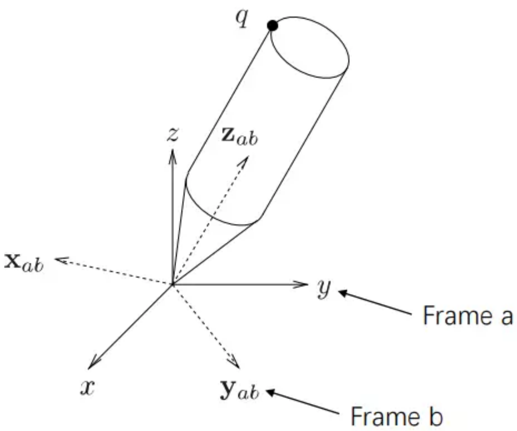
Lecture 01 / 3D Kinematics

1.1 3D rotation / 三维旋转

旋转：运动 / 坐标系变换

计算：新坐标系相对于原坐标系的转动

$$q_a = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \mathbf{x}_{ab} x_b + \mathbf{y}_{ab} y_b + \mathbf{z}_{ab} z_b = \underbrace{\begin{bmatrix} \mathbf{x}_{ab} & \mathbf{y}_{ab} & \mathbf{z}_{ab} \end{bmatrix}}_{R_{ab}} \underbrace{\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}}_{q_b}$$



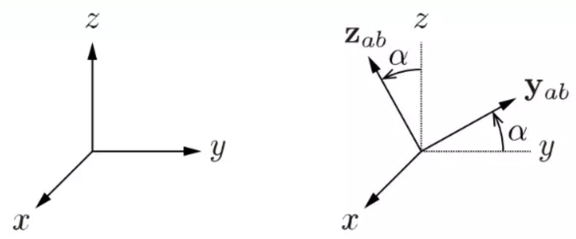
Properties of R_{ab}

$$R_{ab}^T R_{ab} = R_{ab} R_{ab}^T = I$$

$$\det R_{ab} = 1$$

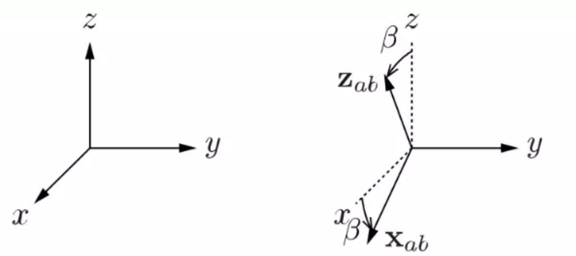
三维空间内的平面旋转：

- $R_{\mathbf{x}}(\alpha)$



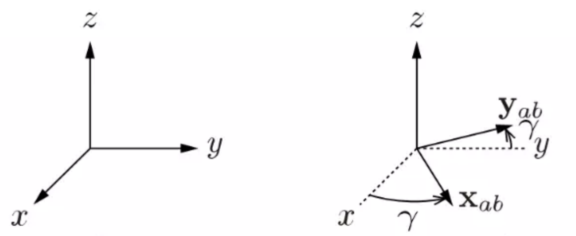
$$R_{\mathbf{x}}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

- $R_{\mathbf{y}}(\beta)$



$$R_{\mathbf{y}}(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- $R_{\mathbf{z}}(\gamma)$



$$R_{\mathbf{z}}(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.1.2 Linear Transformation / 线性变换

Map：

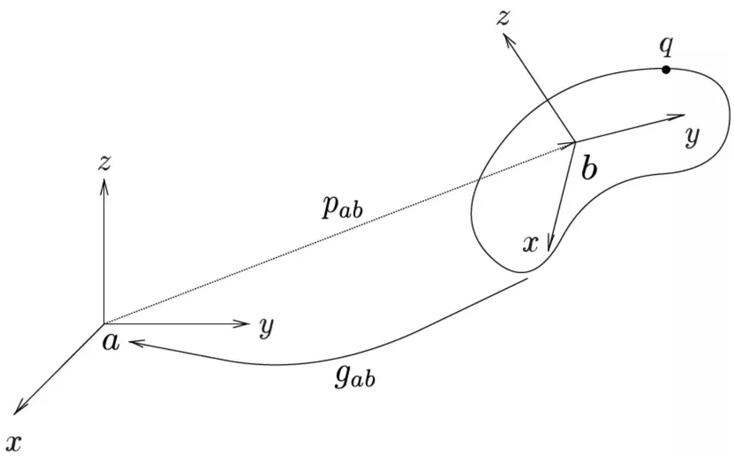
Instructor: course –> prof.

