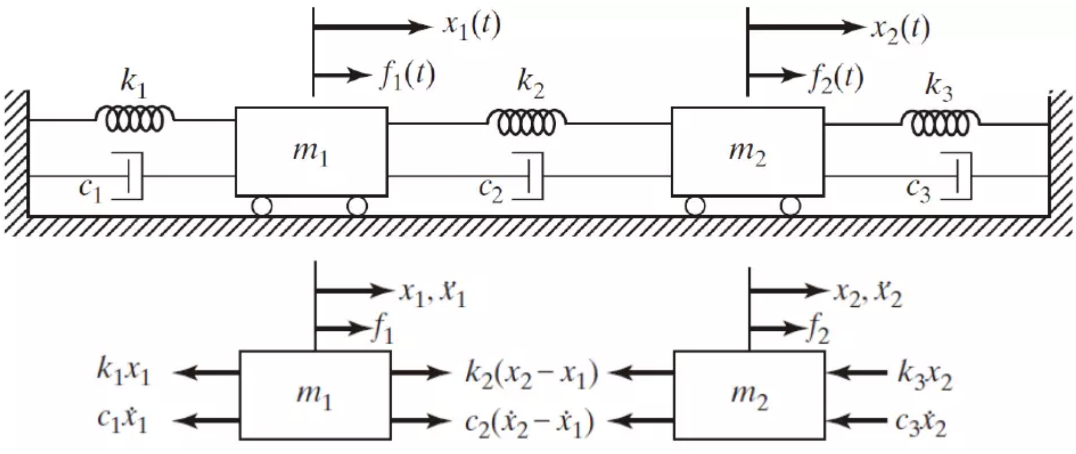
Lecture 12 / Multi-DoF Vibration

Two-DoF vibration

Two-DoF mass-spring-damper systems



$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

⇓ 矩阵化表示

$$M\ddot{\vec{x}}(t) + C\dot{\vec{x}}(t) + K\vec{x}(t) = \vec{f}(t)$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

//优秀的矩阵性质：对称且正定

Undamped system (C=0)

$$M\ddot{\vec{x}}(t) + K\vec{x}(t) = \vec{0}, \qquad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

⇓ 拉氏变换

$$m_1(s^2X_1 - sx_1(0) - \dot{x}_1(0)) + (k_1 + k_2)X_1 - k_2X_2 = 0$$
$$m_2(s^2X_2 - sx_2(0) - \dot{x}_2(0)) + -k_2X_1 + (k_2 + k_3)X_2 = 0$$

⇓ 化简

$$(s^2M + K)\vec{X} = sM\vec{x}_0 + M\dot{\vec{x}}_0$$
$$\vec{X} = (s^2M + K)^{-1}(sM\vec{x}_0 + M\dot{\vec{x}}_0)$$

⇓ 矩阵求逆：伴随矩阵

$$(s^2M+K)^{-1}=\frac{\mathrm{adj}(s^2M+K)}{\Delta}$$

特征方程（characteristic equation）： $\Delta=0$

$$s^2M+K=\begin{bmatrix} s^2m_1+(k_1+k_2) & -k_2 \\ -k_2 & s^2m_2+(k_2+k_3)\end{bmatrix}$$

$$\Delta=\det(s^2M+K)=m_1m_2s^4+((k_1+k_2)m_2+(k_2+k_3)m_1)s^2+(k_1k_2+k_2k_3+k_3k_1)$$

求解特征方程：

$$m_1m_2s^4+((k_1+k_2)m_2+(k_2+k_3)m_1)s^2+(k_1k_2+k_2k_3+k_3k_1)=0$$

$$\Downarrow$$

$$\begin{aligned} s^2&=\frac{-((k_1+k_2)m_2+(k_2+k_3)m_1)\pm\sqrt{((k_1+k_2)m_2+(k_2+k_3)m_1)^2-4(k_1k_2+k_2k_3+k_3k_1)m_1m_2}}{2m_1m_2} \\ &=\frac{-((k_1+k_2)m_2+(k_2+k_3)m_1)\pm\sqrt{((k_1+k_2)m_2-(k_2+k_3)m_1)^2+4k_2^2m_1m_2}}{2m_1m_2}\end{aligned}$$

于是我们得到了两个负实数根！ $\Rightarrow$ 表示为 $-w_1^2,-w_2^2$

$$// \quad s=\pm j\omega \quad\Rightarrow\quad s^2=-\omega^2$$

$$\vec{X}=\frac{1}{(s^2+\omega_1^2)(s^2+\omega_2^2)}\begin{bmatrix} a_0s^3+a_1s^2+a_2s+a_3 \\ b_0s^3+b_1s^2+b_2s+b_3\end{bmatrix}$$

$\Downarrow$ 逆变换

$$\begin{aligned} x_1&=X_1^{(1)}\cos(\omega_1t+\phi_1)+X_1^{(2)}\cos(\omega_2t+\phi_2) \\ x_2&=X_2^{(1)}\cos(\omega_1t+\phi_1')+X_2^{(2)}\cos(\omega_2t+\phi_2')\end{aligned}$$

- $\phi_i=\phi_i'$ 的证明（矩阵对角化）

// 对称阵(Symmetric) 可以进行对角化(Diagonalization)！

$$M\ddot{\vec{x}}(t)+K\vec{x}(t)=\vec{0},\qquad M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix},\quad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3\end{bmatrix}$$

