g(\vec{x}) \sim p_g(\vec{x}) &= g(\vec{x_0}) + 2x_0\Delta x + 2y_0\Delta y. \end{aligned}$$

Approximate Problem. Find local extreme $\vec{x_0} = (x_0, y_0)$ of $p_f = a + y_0\Delta x + x_0\Delta y$ on $p_g = 1$.

By $g(\vec{x_0}) = 1$, the constraint $p_g = 1$ is $2x_0\Delta x + 2y_0\Delta y = 0$
$\implies$ $\Delta y = -\frac{x_0}{y_0}\Delta x$ $\implies$ $p_f = a + \frac{y_0^2 - x_0^2}{y_0}\Delta x$
Consider both $\Delta x > 0$ and < 0, we need $y_0^2 - x_0^2 = 0$.
We also need $g(\vec{x_0}) = x_0^2 + y_0^2 = 1$ $\implies$ $x_0 = \pm\frac{1}{\sqrt{2}}$, $y_0 = \pm\frac{1}{\sqrt{2}}$.

- 二重约束

Problem. Find local extreme $\vec{x}_0$ of $f(\vec{x})$ subject to the constraints $g_1(\vec{x}) = c_1$ and $g_2(\vec{x}) = c_2$.

We have $g_1(\vec{x}_0) = c_1$ and $g_2(\vec{x}_0) = c_2$.
We need linear approximation for all data:

$$\begin{aligned} f(\vec{x}) &\sim p_f(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \Delta\vec{x}, \\ g_1(\vec{x}) &\sim p_{g_1}(\vec{x}) = g_1(\vec{x}_0) + \nabla g_1(\vec{x}_0) \cdot \Delta\vec{x}, \\ g_2(\vec{x}) &\sim p_{g_2}(\vec{x}) = g_2(\vec{x}_0) + \nabla g_2(\vec{x}_0) \cdot \Delta\vec{x}. \end{aligned}$$

Approximate Problem. Find local extreme $\vec{x}_0$ of p_f subject to $p_{g_1} = c_1$ and $p_{g_2} = c_2$.

Necessary condition:

$$\nabla g_1(\vec{x}_0) \cdot \Delta\vec{x} = \nabla g_2(\vec{x}_0) \cdot \Delta\vec{x} = 0 \implies \nabla f(\vec{x}_0) \cdot \Delta\vec{x} = 0.$$

This is equivalent to

$$\nabla f(\vec{x}_0) = \lambda_1 \nabla g_1(\vec{x}_0) + \lambda_2 \nabla g_2(\vec{x}_0).$$

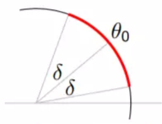
Differentiation on Manifold (流形)

- 圆上的微分
  - 用角度 θ 做参数：(存在映射问题

    Function $f(x,y) = xy$ on the circle $\mathbb{S}^1 = \{(x,y)\colon x^2+y^2=1\}$.
    Parameterise circle by angle θ

    $$x = \cos\theta, \quad y = \sin\theta, \quad f(x,y) = \cos\theta\sin\theta.$$

    The part of the circle corresponding to $|\theta - \theta_0| < \delta$ is an interval.

    Differentiation at $(x_0, y_0) = (\cos\theta_0, \sin\theta_0)$:

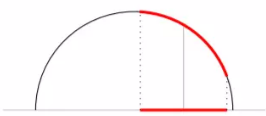
    $$\begin{aligned} f(x,y) &= (\cos\theta_0 - \sin\theta_0\,\Delta\theta + o(\Delta\theta))(\sin\theta_0 + \cos\theta_0\,\Delta\theta + o(\Delta\theta)) \\ &= f(x_0,y_0) + \cos 2\theta_0\,\Delta\theta + o(\Delta\theta). \end{aligned}$$

  - Local extreme $\implies \cos 2\theta_0 = 0 \implies (x_0, y_0) = \frac{1}{\sqrt{2}}(\pm 1, \pm 1)$.
  - 用 x / y 做参数（略复杂

    Parameterise by x (only covering the upper half cricle)

    $$y = \sqrt{1-x^2}, \quad f(x,y) = x\sqrt{1-x^2}.$$

    The part of the circle corresponding to $|x - x_0| < \delta$ is an interval.

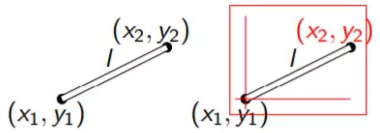
    

    Differentiation at $(x_0, y_0) = (x_0, \sqrt{1-x_0^2})$:

    $$f(x,y) = f(x_0,y_0) + (1-x_0^2)^{\frac{1}{2}}\left(1 - \frac{x_0^2}{1-x_0^2}\right)\Delta x + o(\Delta x)$$

    Local extreme $\implies 1 - \frac{x_0^2}{1-x_0^2} \implies (x_0, y_0) = \frac{1}{\sqrt{2}}(\pm 1, 1)$.
  - Also need to cover the lower half circle by $y = -\sqrt{1-x^2}$.
- 球上的微分
  - 用 (x,y) 或 (θ,ρ)
- 机械系统中的例子:

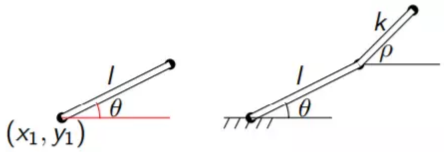
- Configuration of one link：表示一根连杆所有的可能位置



Configuration of one link

$$
\begin{aligned}
\mathcal{C} &= \{(x_1, x_2, x_3, x_4) \colon (x_1 - x_2)^2 + (y_1 - y_2)^2 - l^2 = 0\} && \subset \mathbb{R}^4 \\
&= \{(x_1, y_1, x_2, y_2) \colon {x_2}^2 + {y_2}^2 - l^2 = 0\} && = \mathbb{R}^2 \times \mathbb{S}^1 \\
&= \{(x_1, y_1, \theta)\}.
\end{aligned}
$$

- Configuration of two links：



Configuration of two links

$$\mathcal{C} = \{(\theta, \rho)\} = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2.$$

  - Both are manifolds.

- 旋转矩阵

All rotations in $\mathbb{R}^2$

$$SO(2) = \{R_\theta\}, \quad \text{Note: } R_\theta = R_{\theta+2\pi}.$$

All the rotations of $\mathbb{R}^3$

$$SO(3) = \{R_{\vec{\omega},\theta}\}, \quad \text{Note: } R_{\vec{\omega},\theta} = R_{-\vec{\omega},-\theta} = R_{\vec{\omega},\theta+2\pi}.$$

Here $\vec{\omega} \in \mathbb{S}^2$ is the axis of rotation, and θ is the rotation angle (as measured by the right hand rule).