Composition: $(g \circ f)_{ZX} = g_{ZY} \circ f_{YX}$ ，先做f变换

Denotation: $f = X \to Y \Leftrightarrow f_{yx}$

Notation:

1.2 3D displacement / 三维平动



$$q_a = p_{ab} + R_{ab} q_b \quad \Rightarrow \quad \underbrace{\begin{bmatrix} q_a \\ 1 \end{bmatrix}}_{\bar{q}_a} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \underbrace{\begin{bmatrix} q_b \\ 1 \end{bmatrix}}_{\bar{q}_b}$$

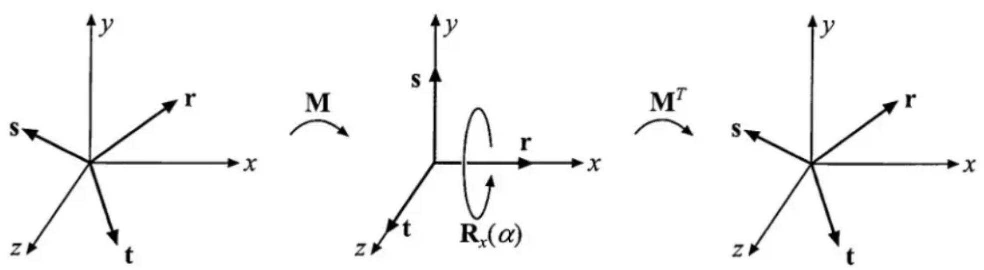
1.3 Composition

$$g_{ac} = \begin{bmatrix} R_{ac} & p_{ac} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix} = g_{ab} g_{bc}$$

1.4 Euler angle

ZYX Euler angle parameterization:

$$R_{ab} = R_{aa'} R_{a'a''} R_{a''b} = R_{\mathbf{z}}(\alpha) R_{\mathbf{y}}(\beta) R_{\mathbf{x}}(\gamma)$$

> 一个物体分别绕自己的xyz坐标轴旋转αβγ角度相当于物体绕世界坐标轴的zyx分别旋转γβα角度。
>
> 
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>
> 这里假设为右手坐标系，xyz为世界坐标系（world space），rst为某一物体P的物体坐标系(local space)，两者原点重合。 如图，如果我们要将P绕自己的x轴（即图中的 r 轴）旋转α角，可以先乘一个旋转矩阵M让xyz和rst重合，再乘以Rx(α)进行实际的旋转，最后乘以M的逆，就可得到所要的结果。

> 欧拉角变换（坐标系{B}相对于世界坐标系{A}的空间变换）：可以理解为坐标系{B}相对于坐标系{A}变化，先做Z(theta1)变换，接着做Y（theta2）变换，最后做X（theta3）变换，得到{A}和{B}的相对变换关系，然后再一步到位变换点的位置,公式表示：
> R = R（theta1）X R（theta2）X R（theta3）
>
> 1.变换顺序：从右至左--->指明运动是相对固定坐标系而言
> 从左至右--->指明运动是相对运动坐标系而言
> 2.两个变换的矩阵形式相同，说明有内在联系，一种是通过坐标旋转得到，一种是通过点的运动的得到，两者的效果达到一致。

https://zhuanlan.zhihu.com/p/28325851 XYZ Fixed Angles 和 ZYX Euler Angles的比较

2.1 3D angular velocity / 三维旋转速度

数学基础知识

- 旋转矩阵的指数形式表示

Theorem. $F(t)F(s) = F(t+s),\ \Omega = F'(0) \implies F(t) = e^{\Omega t}.$

$$F'(0) = \begin{pmatrix} -\omega\sin\omega t & -\omega\cos\omega t \\ \omega\cos\omega t & -\omega\sin\omega t \end{pmatrix}_{t=0} = \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega J.$$

$$\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} = e^{\omega J t}$$

$J = R_{\frac{1}{2}\pi}$. 逆时针旋转90°

- 复数的计算与欧拉公式

Complex number $z = x + iy$,
where $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$ satisfies $i^2 = -1$.

$$(1+2i) \div (3+4i) = \frac{1+2i}{3+4i} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)}$$
$$= \frac{1\cdot 3 - 1\cdot 4i + 2\cdot 3i - 2\cdot 4i^2}{3^2+4^2} = \frac{11}{25} + \frac{2}{25}i.$$