（证明1）存在可逆矩阵A，使得：

$$A^TMA=I_{2\times2},\qquad A^TKA=D=\mathrm{diag}(d_1,d_2)$$

（线性映射）定义 $y(t):x(t)=Ay(t)$

$$M\ddot{x}(t)+K\vec{x}(t)=\vec{0}\quad\Rightarrow\quad I\ddot{\vec{y}}(t)+D\vec{y}(t)=0$$

// 对角化（对角化意味着解耦，即两项之间互不影响）

$$\Downarrow$$

$$(s^2+D)\vec{Y}=s\vec{y}_0+\dot{\vec{y}}_0\quad\Rightarrow\quad\vec{Y}=\begin{bmatrix}\frac{sy_1(0)+\dot{y}_1(0)}{s^2+d_1} \\ \frac{sy_2(0)+\dot{y}_2(0)}{s^2+d_2}\end{bmatrix}$$

$$d_1 = w_1^2, d_2 = w_2^2$$

$$\Downarrow$$

$$\vec{y}(t) = \begin{bmatrix} Y_1 \cos(\omega_1 t + \phi_1) \\ Y_2 \cos(\omega_2 t + \phi_2) \end{bmatrix}$$

$$\vec{x} = A\vec{y}(t) = \begin{bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ X_2^{(1)} \cos(\omega_1 t + \phi_1) + X_2^{(2)} \cos(\omega_2 t + \phi_2) \end{bmatrix}$$

$$\Downarrow$$

- 幅值比(Amplitude ratio) r 的求解：

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{matrix} X_1^{(1)} = a_{11}Y_1, & X_1^{(2)} = a_{12}Y_2 \\ X_2^{(1)} = a_{21}Y_1, & X_2^{(2)} = a_{22}Y_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} r_1 \triangleq \dfrac{X_2^{(1)}}{X_1^{(1)}} = \dfrac{a_{21}}{a_{11}} \\ r_2 \triangleq \dfrac{X_2^{(2)}}{X_1^{(2)}} = \dfrac{a_{22}}{a_{12}} \end{matrix}$$

$$\vec{x} = \underbrace{\begin{bmatrix} X_1^{(1)} \\ X_1^{(1)} r_1 \end{bmatrix} \cos(\omega_1 t + \phi_1)}_{\text{First mode}} + \underbrace{\begin{bmatrix} X_2^{(1)} \\ X_2^{(1)} r_2 \end{bmatrix} \cos(\omega_2 t + \phi_2)}_{\text{Second mode}}$$

$$M\ddot{\vec{x}}(t) + K\vec{x}(t) = \vec{0}, \qquad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$(-m_1\omega_1^2 + (k_1 + k_2) - k_2 r_1) X_1^{(1)} + (-m_1\omega_2^2 + (k_1 + k_2) - k_2 r_2) X_2^{(1)} = 0$$

$$(-m_2\omega_1^2 r_1 + (k_2 + k_3) r_1 - k_2) X_1^{(1)} + (-m_2\omega_2^2 r_2 + (k_2 + k_3) r_2 - k_2) X_2^{(1)} = 0$$

$r1$, $r2$ given by setting all four coefficients to zero.

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

证明1

// 对角阵可以开根号

Proof: Since M is positive definite, we can define $M^{\frac{1}{2}}$, which is given by:

$$M^{\frac{1}{2}} = \begin{bmatrix} m_1^{\frac{1}{2}} & 0 \\ 0 & m_2^{\frac{1}{2}} \end{bmatrix}$$

Next, since K is symmetric, $M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$ is also symmetric, and is given by:

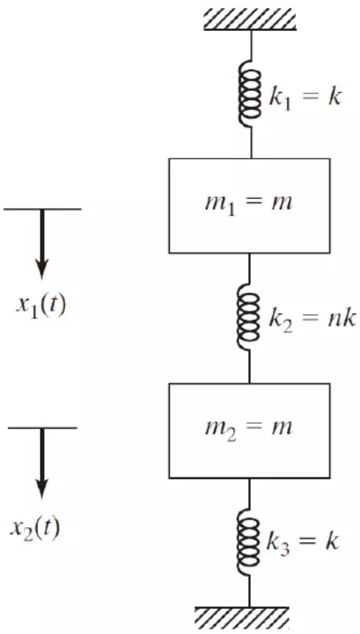
$$M^{-\frac{1}{2}}KM^{-\frac{1}{2}} = \begin{bmatrix} m_1^{-1}(k_1+k_2) & -(m_1m_2)^{-\frac{1}{2}}k_2 \\ -(m_1m_2)^{-\frac{1}{2}}k_2 & m_2^{-1}(k_2+k_3) \end{bmatrix}$$

It is well known that symmetric matrix can be diagonalized by orthogonal matrices, we have:

$$M^{-\frac{1}{2}}KM^{-\frac{1}{2}} = QDQ^T, QQ^T = I, D = \mathrm{diag}(d_1, d_2) \quad \Rightarrow \quad (M^{-\frac{1}{2}}Q)^T K (M^{-\frac{1}{2}}Q) = D$$

We see that if we let $A = M^{-\frac{1}{2}}Q$, then $A^TKA = D$, and $A^TMA = Q^TM^{-\frac{1}{2}}MM^{-\frac{1}{2}}Q = Q^TQ = I$.

