CSE16: Homework 9B

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1. a) $10^8 = 100,000,000$
   b) $(10^7)(1^1) = 10,000,000$
   c) $|\{0,2,4,6,8\}| = 5$, so $(10^6)(5^2) = 25,000,000$
   d) $|\{0,2,4,6,8\}| = 5$ and $|\{1,3,5,7,9\}| = 5$,
   so only odd OR only even ending in 0 is $(5^8) + (5^7)(1^1) = 468,750$
   e) There is a bijection between the set of length-8 strings which are palindromes and the set of length-4 strings. Thus, the number of palindromes $= 10^4 = 10,000$
   f) We'll count the number of strings that start with an even digit and the number of strings that end with two even digits, and then subtract the number of strings for which both are true:
   $(5^1 * 10^7) + (10^6 * 5^2) - (10^5 * 5^2 * 5^1) = 62,500,000$
   g) We'll count the total number of length-8 strings minus the number of strings that start with 000:
   $(10^8) - (1^3 * 10^5) = 99,900,000$
   h) The number of ways to place three 0s in a length-8 string is $\binom{8}{3}$. The number of ways to choose from the remaining five non-zero digits is 9^5. Thus the number of length-8 strings with exactly three 0s is $\binom{8}{3}(9^5) = \frac{8!}{5!3!}9^5 = 56 * 9^5 = 3,306,744$
   i) The number of length-8 strings with three 0s, two 1s, and three 2s is equivalent to $\binom{8}{3}\binom{5}{2}\binom{3}{3} = \frac{8!}{5!3!}\frac{5!}{3!2!}\frac{3!}{3!0!} = 560$.
   j) $10P8 = \frac{10!}{2!} = 1,814,400$.
   k) The number of ways to select four distinct even digits is $5 * 4 * 3 * 2$. The remaining four digits can be anything from 0-9, so 10^4. Thus $10^4 * 5 * 4 * 3 * 2 = 1,200,000$
   l) We have to evaluate the number of possible combinations with n distinct even integers for each case in $0 \leq n \leq 5$. First, we pick n even digits from the 5 available, then we pick n locations from the set of 8 spaces, then we fill the remaining $(8 - n)$ digits with odd digits. Thus the total number of length-8 strings with no even digits that appear more than once is $\sum_{n=0}^{5}\binom{5}{n}\binom{8}{n}5^{8-n} = \sum_{n=0}^{5}\frac{5!}{(5-n)!n!}\frac{8!}{(8-n)!n!}5^{8-n} = 9,866,375$.
   m) We will count a) the number of all length-8 strings with no digits that appear more than once and that contain both 1 and 0, and subtract b) the number of length-8 strings with no digits that appear more than once and where both 1 and 0 appear adjacent to each other:
   a) (# of ways to place 1 and 0 into 8 digits)(# of ways to fill the remaining 6 digits with unique numbers all ≥ 2) $= \binom{8}{2}(8P6) = 28 * \frac{8!}{2!} = 564,480$.
   b) (# of ways to place the digits 1,0 and the digits 0,1 adjacently into a length-8 string)(# of ways to fill the remaining 6 digits with unique numbers all ≥ 2) $= (7+7)(8P6) = 14 * \frac{8!}{2!} = 282240$.
   Final answer: $564480 - 282240 = 282,240$.

n) There is a bijection between the number of ways to pick 8 digits from 0-9 and the number of ways to arrange 8 digits from 0-9 in increasing order. In other words, once 8 digits are picked, there is only one way to arrange them in increasing order. So $\binom{10}{8} = \frac{10!}{2!8!} = 5 * 9 = 45$.

o) This is equivalent to the number of ways to distribute 8 identical ojects (the spaces in our length-8 string) into 10 distrinct bins (the integers from 0-9). Every possible combination has a nondecreasing arrangement. So, $\binom{n+k-1}{n} = \binom{17}{8} = \frac{17!}{9!8!} = 24,310$.

2. a) $|P(\{0,1,2,3,4,5,6,7,8,9\})| = 2^{10} = 1024$

   b) $|P(\{0,1,2,3,4,5,6,7,8,9\})| - |P(\{1,2,3,4,5,6,7,8,9\})| = 2^{10} - 2^9 = 2^9 = 512$

   c) $|P(\{0,1,2,3,4,5,6,7,8,9\})| - |P(\{2,3,4,5,6,7,8,9\})| = 2^{10} - 2^8 = 3 * 2^8 = 768$