Complex conjugation $\bar{z} = x - iy$. We have

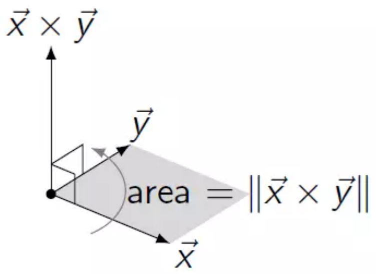
$$z\bar{z} = x^2 + y^2, \quad |z| = \sqrt{z\bar{z}}.$$

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}, \quad \overline{re^{i\theta}} = re^{-i\theta}.$$

- 叉乘的表示

  几何意义：平行四边形的面积



$$\vec{x} \times \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \hat{x}\vec{y}.$$

叉乘的性质：

Bilinear: $(a\vec{u} + b\vec{w}) \times \vec{v} = a\vec{u} \times \vec{v} + b\vec{w} \times \vec{v}$,
Bilinear: $\vec{u} \times (a\vec{v} + b\vec{w}) = a\vec{u} \times \vec{v} + b\vec{u} \times \vec{w}$,
Alternating: $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$,
Double cross: $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

扩展： $(u \times v)\wedge = ?$

Jacobi identity $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$.
Using $\hat{u} = \vec{u}\times$, the Jacobi identity is

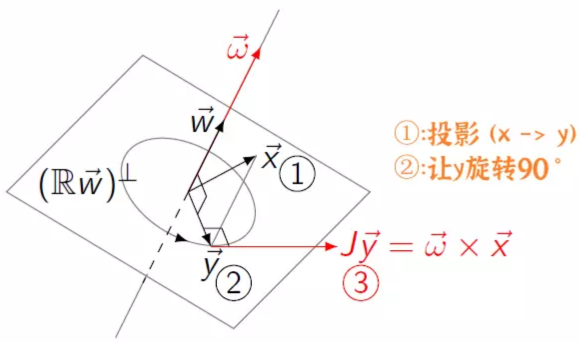
$$(\hat{u}\hat{v} - \hat{v}\hat{u})\vec{w} = \vec{u} \times (\vec{v} \times \vec{w}) - \vec{v} \times (\vec{u} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w} = \widehat{u \times v}\vec{w}.$$

We get the commutator

$$[\hat{u}, \hat{v}] = \hat{u}\hat{v} - \hat{v}\hat{u} = \widehat{u \times v}.$$

- **角速度的表示**

Recall $A = \omega J$ on $(\mathbb{R}\vec{w})^{\perp}$: For any $\vec{x}$, take orthogonal projection $\vec{y}$ of $\vec{x}$ onto $(\mathbb{R}\vec{w})^{\perp}$, then $A\vec{x} = \omega J\vec{y}$.



We have $J\vec{y} = \vec{w} \times \vec{y} = \vec{w} \times \vec{x}$.

*平行四边形： $||\vec{w} \times \vec{x}|| = |\vec{w}||\vec{y}| = |\vec{y}|$

Introduce angular velocity $\vec{\omega} = \omega\vec{w}$. Then

$$A\vec{x} = \omega J\vec{y} = \omega\vec{w} \times \vec{x} = \vec{\omega} \times \vec{x} = \hat{\omega}\vec{x}.$$

We get $A = \hat{\omega}$, and $F(t) = e^{\hat{\omega}t}$.

- **角速度与反对称阵**

Rotation $R(t)$ of rigid body, given by the changing orthonormal frame $\alpha(t)$. Initial frame $\alpha_0 = \alpha(0)$.
A point v on the body has α_0-coordinate $v(t) = R(t)v_0$. Then

$$\dot{v}(t) = \dot{R}(t)v_0 = \dot{R}(t)R(t)^{-1}v(t) = \hat{\omega}(t)v(t).$$

$$\hat{\omega} = \dot{R}R^{-1} = \dot{R}R^T.$$

(Special) orthogonal matrix R satisfies $RR^T = I$. Therefore

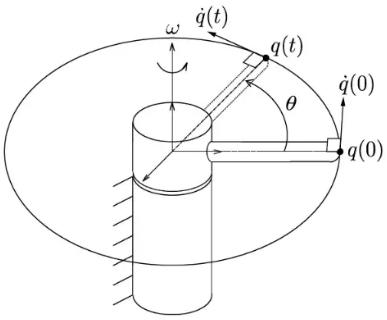
$$O = (RR^T)' = \dot{R}R^T + R\dot{R}^T.$$