  - Both are manifolds.

→ Presenting Manifold / 流形的表示

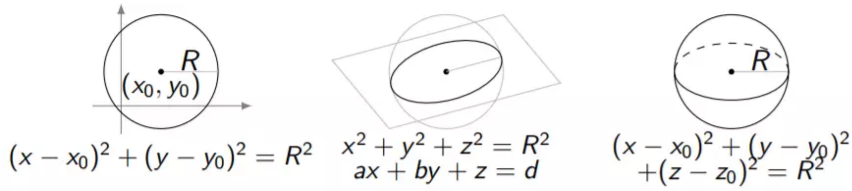
- Parameterisation / 参数化表示

The parameters are generalised coordinates.
Two problems:

  - May need several paramaterisation pieces to cover the whole manifold.
  - Two parameters may correspond to the same point (e.g., θ and $\theta + 2\pi$).

Not affecting differentiation. Can be solved by defining atlas of coordinate charts.

- Holonomic constraints



Subset of $\mathbb{R}^n$ satisfying k independent equations

$$f_1(x_1,\dots,x_n)=0, f_2(x_1,\dots,x_n)=0,\dots,f_k(x_1,\dots,x_n)=0.$$

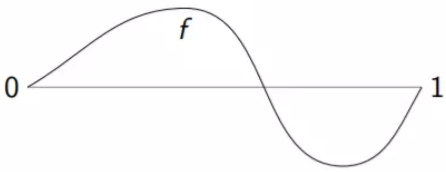
Example

$$SO(n)=\{U\colon U^TU=I\}\subset\mathbb{R}^{n^2}.$$

Variation / 变分

Calculus of Variation: Shortest path / 最小路径问题

Find shortest path between two points.



The path is given by $y=f(x)$, $x\in[0,1]$, $f(0)=f(1)=0$.
The length of the path is a function of f

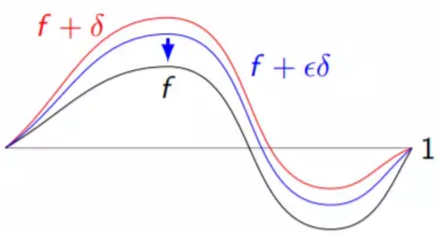
$$\mathcal{L}(f)=\int_0^1\sqrt{1+f'^2}dx.$$

By shortest path, we mean

$$\mathcal{L}(g)\ge\mathcal{L}(f)\quad\text{for any function } g \text{ close to } f.$$

- 从数字到函数（泛函）

Take nearby $g(x)=f(x)+\delta(x)$, where $\delta(x)$ is a small function satisfying $\delta(0)=\delta(1)=0$.



For each $\delta(x)$, further consider the family of nearby $g(x)=f(x)+\epsilon\delta(x)$. Change the problem to

$$\boxed{\mathcal{L}(f(x)+\epsilon\delta(x))\ge\mathcal{L}(f(x)).}$$

Then the function $\mathcal{L}(f(x)+\epsilon\delta(x))$ of ϵ minimizes at $\epsilon=0$.

- 继续用逼近

$$\begin{aligned}\mathcal{L}(f(x)+\epsilon\delta(x)) &= \int_0^1 \sqrt{1+(f'(x)+\epsilon\delta'(x))^2}dx \\ &= \int_0^1 \sqrt{1+f'(x)^2+2\epsilon f'(x)\delta'(x)+o(\epsilon)}dx \\ &= \int_0^1 \sqrt{1+f'(x)^2}\left(1+\epsilon\frac{2f'(x)\delta'(x)}{1+f'(x)^2}+o(\epsilon)\right)^{\frac{1}{2}} dx \\ &= \int_0^1 \sqrt{1+f'(x)^2}\left(1+\frac{1}{2}\epsilon\frac{2f'(x)\delta'(x)}{1+f'(x)^2}+o(\epsilon)\right) dx \\ &= \int_0^1 \left(\sqrt{1+f'(x)^2}+\epsilon\frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}+o(\epsilon)\right) dx \\ &= \mathcal{L}(f(x))+\epsilon\int_0^1 \frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}dx+o(\epsilon).\end{aligned}$$

The red term should vanish.

- 对所有δ积分都等于0 → F’(x)=0 → F(x)是常数 → f(x)是常数

$\mathcal{L}(f(x)+\epsilon\delta(x))$ minimises at $\epsilon=0$.

$$\begin{aligned}&\int_0^1 \frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}dx = \int_0^1 \frac{f'(x)}{\sqrt{1+f'(x)^2}}d\delta(x) \\ &= \frac{f'(x)}{\sqrt{1+f'(x)^2}}\delta(x)\Big|_{x=0}^{x=1} - \int_0^1 \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)\delta(x)dx \\ &= -\int_0^1 \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)\delta(x)dx. \quad (\text{by } \delta(0)=\delta(1)=0)\end{aligned}$$

The above should vanishes *for all* $\delta(x)$

$\implies \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)=0 \implies \frac{f'(x)}{\sqrt{1+f'(x)^2}} = \text{constant}$

$\implies f'(x) = \text{constant, i.e., straight line.}$

- 一般化推广

Problem. For all $f(a)=A$ and $f(b)=B$, minimise

$$\mathcal{L}(f) = \int_a^b L(t,f,f')dt.$$

把t,f,f'都当成L的变量

This becomes minimising the following at $\epsilon=0$

$$\begin{aligned}\mathcal{L}(f+\epsilon\delta) &= \int_a^b L(t,f+\epsilon\delta,f'+\epsilon\delta')dt \\ &= \int_a^b \left[L(t,f,f')+\epsilon\frac{\partial L}{\partial f}(t,f,f')\delta+\epsilon\frac{\partial L}{\partial f'}(t,f,f')\delta'+o(\epsilon)\right]dt \\ &= \mathcal{L}(f)+\epsilon\int_a^b \frac{\partial L}{\partial f}(t,f,f')\delta dt+\epsilon\int_a^b \frac{\partial L}{\partial f'}(t,f,f')d\delta+o(\epsilon) \\ &= \mathcal{L}(f)+\epsilon\int_a^b \frac{\partial L}{\partial f}\delta dt+\epsilon\frac{\partial L}{\partial f'}\delta\Big|_{t=a}^{t=b}-\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)\delta dt+o(\epsilon) \\ &= \mathcal{L}(f)+\epsilon\int_a^b \left(\frac{\partial L}{\partial f}-\frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)\right)\delta dt+o(\epsilon).\end{aligned}$$

