# 时间序列分析 第四次作业

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## 第一题

**模拟一个 MA(2)模型**

### *(a) 计算极大似然估计*

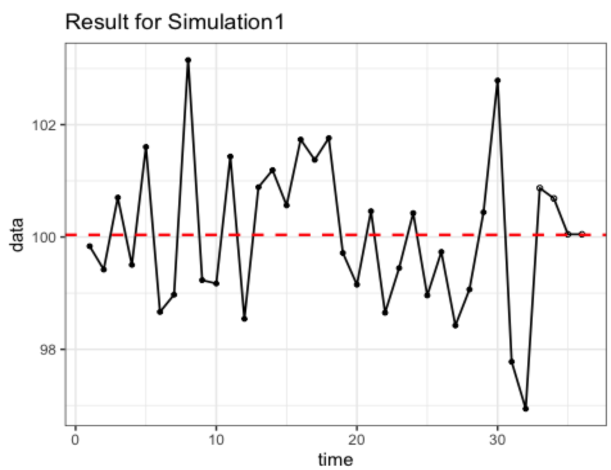
```
rm(list=ls())
# generate random MA(2) data
set.seed(10)
ma_order <- 2
theta_1 <- 1
theta_2 <- -0.6
mu <- 100
offset <- 100
len <- 36
dat_raw <- arima.sim(n = len+offset,
                     list(ma = c(theta_1,theta_2)),
                     innov = rnorm(len+offset))+mu
ar2dat <- ts(dat_raw[(offset+1):(offset+len)])
indat <- ts(ar2dat[1:32])
outdat <- ts(ar2dat[33:36],start=33)
# MLE
ar.mle <- TSA::arima(indat, order=c(0,0,2),method='ML',include.mean = T
RUE)
ar.mle

##
## Call:
## TSA::arima(x = indat, order = c(0, 0, 2), include.mean = TRUE, metho
d = "ML")
##
## Coefficients:
##          ma1      ma2  intercept
##       0.1741  -0.3925   100.0482
## s.e.  0.1834   0.1996     0.1917
##
## sigma^2 estimated as 1.783:  log likelihood = -54.87,  aic = 115.74
```

通过上述输出可以看到，极大似然估计的结果分别为 0.1741,-0.3925,100.0482

### *(b) 预测并绘图*

```
ar.mle.pred <- forecast(indat,model=ar.mle,h=4)
pred <- data.frame(dat=as.matrix(ar.mle.pred$mean))
```

```
indat.df <- data.frame(dat=as.matrix(indat))
mix <- rbind(indat.df,pred)
ggplot()+
  geom_point(data=indat.df,mapping = aes(y=dat,x=c(1:32)),
             size=1)+
  theme_bw()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(33:36)),
             shape=21,size=1.2)+
  geom_line(data=mix, mapping=aes(y=dat,x=c(1:36)),
            linewidth=0.6)+
  labs(x = "time", y = "data")+
  geom_hline(yintercept=mean(mix$dat),
             color='red',linetype="dashed",linewidth=0.8)+
  ggtitle("Result for Simulation1")
```




```
print(ar.mle.pred$mean)

## Time Series:
## Start = 33
## End = 36
## Frequency = 1
## [1] 100.8712 100.6876 100.0482 100.0482
```

预测值以及绘制的时间序列图如上所示。

### *(c) 第三、四个预测值有什么特别之处？*

第三四个预测值都是相同的，并且都基本落在了数据的均值水平线上。

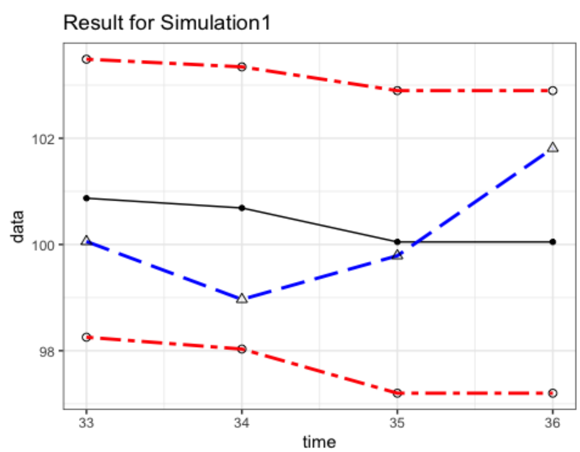
### *(d) 将最后四个预测值与真实值进行比较，计算 MSE 与 MAE*

```
accuracy(ar.mle.pred$mean,outdat)

##                   ME     RMSE      MAE        MPE     MAPE      ACF1
Theil's U
## Test set -0.2584472 1.304165 1.139847 -0.2708068 1.136528 0.1244757
 1.013609
```

MSE 与 MAE 的输出如上所示，分别为 1.304165，1.139847

*(e) 画出预测区间，真实值是否在该区间内？*

```
low <- data.frame(dat=ar.mle.pred$lower[,2])
high <- data.frame(dat=ar.mle.pred$upper[,2])
out <- data.frame(dat=outdat)
ggplot()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(33:36)),
             shape=20,size=2)+
  geom_line(data=pred,mapping=aes(y=dat,x=c(33:36)),
            linetype="solid")+
  geom_point(data=low,mapping=aes(y=dat,x=c(33:36)),
             shape=1,size=2)+
  geom_line(data=low,mapping=aes(y=dat,x=c(33:36)),
            linetype="twodash",linewidth=1,color='red')+
  geom_point(data=high,mapping=aes(y=dat,x=c(33:36)),
             shape=1,size=2)+
  geom_line(data=high,mapping=aes(y=dat,x=c(33:36)),
            linetype="twodash",linewidth=1,color='red')+
  geom_line(data=out,mapping=aes(y=dat,x=c(33:36)),
            linetype="longdash",linewidth=1,color='blue')+
  geom_point(data=out,mapping=aes(y=dat,x=c(33:36)),
             shape=2,size=2)+
  labs(x = "time", y = "data")+
  theme_bw()+
  ggtitle("Result for Simulation1")
```

