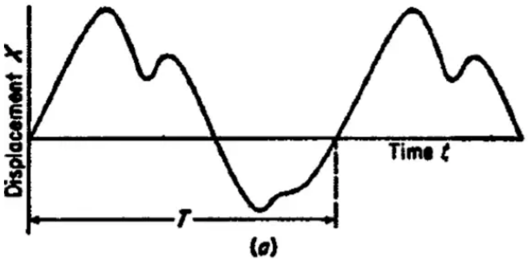
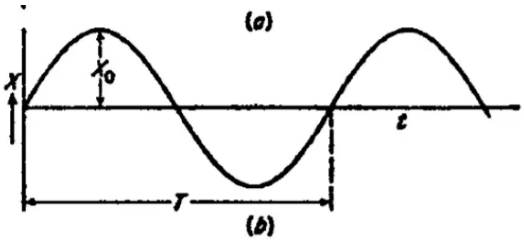
Lecture 11 / Vibration

Vibration/ 振动

振动：**周期运动**





频率：$f = 1/T$

*周期函数都可以分解为正弦/余弦函数分量!（傅里叶级数）

能量：动能/势能能量转换；阻尼(Damp)下能量耗散

振动类型：自然(Free/Natural)振动、受迫(Forced)振动、阻尼(Damped)振动

Harmonic Motion/ 简谐运动

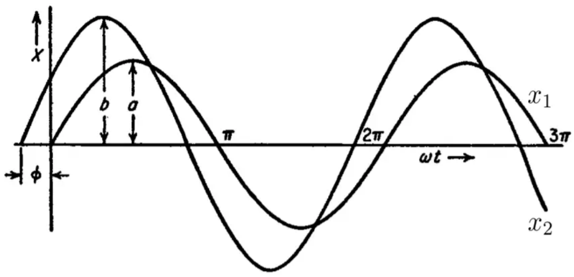
基本特征

ω： circular frequency

x_0： amplitude(振幅)

ϕ： phase angle(相位角)

关系： $T = \frac{2\pi}{w}$ and $f = \frac{w}{2\pi}$



Two harmonic motions with $T = 2\pi$, including the phase angle ϕ

$$x_1 = a\sin t$$
$$x_2 = b\sin(t+\phi) \text{ phase lead by } \phi$$

微分方程

$$x = x_0\sin(\omega t)$$
$$\dot{x} = wx_0\cos(\omega t)$$
$$\ddot{x} = -w^2\sin(\omega t)$$
$$\Rightarrow \ddot{x} + w^2x = 0$$

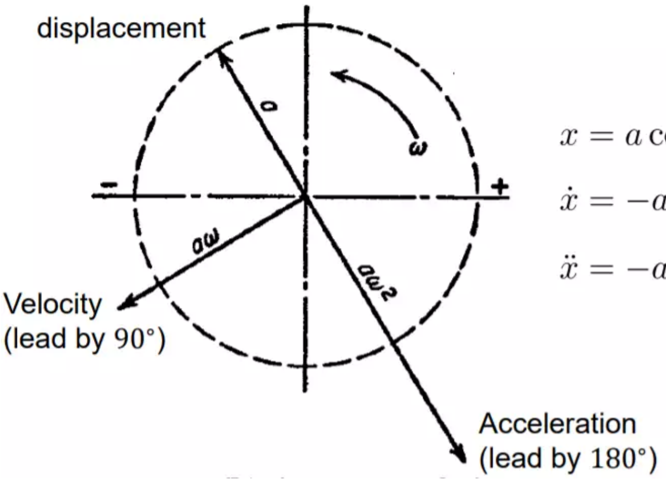
*从能量守恒的角度：

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = Const.$$

$$\dot{E} = \dot{x}(m\ddot{x} + kx) = 0$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0 \quad \text{and} \quad w^2 = \frac{k}{m}$$

向量圆表示



$$x = a\cos\omega t$$

$$\dot{x} = -a\omega\sin\omega t = a\omega\cos(\omega t + \frac{\pi}{2})$$

$$\ddot{x} = -a\omega^2\cos\omega t = a\omega^2\cos(\omega t + \pi)$$

频率和振幅的变化

将相位差分解成两个三角函数：

$$a\cos(wt) + b\cos(wt - \phi) = R\cos(wt - \psi)$$

$$R^2 = (a + b\cos\phi)^2 + (b\sin\phi)^2 = a^2 + b^2 + 2ab\cos\phi$$

$$\cos\psi = \frac{a + b\cos\phi}{R}$$

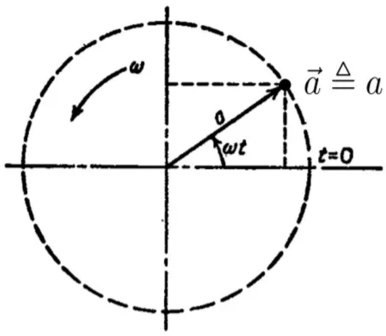
$$\sin\psi = \frac{b\sin\phi}{R}$$

Beats (adding two harmonic motions with slightly different frequency $\omega_1 \approx \omega_2, \omega_2 - \omega_1 = \Delta\omega$):

$$\begin{aligned} a\cos\omega_1 t + b\cos\omega_2 t &= a\cos\omega_1 t + b\cos(\omega_1 + \Delta\omega t) = (a + b\cos\Delta\omega t)\cos\omega_1 t + b\sin\Delta\omega t\sin\omega_1 t \\ &= \underbrace{\sqrt{a^2 + b^2 + 2ab\cos\Delta\omega t}}_{\text{(magnitude changes slowly)}}\cos(\omega_1 t - \phi) \end{aligned}$$

欧拉角和复数表示

复平面下的图像：



$$\vec{a} \triangleq a\cos\omega t + ia\sin\omega t = ae^{i\omega t}$$

$$\dot{\vec{a}} = ai\omega e^{i\omega t} = i\omega\vec{a}$$

$$\ddot{\vec{a}} = a(i\omega)^2 e^{i\omega t} = (i\omega)^2\vec{a}$$

做功计算

Harmonic force: $P = P_0\sin(wt + \phi)$

Track: $x = x_0\sin wt$

$$\int_0^{\frac{2\pi}{w}} P\frac{dx}{dt}dt = \frac{1}{w}\int_0^{2\pi} P\frac{dx}{dt}d(wt)$$

$$= P_0 x_0 \int_0^{2\pi} \sin(wt + \phi)$$

$$= P_0 x_0 \int_0^{2\pi} \cos wt[\sin wt \cos\phi + \cos wt \sin\phi] d(wt)$$

$$\Downarrow$$

$$W = \pi P_0 x_0 \sin\phi$$

*微积分知识：力在速度方向分量做功（和速度同相位）– 持续输入能量

基频和倍频的能量关系

关于基频与倍频：初始相位不变，周期 $T \Rightarrow mT$

Harmonic force: $P = P_0 \sin(mwt + \phi)$

Track: $x = x_0 \sin nwt$

$$\int_0^T P \frac{dx}{dt} dt = \int_0^T P_0 x_0 mw \cdot [\sin nwt \cdot cos(mwt + \phi)] dt = 0 \quad (n \neq m)$$

- 三角函数系的正交性：内积为0
  对于三角函数系{sinnx, cosnx}中任意两个系数不相等的三角函数A、B,
  $$\int_{-\pi}^{\pi} A(x) \cdot B(x) dx = 0$$
  // 证明：用和角/差角公式分解，只要系数不相等（差角不为0），积分就会抵消
  $$\int_{-\pi}^{\pi} \sin(mx) \cdot \cos(nx) dx = 0 \quad (m \neq n)$$
  $$\int_{-\pi}^{\pi} \sin(mx) \cdot \sin(nx) dx = 0 \quad (m \neq n)$$
  $$\int_{-\pi}^{\pi} \cos(mx) \cdot \cos(nx) dx = 0 \quad (m \neq n)$$

傅里叶级数

- For a function to be expressed as a Fourier series it must meet certain requirements the Dirichlet conditions (狄利克莱条件）：
  - f(t) must be single valued everywhere.
  - It must have a finite number of finite discontinuities per period .
  - It must have a finite number of maximum and minima per period
  - For any t_0 , it should be satisfied:
    $$\int_{t_0}^{t_0+T} |f(t)| dt < \infty$$