We find $\hat{\omega}$ is skew-symmetric

$$\hat{\omega}^T = (\dot{R}R^T)^T = R\dot{R}^T = -\dot{R}R^T = -\hat{\omega}.$$

匀角速度旋转的表示

- 前面是一般形式的推导； $||\omega||$ 是角速度



$$q(t) = e^{\hat{\omega}t}q(0)$$
$$= \left(I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|t + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos\|\omega\|t)\right)q(0)$$

Rodrigues formula

$$R(\omega, t) = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|t + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos\|\omega\|t)$$

If we write ω as $\omega\dot{\theta}$, with $\|\omega\| = 1$ and $\dot{\theta}$ constant

Rodrigues formula

$$e^{\hat{\omega}\dot{\theta}t} = e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

一般形式的旋转

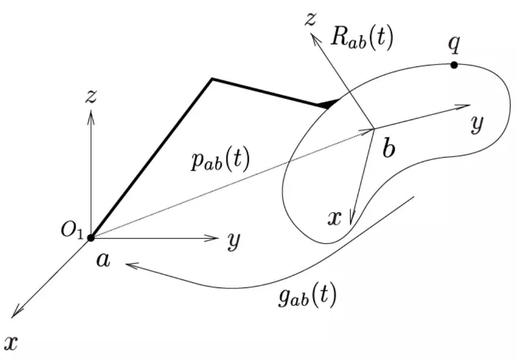
- General angular velocity

$$q_a(t) = R_{ab}(t)q_b \quad \Rightarrow \quad \dot{q}_a(t) = \dot{R}_{ab}q_b = \dot{R}_{ab}R_{ab}^T R_{ab}q_b = \underbrace{\dot{R}_{ab}R_{ab}^T}_{\hat{\omega}_{ab}} q_a$$

- 旋转矩阵 $R_{ab}^T = R_{ab}^{-1}$ ，用于转换成 $\hat{\omega_{ab}}$ （运动学中可以通用的表示）

$$\hat{\omega}_{ab} = \dot{R}_{ab}R_{ab}^T = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

2.2 3D spatial velocity / 三维空间速度



$$q_a(t) = g_{ab}(t) q_b = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} q_b$$

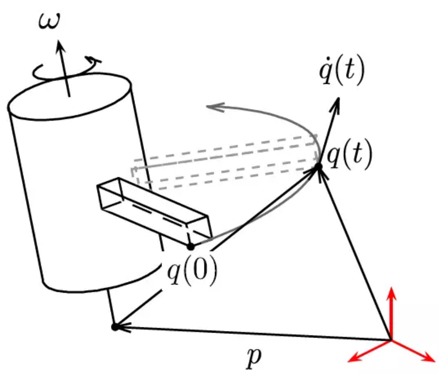
$$\begin{aligned}\dot{q}_a(t) &= \dot{g}_{ab} q_b = \dot{g}_{ab} g_{ab}^{-1} q_a(t) \\ &= \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & \dot{p}_{ab} - \dot{R}_{ab} R_{ab}^T p_{ab} \\ 0 & 0 \end{bmatrix} q_a(t) \\ &= \underbrace{\begin{bmatrix} \hat{\omega}_{ab} & v_{ab} \\ 0 & 0 \end{bmatrix}} q_a(t)\end{aligned}$$

Denoted $\hat{V}_{ab}$ and called **spatial velocity** or **twist**

Vector form of twist

$$V_{ab} = \begin{bmatrix} v_{ab} \\ \omega_{ab} \end{bmatrix} = \begin{bmatrix} \dot{p}_{ab} - \hat{\omega}_{ab} p_{ab} \\ \omega_{ab} \end{bmatrix}$$

Case 1：绕固定轴旋转

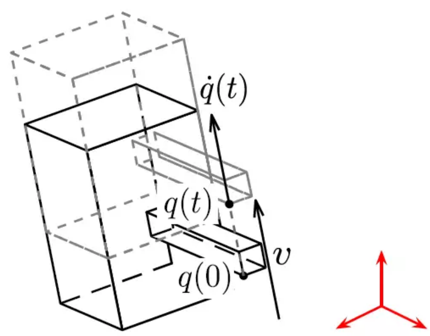


$$\frac{d}{dt}(q(t) - p) = \dot{q}(t) = \omega \times (q(t) - p)$$

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{q}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\hat{\omega} p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ 1 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} e^{\hat{\omega} t} & (I - e^{\hat{\omega} t}) p \\ 0 & 1 \end{bmatrix} q(0)$$