Necessary condition: For all $\delta(t)$, we have

$$\int_a^b \left(\frac{\partial L}{\partial f} - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)\right)\delta(t)dt = 0.$$

This is equivalent to Euler-Lagrange equation

$$\frac{\partial L}{\partial f} - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right) = 0.$$

- 多变量推广

Problem. For $f_1(a) = A_1, f_1(b) = B_1, f_2(a) = A_2, f_2(b) = B_2$, minimise

$$\mathcal{L}(f_1, f_2) = \int_a^b L(t, f_1, f_2, f_1', f_2')dt.$$

This becomes minimising the following at $\epsilon = 0$

$$\begin{aligned}
\mathcal{L}(f_1 + \epsilon\delta_1, f_2 + \epsilon\delta_2) &= \int_a^b L(t, f_1 + \epsilon\delta_1, f_2 + \epsilon\delta_2, f_1' + \epsilon\delta_1', f_2' + \epsilon\delta_2')dt \\
&= \int_a^b \Big[L(t, f_1, f_2, f_1', f_2') + \epsilon\frac{\partial L}{\partial f_1}(t, f_1, f_2.f_1', f_2')\delta_1 + \epsilon\frac{\partial L}{\partial f_2}(t, f_1, f_2.f_1', f_2')\delta_2 \\
&\quad + \epsilon\frac{\partial L}{\partial f_1'}(t, f_1, f_2.f_1', f_2')\delta_1' + \epsilon\frac{\partial L}{\partial f_2'}(t, f_1, f_2.f_1', f_2')\delta_2' + o(\epsilon)\Big]dt \\
&= \mathcal{L}(f_1, f_2) + \epsilon\int_a^b \left(\frac{\partial L}{\partial f_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_1'}\right)\right)\delta_1 dt \\
&\quad + \epsilon\int_a^b \left(\frac{\partial L}{\partial f_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_2'}\right)\right)\delta_2 dt + o(\epsilon).
\end{aligned}$$

The necessary condition for (f_1, f_2) to be a local extreme is the Euler-Lagrange equation

$$\frac{\partial L}{\partial f_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_1'}\right) = 0, \quad \frac{\partial L}{\partial f_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_2'}\right) = 0.$$

Or

$$\frac{\partial L}{\partial (f_1, f_2)} - \frac{d}{dt}\left(\frac{\partial L}{\partial (f_1', f_2')}\right) = \vec{0}.$$

- Least action path / 例题计算

Change $(f_1(t), f_2(t))$ to a path $\vec{x}(t) = (x_1(t), \ldots, x_n(t))$ in $\mathbb{R}^n$, connecting $\vec{x}(a) = \vec{A}$ and $\vec{x}(b) = \vec{B}$. Minimising the action

$$\mathcal{L}(\vec{x}(t)) = \int_a^b L(t, \vec{x}(t), \vec{x}\,'(t))dt.$$

Euler-Lagrange $\dfrac{\partial L}{\partial x_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial x_i'}\right) = 0.$

Example. For $L = t(x_1 x_2' - x_2 x_1')$, x1,x2都要对t求导

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= t x_2', & \frac{d}{dt}\left(\frac{\partial L}{\partial x_1'}\right) &= \frac{d}{dt}(-t x_2) = -x_2 - t x_2', \\
\frac{\partial L}{\partial x_2} &= -t x_1', & \frac{d}{dt}\left(\frac{\partial L}{\partial x_2'}\right) &= \frac{d}{dt}(t x_1) = x_1 + t x_1'.
\end{aligned}$$

Euler-Lagrange: $2t x_2' + x_2 = 0$ and $-2t x_1' - x_1 = 0$. Solution $(x_1, x_2) = \frac{1}{\sqrt{t}}(c_1, c_2)$, straight line passing through the origin.

- 广义坐标下的变分

Use example $x = r\cos\theta$ and $y = r\sin\theta$.
An expression of L in x, y can then be rewritten in terms of r, θ

$$\begin{aligned} L(t, x, y, x', y') &= L(t, r\cos\theta, r\sin\theta, r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta) \\ &= \tilde{L}(t, r, \theta, r', \theta'). \end{aligned}$$

Then

$$\int_a^b L(t, x(t), y(t), x'(t), y'(t))dt = \int_a^b \tilde{L}(t, r(t), \theta(t), r'(t), \theta'(t))dt.$$

Minimising left $\iff$ minimising the right. Therefore solution of

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = 0,$$

is related to the solution of

$$\frac{\partial \tilde{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial r'}\right) = 0, \quad \frac{\partial \tilde{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \theta'}\right) = 0,$$

by $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$.

In general, change of general coordinates $\vec{y} = \phi(\vec{x})$. Then $\vec{y}' = \phi'(\vec{x})\vec{x}'$, and

$$L(t, \vec{y}, \vec{y}') = L(t, \phi(\vec{x}), \phi'(\vec{x})\vec{x}') = \tilde{L}(t, \vec{x}, \vec{x}').$$

Then solutions of

$$\frac{\partial L}{\partial \vec{y}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{y}'}\right) = 0, \quad \frac{\partial \tilde{L}}{\partial \vec{x}} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \vec{x}'}\right) = 0,$$

are related by $\vec{y}(t) = \phi(\vec{x}(t))$.

This means the calculus of variations can be done over manifolds.

Mechanics / 不同观点下的物理公式

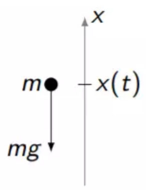
Gravity

First Method: Newton’s second law of motion

$$mx''(t) = F = -mg.$$

This is $x''(t) = -g$, and we get

$$x(t) = a + vt - gt^2.$$



$a = x(0)$ is the initial height of the mass.
$v = x'(0)$ is the initial velocity of the mass.

