Assignment6
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1 Q1

(a) The maximum number of edges is $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$

(b) The minimum number of edges is $n-1$

(c) Assume that the minimum degree of any vertex in G is $\geq (n-1)/2$ and G is not connected

Because G is not connected, then G has at least two unconnected components.

Every vertex in G have degree of $(n-1)/2$ or more.

Thus each component in G have at least $(n-1)/2+1$ vertices

Therefore there exist at least $n+1$ vertices in G, which leads to a contradiction

Therefore G must be connected.

Q.E.D

2 Q2

(a) The adjacent matrix is above

	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
{1,2}								1	1	1
{1,3}						1	1			1
{1,4}					1		1		1	
{1,5}					1	1		1		
{2,3}			1	1						1
{2,4}		1		1					1	
{2,5}		1	1					1		
{3,4}	1			1			1			
{3,5}	1		1			1				
{4,5}	1	1			1					

(b) The degree of each vertex in G_n is C_2^{n-2}

3 Q3

Because G is a simple and self-complementary graph, therefore G must have half of all possible edges.

Set the number of vertices of G is v, then G have $C_2^v/2 = \frac{v(v-1)}{4}$ edges.

Because the number of edges is an integer

Therefore $v \equiv 0 \; or \; 1 \; (mod \; 4)$

4 Q4

(a) We start from choose a random vertex v in set, each time we add a unused vertex. Then except for v and the newest added vertex, every used vertices have degree of 2. Because the set is finite, we have limited vertex, if we repeat the process until the last vertex in set, the last vertex and v have degree of 1, which means we have to connect the last vertex and v. Therefore G must contain at least one cycle.

(b) Assume that G is connected.

If (u, v) is not an edge of G, then it is an edge of $\overline{G}$

If (u, v) is an edge of G, because G is not connected. Then there exist vertex w that neither (w, u) nor (w, v) is in G. Thus (u, w) and (w, v) is exist in $\overline{G}$. Therefore there exist a path from u to v in $\overline{G}$.

Therefore for any two vertices in $\overline{G}$ have a path, $\overline{G}$ is connected.

5 Q5

(1) If $a \in A \cup B$, than $a \in A \vee a \in B$, therefore $N(a) \subseteq N(A) \cup N(B)$, thus $N(A \cup B) \subseteq N(A) \cup N(B)$

If $a \in A$, $a \in A \cup B$, therefore $N(a) \subseteq N(A \cup B)$, thus $N(A) \subseteq N(A \cup B)$

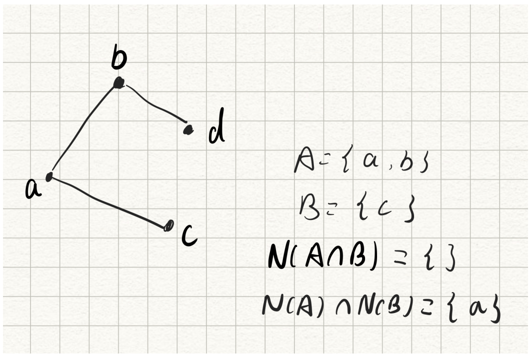
If $b \in B$, $b \in A \cup B$, therefore $N(b) \subseteq N(A \cup B)$, thus $N(B) \subseteq N(A \cup B)$

Therefore $N(A) \cup N(B) \subseteq N(A \cup B)$

Thus $N(A \cup B) = N(A) \cup N(B)$

(2) If $a \in A \cap B$, then $a \in A \wedge a \in B$, thus $N(a) \in N(A) \wedge N(a) \in N(B)$, therefore $N(A \cap B) \subseteq N(A) \cup N(B)$

e.g.