// 所有周期信号都可以表现为简谐函数形式的复合（级数

- 周期为 T 函数 $f(t) = f(t + 2T)$ 的三角函数化表示 / 傅里叶展开
  $$f(t) = a_0 \cdot 1 + b_0 \cdot 0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$
  - 求解 a_0： 直接积分
    $$\int_0^T f(t) dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \right] dt$$
    $$= \int_0^T a_0 dt + \sum_{n=1}^{\infty} \left[\int_0^T a_n \cos n\omega_0 t dt + \int_0^T b_n \sin n\omega_0 t dt \right] dt$$
    $$\Downarrow$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

- 求解 a_n： 左右同乘 $\cos mx$

$$\int_0^T f(t)\cos m\omega_0 tdt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]\cos m\omega_0 tdt$$

$$= \int_0^T a_0 \cos m\omega_0 tdt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t \cos m\omega_0 tdt + \int_0^T b_n \sin n\omega_0 t \cos m\omega_0 tdt\right]dt$$

$$\Downarrow$$

$$\int_0^T f(t)\cos m\omega_0 tdt = a_n \frac{T}{2}, \quad \text{for } m = n$$

$$\Downarrow$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 tdt$$

- 两边同乘 $\sin mx$ ，同理可得：

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 tdt$$

- 复数形式

欧拉公式： $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nw_0 t + b_n \sin nw_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty}(a_n \cdot \frac{1}{2}(e^{i\cdot nw_0 t} + e^{-i\cdot nw_0 t}) + b_n \cdot -\frac{1}{2}(e^{i\cdot nw_0 t} - e^{-i\cdot nw_0 t})$$

$$= \sum_{n=0}^{\infty} a_0 e^{i\cdot nw_0 t} + \sum_{n=1}^{\infty}\frac{a_n - i\cdot b_n}{2}e^{i\cdot nw_0 t} + \sum_{n=-\infty}^{-1}\frac{a_{-n} + i\cdot b_{-n}}{2}e^{i\cdot nw_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\cdot nw_0 t}, \quad c_n = \begin{cases} a_0 \quad n = 0 \quad and \quad a_0 = \frac{1}{T}\int_0^T f(t)dt \\ \frac{a_n - ib_n}{2} \quad n > 0 \quad and \quad a_n = \frac{2}{T}\int_0^T f(t)\cos nw_0 tdt \\ \frac{a_{-n} + ib_{-n}}{2} \quad n < 0 \quad and \quad b_n = \frac{2}{T}\int_0^T f(t)\sin nw_0 tdt \end{cases}$$

Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)dt$$

Limit exists $\Rightarrow$ Converge

基础计算表

// 善用求导和分部积分！

$f(t)=\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$	(unit impulse function) $\mathcal{L}(f(t))=1$
$f(t)=u(t)=\begin{cases}1 & t>0\\ 0 & t\leq 0\end{cases}$	$\mathcal{L}(f(t))=\int_0^{\infty} e^{-st}1dt=\lim_{\tau\to\infty}\left(\frac{e^{-st}}{-s}\Big\vert_0^{\tau}\right)=\lim_{\tau\to\infty}\left(\frac{e^{-s\tau}}{-s}+\frac{1}{s}\right)=\frac{1}{s}$
$f(t)=t$	$\mathcal{L}(f(t))=\int_0^{\infty} e^{-st}tdt=\frac{te^{-st}}{-s}\Big\vert_0^{\infty}+\frac{1}{s}\int_0^{\infty} e^{-st}dt=\frac{1}{s^2}$
$f(t)=e^{at}$	$\mathcal{L}\left(e^{at}\right)=\int_0^{\infty} e^{-st}e^{at}dt=\lim_{\tau\to\infty}\frac{e^{(a-s)t}}{a-s}\Big\vert_0^{\tau}=\frac{1}{s-a}$
$f(t)=\sin\omega t=\frac{1}{2i}\left(e^{i\omega t}-e^{-i\omega t}\right)$	$\mathcal{L}(\sin\omega t)=\frac{1}{2i}\left(\frac{1}{s-i\omega}-\frac{1}{s+i\omega}\right)=\frac{\omega}{s^2+\omega^2}$
$f(t)=\cos\omega t=\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)$	$\mathcal{L}(\cos\omega t)=\frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)=\frac{s}{s^2+\omega^2}$
$f(t)=\sin(\omega t+\theta)$	$\mathcal{L}(\sin(\omega t+\theta))=\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$f(t)=\cos(\omega t+\theta)$	$\mathcal{L}(\cos(\omega t+\theta))=\frac{s\cos\theta+\omega\sin\theta}{s^2+\omega^2}$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta+\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

  ①$t\geqslant 0$；$t<0$ 时，$f(t)=0$。

性质	$f(t)$	$F(S)$
线性性质	$a_1 f_1(t)+a_2 f_2(t)$	$a_1 F_1(s)+a_2 F_2(s)$
尺度变换性质	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
时域平移性质	$f(t-a)u(t-a)$	$e^{-as}F(s)$
频域平移性质	$e^{-at}f(t)$	$F(s+a)$
	$\frac{df}{dt}$	$sF(s)-f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2F(s)-sf(0^-)-f'(0^-)$
时域微分性质	$\frac{d^3 f}{dt^3}$	$s^3F(s)-s^2f(0^-)-sf'(0^-)-\cdots-f^n(0^-)$
	$\frac{d^n f}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-sf'(0^-)-\cdots-f^{n-1}(0^-)$
时域积分性质	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$
频域微分性质	$tf(t)$	$-\frac{d}{ds}F(s)$
频域积分性质	$\frac{f(t)}{t}$	$\int_s^{\infty}F(s)ds$
时域周期性质	$f(t)=f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
初值定理	$f(0)$	$\lim_{s\to\infty} sF(s)$
终值定理	$f(\infty)$	$\lim_{s\to 0} sF(s)$
卷积性质	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$

计算技巧 / 变换操作

1. 尺度变换
   - $f(t)=F(s) \quad \Rightarrow \quad f(at)=\frac{1}{a}F(\frac{s}{a})$
2. 延迟(delay) – 时域平移
   - First translation theorem / 第一平移定理：

     $$F(s)=\mathcal{L}(f(t)) \quad \Rightarrow \quad F(s-a)=\mathcal{L}(e^{at}f(t))$$

   - Second translation theorem / 第二平移定理：

     $$\text{If } F(s)=\mathcal{L}(f(t)), \text{ then } \mathcal{L}\left(u_a(t)f(t-a)\right)=e^{-as}F(s) \quad (a\geq 0)$$

- Application examples / 应用实例
  - $f(t) = u_a(t) = \begin{cases} 1 & t > a \\ 0 & t \le a \end{cases} \quad \Rightarrow \quad \mathcal{L}(u_a(t)) = e^{-as}\frac{1}{s}$
  - $f(t) = te^{at}$

  Example 1: $f(t) = te^{at}, t \ge 0$

  $$\mathcal{L}(t) = \frac{1}{s^2} \quad (\mathcal{R}e(s) > 0) \xrightarrow{1^{st}\text{ translation Thm.}} \mathcal{L}(t\,e^{at}) = \frac{1}{(s-a)^2} \quad (\mathcal{R}e(s) > a)$$

  Note: in general

  $$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \ldots \quad (\mathcal{R}e(s) > a)$$

  $$\Downarrow$$

  $$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) = \frac{1}{n!}t^n e^{at}, \quad t \ge 0$$

  - $f(t) = e^{at}\sin\omega t$

$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$	$\mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2+\omega^2}(\mathcal{R}e(s) > a)$
$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}$	$\mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}(\mathcal{R}e(s) > a)$
$\mathcal{L}(\sinh\omega t) = \frac{\omega}{s^2-\omega^2}$	$\mathcal{L}\left(e^{at}\sinh\omega t\right) = \frac{\omega}{(s-a)^2-\omega^2} \quad (\mathcal{R}e(s) > a)$
$\mathcal{L}(\cosh\omega t) = \frac{s}{s^2-\omega^2}$	$\mathcal{L}\left(e^{at}\cosh\omega t\right) = \frac{s-a}{(s-a)^2-\omega^2} \quad (\mathcal{R}e(s) > a)$