Second Method: Conservation of energy

Kinetic energy

$$K = \frac{1}{2}mv^2.$$

Potential energy

$$V = mgx.$$

Conservation of energy: The total energy is constant

$$T = K + V = \frac{1}{2}mv^2 + mgx = \frac{1}{2}mx'^2 + mgx$$

This is the same as the equation by the first method

$$T' = (mx'' + mg)x' = 0.$$

Third Method: Euler–Lagrange

The dynamics is the path $x(t)$ that minimises the action

$$\mathcal{L}(x(t)) = \int_0^b L(x, x')dt, \quad L = K - V = \frac{1}{2}mx'^2 - mgx.$$

We have

$$\frac{\partial L}{\partial x} = -mg, \quad \frac{\partial L}{\partial x'} = mx', \quad \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = mx''.$$

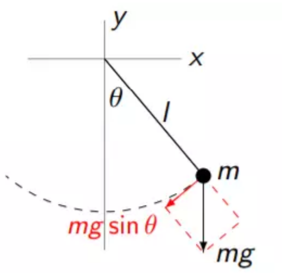
Euler-Lagrange is the same as the equation by the first method

$$-mg - mx'' = 0.$$

Pendulum / 简单运动 – 单摆

Newton’s second law of motion

$$ml\theta'' = -mg\sin\theta.$$



*需要找到运动方向；难以找到精确解…

Conservation of energy

Kinetic energy: The speed is $v = l\theta'$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\theta'^2.$$

Potential energy: Height $y = -l\cos\theta$

$$V = -mgl\cos\theta.$$

Conservation of energy: The total energy is constant

$$T = K + V = \frac{1}{2}ml^2\theta'^2 - mgl\cos\theta$$

This is the same as the equation by the first method

$$T' = (ml\theta'' + mg\sin\theta)l\theta' = 0.$$

Euler–Lagrange

The dynamics is the path $\theta(t)$ that minimises the action

$$\mathcal{L}(x(t)) = \int_0^b L(x, x')dt, \quad L = K - V = \frac{1}{2}ml^2\theta'^2 + mgl\cos\theta.$$

We have

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \quad \frac{\partial L}{\partial \theta'} = ml^2\theta', \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) = ml^2\theta''.$$

Euler-Lagrange is the same as the equation by the first method

$$-mgl\sin\theta - ml^2\theta'' = 0.$$

Euler–Lagrange Equation

D'Alembert principle

→ 达朗贝尔原理：用惯性力(Inertial force)将动力问题转换为静力问题

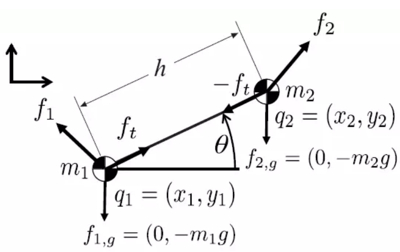
Consider again the example on the right. This time we assume the two mass points are in motion, with $\dot{q}_1, \ddot{q}_1, \dot{q}_2, \ddot{q}_2$.

$$f_1 + f_t + f_{1,g} - m_1\ddot{q}_1 = 0$$
$$f_2 - f_t + f_{2,g} - m_2\ddot{q}_2 = 0$$

D'Alembert says, we can treat $-m_1\ddot{q}_1$,$-m_2\ddot{q}_2$ as forces (which we shall call *inertial forces*), and then we can apply PVW as before:

$$\delta W = (f_1 + f_{1,g} - m_1\ddot{q}_1)^T\delta q_1 + \cdots$$
$$(f_2 + f_{2,g} - m_2\ddot{q}_2)^T\delta q_2 = 0$$

This is known as *d'Alembert's principle*.



Jean d'Alembert
(1717–1783)

From d'Alembert principle to Hamilton's principle

→ $\delta W = 0 \rightarrow \int \delta W = 0$

→ Details:

- 动能 $T = m_i \dot{q}_i^{\ T} \dot{q}_i / 2$
- 势能 $V = m_1 g y_1 + m_2 g y_2$
- 计算过程（分部积分）：

We consider a general case with n particles:

$$
\begin{aligned}
\int_{t_0}^{t_f} \delta W dt &= \int_{t_0}^{t_f} \sum_{i=1}^{n} (-\frac{\partial V}{\partial q_i} - m\ddot{q}_i)^T \delta q_i dt = \int_{t_0}^{t_f} (-\delta V) dt - \int_{t_0}^{t_f} \sum_{i=1}^{n} m_i \delta q_i^T d\dot{q}_i \\
&= \int_{t_0}^{t_f} (-\delta V) dt - \sum_{i=1}^{n} m_i \delta q_i^T \dot{q}_i \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \sum_{i=1}^{n} m_i \dot{q}_i^T \frac{d(\delta q_i)}{dt} dt \\
&= \int_{t_0}^{t_f} (-\delta V) dt - \sum_{i=1}^{n} m_i \delta q_i^T \dot{q}_i \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \sum_{i=1}^{n} m_i \dot{q}_i^T \delta \dot{q}_i dt \\
&= \int_{t_0}^{t_f} (-\delta V) dt - \sum_{i=1}^{n} m_i \delta q_i^T \dot{q}_i \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \sum_{i=1}^{n} \delta \left(\frac{1}{2} m_i \dot{q}_i^T \dot{q}_i \right) dt \\
&= \int_{t_0}^{t_f} \delta \underbrace{(T - V)}_{L} dt - \sum_{i=1}^{n} m_i \dot{q}_i^T \delta q_i \Big|_{t_0}^{t_f} = 0
\end{aligned}
$$

- L is called the Lagrangian function

→ 由拉格朗日函数还原系统方程

- 约束任取的虚位移，固定轨迹端点位置： $\delta q_i(t_0) = \delta q_i(t_f) = 0, \mathrm{i} = 1, \ldots, n$
- Action integral: $A = \int_{t_0}^{t_f} L dt$
- $\delta A = \int_{t_0}^{t_f} \delta L dt = \delta \int_{t_0}^{t_f} L dt = 0$

→ Hamilton's principle

> if the action integral A is stationary for a particular trajectory, i.e. $\delta A = 0$, then it is the actual trajectory of the system!