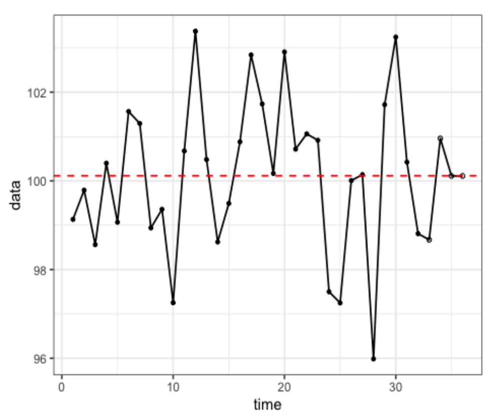



结果如上图所示，红色点线区间为 95%的预测区间，黑色直线为预测值，蓝色长虚线为真实值。由此可见，真实值处在 95%的预测区间内。

*(f) 更改随机数种子，重复该过程*

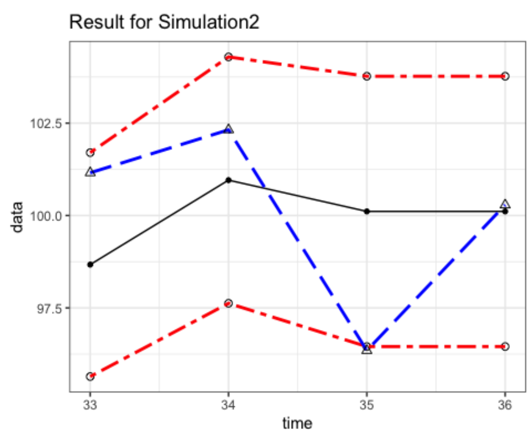
```
rm(list=ls())
# generate random MA(2) data
set.seed(12345)
ma_order <- 2
theta_1 <- 1
```

```
theta_2 <- -0.6
mu <- 100
offset <- 100
len <- 36
dat_raw <- arima.sim(n = len+offset,
                     list(ma = c(theta_1,theta_2)),
                     innov = rnorm(len+offset))+mu
ar2dat <- ts(dat_raw[(offset+1):(offset+len)])
indat <- ts(ar2dat[1:32])
outdat <- ts(ar2dat[33:36],start=33)
# MLE
ar.mle <- TSA::arima(indat, order=c(0,0,2),method='ML',include.mean = T
RUE)
print(ar.mle)

#predict
ar.mle.pred <- forecast(indat,model=ar.mle,h=4)
pred <- data.frame(dat=as.matrix(ar.mle.pred$mean))
indat.df <- data.frame(dat=as.matrix(indat))
mix <- rbind(indat.df,pred)
ggplot()+
  geom_point(data=indat.df,mapping = aes(y=dat,x=c(1:32)),
             size=1)+
  theme_bw()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(33:36)),
             shape=21,size=1.2)+
  geom_line(data=mix, mapping=aes(y=dat,x=c(1:36)),
            linewidth=0.6)+
  labs(x = "time", y = "data")+
  geom_hline(yintercept=mean(mix$dat),
             color='red',linetype="dashed",linewidth=0.6)
```




```
print(ar.mle.pred$mean)

print(accuracy(ar.mle.pred$mean,outdat))

low <- data.frame(dat=ar.mle.pred$lower[,2])
```

```r
high <- data.frame(dat=ar.mle.pred$upper[,2])
out <- data.frame(dat=outdat)
ggplot()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(33:36)),
             shape=20,size=2)+
  geom_line(data=pred,mapping=aes(y=dat,x=c(33:36)),
            linetype="solid")+
  geom_point(data=low,mapping=aes(y=dat,x=c(33:36)),
             shape=1,size=2)+
  geom_line(data=low,mapping=aes(y=dat,x=c(33:36)),
            linetype="twodash",linewidth=1,color='red')+
  geom_point(data=high,mapping=aes(y=dat,x=c(33:36)),
             shape=1,size=2)+
  geom_line(data=high,mapping=aes(y=dat,x=c(33:36)),
            linetype="twodash",linewidth=1,color='red')+
  geom_line(data=out,mapping=aes(y=dat,x=c(33:36)),
            linetype="longdash",linewidth=1,color='blue')+
  geom_point(data=out,mapping=aes(y=dat,x=c(33:36)),
             shape=2,size=2)+
  labs(x = "time", y = "data")+
  theme_bw()+
  ggtitle("Result for Simulation2")
```




```
##
## Call:
## TSA::arima(x = indat, order = c(0, 0, 2), include.mean = TRUE, metho
d = "ML")
##
## Coefficients:
##          ma1      ma2  intercept
##       0.4888  -0.5112   100.1114
## s.e.  0.1971   0.1673     0.2697
##
## sigma^2 estimated as 2.321:  log likelihood = -60.37,  aic = 126.74
## Time Series:
## Start = 33
## End = 36
```

```
## Frequency = 1
## [1]  98.67206 100.95828 100.11142 100.11142
##                     ME      RMSE       MAE         MPE      MAPE        ACF1
 Theil's U
## Test set 0.06393199 2.356877 1.945897 0.01332909 1.966639 -0.1013636
 0.5432556
```

本次模拟的具体预测数值及 MSE、MAE 等见上述输出。从图中可以看出真实值稍有偏差 95%的估计区间之外。但第三四次的预测值仍基本落在水平均值上，该规律没有改变。