  - $\mathcal{L}(\cosh\omega t) = \frac{1}{2}\left[\mathcal{L}\left(e^{\omega t}\right) + \mathcal{L}\left(e^{-\omega t}\right)\right] = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}$

3. Differentiation theorem / 微分定理

$$\frac{d^n}{ds^n}F(s) = \mathcal{L}\left((-1)^n t^n f(t)\right), \quad n = 1, 2, 3, \ldots (s > \alpha)$$

- $\mathcal{L}(t\cos\omega t) = -\frac{d}{ds}\mathcal{L}(\cos\omega t) = -\frac{d}{ds}\frac{s}{s^2+\omega^2} = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$
- $\mathcal{L}(t\sin\omega t) = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$

4. Integration theorem / 积分定理

$$\int_s^\infty F(x)dx = \mathcal{L}\left(\frac{f(t)}{t}\right) \quad (s > \alpha)$$

- $\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{dx}{x^2+1} = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\left(\frac{1}{s}\right) \quad (s > 0).$
- $\mathcal{L}\left(\frac{\sinh\omega t}{t}\right) = \int_s^\infty \frac{\omega dx}{x^2-\omega^2} = \frac{1}{2}\int_s^\infty\left(\frac{1}{x-\omega} - \frac{1}{x+\omega}\right)dx = \frac{1}{2}\ln\frac{s+\omega}{s-\omega} \quad (s > |\omega|)$

5. Partial fractions / 多项式相除

- $F(s)=P(s)/Q(s)$ 的展开方法：
  - $Q(s)$ 中有 $(as+b)$ 的项，可以分解为 $\frac{A}{as+b}$
  - $Q(s)$ 中有 $(as+b)^n$ 的项，可以分解为 $\frac{A_1}{as+b}+\frac{A_1}{(as+b)^2}+\ldots+\frac{A_1}{(as+b)^n}$
  - $Q(s)$ 中有 (as^2+bs+c) 的项，可以分解为 $\frac{As+B}{as^2+bs+c}$
  - $Q(s)$ 中有 $(as^2+bs+c)^n$ 的项，可以分解为
    $\frac{A_1s+B_1}{as^2+bs+c}+\frac{A_2s+B_2}{(as^2+bs+c)^2}+\ldots+\frac{A_ns+B_n}{(as^2+bs+c)^n}$
- 分解项系数的求解 / 例题：

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right)$$
$$\Downarrow$$
$$F(s)=\frac{s+1}{s^2(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}$$

  - 暴力展开法 – 求解方程组 / 矩阵

$$(s+1)=As(s-1)+B(s-1)+Cs^2$$

  - 留数法求解

$$C=(s-1)F(s)\big|_{s=1}=2,\quad B=s^2F(s)\big|_{s=0}=-1,\quad A=\tfrac{d}{ds}\left(s^2F(s)\right)\big|_{s=1}=-2$$
$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right)&=-2\mathcal{L}^{-1}\left(\frac{1}{s}\right)-\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)+2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\\&=-2-t+2e^t.\end{aligned}$$

    - 求解 $(as+b)^n=\frac{A_1}{as+b}+\frac{A_1}{(as+b)^2}+\ldots+\frac{A_1}{(as+b)^n}$ 的第i项系数：

    同乘 $(as+b)^n$ ，然后求i次导数，低阶项会被消掉...

$$A_i=\frac{1}{a^{n-i}(n-i)!}\frac{d^{n-i}}{ds^{n-i}}\left((as+b)^nF(s)\right)\Big|_{s=-\frac{b}{a}}$$

6. Periodic functions / 周期函数

$$f(t)=f(t+T)$$
$$\text{define } F_1(s)=\int_0^T e^{-st}f(t)dt\quad F(s)=\frac{1}{1-e^{-sT}}F_1(s).$$

7. Derivative theorem / 倒数定理
   - Suppose that f is continuous on (0,∞) and of exponential order α and that f' is piecewise continuous on [0,∞). Then:

$$\mathcal{L}\left(f'(t)\right)=s\mathcal{L}(f(t))-f\left(0^+\right)\quad(\mathcal{R}e(s)>\alpha)$$

   - Suppose that f is continuous on [0,∞) except for a jump discontinuity at t=t1>0, and f has exponential order α with f' piecewise continuous on [0,∞). Then:

$$\mathcal{L}\left(f'(t)\right)=s\mathcal{L}(f(t))-f(0)-e^{-t_1s}\left(f\left(t_1^+\right)-f\left(t_1^-\right)\right)\quad(\mathcal{R}e(s)>\alpha)$$

   - General cases: multiple jump discontinuities

$$\mathcal{L}\left(f'(t)\right)=s\mathcal{L}(f(t))-f\left(0^+\right)-\sum_{k=1}^{n}e^{-st_k}\left(f\left(t_k^+\right)-f\left(t_k^-\right)\right)$$

   - General cases: higher order differentiation

$$\mathcal{L}\left(f^{(n)}(t)\right)=s^n\mathcal{L}(f(t))-s^{n-1}f\left(0^+\right)-s^{n-2}f'\left(0^+\right)-\cdots-f^{(n-1)}\left(0^+\right)$$

   - 对于我们而言，需要知道函数的初始条件、间断点条件（如果有），以及原函数的F(s)

Example: $\mathcal{L}(t\sinh\omega t)$

$$(t\sinh\omega t)' = \sinh\omega t + \omega t\cosh\omega t \qquad t\sinh\omega t|_0 = 0, \qquad (t\sinh\omega t)'|_0 = 0$$
$$(t\sinh\omega t)'' = \omega\cosh\omega t + \omega\cosh\omega t + \omega^2 t\sinh\omega t$$

$$\mathcal{L}((t\sinh\omega t)'') = s^2\mathcal{L}(t\sinh\omega t) = \mathcal{L}(2\omega\cosh\omega t)+\mathcal{L}(\omega^2 t\sinh\omega t) \Rightarrow \mathcal{L}(t\sinh\omega t) = \frac{2\omega s}{(s^2-\omega^2)^2}$$

8. Integration theorem – Extended

Integration theorem: Suppose that f is continuous on $(0,\infty)$ and of exponential order α (≥ 0), and $g(t) = \int_0^t f(u)\,du$. Then:

$$\mathcal{L}(g(t)) = \frac{1}{s}\mathcal{L}(f(t)) \qquad (\mathcal{R}e(s) > \alpha)$$

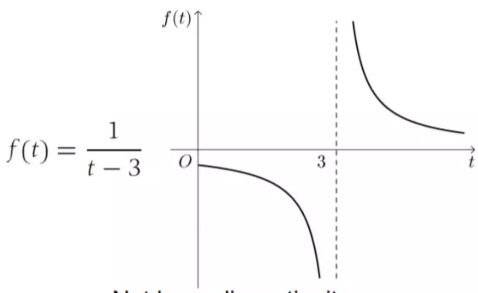
Example: Ordinary differential equations

$$y'' + y = Eu_a(t), \quad y(0) = 0, y'(0) = 1$$

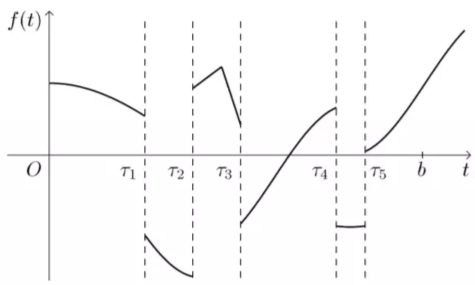
$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{Ee^{-as}}{s}$$
$$\mathcal{L}(y) = \frac{1}{s^2+1} + \frac{Ee^{-as}}{s(s^2+1)} = \frac{1}{s^2+1} + E\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-as}$$
$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + E\mathcal{L}^{-1}\left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-as}\right] = \sin t + Eu_a(t)(1-\cos(t-a))$$

Existence of Laplace transform / 存在变换的条件

- Exponential order / 指数阶：判断函数对比指数形式下的幂，有界函数阶为0；
- Piecewise Continuous / 分段连续：左右极限存在但不相等