From Hamilton's principle to Euler–Lagrange equation

Starting with a given Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ w.r.t. to generalized coordinates $(q_1, \ldots, q_n) \in \mathbb{R}^n$, Hamilton's principle leads to:

$$
\delta A = \int_{t_0}^{t_f} \delta L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) dt = \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt
$$

$$
\begin{aligned}
&= \int_{t_0}^{t_f} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i dt + \int_{t_0}^{t_f} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\delta q_i(t) = \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) dt + \cancelto{0}{\frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_0}^{t_f}} \\
&= \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i dt = 0
\end{aligned}
$$

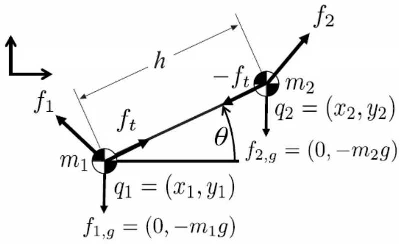
Since δq_i's are arbitrarily chosen other than the fact that $\delta q_i(t_0) = \delta q_i(t_f) = 0$, we conclude that:

$$
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad i = 1, \ldots, n
$$

This is called *Euler-Lagrange equations*, a set of second order ordinary differential equations.

→ 欧拉–拉格朗日方程的优势：消除约束力，选取最少坐标描述系统方程

In Hamilton's principle and Euler-Lagrange equation, $q_1, \dots q_n$; $\dot{q}_1, \dots, \dot{q}_n$ can be chosen to be any generalized coordinates, not just Cartesian coordinates!



Example: in the two-particle system, q_1, q_2 are not independent:

$$q_2 = q_1 + h \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and its Lagrangian becomes a function of $q_1, \dot{q}_1, \theta, \dot{\theta}$; we have:

$$\int_{t_0}^{t_f} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right)^T \delta q_1 + \left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\right) \delta\theta dt = 0$$

Since δq_1 (it is a vector and has the component $\delta x_1, \delta y_1$) and $\delta\theta$ are linearly independent:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

For systems with non-conservative generalized forces Q_i's (we will see how to derive Q_i's in a moment) associated with generalized coordinates q_i's:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + Q_i = 0, \quad i = 1, \dots, n \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, \dots, n$$

→ 拉格朗日乘子（Lagrange multipliers）：消除多余的虚位移

For systems with holonomic constraints $h_1(q_1, \dots, q_n) = 0, \dots, h_m(q_1, \dots, q_n) = 0$, not all virtual displacements δq_i's are linear independent; we use *Lagrange multipliers $\lambda_1, \dots, \lambda_m$ (they are new unknown variables now!)* to eliminate redundant virtual displacements.

$$\int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i}\right) \delta q_i dt = 0$$

which is the same as (because $\delta(\lambda_j h_j) = \lambda_j \delta h_j + \delta\lambda_j h_j = \lambda_j \delta h_j$ since $h_j = 0$):

$$\int_{t_0}^{t_f} \delta L + \sum_{j=1}^{m} \lambda_j \delta h_j dt = \delta \int_{t_0}^{t_f} \underbrace{(L + \sum_{j=1}^{m} \lambda_j h_j)}_{\bar{L}} dt = 0, \qquad h_1 = h_2 = \cdots = h_m = 0$$

We define *augmented Lagrangian* $\bar{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, \lambda_1, \dots, \lambda_m) \triangleq L + \lambda_1 h_1 + \dots + \lambda_m h_m$, then Hamilton's principle still holds for the augmented Lagrangian $\bar{L}$ in place of L:

$$\int_{t_0}^{t_f} (\delta L + \sum_{j=1}^{m} \lambda_j \delta h_j) dt = \delta \int_{t_0}^{t_f} \underbrace{(L + \sum_{j=1}^{m} \lambda_j h_j)}_{\bar{L}} dt = 0, \qquad h_1 = h_2 = \cdots = h_m = 0$$

Now the Euler-Lagrange equation becomes (along with $h_1 = \dots = h_m = 0$):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + Q_i + \lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i} = 0, \quad i = 1, \dots, n$$

or

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}_{\text{All inertial force and conservative forces go here}} = \underbrace{Q_i}_{\text{Whatever forces not covered by } L} + \underbrace{\lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i}}_{\text{Constraint forces}}, \quad i = 1, \dots, n$$

Example – Approach 1

1. 用广义坐标(Generalized coordinates)表示 *L*

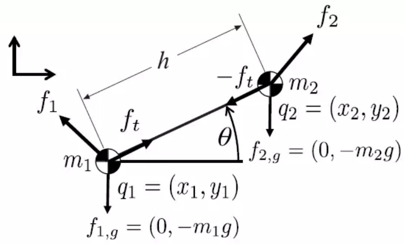
We now recover Newton's equation for the example with two particles. We have:

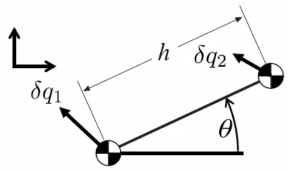
$$L = T - V = \frac{1}{2}m_1\dot{q}_1^T\dot{q}_1 + \frac{1}{2}m_2\dot{q}_2^T\dot{q}_2 - m_1gy_1 - m_2gy_2$$

We first express L as function of the generalized coordinates x_1, y_1, θ:

$$\dot{q}_2 = \begin{bmatrix}\dot{x}_1 - h\sin\theta\dot{\theta}\\ \dot{y}_1 + h\cos\theta\dot{\theta}\end{bmatrix}, \qquad y_2 = y_1 + h\sin\theta$$

$$L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2((\dot{x}_1 - h\sin\theta\dot{\theta})^2 + \cdots$$
$$(\dot{y}_1 + h\cos\theta\dot{\theta})^2) - m_1gy_1 - m_2g(y_1 + h\sin\theta)$$





2. 求广义力(Generalized force)

To derive generalized force from f_1 and f_2 , we go back one step in Hamilton's principle:

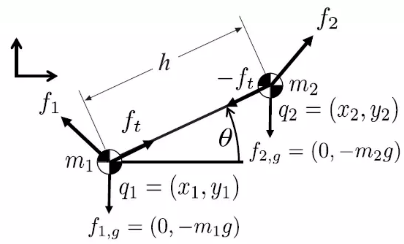
$$\int_{t_0}^{t_f}(\delta L + f_1^T\delta q_1 + f_2^T\delta q_2)dt = 0$$

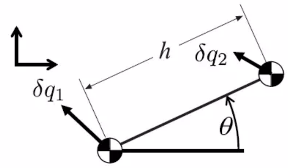
We recall that:

$$\delta q_2 = \delta q_1 + h\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\delta\theta$$

Then we

$$\int_{t_0}^{t_f}\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) + f_1 + f_2\right)^T\delta q_1 + \cdots$$
$$\left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + f_2^T h\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right)\delta\theta dt = 0$$