Example



$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

$$\omega_1 = \left\{ \frac{4km - [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \left\{ \frac{4km + [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{3k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = 1$$

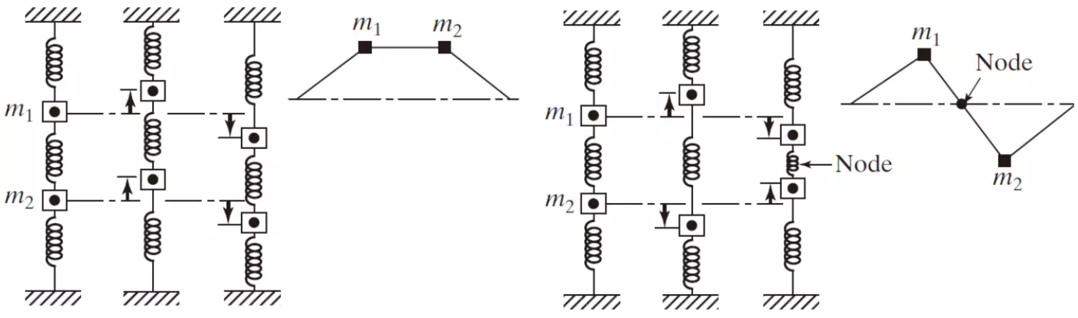
$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-m\omega_2^2 + 2k} = -1$$

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \end{Bmatrix}$$

$$x_1(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right)$$

$$x_2(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right)$$



(a) First mode
(b) Second mode

Damped System (C≠0)

$$M\ddot{\vec{x}}(t)+C\dot{\vec{x}}(t)+K\vec{x}(t)=\vec{f}(t)$$

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\qquad C=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix},\qquad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

$$\Delta=\det\begin{bmatrix} m_1s^2+(c_1+c_2)s+(k_1+k_2) & -c_2s-k_2 \\ -c_2s-k_2 & m_2s^2+(c_2+c_3)s+(k_2+k_3) \end{bmatrix}=0$$

$$(s^2+2\zeta_1\omega_1s+\omega_1^2)(s^2+2\zeta_2\omega_2s+\omega_2^2)=0$$

$$x_1=e^{-\zeta_1\omega_1t}X_1^{(1)}\cos(\sqrt{1-\zeta_1^2}\omega_1t+\phi_1)+e^{-\zeta_2\omega_2t}X_1^{(2)}\cos(\sqrt{1-\zeta_2^2}\omega_2t+\phi_2)$$

$$x_2=e^{-\zeta_1\omega_1t}X_2^{(1)}\cos(\sqrt{1-\zeta_1^2}\omega_1t+\phi_1')+e^{-\zeta_2\omega_2t}X_2^{(2)}\cos(\sqrt{1-\zeta_2^2}\omega_2t+\phi_2')$$

要想求解模态，需要能同时对角化 M 、C 、$K \Rightarrow$ Proportional damping condition

$$CM^{-1}K=KM^{-1}C$$

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\qquad C=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix},\qquad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

> https://engweb.swan.ac.uk/~adhikaris/TeachingPages/DampedVibration.pdf

proof. Assuming $\mathbf{M}$ is not singular, premultiplying equation (4.2) by $\mathbf{M}^{-1}$ we have

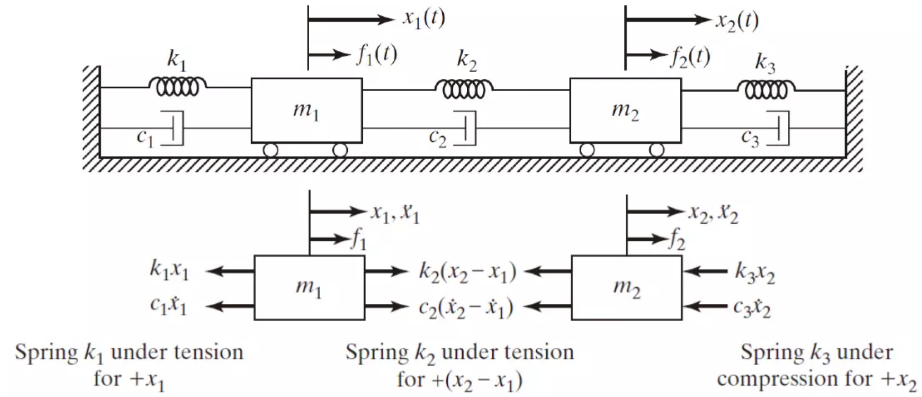
$$\mathbf{I}\ddot{\mathbf{q}}(t)+[\mathbf{M}^{-1}\mathbf{C}]\dot{\mathbf{q}}(t)+[\mathbf{M}^{-1}\mathbf{K}]\mathbf{q}(t)=\mathbf{M}^{-1}\mathbf{f}(t). \tag{4.8}$$

For classical normal modes, (4.8) must be diagonalized by an orthogonal transformation. Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be diagonalized by an orthogonal transformation if and only if the commute in product, *i.e.*, $\mathbf{AB}=\mathbf{BA}$. Using this condition in (4.8) we have

$$\begin{aligned} &[\mathbf{M}^{-1}\mathbf{C}][\mathbf{M}^{-1}\mathbf{K}]=[\mathbf{M}^{-1}\mathbf{K}][\mathbf{M}^{-1}\mathbf{C}], \\ \text{or}\quad &\mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{K}\mathbf{M}^{-1}\mathbf{C}\quad\text{(premultiplying both sides by }\mathbf{M}). \end{aligned} \tag{4.9}$$

Forced System

// 建议看project学习这一部分



$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = f_1\cos\omega_1 t$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2+c_3)\dot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = f_2\cos\omega_2 t$$