Not jump discontinuity



A piecewise continuous function

- Existence of Laplace transform

  函数 $f(t)$ 在[0,∞)分段连续且指数阶为α $\Rightarrow$ 在 $Re(s) > \alpha$ 时 $\mathcal{L}(f(t))$ 存在并绝对收敛

$$X(s) \triangleq \int_0^\infty x(t)e^{-st}dt = \int_0^\infty Ae^{\alpha t}e^{-st}dt = A\int_0^\infty e^{(\alpha-s)t}dt$$
$$= \frac{A}{\alpha-s}e^{(\alpha-s)t}\Big|_0^\infty = \frac{A}{\alpha-s}e^{(\alpha-\sigma-j\omega)\infty} - \frac{A}{\alpha-s}$$
$$= \begin{cases} \frac{A}{s-\alpha}, & \sigma > \alpha \\ \text{(indeterminate)}, & \sigma = \alpha \\ \infty, & \sigma < \alpha \end{cases}$$

// $s = \sigma + j\omega$,其中σ是实数部分，ω是虚数部分

Basic properties of Laplace transform

- Linearity / 线性：$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2)$
- $\mathcal{L}(f) \to 0 \quad \text{as} \quad Re(s) \to \infty$

Inverse Laplace transform

$$\mathcal{L}^{-1}\big(F(s)\big) = f(t), \qquad t \geq 0$$

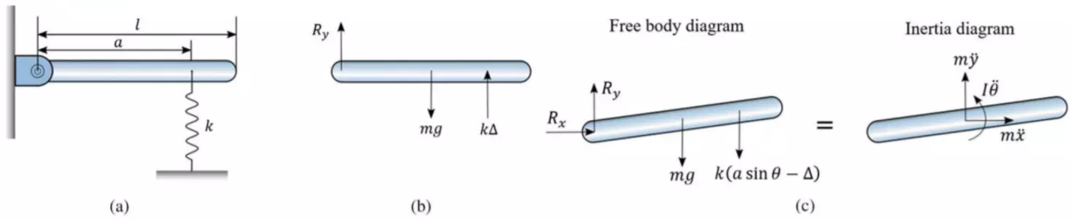
- Linearity: $\mathcal{L}^{-1}\big(a\,F(s) + b\,G(s)\big) = a\,f(t) + b\,g(t)$

Free vibration

Undamped vibration: Conservation of energy !

两个方程：静态（势能）和动态（动能） = 能量守恒

Example 1:



Static equilibrium: $k\Delta a - mg\dfrac{l}{2} = 0$

Applied moment: $M_a = -mg\dfrac{l}{2}\cos\theta - k(a\sin\theta - \Delta)a\cos\theta \;\Rightarrow\; M_a = -ka^2\sin\theta\cos\theta$

(Effective) Inertial moment: $M_{\text{eff}} = I\ddot{\theta} - m\ddot{x}\dfrac{l}{2}\sin\theta + m\ddot{y}\dfrac{l}{2}\cos\theta$

Kinematics: $\ddot{x} = -(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\dfrac{l}{2} \qquad \ddot{y} = (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\dfrac{l}{2}$

$$M_{\text{eff}} = I\ddot{\theta} + m(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\left(\frac{l}{2}\right)^2\sin\theta + m(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\left(\frac{l}{2}\right)^2\cos\theta$$

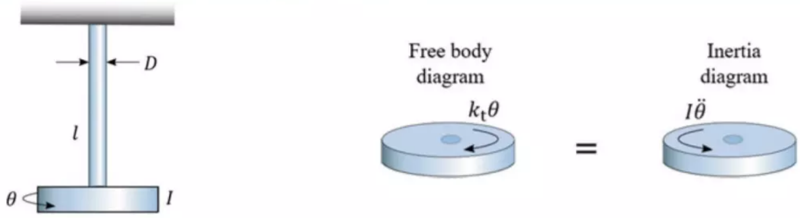
$$= I\ddot{\theta} + m\frac{l^2}{4}\ddot{\theta} = \left(I + \frac{ml^2}{4}\right)\ddot{\theta} = \left(\frac{ml^2}{12} + \frac{ml^2}{4}\right)\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta}$$

Equation of motion: $M_a = M_{\text{eff}}$

$$\frac{ml^2}{3}\ddot{\theta} + ka^2\theta = 0 \qquad \omega = \sqrt{\frac{ka^2}{ml^2/3}}$$

// 相位角与初始状态有关，自然频率与初始状态无关…

Example 2: equivalent torsional system (ignore rod inertia)



$$M_a = -k_t\theta = -\frac{GJ}{l}\theta \qquad I\ddot{\theta} + k_t\theta = 0 \quad \omega = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{GJ}{Il}}$$

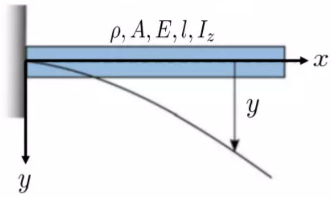
Solution: (harmonic motion) $\theta(t) = \Theta\sin(\omega t + \phi)$

Note: harmonic motion is a result of conservation of energy.

$$\dot{E} = \frac{d}{dt}(T + V) = \frac{d}{dt}\left(\frac{1}{2}m_e\dot{q}^2 + \frac{1}{2}k_e q^2\right) = 0 \;\Rightarrow\; m_e\ddot{q} + k_e q = 0$$

// 形状函数：由静态（已知的离散情况）”还原”到动态（建立函数式的联系）

Example 3: equivalent bending vibration (cantilever beam)



ρ: density
A: cross-section area
E: Young's modulus
l: length
I_z: area moment

Kinetic energy:

$$T = \frac{1}{2}\int_0^l \rho A \dot{y}^2 \, dx = \frac{1}{2} m_e \dot{q}^2, \quad m_e = \int_0^l \rho A \phi^2(x) \, dx = \frac{33}{140}\rho A l$$

Potential (strain) energy: (recall $\sigma_x = Eyy''$)

$$V = \frac{1}{2}\int_V \sigma_x \epsilon_x dV = \frac{1}{2}\int_0^l \int_A \frac{\sigma_x^2}{E}\rho dA \, dx = \frac{1}{2}\int_0^l E \underbrace{\int_A y^2 dA}_{I_z} (y'')^2 dx = \frac{1}{2}\int_0^l EI_z (y'')^2 dx$$

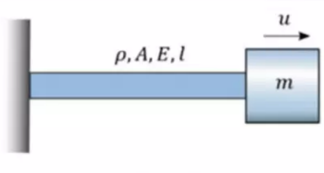
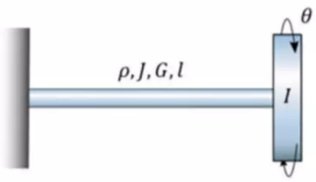
Potential (strain) energy: (continued)

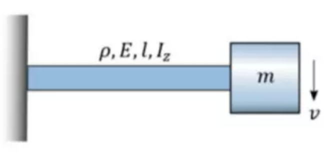
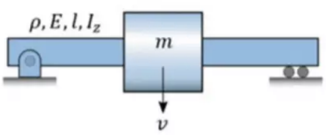
$$V = \frac{1}{2} k_e q^2(t), \qquad k_e = \int_0^l EI_z \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 dx = \frac{3EI_z}{l^3}$$

Equation of motion ($\dot{E} = \dot{T} + \dot{V} = 0$) and natural frequency:

$$m_e \ddot{q} + k_e q = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{k_e}{m_e}} = \frac{3.568}{l^2}\sqrt{\frac{EI_z}{\rho A}}$$

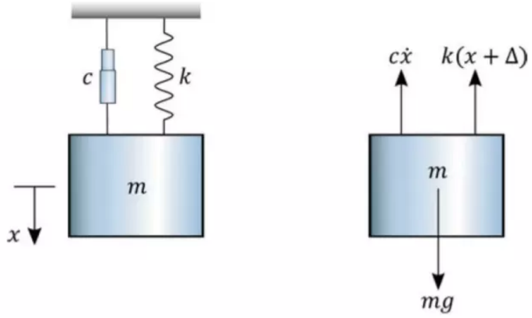
A list of continuous systems:

System	Shape Function	m_e	k_e
 Longitudinal vibration	$\phi(x) = \frac{x}{l}$	$m + \frac{\rho A l}{3}$	$\frac{EA}{l}$
 Torsional vibration	$\phi(x) = \frac{x}{l}$	$I + \frac{\rho J l}{3}$	$\frac{GJ}{l}$

System	Shape Function	m_e	k_e
 Bending of a cantilever beam	$\phi(x) = \frac{3x^2 l - x^3}{2l^3}$	$m + \frac{33\rho A l}{140}$	$\frac{3EI_z}{l^3}$
 Bending of a simply supported beam	$\phi(x) = \sin\left(\frac{\pi x}{l}\right)$	$m + \frac{\rho A l}{2}$	$\frac{\pi^4 EI_z}{2l^3}$

Free – damped vibration

Free damped vibration:



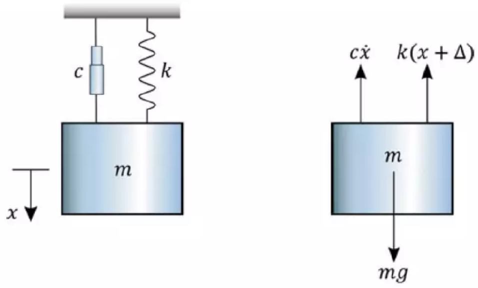
$$ms^2\mathcal{L}(x) - msx(0) - m\dot{x}(0) + cs\mathcal{L}(x) - cx(0) + k\mathcal{L}(x) = 0$$

$$\mathcal{L}(x) = \frac{sx_0 + \dot{x}_0 + \frac{c}{m}x_0}{(s+\frac{c}{2m})^2 + \frac{k}{m} - \frac{c^2}{4m^2}} = \frac{x_0(s+2\xi\omega_n)+\dot{x}_0}{(s+\xi\omega_n)^2 + (1-\xi^2)\omega_n^2}$$

where $\xi = \frac{c}{2\sqrt{km}}$ is called the damping ratio, and $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency.

Free – damped vibration

Free damped vibration:



$$m\ddot{x} = mg - c\dot{x} - k(x+\Delta)$$

$$mg = k\Delta$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Laplace transform

$$ms^2\mathcal{L}(x) - msx(0) - m\dot{x}(0) + cs\mathcal{L}(x) - cx(0) + k\mathcal{L}(x) = 0$$

$$\mathcal{L}(x) = \frac{sx_0 + \dot{x}_0 + \frac{c}{m}x_0}{(s+\frac{c}{2m})^2 + \frac{k}{m} - \frac{c^2}{4m^2}} = \frac{x_0(s+2\zeta\omega_n)+\dot{x}_0}{(s+\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}$$

where $\zeta = \frac{c}{2\sqrt{km}}$ is called the damping ratio, and $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency.

1. Underdamped $(\xi < 1)$

$$\mathcal{L}(x) = \frac{x_0\left(s+2\xi\omega_n\right)+\dot{x}_0}{\left(s+\xi\omega_n\right)^2 + (1-\xi^2)\omega_n^2} \quad \Rightarrow \quad x(t) = Ce^{-\xi\omega_n t}\cos\left(\sqrt{1-\xi^2}\omega_n t + \phi\right)$$

（直观概念）Damped frequency / 阻尼频率:

$$\omega_d = \sqrt{1-\xi^2}\omega_n$$

// C 和 ϕ 仍然由初始状态（ $x, \dot{x}$ ）决定

2. Critically damped $(\xi = 1)$

$$\mathcal{L}(x) = \frac{x_0\left(s+2\omega_n\right)+\dot{x}_0}{\left(s+\omega_n\right)^2} \quad \Rightarrow \quad x(t) = (C_1 + C_2)e^{-\xi\omega_n t}$$

where:

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

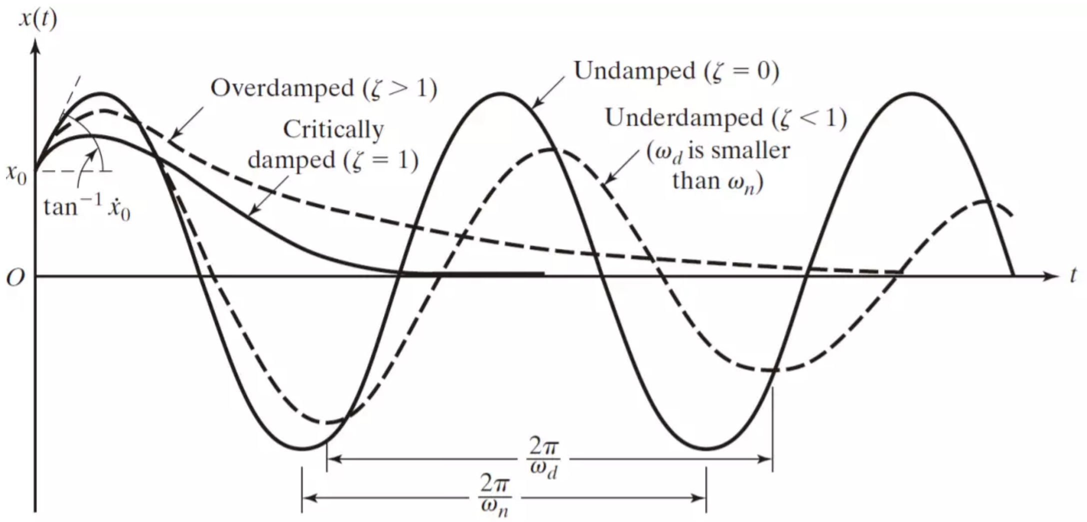
3. Overdamped $(\xi > 1)$

$$\mathcal{L}(x) = \frac{x_0\left(s+2\xi\omega_n\right)+\dot{x}_0}{\left(s+\xi\omega_n\right)^2-(\xi^2-1)\omega_n^2} \Rightarrow x(t) = Ce^{-\xi\omega_n t}\cosh\left(\sqrt{\xi^2-1}\omega_n+\phi\right)$$
$$= C_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + C_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}$$

where:

$$C_1 = \frac{x_0\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)+\dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$
$$C_2 = \frac{-x_0\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)-\dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

经典图片重出江湖...



Characteristic roots / 特征根

Defined as the roots of $s^2+2\xi\omega_n s+\omega_n^2=0$:

1. harmonic oscillation $(\xi = 0)$

$$s_1 = i\omega_n, \quad s_2 = -i\omega_n$$

2. underdamped $(\xi < 1)$

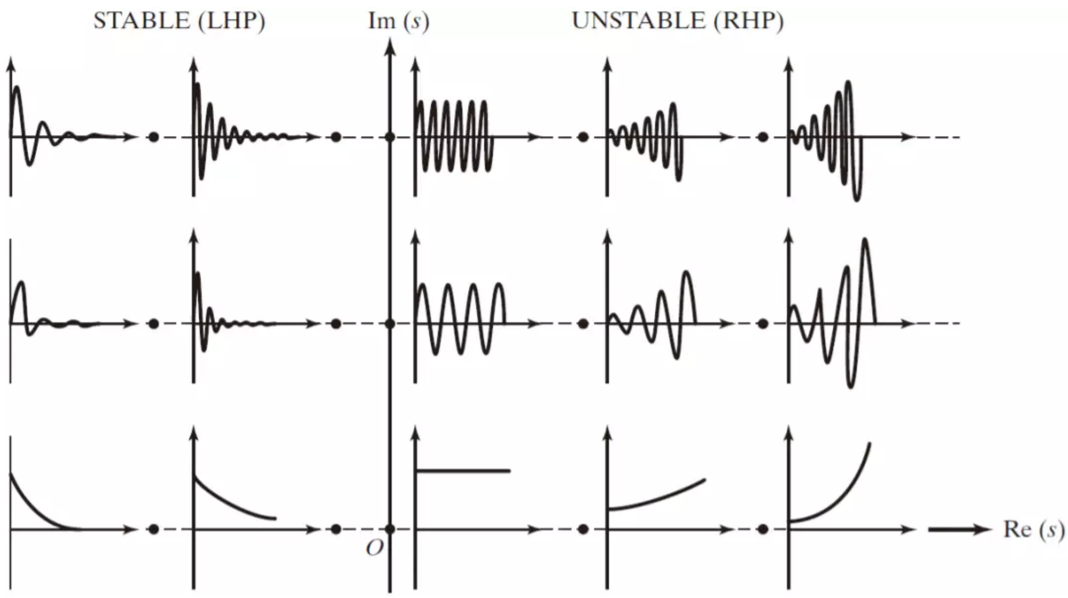
$$s_1 = \left(-\xi+i\sqrt{1-\xi^2}\right)\omega_n, \quad s_2 = \left(-\xi-i\sqrt{1-\xi^2}\right)\omega_n$$