Case 2：平动

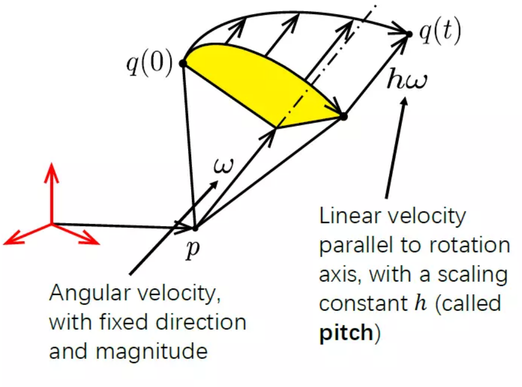


$$\dot{q}(t) = v$$

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{q}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ 1 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} I & vt \\ 0 & 1 \end{bmatrix} q(0)$$

Case 3：螺旋（丝杠坐标系）



$$\frac{d}{dt}(q(t)-p) = \dot{q}(t) = \omega \times (q(t)-p) + h\omega$$

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{q}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & h\omega - \hat{\omega}p \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q(t) \\ 1 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} e^{\hat{\omega}t} & (I - e^{\hat{\omega}t})p + h\omega t \\ 0 & 1 \end{bmatrix} q(0)$$

2.3 3D Acceleration / 三维空间加速度

从速度到加速度

转动速度

$$\dot{q} = \hat{\omega}q = \omega \times q$$

$$\boxed{\begin{aligned} \ddot{q} &= \dot{\omega} \times q + \omega \times \dot{q} \\ &= \dot{\omega} \times q + \omega \times (\omega \times q) \end{aligned}}$$

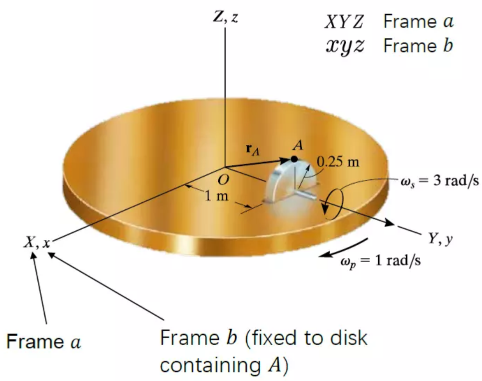
空间速度

$$\dot{q} = \hat{V}q = \omega \times q + v$$

$$\boxed{\begin{aligned} \ddot{q} &= \dot{\omega} \times q + \omega \times (\omega \times q + v) + \dot{v} \\ &= \dot{\hat{V}}q + \hat{V}^2 q \end{aligned}}$$

Example 1:

- Compute $\dot{A}_a$ and $\ddot{A}_a$



$$A_a = R_{ab}A_b = R_{\mathbf{z}}(\alpha)R_{\mathbf{y}}(\beta)A_b,$$
$$\alpha(0) = \beta(0) = 0, \dot{\alpha} = -1, \dot{\beta} = 3$$

$$\hat{\omega}_{ab} = \hat{\mathbf{z}}_{aa}\dot{\alpha} + R_{\mathbf{z}}(\alpha)\hat{\mathbf{y}}_{bb}R_{\mathbf{z}}(\alpha)^T\dot{\beta}$$

$$R\hat{\omega}R^T = (R\omega)^{\wedge}$$

$$\hat{\omega}_{ab} = \hat{\mathbf{z}}_{aa}\dot{\alpha} + (R_{\mathbf{z}}(\alpha)\mathbf{y}_{bb})^{\wedge}\dot{\beta} = \hat{\mathbf{z}}_{aa}\dot{\alpha} + \hat{\mathbf{y}}_{ab}\dot{\beta}$$

$$\omega_{ab} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\alpha} + R_{\mathbf{z}}(\alpha)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot{\beta} = \begin{bmatrix} -3\sin\alpha \\ 3\cos\alpha \\ -1 \end{bmatrix}$$

$$\vdots$$

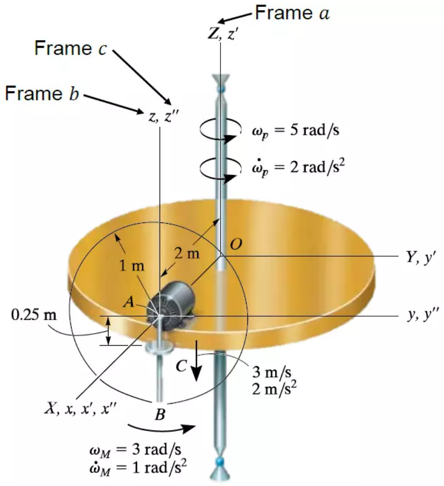
$$\dot{\omega}_{ab} = \begin{bmatrix} -3\dot{\alpha}\cos\alpha \\ -3\dot{\alpha}\sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 3\cos\alpha \\ 3\sin\alpha \\ 0 \end{bmatrix}$$

