CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment #1
Due: 23:59 on Mar. 9th, 2022, please submit through Sakai

Q.1 (5 points) Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You do not get 100 marks on the final.

(b) You get 100 marks on the final, but you do not get an A in this course.

(c) You will get an A in this course if you get 100 marks on the final.

(d) If you do not get 100 marks on the final, then you will not get an A in this course.

(e) Getting 100 marks on the final is sufficient for getting an A in this course.

(f) You get an A in this course, but you do not get 100 marks on the final.

(g) Whenever you get an A in this course, you got 100 marks on the final.

Q.2 (5 points) Construct a truth table for each of these compound propositions.

(a) $(p \oplus q) \rightarrow (p \wedge q)$

(b) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(c) $(p \oplus q) \rightarrow (p \oplus \neg q)$

Q.3 (5 points) Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $p \rightarrow q$ and $\neg p \vee q$ (This is the Useful Law)

(b) $p \oplus q$ and $\neg p \vee \neg q$

(c) $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$

(d) $(\neg q \wedge \neg(p \rightarrow q))$ and $\neg p$

(e) $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$

Q.4 (10 points) Use logical equivalences to prove the following statements.

$r \rightarrow p = T$
$r \rightarrow q = F$ $r = T$ $q = F$

(a) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

$p : T$

(b) $\neg(p \rightarrow q) \rightarrow \neg q$ is a tautology.

$q : F$

(c) $(p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$ is a tautology.

$r = T$

$p\bar{q} + q\bar{r} + r\bar{p} = (p\bar{q} + q\bar{r} + r)(p\bar{q} + q\bar{r} + \bar{p})$

$= (p\bar{q}+q+r)$

Q.5 (5 points) Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true, when p, q, and r have the same truth value and it is false otherwise.

Q.6 (10 points) Determine whether or not the following two are logically equivalent, and explain your answer.

(a) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(b) $\neg p \to (q \to r)$ and $q \to (p \vee r)$

(c) $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$.

(d) $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$.

Q.7 (5 points) Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret." Express each of these sentences in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Q.8 (5 points) Let $F(x, y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express each of these statement.

(a) Everybody can fool Fred.

(b) Evelyn can fool everybody.

(c) There is no one who can fool everybody.

(d) Nancy can fool exactly two people.

(e) There is exactly one person whom everybody can fool.

(f) There is someone who can fool exactly one person besides himself or herself.

Q.9 (5 points) Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

(b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

(c) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

(d) $\forall x \exists y \forall z T(x, y, z)$

Q.10 (5 points) Prove that if $p \wedge q$, $p \rightarrow \neg(q \wedge r)$, $s \rightarrow r$, then $\neg s$.

Q.11 (10 points) For each of these arguments, explain which rules of inference are used for each step.

(a) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

(b) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

(c) "Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Q.12 (5 points)

(a) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Q.13 (5 points) Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.

Q.14 (5 points) Prove or disprove that if a and b are rational numbers, then a^b is also rational.

Q.15 (5 points) Prove that $\sqrt[3]{2}$ is irrational.

Q.16 (5 points) Suppose that we have a theorem: "$\sqrt{n}$ is irrational whenever n is a positive integer that is *not* a perfect square." Use this theorem to prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Q.17 (5 points) Please read the following description carefully and answer questions. We consider a single object sealed-bid second-price auction (to be explained). In this auction, an auctioneer is responsible for selling a product, and bidders bid for the product. The winner of the auction wins the product and pays for it. The detailed settings are as follows:

- There is one product to be sold.
- There are N bidders, denoted by $\mathcal{N} = \{1, 2, ..., N\}$. Bidder $n \in \mathcal{N}$ has a valuation over the product of v_n.
- Every bidder submits his or her bid in a sealed envelope, so other bidders do not know his or her bid. Bidder $n \in \mathcal{N}$ submits a bid of b_n.

- After receiving the bids from all bidders, the auctioneer announces the winner and payment. The winner is the bidder who submits the highest bid. The payment of this winner is the second highest bid. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 4$, $b_3 = 5$. Then, the winner is bidder 3, and the payment is the second highest bid 4.

- If multiple bidders have the same bid, then they draw a lottery. Each of them has equally probability of winning. In this case, the payment is equal to their bids. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 5$, $b_3 = 5$. The winner is either 2 or 3 with equal probability. The payment is 5.

- After the auction, the payoffs of the bidders are as follows:

  - If bidder n loses, his or her payoff is zero.
  - If bidder n wins, his or her payoff is equal to its valuation v_n minus the payment.

  For bidder n, the higher payoff, the better.

Now, suppose you are a bidder in this auction, e.g., bidder n, and you do not know any other bidders' valuations and bids. You know your valuation v_n. You can choose your bid b_n to maximize your payoff. Prove that for an arbitrary bidder $n \in \mathcal{N}$, submitting a bid $b_n = v_n$ will always lead to a payoff that is no smaller than submitting a bid with $b_n \neq v_n$.

(Note: This second-price auction is commonly used, due to the property that bidders are willing to submit their valuation as their bid.)

(Hint: Use proof by cases; consider the highest bid of the others, and compare it with your valuation v_n; enumerate all possibilities.)