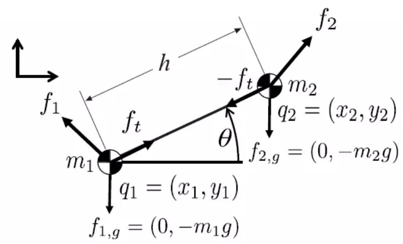




3. 得到方程

The corresponding Euler-Lagrange equations are (they are three equations):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = f_1 + f_2$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = f_2^T h\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$



which leads to

$$\begin{aligned}
m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) &= f_{1,x} + f_{2,x}\\
m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) &= f_{1,y} + f_{2,y} - (m_1 + m_2)g\\
m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) &= -f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2gh\cos\theta
\end{aligned}$$

We can see that:

$$\begin{aligned}
m_1\ddot{x}_1 + m_2\underbrace{(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta})}_{\ddot{x}_2} &= f_{1,x} + f_{2,x}\\
m_1\ddot{y}_1 + m_2\underbrace{(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta})}_{\ddot{y}_2} &= f_{1,y} + f_{2,y} - (m_1 + m_2)g\\
\underbrace{m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta})}_{-m_2\ddot{x}_2 h\sin\theta + m_2\ddot{y}_2 h\cos\theta} &= -f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2gh\cos\theta
\end{aligned}$$

Example – Approach 2

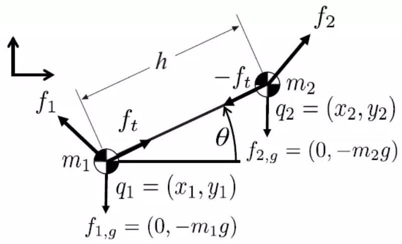
使用拉格朗日乘子法可以恢复约束力！

1.

Just use Cartesian coordinates q_1, q_2, along with the constraint:

$$(q_1 - q_2)^T (q_1 - q_2) - h^2 = 0$$

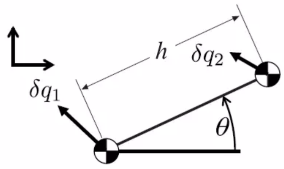
两点间距离



Now we have to use augmented Lagrangian:

$$\bar{L} = \frac{1}{2} m_1 \dot{q}_1^T \dot{q}_1 + \frac{1}{2} m_2 \dot{q}_2^T \dot{q}_2 - m_1 g y_1 - m_2 g y_2 + \cdots$$
$$\lambda((q_1 - q_2)^T (q_1 - q_2) - h^2)$$

We will get four Euler-Lagrange equations:



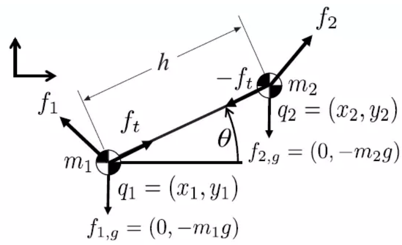
$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_1}\right) - \frac{\partial \bar{L}}{\partial q_1} = f_1$$
$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_2}\right) - \frac{\partial \bar{L}}{\partial q_2} = f_2$$

2.

The Euler-Lagrange equations become:

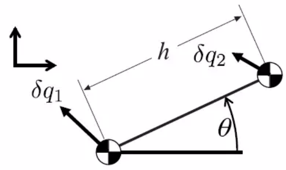
Ft, 大小为hλ

$$m_1 \ddot{q}_1 = f_1 + \boxed{\lambda(q_1 - q_2)} - \begin{bmatrix} 0 \\ m_1 g \end{bmatrix}$$
$$m_2 \ddot{q}_2 = f_2 - \lambda(q_1 - q_2) - \begin{bmatrix} 0 \\ m_2 g \end{bmatrix}$$



You can easily see now that $f_t = \lambda(q_1 - q_2)$!

Question: But how do we solve λ?

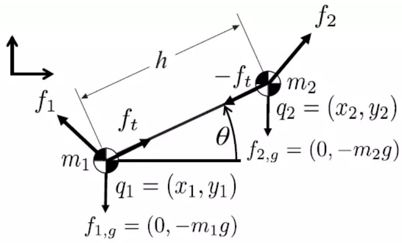


$$(q_1 - q_2)^T (q_1 - q_2) - h^2 = 0 \Rightarrow$$
$$(q_1 - q_2)^T (\ddot{q}_1 - \ddot{q}_2) = -(\dot{q}_1 - \dot{q}_2)^T (\dot{q}_1 - \dot{q}_2)$$

3.

From the E-L equations:

$$\ddot{q}_1 - \ddot{q}_2 = \frac{f_1}{m_1} - \frac{f_2}{m_2} + \lambda\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(q_1 - q_2)$$



Multiply on both sides $(q_1 - q_2)^T$:

$$-(\dot{q}_1 - \dot{q}_2)^T (\dot{q}_1 - \dot{q}_2) = (q_1 - q_2)^T \frac{m_2 f_1 - m_1 f_2}{m_1 m_2} + \lambda \frac{h^2 (m_1 + m_2)}{m_1 m_2}$$

$$\dot{q}_2 - \dot{q}_1 = h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \dot{\theta}$$

$$\lambda = -\frac{m_1 m_2}{m_1 + m_2} \dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x}) \cos\theta + (m_2 f_{1,y} - m_1 f_{2,y}) \sin\theta}{h(m_1 + m_2)}$$

4.

leading to:

$$m_1 \ddot{q}_1 = f_1 + \left(-\frac{m_1 m_2}{m_1 + m_2} \dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x}) \cos\theta + (m_2 f_{1,y} - m_1 f_{2,y}) \sin\theta}{h(m_1 + m_2)}\right)(q_1 - q_2) + \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix}$$
$$m_2 \ddot{q}_2 = f_2 - \left(-\frac{m_1 m_2}{m_1 + m_2} \dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x}) \cos\theta + (m_2 f_{1,y} - m_1 f_{2,y}) \sin\theta}{h(m_1 + m_2)}\right)(q_1 - q_2) + \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

But is it equivalent to our previous derivations? √

$$\begin{aligned}
m_1 \ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) &= f_{1,x} + f_{2,x} \\
m_1 \ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) &= f_{1,y} + f_{2,y} - (m_1 + m_2) g \\
m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) &= -f_{2,x} h\sin\theta + f_{2,y} h\cos\theta - m_2 g h\cos\theta
\end{aligned}$$

You are about to find out in the written assignment… ;-)

Newton–Euler Equation

Spatial velocity v.s. Body velocity

关于b-frame：跟随刚体，瞬时状态下关于原点坐标系静止（打点计时器?