6 Q6

Set one group of G has n vertices, other group has $v-n$ vertices. Then the max number of edges is

$f(n)=n(v-n)$. Take derivative to f, we have $\frac{df(n)}{dn}=v-2n$. When $\frac{df(n)}{dn}=0$, we will find the max

value $f(v/2)=v^2/4$ Therefore the max value of e is $v^2/4$

7 Q7

(a) Because $rad(G)=min_{v\in V}[max_{u\in V}d_G(u,v)]$, clearly $rad(G)\le diam(G)$

Set $diam(G)=d_G(u,v)$, $rad(G)$ start from r, then $diam(G)\le d_G(u,r)+d_G(r,v)$, for $diam(G)$ is the

shortest path from u to v

Because $rad(G)$ start from r, $d_G(u,r)+d_G(r,v)\le 2rad(G)$

Therefore $rad(G)\le diam(G)\le 2rad(G)$

(b)A circle with $2n$ or $2n+1$ vertices will always have $diam(G)=rad(G)=n$

A line with $2n+1$ vertices will always have $diam(G)=2rad(G)=2n$

8 Q8.

In G, u_5 have 1 path of length 4 to u_6. But in H, v_5 have 2 path of length 4 to v_7. In both graph, u_5,u_6,v_5,v_7 are all have degree of 3. Therefore G and H are not isomorphic.

9 Q9.

Assume that a a directed multigraph G having no isolated vertices has an Euler circuit.

Because Euler circuit is a simple circuit containing every edges of G, then the in-degree and

out-degree of every vertices must be equal. By going through Euler circuit, we can find a

path between every two vertices. Therefore G must be weakly connected.

The first part is proved.

Assume that for a weakly connected graph G, every vertices $v\in G$ has

$deg^+(v)=deg^-(v)$

We arbitarily choose a vertex u and any out edge. Because every vertices in G have

$deg^+(v)=deg^-(v)$, then we can always find another unvisited out edge in next

vertex. In the end the only vertex which have unvisited out is u. Since there is

always an out edge we can visit any vertex other than u. Therefore there must exist

a cycle contain u

Then we chose a random vertex v and find a circuit begin and end at v. We delete

all edges contained in the cycle in G. After deletion, all vertices' out-degree still equal to in-degree. By repeating the deletion, we can find cycles that have common

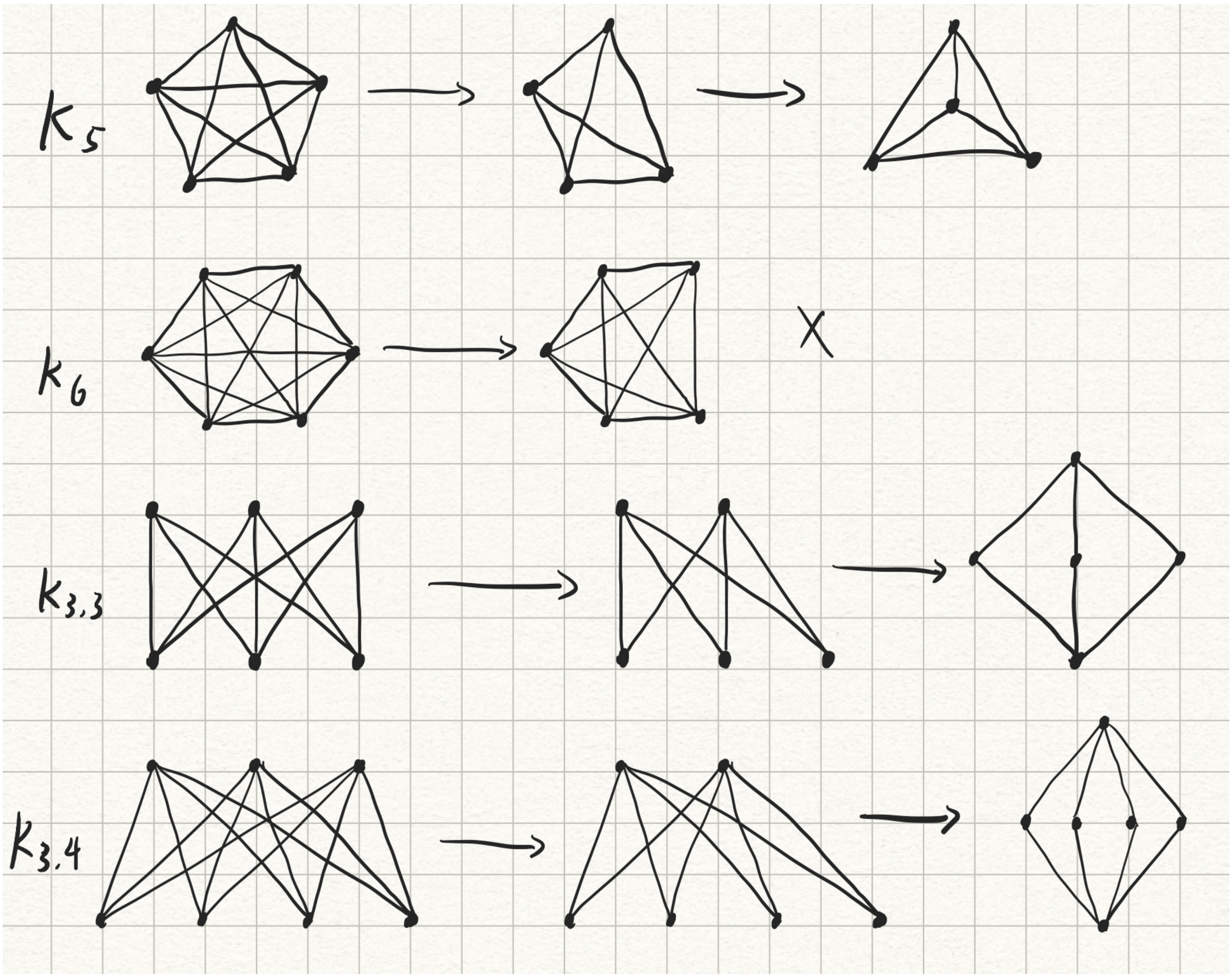
vertices. At last we can build a cycle goes through all other edges and back to itself