## 第二题

### 模拟一个 ARIMA(1,1)模型

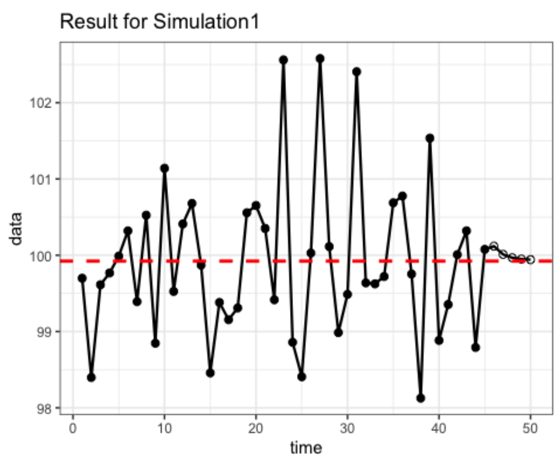
#### *(a) 计算极大似然估计*

```
rm(list=ls())
# generate random MA(2) data
set.seed(100)
offset <- 200
len <- 50
mu <- 100
dat_raw <- arima.sim(n = len+offset,
                     list(ma = c(-0.5),ar=c(0.7)),
                     innov = rnorm(len+offset))+mu
arimadat <- ts(dat_raw[(offset+1):(offset+len)])
indat <- ts(arimadat[1:45])
outdat <- ts(arimadat[46:50],start=46)
# MLE
fit <- TSA::arima(indat, order=c(1,0,1),method='ML',include.mean = TRUE
)
fit

##
## Call:
## TSA::arima(x = indat, order = c(1, 0, 1), include.mean = TRUE, metho
d = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.4065  -0.6848    99.9372
## s.e.  0.3701   0.3094     0.0835
##
## sigma^2 estimated as 0.9496:  log likelihood = -62.77,  aic = 131.54
```

#### *(b) 预测并绘图*

```
mle.pred <- forecast(indat,model=fit,h=5)
pred <- data.frame(dat=as.matrix(mle.pred$mean))
indat.df <- data.frame(dat=as.matrix(indat))
mix <- rbind(indat.df,pred)
```

```
ggplot()+
  geom_point(data=indat.df,mapping = aes(y=dat,x=c(1:45)),
             size=2)+
  theme_bw()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(46:50)),
             shape=21,size=2)+
  geom_line(data=mix, mapping=aes(y=dat,x=c(1:50)),
            size=0.8)+
  labs(x = "time", y = "data")+
  geom_hline(yintercept=mean(mix$dat),
             color='red',linetype="dashed",size=1)+
  ggtitle("Result for Simulation1")

## Warning: Using `size` aesthetic for lines was deprecated in ggplot2
3.4.0.
## ℹ Please use `linewidth` instead.
```



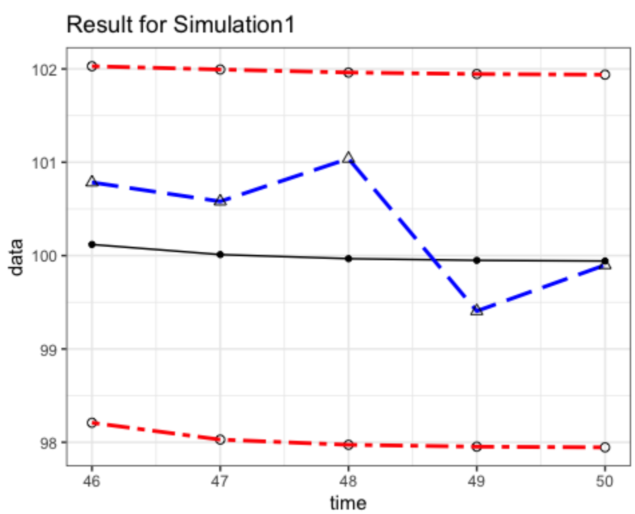

```
print(mle.pred$mean)

## Time Series:
## Start = 46
## End = 50
## Frequency = 1
## [1] 100.11913 100.01114  99.96725  99.94941  99.94216
```

*(c) 将最后四个预测值与真实值进行比较，计算 MSE 与 MAE*

```
accuracy(mle.pred$mean,outdat)

##                 ME      RMSE       MAE       MPE      MAPE        AC
F1
## Test set 0.3438047 0.6652058 0.5786785 0.3392496 0.5754412 -0.040344
75
##          Theil's U
## Test set 0.7496215
```

*(d) 画出预测区间，真实值是否在该区间内？*

```
low <- data.frame(dat=mle.pred$lower[,2])
high <- data.frame(dat=mle.pred$upper[,2])
out <- data.frame(dat=outdat)
ggplot()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(46:50)),
             shape=20,size=2)+
  geom_line(data=pred,mapping=aes(y=dat,x=c(46:50)),
            linetype="solid")+
  geom_point(data=low,mapping=aes(y=dat,x=c(46:50)),
             shape=1,size=2)+
  geom_line(data=low,mapping=aes(y=dat,x=c(46:50)),
            linetype="twodash",size=1,color='red')+
  geom_point(data=high,mapping=aes(y=dat,x=c(46:50)),
             shape=1,size=2)+
  geom_line(data=high,mapping=aes(y=dat,x=c(46:50)),
            linetype="twodash",size=1,color='red')+
  geom_line(data=out,mapping=aes(y=dat,x=c(46:50)),
            linetype="longdash",size=1,color='blue')+
  geom_point(data=out,mapping=aes(y=dat,x=c(46:50)),
             shape=2,size=2)+
  labs(x = "time", y = "data")+
  theme_bw()+
  ggtitle("Result for Simulation1")
```