The steady-state solution under principal coordinates should look like:

$$\vec{y}_{\text{ss}}(t) = \begin{bmatrix} Y_1^{(1)}\cos(\omega_1 t+\phi_1) + Y_2^{(1)}\cos(\omega_2 t+\phi_2) \\ Y_1^{(2)}\cos(\omega_1 t+\phi_1') + Y_2^{(2)}\cos(\omega_2 t+\phi_2') \end{bmatrix}$$

which gives:

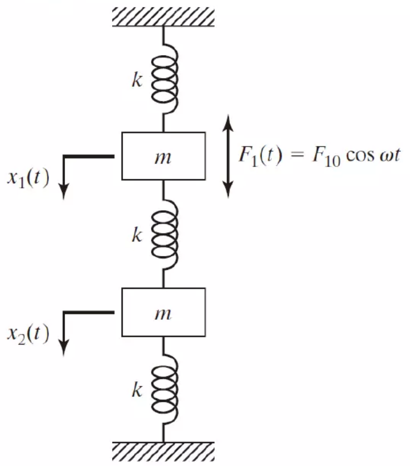
$$\vec{x}_{\text{ss}}(t) = A\vec{y}_{\text{ss}}(t) = \begin{bmatrix} X_1^{(1)}\cos(\omega_1 t+\tilde{\phi}_1) + X_2^{(1)}\cos(\omega_2 t+\tilde{\phi}_2) \\ X_1^{(2)}\cos(\omega_1 t+\tilde{\phi}_1') + X_2^{(2)}\cos(\omega_2 t+\tilde{\phi}_2') \end{bmatrix}$$

The close-form solution for amplitudes and phase angles are indeed very difficult to derive… But you are encouraged to have a try and verify by MATLAB ;-).

Example – 1

// 自然频率与外界输入无关

Example: Undamped



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{10}\cos\omega t \\ 0 \end{Bmatrix}$$

$$X_1(\omega) = \frac{(-\omega^2 m + 2k)F_{10}}{(-\omega^2 m + 2k)^2 - k^2} = \frac{(-\omega^2 m + 2k)F_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)}$$

$$X_2(\omega) = \frac{kF_{10}}{(-m\omega^2 + 2k)^2 - k^2} = \frac{kF_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)}$$

Define $\omega_1^2 = \frac{k}{m}, \omega_2^2 = \frac{3k}{m}$, we have:

$$X_1(\omega) = \frac{\left\{2 - \left(\frac{\omega}{\omega_1}\right)^2\right\}F_{10}}{k\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]}$$

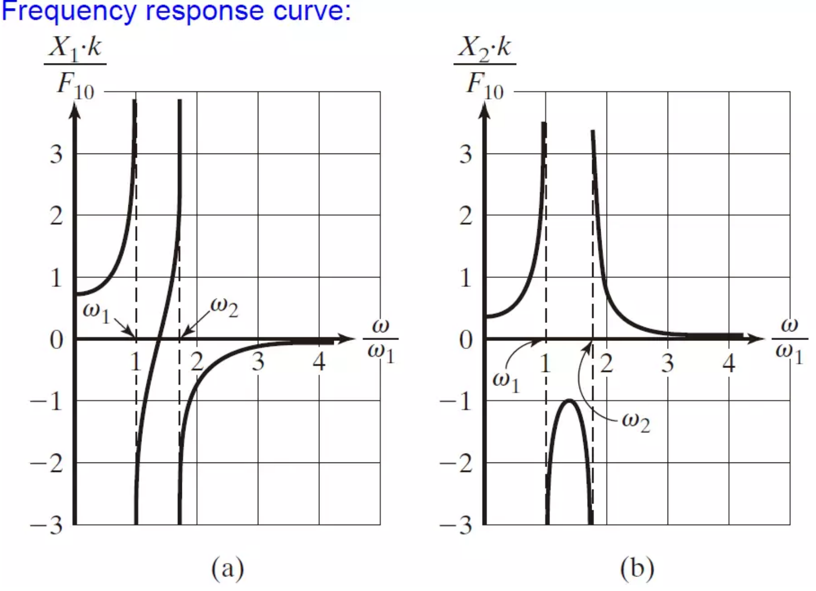
$$x_1 = X_1\cos\omega t$$
$$x_2 = X_2\cos\omega t$$

$$X_2(\omega) = \frac{F_{10}}{k\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]}$$

接近自然频率又没有阻尼，就会爆掉（共振）……

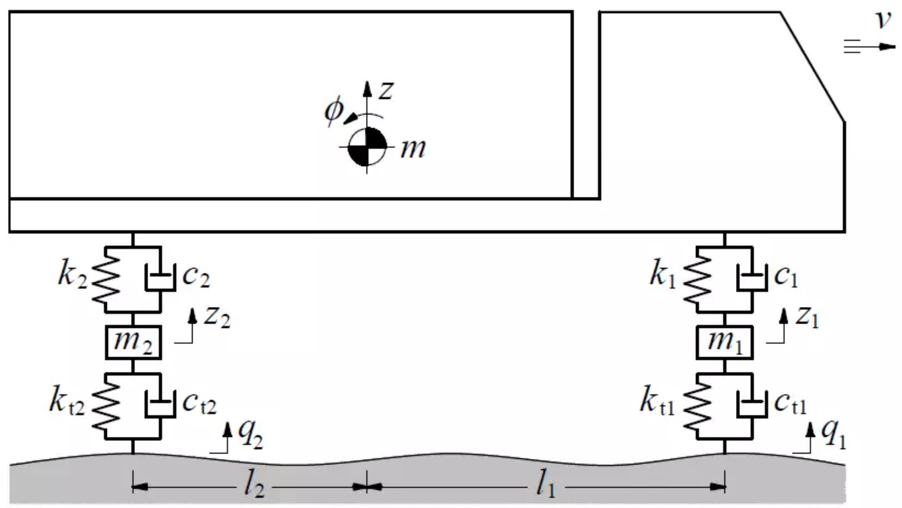
控制考虑可以取绝对值（在两个自然频率处趋于正无穷）

Frequency response curve:



Example – 2

> 基本小车模型，大部分内容引自:
>
> Nguyen Vanliem, Le Vanquynh, Jiao Renqiang, Yuan Huan. Low-frequency vibration analysis of heavy vehicle suspension system under various operating conditions. Mathematical Models in Engineering, Vol. 6, Issue 1, 2020, p. 13–22.
>
> 在推导完原文内容的基础上，加入了分块矩阵。



$$
\begin{cases}
m\ddot{z} + (c_1 + c_2)\dot{z} + (k_1 + k_2)z + (c_1 l_1 + c_2 l_2)\dot{\phi} + (k_1 l_1 + k_2 l_2)\phi \\
\quad -c_1\dot{z}_1 - k_1 z_1 - c_2\dot{z}_2 - k_2 z_2 = 0, \\
(c_1 l_1 - c_2 l_2)\dot{z} + (k_1 l_1 - k_2 l_2)z + I\ddot{\phi} + (c_1 l_1^2 + c_2 l_2^2)\dot{\phi} + (k_1 l_1^2 + k_2 l_2^2)\phi \\
\quad -c_1 l_1 \dot{z}_1 - c_1 l_1 z_1 + c_2 l_2 \dot{z}_2 + c_2 l_2 z_2 = 0, \\
-c_1\dot{z} - k_1 z - c_1 l_1 \dot{\phi} - k_1 l_1 \phi + m_1\ddot{z}_1 + (c_1 + c_{t1})\dot{z}_1 + (k_1 + k_{t1})z_1 = c_{t1}\dot{q}_1 + k_{t1}q_1, \\
-c_2\dot{z} - k_2 z + c_2 l_2 \dot{\phi} + k_2 l_2 \phi + m_2\ddot{z}_2 + (c_2 + c_{t2})\dot{z}_2 + (k_2 + k_{t2})z_2 = c_{t2}\dot{q}_2 + k_{t2}q_2.
\end{cases}
$$

$\Downarrow$ Laplace transform

$$
\begin{cases}
[ms^2+(c_1+c_2)s+(k_1+k_2)]Z(s)+[(c_1l_1+c_2l_2)s+(k_1l_1+k_2l_2)]\Phi(s) \\
\quad +[-c_1s-k_1]Z_1(s)+[-c_2s-k_2]Z_2(s)=0, \\
(c_1l_1-c_2l_2)s+(k_1l_1-k_2l_2)Z(s)+\left[Is^2+(c_1l_1^2+c_2l_2^2)s+(k_1l_1^2+k_2l_2^2)\right]\Phi(s) \\
\quad +[-c_1l_1s-k_1l_1]Z_1(s)+[c_2l_2s+k_2l_2]Z_2(s)=0, \\
[-c_1s-k_1]Z(s)+[-c_1l_1s-k_1l_1]\Phi(s)+[m_1s^2+(c_1+c_{t1})]s \\
\quad +(k_1+k_{t1})Z_1(s)+0=[c_{t1}s+k_{t1}]Q_1(s), \\
[-c_2s-k_2]Z(s)+[c_2l_2s+k_2l_2]\Phi(s)+[m_2s^2+0+(c_2+c_{t2})s \\
\quad +(k_2+k_{t2})]Z_2(s)=[c_{t2}s+k_{t2}]Q_2(s),
\end{cases}
$$

$$
// \quad s=\pm j\omega \quad \Rightarrow \quad s^2=-\omega^2
$$

$\Downarrow$ Matrix expression

$$
\begin{cases}
a_{11}Z(j\omega)+a_{12}\Phi(j\omega)+a_{13}Z_1(j\omega)+a_{14}Z_2(j\omega)=0, \\
a_{21}Z(j\omega)+a_{22}\Phi(j\omega)+a_{23}Z_1(j\omega)+a_{24}Z_2(j\omega)=0, \\
a_{31}Z(j\omega)+a_{32}\Phi(j\omega)+a_{33}Z_1(j\omega)+0=b_3Q_1(j\omega), \\
a_{41}Z(j\omega)+a_{42}\Phi(j\omega)+0+a_{44}Z_2(j\omega)=b_4Q_2(j\omega)
\end{cases}
$$

where:

$$
\begin{aligned}
&a_{11}=-m\omega^2+(c_1+c_2)j\omega+(k_1+k_2), \quad a_{23}=a_{32}=-c_1l_1j\omega-k_1l_1, \\
&a_{12}=(c_1l_1+c_2l_2)j\omega+(k_1l_1+k_2l_2), \quad a_{24}=a_{42}=c_2l_2j\omega+k_2l_2, \\
&a_{13}=a_{31}=-c_1j\omega-k_1, \quad a_{33}=-m_1\omega^2+(c_1+c_{t1})j\omega+(k_1+k_{t1}), \\
&a_{14}=a_{41}=-c_2j\omega-k_2, \quad a_{44}=-m_2\omega^2+(c_2+c_{t2})j\omega+(k_2+k_{t2}), \\
&a_{21}=(c_1l_1-c_2l_2)j\omega+(k_1l_1-k_2l_2), \quad b_3=c_{t1}j\omega+k_{t1}, \\
&a_{22}=-I\omega^2+(c_1l_1^2+c_2l_2^2)j\omega+(k_1l_1^2+k_2l_2^2), \quad b_4=c_{t2}j\omega+k_{t2}.
\end{aligned}
$$