Thus we find an Euler circuit .

Therefore we proved a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in- degree and out-degree of each vertex are equal.

10 Q10.

$K_5, K_{3,3}, K_{3,4}$



11 Q11.

When $m = 1 \; or \; n = 1$

Because when $m \neq 1 \wedge n \neq 1$

There must exist a simple circuit between any $u \in M, v \in N$, it's not a tree.

Therefore any complete bipartite graphs $K_{m,n}$ with $m = 1 \; or \; n = 1$ is a tree

12 Q12.

Using mathematical induction to prove

Set $f(n)$ to be n-cube has a Hamilton circuit

Basic step: Obviously Q_1 and Q_2 have a Hamilton path. $f(1)$ and $f(2)$ are true

Inductive step: Assume $f(n)$ is true. Because we can construct Q_{n+1} from Q_n with two Q_n and 2^n new edges. We can go through the one of the Hamilton path in Q_n , then go through any edges that connect two Q_n. Then go through the Hamilton path of another Q_n start from that point. Thus we go through all vertices once, the Hamilton path of Q_{n+1} is constructed.

$f(n+1)$ is true

Therefore we proved that every n-cube has a Hamilton path.

13 Q13.

(1) G is bipartite, H is not

(2) No, vertex h in H has a circuit of length 3, but vertex g in G doesn't have.

(3)Neither of them have an Euler circuit

14 Q14.

Vertex v have 16 edges out. According to Pigeon hole principle, there must be $\lceil 16/3 \rceil = 6$ students are talking about same problem A with v. Consider these 6 students to be K_6.

If there exist a pair students (p, q) in K_6 are talking about A, then there exist (v, p, q) are discussing the same problem A.

If there are no pairwise students in K_6 are talking about A. Choosing a vertex$u$ in K_6. Then according to PHP, there must be $\lceil 5/2 \rceil = 3$ students talking about the same problem with u. If all students in K_3 all pairewise talking about the same question other than they talking with u, then we find that 3 students. Others, their exist other 2 students pairwise talking the same problem with u.

$Q.E.D$

15 Q15

(a) 3

(b) 16

(c) 4

(d) 5

16 Q16

(a) 1

(b) 2

(c) 3