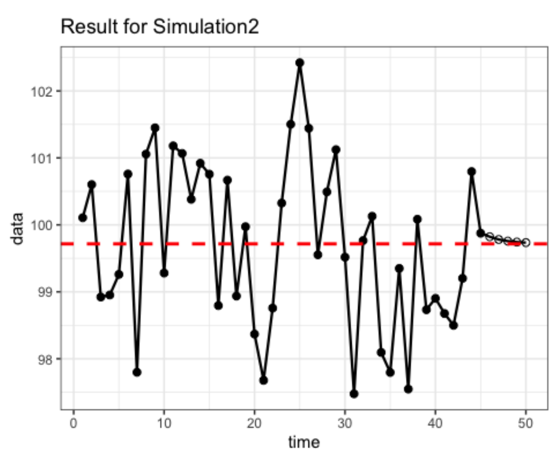

*(e) 更改随机数种子，重复该过程*

```
rm(list=ls())
# generate random MA(2) data
set.seed(999)
offset <- 200
len <- 50
mu <- 100
dat_raw <- arima.sim(n = len+offset,
                     list(ma = c(-0.5),ar=c(0.7)),
```

```
                    innov = rnorm(len+offset))+mu
arimadat <- ts(dat_raw[(offset+1):(offset+len)])
indat <- ts(arimadat[1:45])
outdat <- ts(arimadat[46:50],start=46)
# MLE
fit <- TSA::arima(indat, order=c(1,0,1),method='ML',include.mean = TRUE
)
print(fit)
```

```
##
## Call:
## TSA::arima(x = indat, order = c(1, 0, 1), include.mean = TRUE, metho
d = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.5824  -0.3256    99.7234
## s.e.  0.3155   0.3621     0.2741
##
## sigma^2 estimated as 1.347:  log likelihood = -70.6,  aic = 147.2
```

```
mle.pred <- forecast(indat,model=fit,h=5)
pred <- data.frame(dat=as.matrix(mle.pred$mean))
indat.df <- data.frame(dat=as.matrix(indat))
mix <- rbind(indat.df,pred)
ggplot()+
  geom_point(data=indat.df,mapping = aes(y=dat,x=c(1:45)),size=2)+
  theme_bw()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(46:50)),shape=21,size=2)+
  geom_line(data=mix, mapping=aes(y=dat,x=c(1:50)),size=0.8)+
  labs(x = "time", y = "data")+
  geom_hline(yintercept=mean(mix$dat),color='red',linetype="dashed",siz
e=1)+
  ggtitle("Result for Simulation2")
```



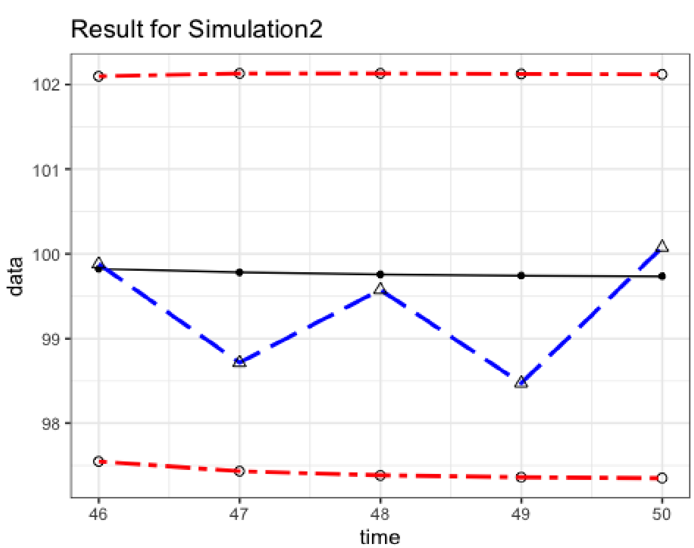

```
print(mle.pred$mean)
```

```
## Time Series:
## Start = 46
## End = 50
## Frequency = 1
## [1] 99.82281 99.78129 99.75711 99.74303 99.73482
```

```
print(accuracy(mle.pred$mean,outdat))
```

```
##                    ME      RMSE       MAE       MPE      MAPE        ACF
1 Theil's U
## Test set -0.423807 0.7618719 0.5833369 -0.430714 0.5901691 -0.657211
1 0.6977817
```

```
low <- data.frame(dat=mle.pred$lower[,2])
high <- data.frame(dat=mle.pred$upper[,2])
out <- data.frame(dat=outdat)
ggplot()+
  geom_point(data=pred,mapping=aes(y=dat,x=c(46:50)),shape=20,size=2)+
  geom_line(data=pred,mapping=aes(y=dat,x=c(46:50)),linetype="solid")+
  geom_point(data=low,mapping=aes(y=dat,x=c(46:50)),shape=1,size=2)+
  geom_line(data=low,mapping=aes(y=dat,x=c(46:50)),linetype="twodash",
            size=1,color='red')+
  geom_point(data=high,mapping=aes(y=dat,x=c(46:50)),shape=1,size=2)+
  geom_line(data=high,mapping=aes(y=dat,x=c(46:50)),linetype="twodash",
            size=1,color='red')+
  geom_line(data=out,mapping=aes(y=dat,x=c(46:50)),linetype="longdash",
            size=1,color='blue')+
  geom_point(data=out,mapping=aes(y=dat,x=c(46:50)),shape=2,size=2)+
  labs(x = "time", y = "data")+
  theme_bw()+
  ggtitle("Result for Simulation2")
```



## 第三题

### *(a) 用 IMA(1,1) 预测未来 3 个时刻的值，计算 95% 区间*

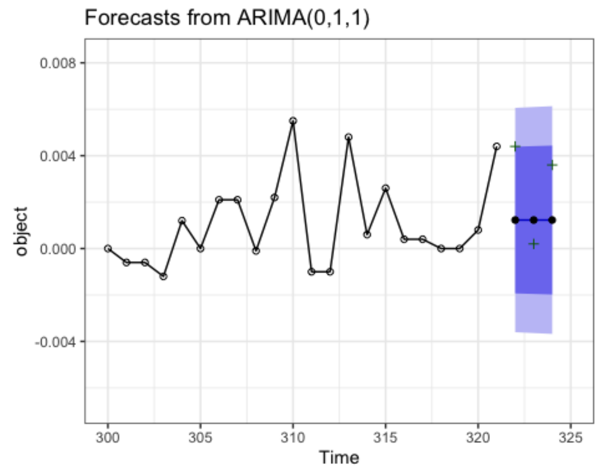
```
rm(list=ls())
data("robot")
robot.inner <- ts(robot[1:321],start=1)
robot.outer <- ts(robot[322:324],start=322)
fit.ima <- TSA::arima(robot.inner,order = c(0,1,1),
                      include.mean = TRUE, method = "ML")
robot.pred <- forecast(model = fit.ima,robot.inner,h=3)
robot.pred$mean
robot.pred$lower[,2]
robot.pred$upper[,2]
```

```
## Time Series:
## Start = 322
## End = 324
## Frequency = 1
## [1] 0.001227982 0.001227982 0.001227982
## Time Series:
## Start = 322
## End = 324
## Frequency = 1
## [1] -0.003602297 -0.003639542 -0.003676504
## Time Series:
## Start = 322
## End = 324
## Frequency = 1
## [1] 0.006058261 0.006095506 0.006132468
```