$\Downarrow$ Block matrix

$$
\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}
\begin{bmatrix} H_z(j\omega) \\ H_\phi(j\omega) \\ H_{z_1}(j\omega) \\ H_{z2}(j\omega) \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ b_3 \\ \frac{b_4Q_2(j\omega)}{Q_1(j\omega)} \end{bmatrix}
$$

where:

$$
A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} -m\omega^2+(c_1+c_2)j\omega+(k_1+k_2) & (c_1l_1+c_2l_2)j\omega+(k_1l_1+k_2l_2) \\ (c_1l_1-c_2l_2)j\omega+(k_1l_1-k_2l_2) & -I\omega^2+\left(c_1l_1^2+c_2l_2^2\right)j\omega+\left(k_1l_1^2+k_2l_2^2\right) \end{bmatrix}
$$

$$
B=\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}=\begin{bmatrix} -c_1j\omega-k_1 & -c_2j\omega-k_2 \\ -c_1l_1j\omega-k_1l_1 & c_2l_2j\omega+k_2l_2 \end{bmatrix}
$$

$$
D=\begin{bmatrix} a_{33} & 0 \\ 0 & a_{44} \end{bmatrix}=\begin{bmatrix} -m_1\omega^2+(c_1+c_{t1})j\omega+(k_1+k_{t1}) & 0 \\ 0 & -m_2\omega^2+(c_2+c_{t2})j\omega+(k_2+k_{t2}) \end{bmatrix}
$$

$$
H_z(j\omega)=\frac{Z(j\omega)}{Q_1(j\omega)}
$$

$$
H_\phi(j\omega)=\frac{\Phi(j\omega)}{Q_1(j\omega)}
$$

$$
H_{z1}(j\omega)=\frac{Z_1(j\omega)}{Q_1(j\omega)}
$$

$$
H_{z2}(j\omega)=\frac{Z_2(j\omega)}{Q_1(j\omega)}
$$

$$
b_3=c_{t1}j\omega+k_{t1}
$$

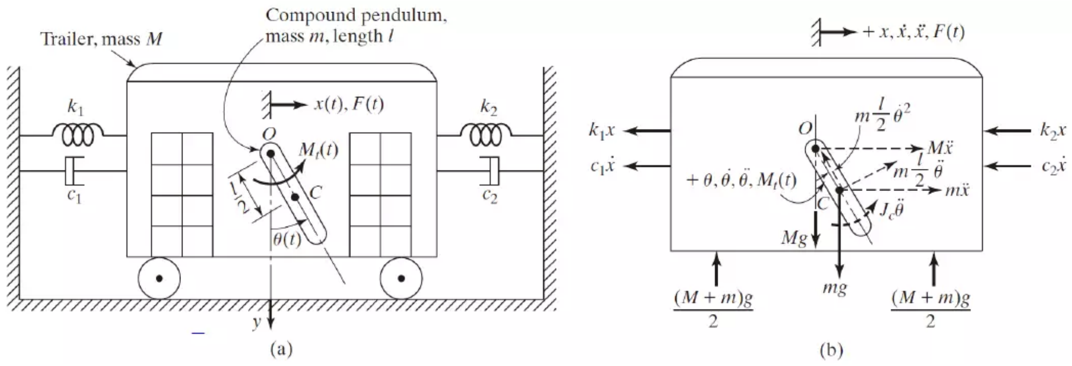
$$
b_4=c_{t2}j\omega+k_{t2}
$$

Multi-DoF vibration

多自由度 ⇒ 多维矩阵：很多时候不能求解…

Example: 单摆小车+弹簧系统，两个自由度（ x 、θ ）

// 非线性系统的线性化近似： $\sin\theta \approx \tan\theta \approx \theta$



Newton – Eular equations:

$$
M\ddot{x} + m\ddot{x} + m\frac{l}{2}\ddot{\theta}\cos\theta - m\frac{l}{2}\dot{\theta}^2\sin\theta = -k_1x - k_2x - c_1\dot{x} - c_2\dot{x} + F(t)
$$

$$
\left(m\frac{l}{2}\ddot{\theta}\right)\frac{l}{2} + \left(m\frac{l^2}{12}\right)\ddot{\theta} + (m\ddot{x})\frac{l}{2}\cos\theta = -(mg)\frac{l}{2}\sin\theta + M_t(t)
$$

$$\Downarrow$$

$$
(M + m)\ddot{x} + \left(m\frac{l}{2}\right)\ddot{\theta} + (k_1 + k_2)x + (c_1 + c_2)\dot{x} = F(t)
$$

$$
\left(\frac{ml}{2}\right)\ddot{x} + \left(\frac{ml^2}{3}\right)\ddot{\theta} + \left(\frac{mgl}{2}\right)\theta = M_t(t)
$$

