Astronomy 4303 Homework 2

Your Name

Due 10/2

Problem 1

In this problem you will construct a code to track the movements of a small cluster of 5 stars. We will use distance units in AU and time units in years throughout this problem.

(a)

You will have to determine a timestep to move your code along. Calculate the dynamical time ($t_{\rm dyn}$) associated with a cluster of stars where the cluster has radius r and contains n stars, each of mass M. When you actually need to find $t_{\rm dyn}$ you are going to have to decide what r is for that case. We will all likely define that slightly differently, but the results will all have the same answer in absolute time units if the codes are done correctly.

Solution:

We can use the virial theorem to estimate the time it would take for a star to travel across the radius of the cluster, which we define as the dynamical time.

Each star has mass M_* where the total mass $M_{\rm tot} = nM_*$.

The virial theorem relates the kinetic and potential energies of the system:

$$2K + U = 0 \tag{1}$$
$$K = -\frac{1}{2}U \tag{2}$$

The gravitational potential energy for a sphere of uniform density is:

$$U = -\frac{3}{5}\frac{GM_r^2}{R}$$

Where M_r is the mass enclosed at radius R. We take the radius as the cluster radius, so that all our stars are within this radius. To apply this to code, I'll use the average radius as it is only an approximation.

$$K = -\frac{U}{2} = \frac{3}{10}\frac{GM_{tot}^2}{R} \tag{3}$$

$$K = \frac{1}{2}M_{tot}\langle v^2\rangle \tag{4}$$

$$\frac{1}{2}M_{tot}\langle v^2\rangle = \frac{3}{10}\frac{GM_{tot}^2}{R} \tag{5}$$

$$\langle v^2\rangle = \frac{3}{10}\frac{GnM}{R} \tag{6}$$

$$v_{\text{rms}} = \sqrt{\frac{3}{10}\frac{GnM}{R}} \tag{7}$$

The dynamical time can then be calculated by:

$$t_{\text{dyn}} = \frac{R}{v} = \sqrt{\frac{10R^3}{3nGM}} \approx \sqrt{\frac{R^3}{nGM}} \tag{8}$$

This is equivalent to the expression derived in class:

$$t_{\text{dyn}} = \sqrt{\frac{R^3}{GM_{\text{tot}}}} \tag{9}$$

Where $M_{\text{tot}} = nM$.

(b)

Rather than begin with the full problem, let's get a code to work on the orbits of two solar mass stars. Take the stars to have positions $(0, 10, 0)$ and

$(0, -10, 0)$ with velocities $(-V_o, 0, 0)$ and $(V_o, 0, 0)$ respectively. Hence, the motion of the components of this binary ought to be solely in the XY plane. For these units, derive what V_o needs to be for these stars to travel in exactly circular orbits.

Solution:

The gravitational force given by the two stars is

$$\vec{F}_{1,2} = -\frac{GM_1M_2}{r_{1,2}^2}\hat{r_{1,2}} \tag{10}$$

With $r = 10AU$, $M = M_\odot$, and using acceleration of circular orbits

$$F_c = \frac{Mv^2}{r} = \frac{GM^2}{(2r)^2} \tag{11}$$

$$v = \sqrt{\frac{GM}{4r}} \tag{12}$$

Then using the velocity to get the period T

$$T = \frac{2\pi r}{v}$$

so $v_0 = .99346AU/yr$ and $T = 63.25yr$

(c)

Write a code that uses the gravitational force to move the stars around. Experiment with the timestep to see what fraction of the dynamical time it needs to be in order to keep the stars in circular orbits around their center of mass. Run the code for at least 10 orbits to make sure that the code is stable. Show a plot of the XY positions of both stars. Now decrease the velocity and show the XY plot. Does the code behave as you expect?

Solution:

I found that the timestep needs to be around .1 or $t_{dyn}/200$ to keep the error reasonable low and on clean circular orbits.

The code behaves as expected, with my calculated velocity producing circular orbits. Reducing the speed places the 2 stars on elliptical orbits, bringing the 2 stars closer.