### *(b) 绘图展示预测值、95% 预测区间、真实值，解释结果*

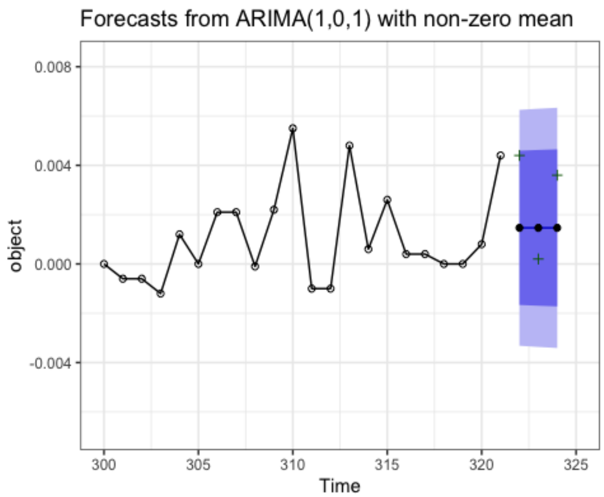
```
autoplot(robot.pred)+
  theme_bw()+
  xlim(c(300,325))+
  geom_point(aes(x=time(robot.pred$mean),y=robot.pred$mean))+
  geom_point(aes(x=time(robot.outer),y=robot.outer),
             shape=3,color='darkgreen')+
  geom_point(aes(x=time(robot.inner),y=robot.inner),
             shape=21)
```

```
## Scale for x is already present.
## Adding another scale for x, which will replace the existing scale.

## Warning: Removed 299 rows containing missing values (`geom_line()`).

## Warning: Removed 299 rows containing missing values (`geom_point()`)
.
```




*(c) 用 ARMA(1,1)再次进行预测并比较两个模型的结果*

```
fit.arima <- TSA::arima(robot.inner, order = c(1,0,1),
                        include.mean = TRUE, method="ML")
robot.pred.arima <- forecast(model = fit.arima, robot.inner, h=3)

robot.pred.arima$mean
robot.pred.arima$lower[1]
robot.pred.arima$upper[1]

autoplot(robot.pred.arima)+
  theme_bw()+
  xlim(c(300,325))+
  geom_point(aes(x=time(robot.pred.arima$mean),y=robot.pred.arima$mean)
)+
  geom_point(aes(x=time(robot.outer),y=robot.outer),
             shape=3,color='darkgreen')+
  geom_point(aes(x=time(robot.inner),y=robot.inner),
             shape=21)

## Scale for x is already present.
## Adding another scale for x, which will replace the existing scale.

## Warning: Removed 299 rows containing missing values (`geom_line()`).

## Warning: Removed 299 rows containing missing values (`geom_point()`)
.
```




```
## Time Series:
## Start = 322
## End = 324
## Frequency = 1
## [1] 0.001465155 0.001464246 0.001463384
## [1] -0.001662404
## [1] 0.004592714
```

## 第四题

*(a) 推导 MA 展示对应的前四个 ψ 权重*

*(b) 计算未来四个季度的预测值及 95%预测区间*

4. (a) MA 展式有如下形式: $z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$. 将其代入季节模型中:

$\Rightarrow \sum_{j=0}^{\infty} \psi_j a_{t-j} = \sum_{j=0}^{\infty} \psi_j a_{t-4-j} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$

$\Rightarrow \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \psi_4 a_{t-4} + \psi_5 a_{t-5} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} + 0 + \psi_0 a_{t-4} + \psi_1 a_{t-5}$

经比对有: $\begin{cases} \psi_0 = 1 \\ \psi_1 = -\theta_1 \\ \psi_2 = -\theta_2 \\ \psi_3 = 0 \\ \psi_4 = \psi_0 = 1 \end{cases}$ $\Rightarrow \forall j \geq 4.\ \psi_j = \psi_{j-4}$

(b) $z_t = z_{t-4} + a_t - \frac{1}{2} a_{t-1} + \frac{1}{4} a_{t-2}$

① $\hat{z}_t(1) = E(z_{t+1} | F_t) = E(z_{t-3} + a_{t+1} - \frac{1}{2} a_t + \frac{1}{4} a_{t-1} | F_t) = z_{t-3} - \frac{1}{2} a_t + \frac{1}{4} a_{t-1} = 24$

$a_t(1) = z_{t+1} - \hat{z}_t(1) = a_{t+1} \Rightarrow Var[a_t(1)] = \sigma_a^2 = 1$

故区间为: $\hat{z}_t(1) \pm 1.96\sqrt{Var} \Rightarrow [22.04, 25.96]$

② $\hat{z}_t(2) = E(z_{t+2} | F_t) = E(z_{t-2} + a_{t+2} - \frac{1}{2} a_{t+1} - \frac{1}{4} a_t | F_t) = z_{t-2} - \frac{1}{4} a_t = 20.75$

$a_t(2) = z_{t+2} - \hat{z}_t(2) = a_{t+2} - \frac{1}{2} a_{t+1} \Rightarrow Var[a_t(2)] = 1.25\sigma_a^2 = 1.25$

故区间: $\hat{z}_t(2) \pm 1.96\sqrt{Var} = 20.75 \pm 1.96\sqrt{1.25} \Rightarrow [18.55865, 22.94135]$

③ $\hat{z}_t(3) = E(z_{t+3} | F_t) = E(z_{t-1} + a_{t+3} - \frac{1}{2} a_{t+2} + \frac{1}{4} a_{t+1} | F_t) = z_{t-1} = 25$