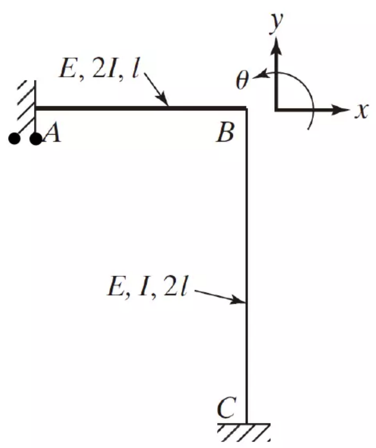
Stiffness matrix method / 刚度矩阵法

广义胡克定律：

$$
F_i = \sum_{j=1}^{n} k_{ij}x_j, \qquad i = 1, 2, \dots, n
$$

$$
\vec{F} = K\vec{x}, \qquad K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}
$$

平面刚架分析：



DOFs: x, y, θ (at B)
Forces: F_x, F_y, M_θ (at B)
The stiffness matrix we are looking for is of the form:

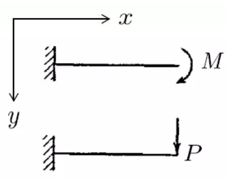
$$
\begin{bmatrix} F_x \\ F_y \\ M_\theta \end{bmatrix} = K \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}
$$

In order to derive K, we consider the forces required to exert the following three displacements:

$$
x = 1, y = 0, \theta = 0
$$
$$
x = 0, y = 1, \theta = 0
$$
$$
x = 0, y = 0, \theta = 1
$$

For that purpose, we need to first compute the force and moment required to generate:

- tip deflection of w_1 without tip slope
- tip slope of θ_1 without tip deflection



	Angle θ	Deflection y
M	$\dfrac{Ml}{EI}$	$\dfrac{Ml^2}{2EI}$
P	$\dfrac{Pl^2}{2EI}$	$\dfrac{Pl^3}{3EI}$

According to the figure (Lecture 07), we need to have proper values of M and P to achieve;

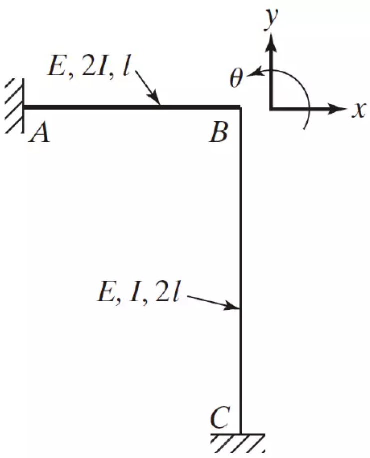
Case i) $y = 1, \theta = 0$

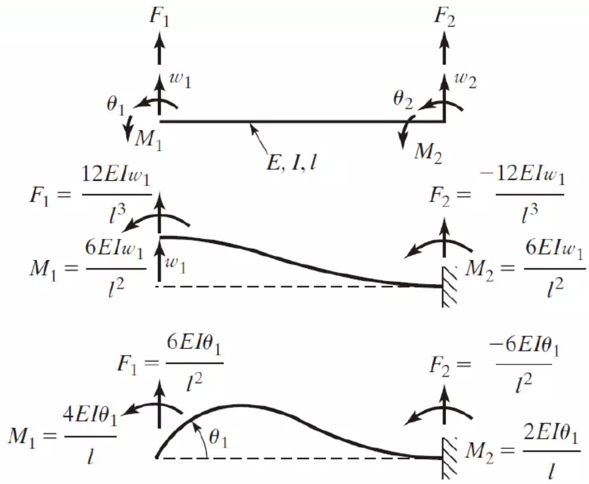
$$\begin{aligned} \frac{Ml}{EI} + \frac{Pl^2}{2EI} = 0 \quad &\Rightarrow \quad M = -\frac{Pl}{2} \\ \frac{Ml^2}{2EI} + \frac{Pl^3}{3EI} = 1 \quad &\Rightarrow \quad \frac{Pl^3}{12EI} = 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} P &= \frac{12EI}{l^3} \\ M &= -\frac{6EI}{l^2} \end{aligned}$$

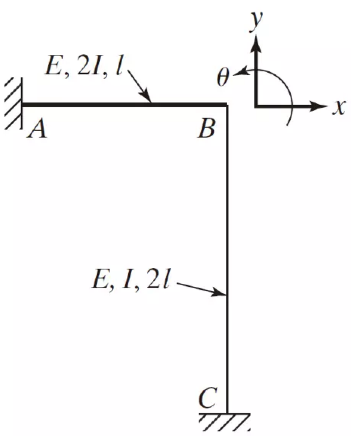
According to the figure (Lecture 07), we need to have proper values of M and P to achieve;

Case ii) $y = 0, \theta = 1$

$$\begin{aligned} \frac{Ml}{EI} + \frac{Pl^2}{2EI} = 1 \quad &\Rightarrow \quad -\frac{Pl^2}{6EI} = 1 \\ \frac{Ml^2}{2EI} + \frac{Pl^3}{3EI} = 0 \quad &\Rightarrow \quad M = -\frac{2Pl}{3} \end{aligned} \quad \Rightarrow \quad \begin{aligned} P &= -\frac{6EI}{l^2} \\ M &= \frac{4EI}{l} \end{aligned}$$







Case $x = 1, y = 0, \theta = 0$:

$$F_x = \left(\frac{12EI}{l^3}\right)_{BC} = \frac{3EI}{2l^3}, \qquad F_y = 0, \qquad M_\theta = \left(\frac{6EI}{l^2}\right)_{BC} = \frac{3EI}{2l^2}$$

Case $x = 0, y = 1, \theta = 0$:

$$F_x = 0, \qquad F_y = \left(\frac{12EI}{l^3}\right)_{BA} = \frac{24EI}{l^3}, \qquad M_\theta = -\left(\frac{6EI}{l^2}\right)_{BA} = -\frac{12EI}{l^2}$$