   d) $|P(\{0,2,4,6,8\})| = 2^5 = 32$

   e) Using the inclusion-exclusion principle for three sets:
   $|P(\{1,2,3,4,5,6,7,8,9\})| + |P(\{0,2,3,4,5,6,7,8,9\})| + |P(\{0,1,3,4,5,6,7,8,9\})|$
   $- |P(\{2,3,4,5,6,7,8,9\})| - |P(\{0,3,4,5,6,7,8,9\})| - |P(\{1,3,4,5,6,7,8,9\})|$
   $+ |P(\{3,4,5,6,7,8,9\})| = 3 * 2^9 - 3 * 2^8 + 1 * 2^7 = 896$

   f) $\binom{10}{5} = \frac{10!}{(10-5)!5!} = 252$

   g) $\binom{9}{4} = \frac{9!}{(9-4)!4!} = 126$

   h) We will find the number of subsets of cardinality 5 and subtract the number of subsets of cardinality 5 that contain $\{0,1\}$ as a subset: $\binom{10}{5} - \binom{8}{3} = 252 - \frac{8!}{5!3!} = 252 - 56 = 196$

   i) There's 5 even and 5 odd numbers to choose from. So, $\binom{5}{3}\binom{5}{2} = \frac{5!}{2!3!}\frac{5!}{3!2!} = 10 * 10 = 100$.

3. a) n indistinguishable balls in m distinguishable bins $\Rightarrow \binom{n+m-1}{m-1}$. So,
   $\binom{20+4-1}{4-1} = \binom{23}{3} = \frac{23!}{20!3!} = 1,771$

   b) Start by placing 3 balls in bin 1. We have $20 - 3 = 17$ left to distribute however we want. So,
   $\binom{17+4-1}{4-1} = \binom{20}{3} = \frac{20!}{17!3!} = 1140$.

   c) The number of solutions that have $x_1 \leq 10$ is equivalent to the total number of solutions minus the number of solutions that have $x_1 \geq 11$. So, using our answer for a) and the same method as in b), $1771 - \binom{9+4-1}{4-1} = 1771 - \binom{12}{3} = 1771 - \frac{12!}{9!3!} = 1771 - 220 = 1551$.

   d) Equivalent to the number of solutions that have $x_1 \geq 3$ minus the number of solutions that have $x_1 > 10$. So, $1140 - 220 = 920$.

   e) Equivalent to the number of solutions minus the number of solutions that have $x_1 \geq 11$ minus the number of solutions that have $x_2 \geq 6$ plus thus number of solutions for which both are true. So, using the same logic as in b) (with 9 balls left to distribute, then 14, then 3),
   $1771 - \binom{9+4-1}{4-1} - \binom{14+4-1}{4-1} + \binom{3+4-1}{4-1} = 1771 - \frac{12!}{9!3!} - \frac{17!}{14!3!} + \frac{6!}{3!3!} = 891$.

   f) If $x_1 + x_2 = 10$, then $x_3 + x_4 = 10$ too. So, the total number of solutions that have $x_1 + x_2 = 10$ is the same as (10 indistinguishable balls into 2 distinguishable bins)$^2 = \binom{10+2-1}{2-1}^2 = \left(\frac{11!}{10!1!}\right)^2 = 11^2 = 121$

   g) None: there are more than $4 * 3 = 12$ balls to distribute, so at least one bin must have > 3 balls.

   h) Equivalent to the total number of solutions minus the number of solutions where $x_1 + x_2 \geq 11$. The number of ways to distribute 11 indistinguishable balls into 2 distinguishable bins is $\binom{11+2-1}{2-1} = \frac{12!}{11!1!} = 12$. The number of ways to distribute the remaining 9 balls however we want into the four bins is $\binom{9+4-1}{4-1} = \frac{12!}{9!3!} = 220$. So, $1771 - 12 * 220$ = −869 (Unfortunately Im getting a negative number here? Not sure why.)

4. a) Equivalent to 12 indistinguishable balls in 10 distinguishable bins:
   $\binom{12+10-1}{10-1} = \frac{21!}{12!9!} = 293,930$.

b) At most one indistinguishable ball per distinguishable bin $\Rightarrow \binom{10}{6} = \frac{10!}{4!6!} = 210.$

c) This question is illegible, I cannot understand what it's asking.

d) (4 indistinguishable balls into 10 bins)(8 indistinguishable balls into 10 bins) = $\binom{4+10-1}{10-1}\binom{8+10-1}{10-1} = \frac{13!}{4!9!}\frac{17!}{8!9!} = 17,381,650.$

e) (10 choose 4)(8 indistinguishable balls into 10 bins) = $\binom{10}{4}\binom{8+10-1}{10-1} = \frac{10!}{6!4!}\frac{17!}{8!9!} = 5,105,100.$

f) There are six possible outcomes for each plant: $|\{0, 2, 2\}| * 2 = 6$. Since there are 10 plants. the total possible ways to fertilize them is $6^{10} = 60,466,176.$

Questions I wasn't able to get correct: 1l, 1m, 3h (and 4c was illegible).