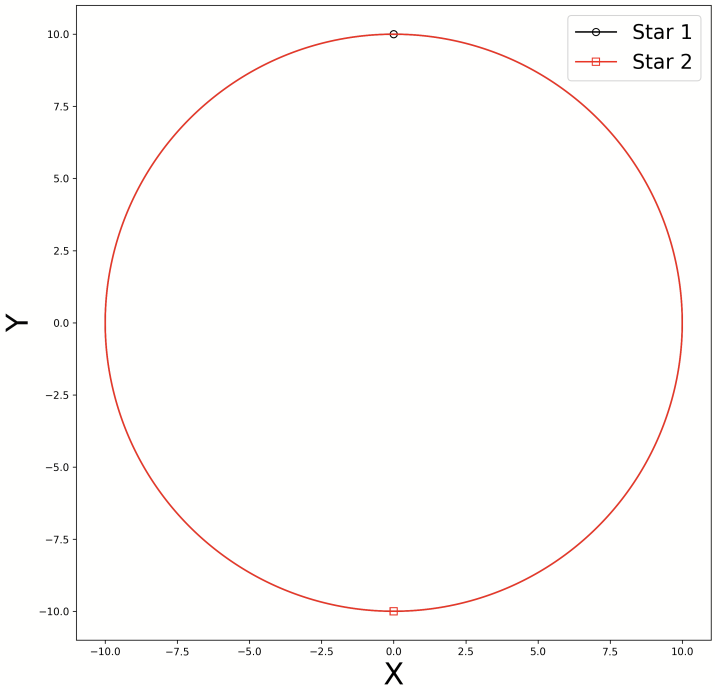

Figure 1: Circular orbit

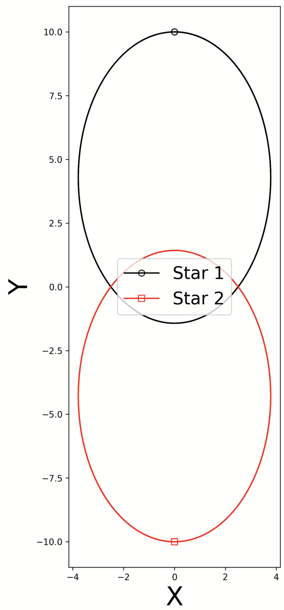

Figure 2: Orbit with $V_0/2$

(d)

Once you are confident that your code works with two stars, you are ready to put in a cluster. With velocities in AU/yr and distances in AU, I've given you the initial positions and velocities of 5 stars. Each star is one solar mass. The origin of the coordinate system is set to be the center of mass of the system and the cluster as a whole is stationary with respect to the origin. N-body codes can be tricky: upon close approaches, timesteps must become smaller to actually track the orbits. However, doing so all the time can bog down your code. You will have to decide upon some scheme to choose a reasonable timestep. Describe and justify your scheme.

Solution:

I am using a relatively simple numerical integrator, the euler-cromer method, the integrates each stars trajectory using:

To do an adaptive time step, I take steps from $t + \Delta t$ at different intervals, with timesteps Δt and $\Delta t/2$ If they agree within a tolerance parameter, then I continue on the next step. If they don't, I reduce the size of Δt for each step until they do, saving the outputs only at the original Δt. There is a tricky interaction at around 4 years which is particulary hard on this code. This integration method is not particularly efficient, so I set the tolerance

Algorithm 1 Euler-Cromer Method

```
for n = 0 to N − 1 do
    a_n = f(t_n, x_n, v_n)          ▷ Calculate acceleration
    v_{n+1} = v_n + a_n Δt          ▷ Update velocity
    x_{n+1} = x_n + v_{n+1} Δt      ▷ Update position
    t_{n+1} = t_n + Δt              ▷ Update time
end for
```

around 1×10^{-7}. For longer simulations, a more accurate method should to be used, such as Runge Kutta.

(e)

Run your code for three dynamical times and plot the positions of the stars. Be sure that the trajectories look right- no sudden jumps, gravity pulls stars together etc. For plotting we will all use the same conventions. Plot the positions of the stars on two side-by-side graphs: one of Y vs X and the other of Z vs X. At the beginning of the time interval, plot a symbol for the star and follow the trajectory with a colored line. Conventions: Star 1 is an open circle with a black line, star 2 is an open square with a red line, star 3 is a filled triangle with a blue line, star 4 is an open triangle with a purple line and star 5 is a filled square with a dashed black line.

You need to provide both your code and the answers to each part in order to receive full credit.