At current configuration, i.e., $\alpha = 0$

$$\dot{A}_a = \omega_{ab} \times A_a = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 0 \\ 0 \end{bmatrix}$$

$$\ddot{A}_a = \dot{\omega}_{ab} \times A_a + \omega_{ab} \times (\omega_{ab} \times A_a)$$
$$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1.75 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2.5 \\ -2.25 \end{bmatrix}$$

Example 2:



Given conditions:

$$g_{ab}(\alpha) = \begin{bmatrix} R_{\mathbf{z}}(\alpha) & R_{\mathbf{z}}(\alpha)2\mathbf{x}_{bb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{\mathbf{z}}(\alpha) & 2\mathbf{x}_{ab} \\ 0 & 1 \end{bmatrix}$$

$$\dot{\alpha} = 5, \ddot{\alpha} = 2$$

$$g_{bc}(\beta) = \begin{bmatrix} R_{\mathbf{x}}(\beta) & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{\beta} = 3, \ddot{\beta} = 1$$

$$C_c = \begin{bmatrix} 0 \\ 0 \\ -0.25 \end{bmatrix}, \dot{C}_c = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \ddot{C}_c = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$\dot{C}_a = \frac{d}{dt}(g_{ac}C_c) = \hat{V}_{ac}C_a + g_{ac}\dot{C}_c$$

where:

$$\begin{aligned}\hat{V}_{ac} &= \frac{d}{dt}\left(g_{ab}g_{bc}\right)\left(g_{ab}g_{bc}\right)^{-1} \\ &= \dot{g}_{ab}g_{ab}^{-1} + g_{ab}\dot{g}_{bc}g_{bc}^{-1}g_{ab}^{-1} \\ &= \hat{V}_{ab} + g_{ab}\hat{V}_{bc}g_{ab}^{-1} \\ &= \begin{bmatrix} \dot{\alpha}\hat{\mathbf{z}}_{aa} & 0 \\ 0 & 0 \end{bmatrix} + g_{ab}\begin{bmatrix} \dot{\beta}\hat{\mathbf{x}}_{bb} & 0 \\ 0 & 0 \end{bmatrix} g_{ab}^{-1}\end{aligned}$$

At current configuration:

$$\alpha = 0, \beta = 0, g_{ab} = \begin{bmatrix} I & 2\mathbf{x}_{aa} \\ 0 & 1 \end{bmatrix}, g_{bc} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_a = g_{ac}C_c = \begin{bmatrix} 2 \\ 0 \\ -0.25 \end{bmatrix} \qquad \hat{V}_{ac} = \begin{bmatrix} 5\hat{\mathbf{z}}_{aa} + 3\hat{\mathbf{x}}_{aa} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned}\dot{C}_a &= \hat{V}_{ac}C_a + g_{ac}\dot{C}_c \\ &= \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ -0.25 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10.75 \\ -3 \end{bmatrix}\end{aligned}$$

$$\hat{V}_{ac} = \hat{V}_{ab} + g_{ab}\hat{V}_{bc}g_{ab}^{-1}$$

$$\begin{aligned}\dot{\hat{V}}_{ac} &= \dot{\hat{V}}_{ab} + \dot{g}_{ab}\hat{V}_{bc}g_{ab}^{-1} \cdots \\ &\quad + g_{ab}\dot{\hat{V}}_{bc}g_{ab}^{-1} + g_{ab}\hat{V}_{bc}\frac{d}{dt}(g_{ab}^{-1}) \\ &= \dot{\hat{V}}_{ab} + \hat{V}_{ab}g_{ab}\hat{V}_{bc}g_{ab}^{-1} \cdots \\ &\quad + g_{ab}\dot{\hat{V}}_{bc}g_{ab}^{-1} - g_{ab}\hat{V}_{bc}g_{ab}^{-1}\hat{V}_{ab}\end{aligned}$$

$$\boxed{\frac{d}{dt}\left(M^{-1}\right) = -M^{-1}\dot{M}M^{-1}}$$