Recall the definition of spatial velocity of a rigid body:

$$q_a(t) \mapsto \dot{q}_a(t) = \hat{V}_{ab}(t)q_a(t) = (\dot{g}_{ab}(t)g_{ab}^{-1}(t))q_a = \hat{\omega}_{ab}(t)q_a + v_{ab}(t)$$

Sometimes, its easier to work with the b-frame (body frame). We define:

$$\dot{q}_b(t) \triangleq g_{ab}^{-1}\dot{q}_a(t) = \underbrace{g_{ab}^{-1}\dot{g}_{ab}}_{\hat{V}^b_{ab}}\,q_b$$

That is, $\dot{q}_b$ is the velocity of point q w.r.t. to a-frame (ref. frame) but observed in b-frame. Here, the *body velocity* $\hat{V}^b_{ab} = g_{ab}^{-1}\dot{g}_{ab}$ is defined similarly to the *spatial velocity* $\hat{V}_{ab} = \dot{g}_{ab}g_{ab}^{-1}$:

$$\hat{V}^b_{ab} = g_{ab}^{-1}\dot{g}_{ab} = \begin{bmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{R}_{ab} & R^T_{ab}\dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\hat{V}^{s(patial)}_{ab}$$

$$\hat{V}^{b(ody)}_{ab}$$

Note: In MLS94, $\hat{V}_{ab}$ is written as $\hat{V}^s_{ab}$ to further distinguish from $\hat{V}^b_{ab}$. We didn't write it that way until now to avoid confusion. From now on, we shall write $\hat{V}^s_{ab}$ in place of $\hat{V}_{ab}$.

两者之间的关系与转换（伴随变换）

body velocity $\hat{V}^b_{ab} = g_{ab}^{-1}\dot{g}_{ab}$

spatial velocity $\hat{V}_{ab} = \dot{g}_{ab}g_{ab}^{-1}$

$$\hat{V}^s_{ab} = \dot{g}_{ab}g_{ab}^{-1} = g_{ab}g_{ab}^{-1}\dot{g}_{ab}g_{ab}^{-1} = g_{ab}\hat{V}^b_{ab}g_{ab}^{-1}$$

$$\Downarrow$$

$$\hat{V}^s_{ab} = \begin{bmatrix} \hat{\omega}^s_{ab} & v^s_{ab} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R^T_{ab} & -R^T_{ab}p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab}\hat{\omega}^b_{ab}R^T_{ab} & -R_{ab}\hat{\omega}^b_{ab}R^T_{ab}p_{ab} + R_{ab}v^b_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}\begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \underbrace{Ad_{g_{ab}}}_{\text{Adjoint transformation}} V^b_{ab}$$

熟悉的二维版本：

Planar case:

$$\hat{V}^s_{ab} = \begin{bmatrix} J\dot{\theta} & \dot{p}_{ab} - \dot{\theta}Jp_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\hat{V}^b_{ab} = \begin{bmatrix} J\dot{\theta} & R^T_{ab}\dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$v^s_{ab} = \dot{p}_{ab} - \dot{\theta}Jp_{ab}, \omega^s_{ab} = \dot{\theta}$$

$$v^b_{ab} = R^T_{ab}\dot{p}_{ab}, \omega^b_{ab} = \dot{\theta}$$

$$J = R(\frac{\pi}{2})$$

$$\begin{aligned} V^s_{ab} &= \begin{bmatrix} v^s_{ab} \\ \dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} R_{ab} & -Jp_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^b_{ab} \\ \dot{\theta} \end{bmatrix} \\ &= Ad_{g_{ab}} V^b_{ab} \end{aligned}$$

Newton–Euler equation

Generalized inertia matrix and momentum

*求导：动能 –> 动量 –> 力

The kinetic energy of a rigid in motion:

$$\begin{aligned} T &= \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV \\ &= \frac{1}{2}\int_V \rho(\hat{\omega}^s_{ab} q_a + v^s_{ab})^T(\hat{\omega}^s_{ab} q_a + v^s_{ab})dV \\ &= \frac{1}{2}\begin{bmatrix} {v^s_{ab}}^T & {\omega^s_{ab}}^T \end{bmatrix} \underbrace{\begin{bmatrix} mI_{3\times3} & -\int_V \rho\hat{q}_a dV \\ \int_V \rho\hat{q}_a dV & -\int_V \rho\hat{q}_a^2 dV \end{bmatrix}}_{M^s} \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} \\ &= \frac{1}{2}{V^s_{ab}}^T \boxed{M^s V^s_{ab}} \qquad m = \int_V \rho dV \end{aligned}$$

广义惯量矩阵与广义动量

M^s is called the *generalized inertia matrix* (superscript s means computed in spatial frame a).

We define *generalized momentum* by $M^s V^s_{ab}$.

$$M^s = \begin{bmatrix} mI_{3\times3} & -\int_V \rho\hat{q}_a dV \\ \int_V \rho\hat{q}_a dV & -\int_V \rho\hat{q}_a^2 dV \end{bmatrix}$$

Newton–Euler equation

牛顿–欧拉法可以直接得到约束力！

The Newton-Euler equation is given by:

$$\frac{d}{dt}(M^s V^s_{ab}) = M^s \dot{V}^s_{ab} + \dot{M}^s V^s_{ab} = F^s = \begin{bmatrix} f^s \\ \tau^s \end{bmatrix}$$

(F^s denotes wrench in a-frame)

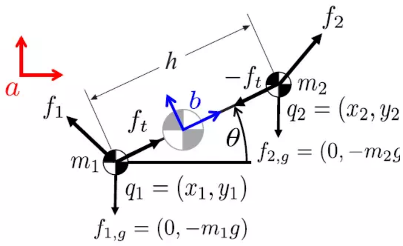
which leads to:

$$M^s \dot{V}^s_{ab} - \begin{bmatrix} \hat{\omega}^s_{ab} & \hat{v}^s_{ab} \\ 0 & \hat{\omega}^s_{ab} \end{bmatrix}^T M^s V^s_{ab} = F^s \quad \text{(3-dimension)}$$

$$M^s \dot{V}^s_{ab} - \begin{bmatrix} J\dot{\theta} & -Jv^s_{ab} \\ 0 & 0 \end{bmatrix}^T M^s V^s_{ab} = F^s \quad \text{(2-dimension)}$$