I plot my orbit in Figure 3, and my code can be found here:

`https://github.com/Bvogel4/730_repo/tree/master/HW_2`

After the tricky interaction, energy is reasonable conserved, staying within an error < .001.

Solution:

Problem 2

The nuclear timescale.

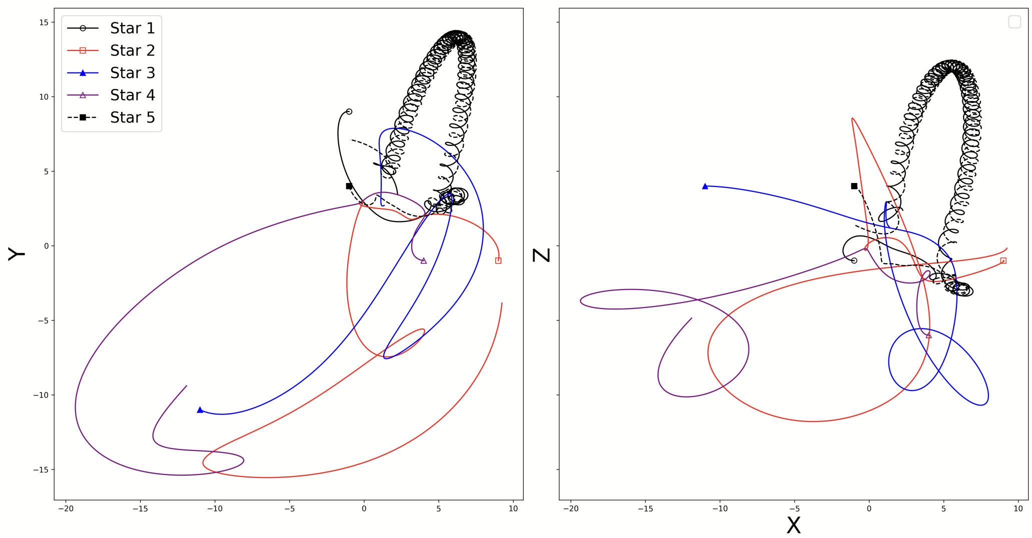

Figure 3: Chaotic orbits for 5 stars

(a)

Calculate the total mass of hydrogen available for fusion over the lifetime of the sun, if 70 percent of its mass was hydrogen when the Sun formed and only 13 percent of all the hydrogen is in the layers where the temperature is high enough for fusion.

Solution: The total mass of hydrogen available for fusion can be calculated as follows:

$$
\begin{aligned}
H_{\text{nuc}} &= M_{\odot} \times \text{initial hydrogen fraction} \times \text{fraction in fusion layers} &(13)\\
&= M_{\odot} \times 0.7 \times 0.13 &(14)\\
&= 0.091 M_{\odot} &(15)
\end{aligned}
$$

(b)

Derive the expression for nuclear time scale in solar units (i.e. in terms of $R/R_{\odot}$ etc).

Solution:

The nuclear timescale is derived by calculating the total energy available from fusion and dividing it by the luminosity:

1. Total energy available from fusion:

$$E_{\text{nuc}} = \phi H_{\text{nuc}} c^2 \tag{16}$$

where ϕ is how much mass is converted into energy (.007) for hydrogen

2. Nuclear timescale in cgs units:

$$t_{\text{nuc}} = \frac{E_{\text{nuc}}}{L} = 0.091\phi c^2 \frac{M}{L} \frac{\text{cm}^2\ \text{s}^{-2}\text{g}}{\text{erg s}^{-1}} \tag{17}$$

3. Convert to solar units and years:

$$\begin{aligned} t_{\text{nuc}} &= 0.091\phi c^2 \frac{M}{L} \frac{L_\odot}{L_\odot} \frac{M_\odot}{M_\odot} \times \frac{1}{365 \times 24 \times 60^2} && (18) \\ &= 0.091\phi c^2 \frac{M/M_\odot}{L/L_\odot} \frac{M_\odot}{L_\odot} && (19) \\ &= 8.96 \times 10^9 \frac{M/M_\odot}{L/L_\odot} \text{ years} && (20) \end{aligned}$$

Where $M_\odot = 1.89 \times 10^{33} g,\ L_\odot = 3.846 \times 10^{33} ergs/s$

Thus, the final expression for the nuclear timescale in solar units is:

$$t_{\text{nuc}} = 8.96 \times 10^9 \left(\frac{M/M_\odot}{L/L_\odot} \right) \text{ years} \tag{21}$$

(c)

Describe in your own words the meaning of the nuclear timescale.