3. critically damped $(\xi = 1)$

$$s_1 = s_2 = -\xi\omega_n$$

4. overdamped $(\xi > 1)$

$$s_1 = \left(-\xi+\sqrt{\xi^2-1}\right)\omega_n, \quad s_2 = \left(-\xi-\sqrt{\xi^2-1}\right)\omega_n$$



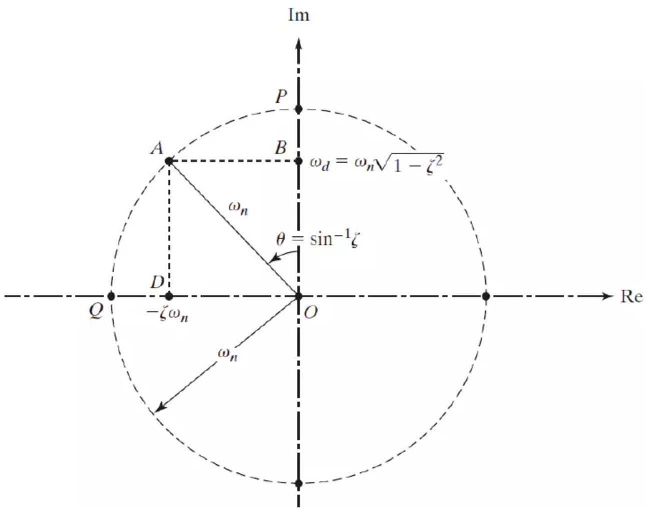
远离实轴 $\Rightarrow$ 自然频率变高

Root locus / 根轨迹

半径是自然频率 $\Rightarrow$ 同心圆上频率相同

虚轴分量是阻尼自然频率 $\Rightarrow$ 水平向平行线上阻尼自然频率相同

角度和damping ratio 有关 = 同一射线上damping ratio相同

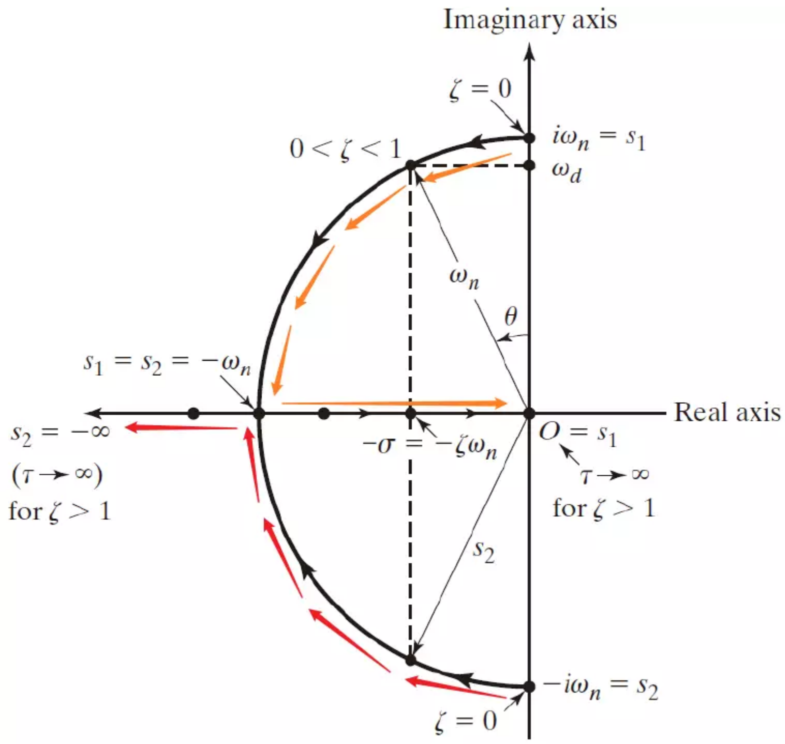


$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\xi = \frac{c}{2\sqrt{km}}$$

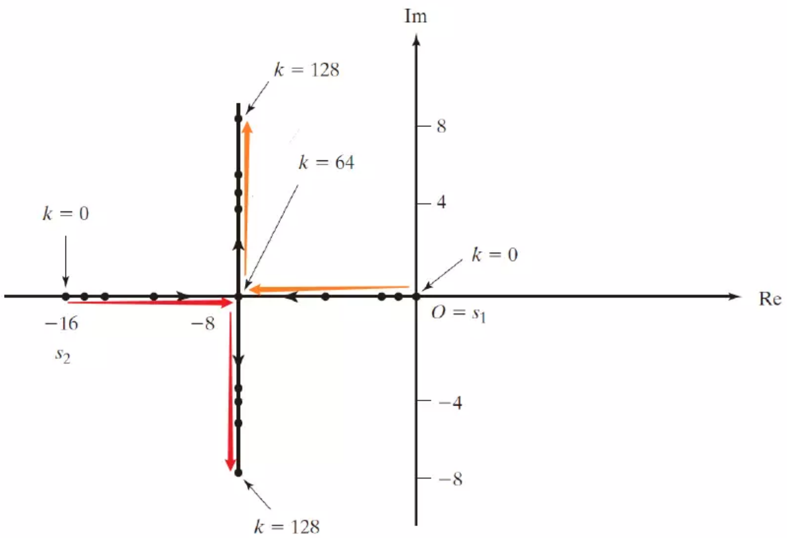
// 以下图片中特征根沿双色箭头方向移动

1. 改变 ξ（Damping ratio / 阻尼比）
    - 特征根越靠左衰减越快
    - 特征根有正的实部 $\Rightarrow$ 指数增长（不稳定）
    - $\xi = 0 \quad \Rightarrow$ naturally stable
    - 特征根没有虚部 $(s = \sigma + j\omega, \quad \omega = 0) \Rightarrow$ 不会震荡
    - 虚部越大震荡频率越高



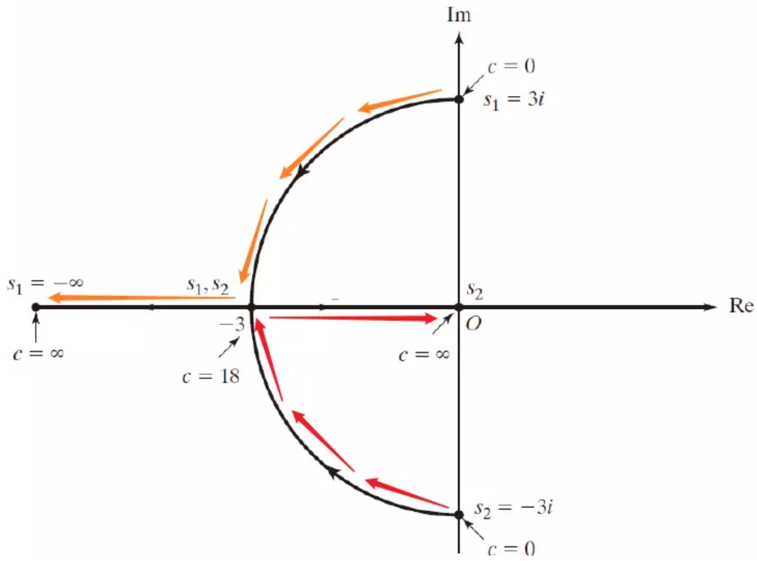
2. 改变 k（Stiffness / 刚度）

$$s^2 + 16s + k = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-16 \pm \sqrt{256 - 4k}}{2} = -8 \pm \sqrt{64 - k}$$