Case $x = 0, y = 0, \theta = 1$:

$$F_x = \left(\frac{6EI}{l^2}\right)_{BC} = \frac{3EI}{2l^2}, \qquad F_y = -\left(\frac{6EI}{l^2}\right)_{BA} = -\frac{12EI}{l^3}$$

$$M_\theta = \left(\frac{4EI}{l}\right)_{BC} + \left(\frac{4EI}{l}\right)_{BA} = \frac{2EI}{l} + \frac{8EI}{l} = \frac{10EI}{l}$$

矩阵性质与物理意义

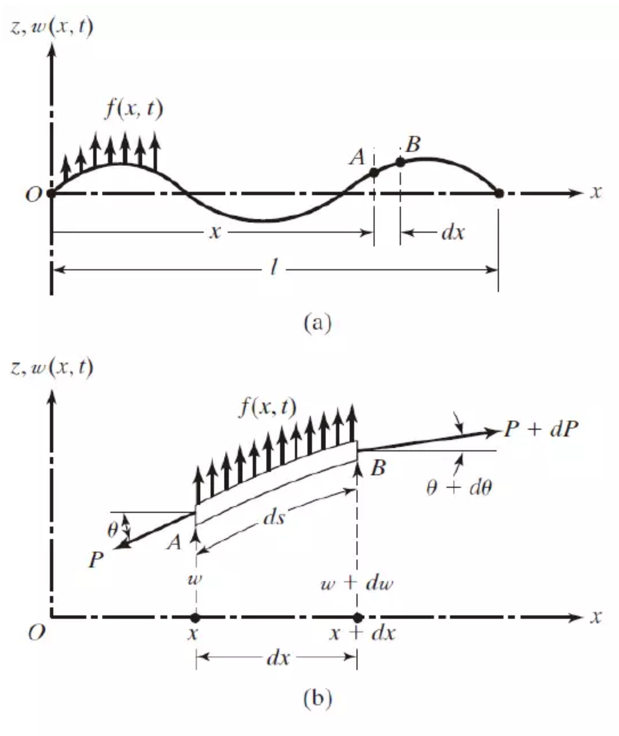
刚度矩阵的特征值：在特征向量的方向下物体抵抗变形的能力（刚度）

Continuous system

Example: String with fixed ends

//波动方程求解

假设tension在弦上连续变化，对力进行线积分：



$$(P+dP)\sin(\theta+d\theta)+fdx-P\sin\theta=\rho dx\frac{\partial^2 w}{\partial t^2}$$

$$\Downarrow$$

$$\sin(\theta+d\theta)\simeq\tan(\theta+d\theta)=\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}dx$$

$$\Downarrow$$

$$P\frac{\partial^2 w(x,t)}{\partial x^2}+f(x,t)=\rho\frac{\partial^2 w(x,t)}{\partial t^2}$$

$$\Downarrow$$

$$c^2\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial t^2}\quad c=\left(\frac{P}{\rho}\right)^{1/2}$$

// 仍然有以下近似： $\sin\theta\approx\tan\theta\approx\theta$

// P与振幅有关，振幅越大张的越紧

变量分离 / separation of variable (shape func. * magnitude)

$$w(x,t)=W(x)*T(t)$$

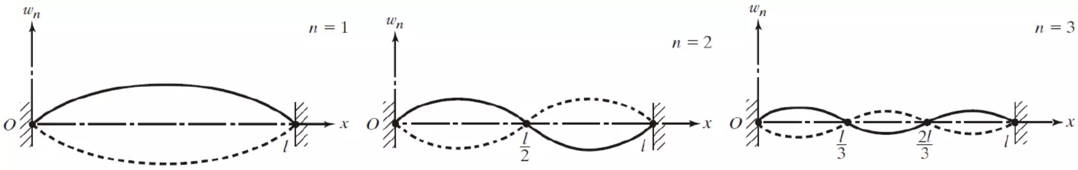
$$W(x)=A\cos\frac{\omega x}{c}+B\sin\frac{\omega x}{c}\quad T(t)=C\cos\omega t+D\sin\omega t$$

边界条件 / Boundary conditions

$$\begin{cases} w(0,t)=w(l,t)=0 \\ B\sin\frac{\omega l}{c}=0 \end{cases}\Rightarrow\begin{cases} W(0)=0\quad W(l)=0 \\ \omega_n=\frac{nc\pi}{l},\quad n=1,2,\ldots \end{cases}$$

$$\Downarrow$$

$$w_n(x,t) = W_n(x)T_n(t) = \sin\frac{n\pi x}{l}\left[C_n\cos\frac{nc\pi t}{l} + D_n\sin\frac{nc\pi t}{l}\right]$$



General solution:

$$w(x,t) = \sum_{n=1}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n\cos\frac{nc\pi t}{l} + D_n\sin\frac{nc\pi t}{l}\right]$$

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = w_0(x) \qquad \sum_{n=1}^{\infty} \frac{nc\pi}{l} D_n \sin\frac{n\pi x}{l} = \dot{w}_0(x)$$

$$C_n = \frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx \qquad D_n = \frac{2}{nc\pi}\int_0^l \dot{w}_0(x)\sin\frac{n\pi x}{l}\,dx$$

Example: Lateral vibration of beams

梁的振动和线类似，都有震动频率；近似考虑的时候只考虑了低阶模态(Mode)…