Example 1

Recall the planar spatial velocity ($J = R(\pi/2)$):



$$\hat{V}^s_{ab} = \begin{bmatrix} J\dot{\theta} & v^s_{ab} \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} \dot{q}_{1a} &= J\dot{\theta} q_{1a} + v^s_{ab} \\ \dot{q}_{2a} &= J\dot{\theta} q_{2a} + v^s_{ab} \end{aligned}$$

which leads to:

$$\begin{aligned} T &= \frac{1}{2} m_1 (J\dot{\theta} q_{1a} + v^s_{ab})^T (J\dot{\theta} q_{1a} + v^s_{ab}) + \frac{1}{2} m_2 (J\dot{\theta} q_{2a} + v^s_{ab})^T (J\dot{\theta} q_{2a} + v^s_{ab}) \\ &= \frac{1}{2} m_1 q_{1a}^T q_{1a} \dot{\theta}^2 + \frac{1}{2} m_2 q_{2a}^T q_{2a} \dot{\theta}^2 + \frac{1}{2}(m_1 + m_2) {v^s_{ab}}^T v^s_{ab} + m_1 {v^s_{ab}}^T J q_{1a} \dot{\theta} + m_2 {v^s_{ab}}^T J q_{2a} \dot{\theta} \\ &= \frac{1}{2} \underbrace{\begin{bmatrix} v^s_{ab} & \dot{\theta} \end{bmatrix}}_{{V^s_{ab}}^T} \underbrace{\begin{bmatrix} (m_1+m_2)I & m_1 J q_{1a} + m_2 J q_{2a} \\ (m_1 J q_{1a} + m_2 J q_{2a})^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}_{\text{这是对称阵！} \rightarrow M^s} \underbrace{\begin{bmatrix} v^s_{ab} \\ \dot{\theta} \end{bmatrix}}_{V^s_{ab}} \end{aligned}$$

Recall the center of mass c is given by:

$$c_a = \frac{m_1 q_{1a} + m_2 q_{2a}}{m_1 + m_2}$$

Also denote $m_1 + m_2$ by m for simplicity; then:

耦合项

$$T = \frac{1}{2} {V^s_{ab}}^T \underbrace{\begin{bmatrix} mI & \boxed{mJc_a} \\ (mJc_a)^T & \boxed{m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a}} \end{bmatrix}}_{M^s} V^s_{ab}$$

惯量相关

The N-E equation becomes:

$$M^s \dot{V}^s_{ab} + \dot{M}^s V^s_{ab} = F_1 + F_2 + F_{1,g} + F_{2,g}$$

Where $F_1, F_2, F_{1,g}, F_{2,g}$ are the wrenches corresponding to $f_1, f_2, f_{1,g}, f_{2,g}$.

The N-E equation becomes:

$$\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix} \begin{bmatrix} \dot{v}^s_{ab} \\ \ddot{\theta} \end{bmatrix} + \cdots$$

$$\begin{bmatrix} mI & mJ\dot{c}_a \\ (mJ\dot{c}_a)^T & 2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a} \end{bmatrix} \begin{bmatrix} v^s_{ab} \\ \dot{\theta} \end{bmatrix} = \cdots$$

$$\begin{bmatrix} f_1 \\ f_1^T J q_{1a} \end{bmatrix} + \begin{bmatrix} f_2 \\ f_2^T J q_{2a} \end{bmatrix} + \begin{bmatrix} f_{1,g} \\ f_{1,g}^T J q_{1a} \end{bmatrix} + \begin{bmatrix} f_{2,g} \\ f_{2,g}^T J q_{2a} \end{bmatrix}$$

Remember we say:

$$\dot{M}^s V^s_{ab} = -\begin{bmatrix} J\dot{\theta} & -Jv^s_{ab} \\ 0 & 0 \end{bmatrix}^T M^s V^s_{ab}$$

$\dot{q}_{1a} = \dot{\theta} T q_{1a} + V^s_{ab}$

$X^T J X = 0$

Inertia Matrix

The generalized inertia matrix M^s is not constant; it is easier to work with b-frame in this case.

$$T = \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV = \frac{1}{2}\int_V \rho \dot{q}_b^T R_{ab}^T R_{ab} \dot{q}_b dV = \frac{1}{2}\int_V \rho \dot{q}_b^T \dot{q}_b dV$$
$$= \frac{1}{2}\int_V \rho(\hat{\omega}_{ab}^b q_b + v_{ab}^b)^T (\hat{\omega}_{ab}^b q_b + v_{ab}^b) dV$$
$$= \frac{1}{2} m {v_{ab}^b}^T v_{ab}^b + \frac{1}{2}{\omega_{ab}^b}^T \underbrace{\int_V -\rho \hat{q}_b^2 dV}_{\mathcal{I}^b} \omega_{ab}^b - {v_{ab}^b}^T \int_V \rho \hat{q}_b dV \omega_{ab}^b$$

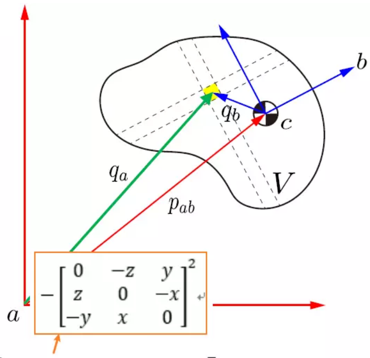
To proceed, we first define the *center of mass*:

$$c_a = \frac{1}{m}\int_V \rho q_a dV, \quad m = \int_V \rho q_a dV$$

If we attach b-frame to the center of mass, we have $c_b = 0$:

$$m\hat{c}_b = \int_V \rho \hat{q}_b dV = 0$$

$$T = \frac{1}{2}\begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}^T \underbrace{\begin{bmatrix} mI_{3\times 3} & \\ & \mathcal{I}^b \end{bmatrix}}_{M^b} \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \frac{1}{2}{V_{ab}^b}^T M^b V_{ab}^b$$



Since $V_{ab}^s = Ad_{g_{ab}} V_{ab}^b$, we have:

$$M^s = Ad_{g_{ab}}^{-T} M^b Ad_{g_{ab}}^{-1} \qquad \mathcal{I}^b = \int_V -\rho \hat{q}_b^2 dV = \int_V \rho \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dxdydz$$

This is the usual inertia matrix you see in your physics class.