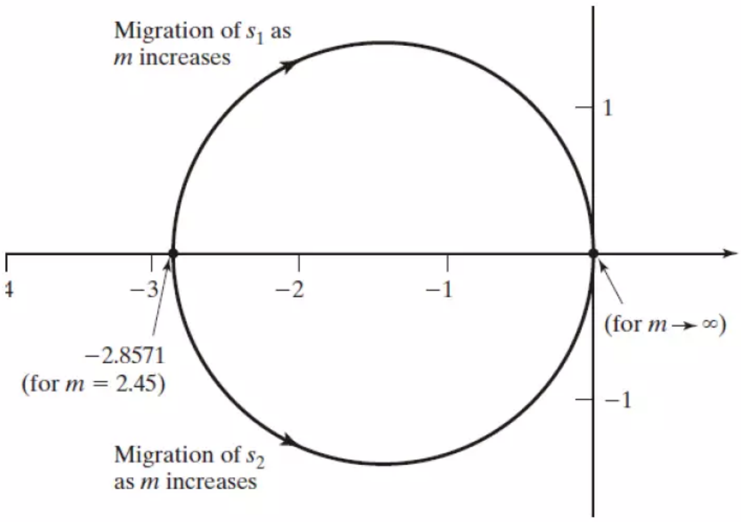
2. 改变 c（Damping / 阻尼）

$$3s^2 + cs + 27 = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-c \pm \sqrt{c^2 - 324}}{6}$$



3. 改变 m（Mass / 质量）

$$ms^2 + 14s + 20 = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-14 \pm \sqrt{196 - 80m}}{2m}$$

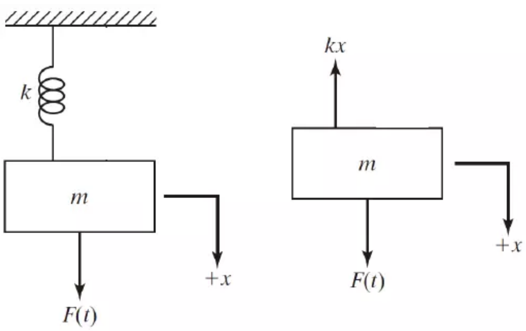


Forced vibration

Undamped System

// 我们关心稳态解（因为主要是让它消失）

Response of an undamped system under harmonic force



(a) (b) Free-body diagram

$$m\ddot{x} + kx = F_0 \cos \omega t$$
$$\Downarrow$$
$$\mathcal{L}(x) = \frac{F_0}{m}\frac{s}{(s^2+\omega^2)(s^2+\omega_n^2)} + \frac{x_0 s + \dot{x}_0}{s^2+\omega_n^2}, \quad \omega_n = \sqrt{\frac{k}{m}}$$

（接上，分解以进行拉普拉斯逆变换）

$$\mathcal{L}(x) = \frac{F_0}{m(\omega_n^2-\omega^2)}(\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega_n^2}) + x_0\frac{s}{s^2+\omega_n^2} + \frac{\dot{x}_0}{\omega_n}\frac{\omega_n}{s^2+\omega_n^2}$$
$$\mathcal{L}_t = \frac{F_0}{m(w_n^2-w^2)}\frac{s}{s^2+w^2}$$
$$\mathcal{L}_s = \frac{(-\frac{F_0}{m(w_n^2-w^2)}+x_0)s+\dot{x}_0}{s^2+w_n^2}$$
$$\Downarrow$$
$$x(t) = (x_0 - \frac{\delta_{st}}{m(\omega_n^2-\omega^2)/k})\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{\delta_{st}}{m(\omega_n^2-\omega^2)/k}\cos\omega t$$
$$= (x_0 - \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2})\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cos\omega t$$

灰色部分（齐次解）：瞬态解

黄色部分（特解）：稳态解（L(x)的第一项）

// 几个基本概念和符号

$X = \frac{\delta_{st}}{1-(\frac{w}{w_n})^2}$ 特解项系数

Static deflection / 静态形变: $\delta_{st} = F/k$

Frequency ratio / 频率比: $r = w/w_n$

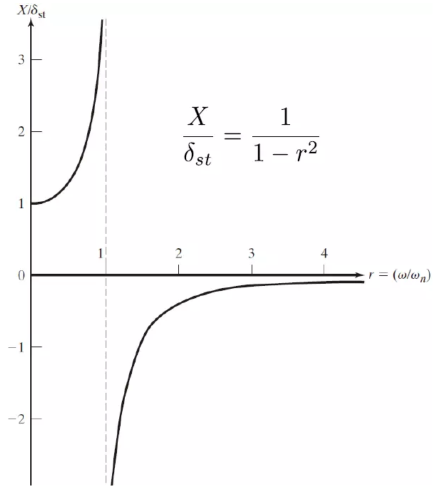
Amplitude ratio / 幅值比: $M = X/\delta_{st} = \frac{1}{1-r^2}$

- 根据不同的 r 进行讨论：

1. $0 < r < 1 \quad \Rightarrow \quad X/\delta_{st} > 0$（同相位）
2. $r > 1 \quad \Rightarrow \quad X/\delta_{st} < 0$（异相位）

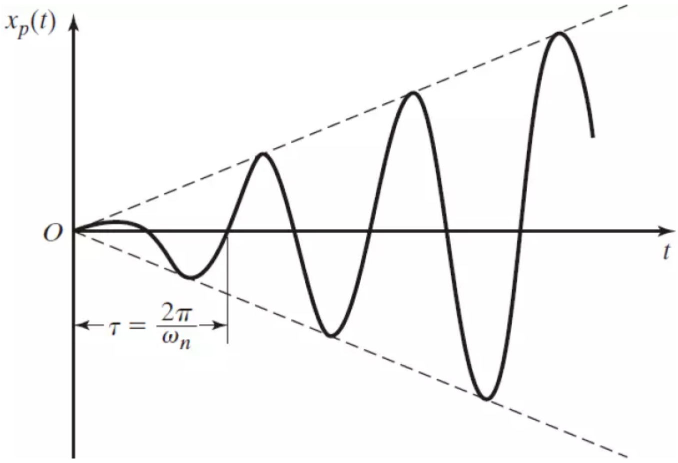
   // $r > \infty \quad \Rightarrow \quad X \to 0$ 高频的响应趋于0，反应跟不上
3. $r = 1 \quad \Rightarrow \quad w = w_n \quad and \quad X \to \infty$

$$x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \delta_{st}\frac{\cos\omega t - \cos\omega_n t}{1-\left(\frac{\omega}{\omega_n}\right)^2}\Bigg|_{\omega\to\omega_n}$$
$$= x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{\delta_{st}\omega_n t}{2}\sin\omega_n t$$



// Particular solution x_p for resonance $(r=1)$:

$X \to \infty$ when $t \to \infty$ (steady state)



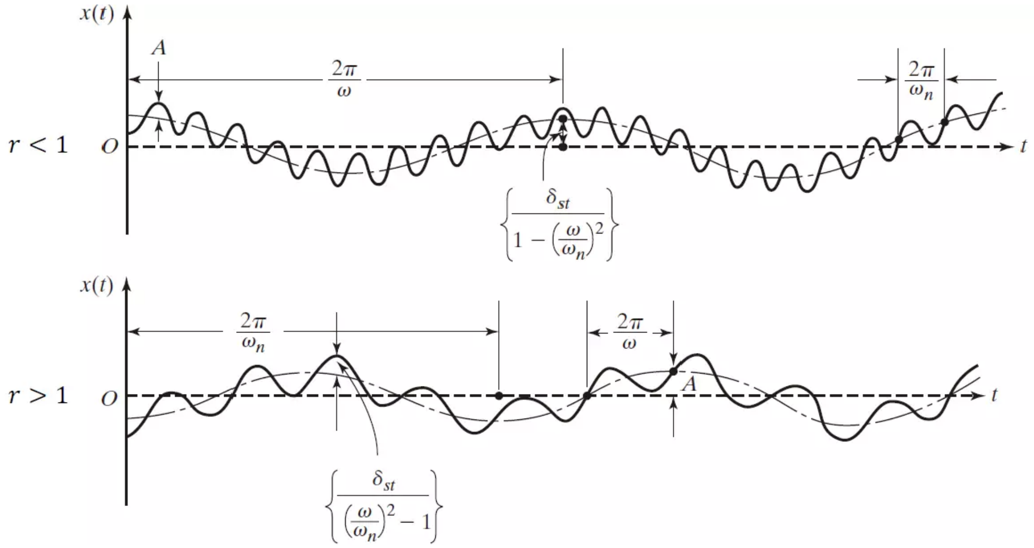
- Total response:

$$x(t) = (x_0 - \frac{\delta_{st}}{m(\omega_n^2 - \omega^2)/k}) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}}{m(\omega_n^2 - \omega^2)/k} \cos \omega t$$

$$= A \cos(\omega_n t - \phi) + \delta_{st} \cos \omega t \,/ (1 - r^2)$$

// 三角函数变换， A 和 ϕ 与初始状态 $x_0, \dot{x_0}$ 和自然频率 w_n 有关

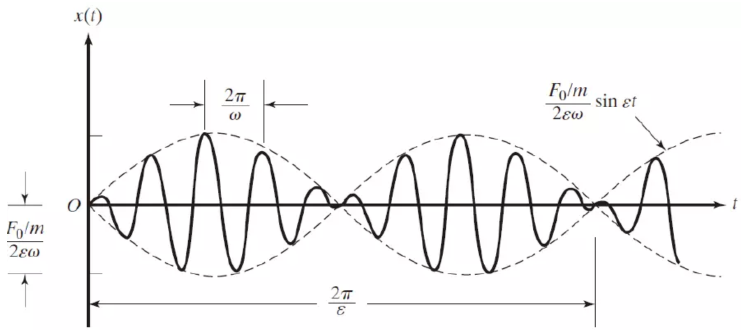
// 这里的实线只是表示周期较小的项，并不对应瞬态解和稳态解



- Beating phenomenon

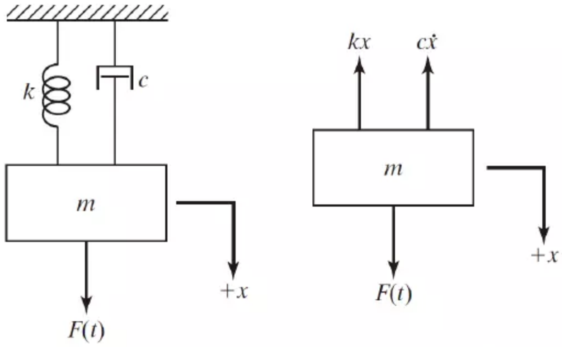
// $x_0 = \dot{x_0} = 0,$ $w_n - 2 = 2\varepsilon$ 为小量 $\Rightarrow$ $w + w_n = 2w,$ $w_n^2 - w^2 = 4\varepsilon w$

$$x(t)=\frac{\delta_{st}}{1-r^2}\left[2\sin\frac{\omega+\omega_n}{2}t\ \sin\frac{\omega_n-\omega}{2}t\right]$$
$$=\underbrace{\left(\frac{F_0/m}{2\varepsilon\omega}\sin\varepsilon t\right)}_{X \text{ Envelope (dashed line)}}\sin\omega t$$



Damped System

// 就多一项 $c\dot{x}$ ，步骤类似（拉普拉斯yyds



(a) (b) Free-body diagram

$$m\ddot{x}+c\dot{x}+kx=F_0\cos\omega t$$
$$\Downarrow$$
$$\mathcal{L}(x)=\frac{F_0}{m}\frac{s}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)}+\frac{x_0(s+2\zeta\omega_n)+\dot{x}_0}{s^2+2\zeta\omega_n s+\omega_n^2},\quad \omega_n=\sqrt{\frac{k}{m}},\zeta=\frac{c}{2\sqrt{mk}}$$
$$\Downarrow$$

（对第一项进行整理，留数法）

$$\frac{F_0}{m}\frac{s}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)}$$
$$\Downarrow$$
$$\begin{aligned}a_1&=\frac{\omega_n^2-\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}, & a_2&=\frac{2\zeta\omega_n\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}\\ a_3&=-\frac{\omega_n^2-\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}, & a_4&=-\frac{2\zeta\omega_n^3}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}\end{aligned}$$
$$\Downarrow$$

$$X(s) = \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}\left[(\omega_n^2 - \omega^2)\left(\frac{s}{s^2 + \omega^2}\right) + (2\zeta\omega_n\omega)\left(\frac{\omega}{s^2 + \omega^2}\right)\right.$$
$$\left. - (\omega_n^2 - \omega^2)\left(\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) - (2\zeta\omega_n)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\right]$$

$\Downarrow$

（拉普拉斯逆变换和稳态成分分析）

Particular solution:

$$\begin{aligned} x_p(t) = & \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega_n\omega\sin\omega t \\ & + \frac{\omega_n^2 - \omega^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)) \\ & - \frac{2\zeta\omega_n^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)] \end{aligned}$$

Steady-state solution:

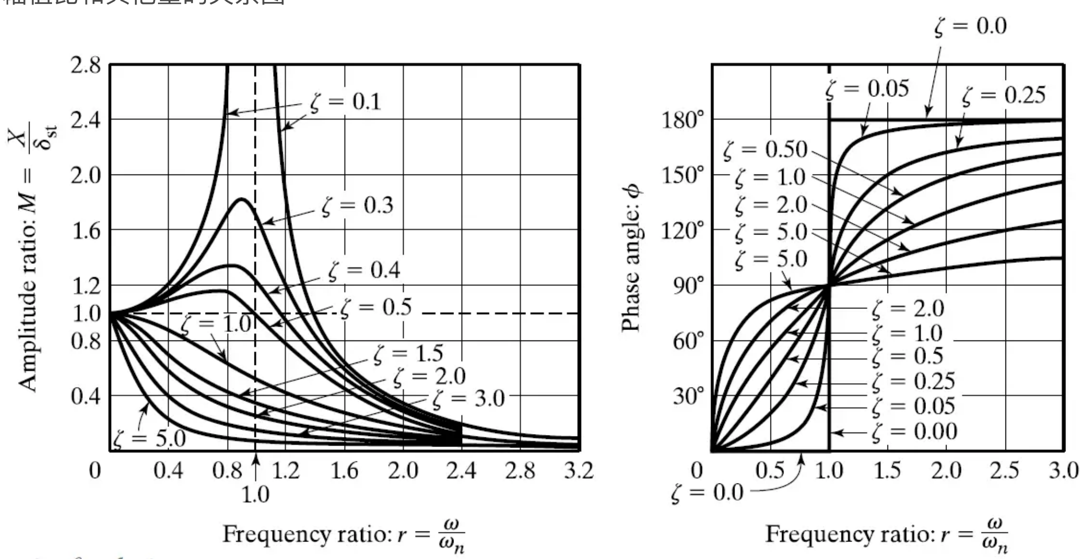
$$\begin{aligned} x_{ss}(t) & = \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega_n\omega\sin\omega t] \\ & = \underbrace{\frac{F_0}{\sqrt{c^2\omega^2 + (k - m\omega^2)^2}}}_{X}\cos(\omega t - \underbrace{\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)}_{\phi}) \end{aligned}$$

（第二项和前面的一样，直接代入）

Amplitude ratio:

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

幅值比和其他量的关系图：



// 相位90°时不做功，>90°做负功；由于因果关系相位一般是落后…

Summary of amplitude ratio:

- For an undamped system ($\zeta = 0$), reduce to undamped case (previous); $M \to \infty$ as $r \to 0$.
- Any amount of damping ($\zeta > 0$) reduces the amplitude ratio (M) for all values of the forcing frequency ω.
- For any specified value of r, a higher value of damping ζ reduces the value of M.
- In the degenerate case of a constant force (when $r = 0$), the value of $M = 1$.
- The reduction in M in the presence of damping is very significant at or near resonance.
- The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency ω (i.e., $M \to 0$ as $r \to \infty$).

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

Summary of phase angle:

- For an undamped system ($\zeta = 0$), the phase angle is 0 for $0 < r < 1$ and 180° for $r > 1$.
- For $\zeta > 0$ and $0 < r < 1$, the phase angle is given by $0^\circ < \phi < 180^\circ$.
- For $\zeta > 0$ and $r > 1$, the phase angle is given by $90^\circ < \phi < 180^\circ$.
- For $\zeta > 0$ and $r = 1$, the phase angle is given by $\phi = 90^\circ$.
- For $\zeta > 0$ and large values of r, the phase angle approaches 180°.

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$