Solution:

This is the approximate timescale it takes for a star to fuse all the available mass of hydrogen, though it could also describe other elements like helium, with modifications. This time describes how long a star will remain on the main sequence.

(d)

Calculate the nuclear timescale for a 10 solar mass main-sequence star and a 25 solar mass main-sequence star. What does this tell you about main sequence lifetime as a function of mass?

Solution:

using the mass luminosity relationship from pols

$$L \propto M^{3.8}$$

and substituting in for mass:

$$t_{\text{nuc}} = 8.96 \times 10^9 \left(\frac{M_\odot}{M}\right)^{2.8} \text{ years} \tag{22}$$

For a 10 solar mass star and 25 solar mass star, the nuclear timescales are 1.4×10^7 and 1.07×10^6 years.

Nuclear timescales get shorter for higher mass stars $T_{nuc} \propto M^{-2.8}$. In other words, higher mass stars burn through their fuel faster and spend less time on the main sequence.

Problem 3

Degeneracy in the cores of stars.

(a)

Explain qualitatively why for degenerate matter, the pressure increases with density?

Solution:

As density increases, electrons are increasingly packed into smaller spaces. Increasing the overlap of the system will push the electron into higher energy states due to the Pauli exclusion principle, where these higher energy states have more momentum, thereby increasing the pressure generated by degeneracy.

(b)

Show quantitatively that the core of a massive star (say, 30 solar masses) is more degenerate than the core of the sun.

Solution:

Stars become increasingly degenerate when the de brogile wavelengths starts to be larger than the mean free path.

The de broglie wavelength of electrons, for an ionized gas we can assume they dominate degeneracy, as hydrogen ions are significantly more massive, and thus have much shorter de broglie wavelengths.

$$\lambda_Q = \frac{h}{\sqrt{3m_e k_B T}}$$

and the mean free path

$$l_\gamma = \frac{1}{\kappa\rho}$$

where ρ is the density and κ is opacity coefficient, for hydrogen $= 0.4gcm^{-1}$

taking the ratio of these: we can quantitatively compare how degenerate a 30 solar mass star is compared to the sun.

$$\mathcal{D} = \frac{\lambda_Q}{l_\gamma} = \frac{h}{\kappa\rho\sqrt{3mk_BT}} \tag{23}$$

we can then compare how the sun compares to a 30 solar mass star

$$\mathcal{D}_1/\mathcal{D}_{30} = \frac{\rho_{30}}{\rho}\sqrt{\frac{T_{30}}{T_1}} \tag{24}$$

Now get things in just terms of the mass of the star:

The temperature of a star

$$T = \frac{\mu m_H GM}{3k_BR}$$

and the average density of a star

$$\bar{\rho} = \frac{3M}{4\pi R^3}$$

$$\mathcal{D}_1/\mathcal{D}_{30} = \frac{M_1}{M_{30}}\frac{R_{30}^3}{R_1^3}\sqrt{\frac{M_{30}}{M_1}\frac{R_1}{R_{30}}} = \sqrt{\frac{M_1}{M_{30}}\frac{R_{30}^5}{R_1^5}} \tag{25}$$

From empirical relations in pols we also know that

$$R \propto M^{0.7}$$

$$\mathcal{D}_1/\mathcal{D}_{30} = \sqrt{\frac{M_1}{M_{30}}\left(\frac{M_{30}}{M_1}\right)^{3.5}} = \left(\frac{M_{30}}{M_1}\right)^{1.25} = 30^{1.25} \approx 70$$

Therefore, a star with $30M_\odot$ is about 70 times as degenerate as the sun.

I have made a lot of simplifying assumptions, averaging temperature and density across the entire star, as well as using an approximate form of temperature that scales with mass. These assumptions break down for increasing degeneracy, and only provide an approximation of the ratio of degeneracy when assuming partial degeneracy regime. This also compares the degeneracy of the entire star, which I believe will translate to the core of each star, as averaging the densities over the stars scales will translate to the density in the core, as long as both stars follow a similar relation to density, like the one presented in homework 1:

$$\rho = \rho_c[1 - (r/R)^2]$$

Given the simplifying assumptions I have made, I think this is a reasonable approximation.