$a_t(3) = z_{t+3} - \hat{z}_t(3) = a_{t+3} - \frac{1}{2} a_{t+2} - \frac{1}{4} a_{t+1} \Rightarrow Var[a_t(3)] = 1.3125\sigma_a^2 = 1.3125$

故区间: $[22.75454, 27.24546]$

④ $\hat{z}_t(4) = E(z_{t+4} | F_t) = E(z_t + a_{t+3} - \frac{1}{2} a_{t+4} + \frac{1}{4} a_{t+2} | F_t) = z_t = 40$

$a_t(4) = a_t(3) = 1.3125\sigma_a^2 \Rightarrow Var[a_t(4)] = 1.3125$

区间: $[37.75454, 42.24546]$

## 第五题

*请问* $W_t = Z_t - Z_{t-s}$ *是什么模型*

5. $W_t = Z_t - Z_{t-s} = b_s + S_t - S_{t-s} + X_t - X_{t-s}$

由于 $S_t$ 平稳. $S_t = S_{t-s}$

由于 $X_t$ 为 ARIMA $(p,0,q)\times(P,1,Q)_s$

$\Rightarrow X_t - X_{t-s} = \nabla_s X_t$ ，为 ARIMA $(p,0,q)\times(P,0,Q)_s$

又知 $b_s$ 为常量.

$\Rightarrow W_t$ 为含常量的季节 ARMA $(p,q)\times(P,Q)_s$

## 第六题

*(a)* $Z_t = 0.5Z_{t-1} + Z_{t-4} - 0.5Z_{t-5} + a_t - 0.3a_{t-1}$

*(b)* $Z_t = Z_{t-1} + Z_{t-12} - Z_{t-13} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13}$

(a) $(1-0.5B - B^4 + 0.5B^5)Z = (1-0.3B)a_t$.

$\Rightarrow (1-0.5B)(1-B^4)Z_t = (1-0.3B)a_t$

故为 ARIMA $(1,0,1)\times(0,1,0)_4$.

其中 $\phi = 0.5$. $\theta = 0.3$

(b) $(1-B-B^{12}+B^{13})Z_t = (1-0.5B-0.5B^{12}+\frac{1}{4}B^{13})a_t$

$(1-B)(1-B^{12})Z_t = (1-0.5B)(1-0.5B^{12})a_t$

$\Rightarrow$ ARIMA $(0,1,1)\times(0,1,1)_{12}$.

其中. $\theta = 0.5$ $\Theta = 0.5$.

## 第七题

*(a) 求方差函数*

*(b) 求自相关系数函数*

*(c) 证明$\{Z_t\}$是季节ARIMA*

7. (a) $EZ_t = 0$

$EZ_t^2 = EZ_tZ_{t-4} + EZ_ta_t = EZ_{t-4}^2 + Ea_tZ_{t-4} + EZ_{t-4}a_t + Ea_t^2 = EZ_{t-4}^2 + \sigma_a^2$

$EZ_{t-4}^2 = EZ_{t-8}^2 + \sigma_a^2$

…

综上：递归地可得 $EZ_t^2 = EZ_{t-4k}^2 + k\sigma_a^2$

故 $EZ_t^2 = (1+k_t)\sigma_a^2$，其中 $\frac{t-4}{4} \le k_t \le \frac{t-1}{4}$，$t \in Z^+$

(b) 不妨先令 $s<t$.

$Z_t = \psi_0 a_t + \psi_1 a_{t-1} + \cdots = a_t + \psi_0 a_{t-4} + \psi_1 a_{t-5} + \psi_2 a_{t-6} + \cdots$

$\Rightarrow$ 归纳整理有：$\begin{cases} \psi_0 = \psi_4 = 1 \\ \psi_1 = \psi_2 = \psi_3 = 0 \\ \psi_j = \psi_{j-4}, \forall j \ge 4. \end{cases}$ $\Rightarrow Z_t = a_t + a_{t-4} + a_{t-8} + \cdots$

故 $E(Z_tZ_s) = E(a_t + a_{t-4} + \cdots)(a_s + a_{s-4} + \cdots) = Ea_ta_s + Ea_{t-4}a_s + \cdots$

若 $s = t-4n_0$. $n_0 \in N^+$. $\Rightarrow E(Z_tZ_s) = (1+k_s)\sigma_a^2$

故 $\rho = \sqrt{\frac{(1+k_s)}{(1+k_t)}}$ 其中 $\frac{t-4}{4} \le k_s, k_t \le \frac{t-1}{4}$，$k_s, k_t \in N^+$

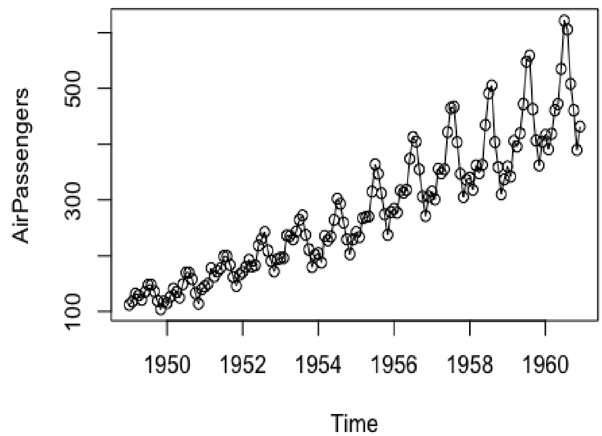
综上：if $\exists n_0 \in N^+$ s.t. $s = t-4n_0$ ：$\rho = \sqrt{\frac{(1+k_{min(s,t)})}{(1+k_{max(s,t)})}}$

else: $\rho = 0$.

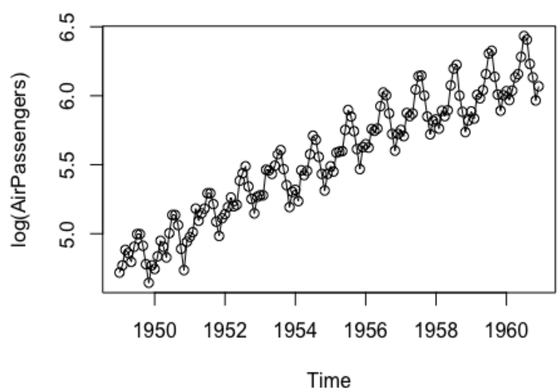
## 第八题

*(a) 绘制原始时序图和对数时序图，说明对数变换的恰当性*

```
rm(list=ls())
data("AirPassengers")
plot(AirPassengers,type='o',main="")
```

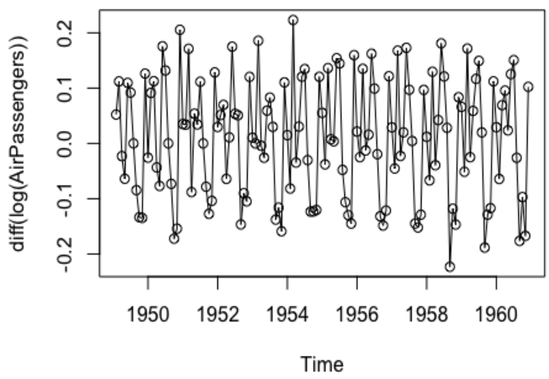



```
plot(log(AirPassengers),type='o',main="")
```



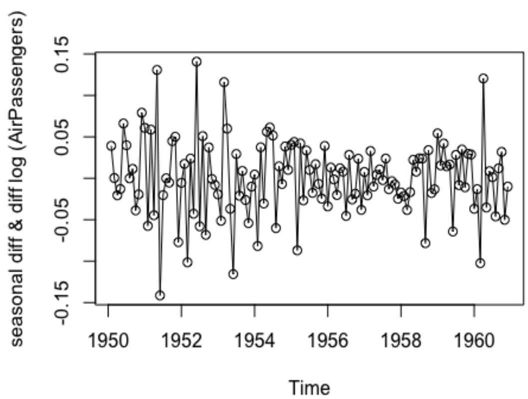

*(b) 绘制并解释对数差分序列的时序图*

```
plot(diff(log(AirPassengers)),type='o',main='')
```



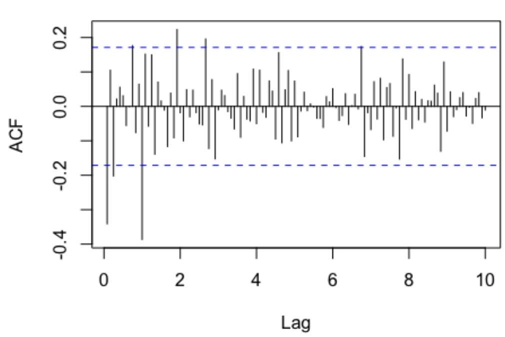

*(c) 绘制并解释对数序列经一次差分和一次季节差分后的时序图;*

```
plot(diff(diff(log(AirPassengers)),lag=12),type='o',
     ylab='seasonal diff & diff log (AirPassengers)')
```




*(d) 计算并解释对数序列经一次差分和一次季节差分后的样本 ACF*

```
acf(diff(diff(log(AirPassengers)),lag=12),main="",lag.max = 120)
```



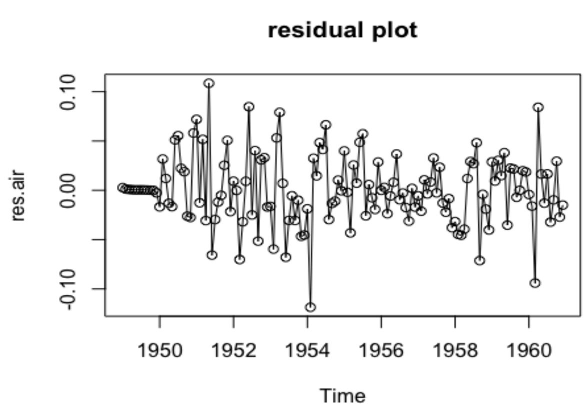
*(e) 用“航线模型” ARIMA (0, 1, 1) × (0, 1, 1)12 拟合对数序列*

```
fit.air <- arima(log(AirPassengers),
                 order = c(0,1,1),
                 seasonal = list(order=c(0,1,1), period = 12),
                 method = "ML", include.mean = TRUE)
fit.air
```

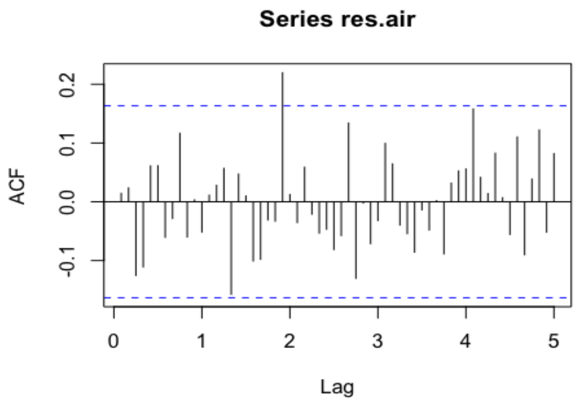
```
##
## Call:
## arima(x = log(AirPassengers), order = c(0, 1, 1), seasonal = list(or
der = c(0,
##     1, 1), period = 12), include.mean = TRUE, method = "ML")
##
## Coefficients:
##           ma1     sma1
##       -0.4018  -0.5569
## s.e.   0.0896   0.0731
##
## sigma^2 estimated as 0.001348:  log likelihood = 244.7,  aic = -485.
4
```

*(f) 基于残差序列对拟合模型进行诊断，包括诊断残差的自相关性和正态性;*

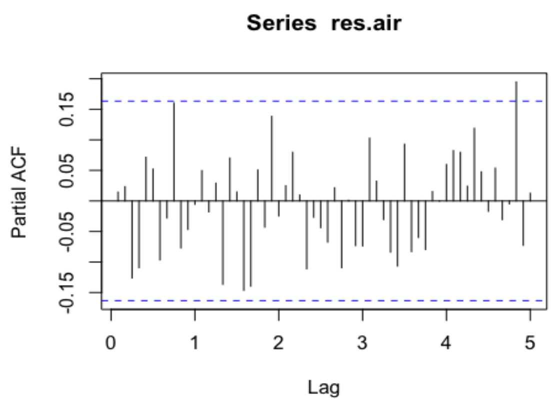
```
res.air <- fit.air$residuals
plot(res.air,type='o', main='residual plot')
```



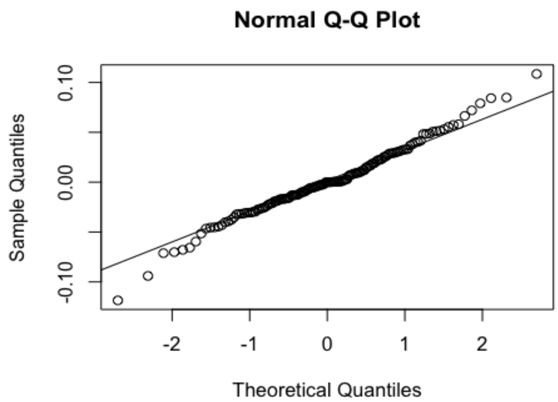

```
# relationship
acf(res.air, lag.max = 60)
```




```
pacf(res.air, lag.max = 60)
```




```
Box.test(res.air, type="Ljung-Box", fitdf = 2, lag = 6)
Box.test(res.air, type="Ljung-Box", fitdf = 2, lag = 12)
Box.test(res.air, type="Ljung-Box", fitdf = 2, lag = 18)
Box.test(res.air, type="Ljung-Box", fitdf = 2, lag = 24)
Box.test(res.air, type="Ljung-Box", fitdf = 2, lag = 30)

# normality
qqnorm(res.air);qqline(res.air)
```

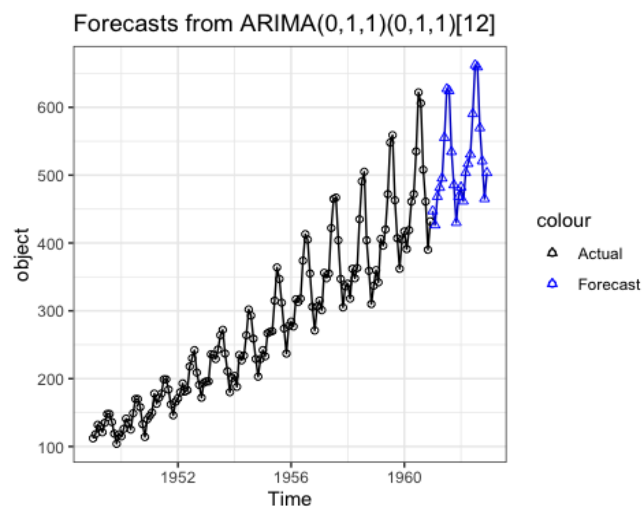



```
shapiro.test(res.air)

##
##  Box-Ljung test
##
## data:  res.air
## X-squared = 5.4434, df = 4, p-value = 0.2447
##
##
##  Box-Ljung test
##
## data:  res.air
## X-squared = 9.2333, df = 10, p-value = 0.5101
##
##
##  Box-Ljung test
##
## data:  res.air
## X-squared = 14.358, df = 16, p-value = 0.5721
##
##
##  Box-Ljung test
##
## data:  res.air
## X-squared = 26.446, df = 22, p-value = 0.233
##
##
##  Box-Ljung test
##
## data:  res.air
## X-squared = 29.494, df = 28, p-value = 0.3878
##
##
##  Shapiro-Wilk normality test
##
```

```
## data:  res.air
## W = 0.98637, p-value = 0.1674
```

*(g) 假设前置时间为两年，对此序列进行预测，给出点预测和95% 预测区间*

```
air.pred <- forecast(model=fit.air, h=24, AirPassengers, level=c(95))
autoplot(air.pred, prediction_intervals = TRUE) +
  geom_point(aes(x = time(AirPassengers), y = AirPassengers, color = "A
ctual"),
             shape = 21) +
  geom_point(aes(x = time(air.pred$mean), y = air.pred$mean, color = "F
orecast"),
             shape = 2) +
  scale_color_manual(values = c("Actual" = "black", "Forecast" = "blue"
)) +
  theme_bw()
```




```
air.pred$mean
air.pred$lower
air.pred$upper

##           Jan      Feb      Mar      Apr      May      Jun      Jul
     Aug
## 1961 447.0625 426.3548 468.4009 481.6704 495.3550 555.2917 627.2105
624.1583
## 1962 482.2070 461.4992 503.5454 516.8149 530.4995 590.4362 662.3550
659.3028
##           Sep      Oct      Nov      Dec
## 1961 534.4697 485.5144 429.8967 468.3593
## 1962 569.6141 520.6589 465.0411 503.5037
##           Jan      Feb      Mar      Apr      May      Jun      Jul
     Aug
## 1961 446.9906 426.2709 468.3067 481.5668 495.2428 555.1715 627.0829
624.0236
## 1962 482.0304 461.3121 503.3482 516.6081 530.2836 590.2115 662.1219
659.0616
```

```
##           Sep      Oct      Nov      Dec
## 1961 534.3282 485.3666 429.7427 468.1994
## 1962 569.3651 520.4022 464.7770 503.2324
##           Jan      Feb      Mar      Apr      May      Jun      Jul
     Aug
## 1961 447.1345 426.4386 468.4952 481.7740 495.4672 555.4119 627.3382
624.2930
## 1962 482.3836 461.6864 503.7426 517.0217 530.7154 590.6608 662.5881
659.5440
##           Sep      Oct      Nov      Dec
## 1961 534.6111 485.6622 430.0507 468.5191
## 1962 569.8632 520.9156 465.3053 503.7751
```