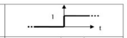
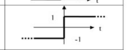
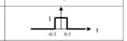
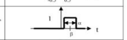
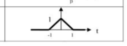
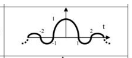
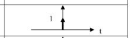
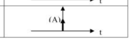
Signal Definitions:

EE228 – Common Signals

Unit Step	$u(t)=\begin{cases}0, t<0\\1, t>0\end{cases}$, undefined for t = 0. Heaviside unit step: $u(0)=0.5$	
Unit Ramp	$r(t)=\begin{cases}0, t<0\\t, t>=0\end{cases}$	
Signum (sign of argument)	$\text{sgn}(t)=\begin{cases}+1, & t>0\\0, & t=0\\-1, & t<0\end{cases}$	
Rectangle or pulse	$rect(t)=\begin{cases}1, & \lvert t\rvert<0.5\\0, & else\end{cases}$, width = 1	
Rectangle or pulse	$rect\left(\frac{t-\beta}{\alpha}\right)=\begin{cases}1, & \beta-(\alpha/2)<t<\beta+(\alpha/2)\\0, & else\end{cases}$, width = α	
Triangle	$tri(t)=\begin{cases}1-\lvert t\rvert, & \lvert t\rvert<1\\0, & else\end{cases}$, width = 2	

Sinc	sinc(t) = sin(πt) / πt sinc(t) = 0, t = ±1, ±2, ±3,... (zero crossings) sinc(0) = 1	
Impulse	$\delta(t)=\begin{cases}0, & t\neq 0\\\infty, & t=0\end{cases}$ Area = 1	
Weighted Impulse	$A\,\delta(t)=\begin{cases}0, & t\neq 0\\\infty, & t=0\end{cases}$ Area = A	

$$e^{j\theta}=\cos\theta+j\sin\theta$$

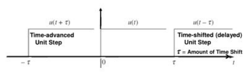
$$\cos\theta=\mathcal{RE}\left\{e^{j\theta}\right\}=\frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta=\mathcal{IM}\left\{e^{j\theta}\right\}=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

UNIT - STEP:

Unit-Step Function

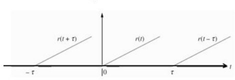
$$u(t)=\begin{cases}1, & \text{for } t\geq 0\\0, & \text{for } t<0\end{cases}$$



$u(t+\tau)=\begin{cases}1, & \text{for } t\geq \\0, & \text{for } t<\end{cases}$ $u(t-\tau)=\begin{cases}1, & \text{for } t\geq \\0, & \text{for } t<\end{cases}$

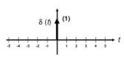
Unit-Ramp Function

$$r(t)=\begin{cases}t, & \text{for } t\geq 0\\0, & \text{for } t<0\end{cases}$$



Unit Impulse (Delta Function)

$\delta(t)=\begin{cases}\_\_, & t\neq 0\\\_\_, & t=0\end{cases}$



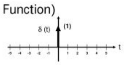
- Zero Duration (Infinitely narrow)
- Infinite Amplitude (height) @ $t=0$
- Finite Area = ___ $Area=\int_{-\infty}^{\infty}\delta(t)dt=$
- Infinite Energy $Energy=\int_{-\infty}^{\infty}\left|\delta^2(t)\right|dt=$

OPERATIONS ON IMPULSES

a) Unit Impulse (Delta Function)

$\delta(t)=\begin{cases}0, & t\neq 0\\\infty, & t=0\end{cases}$

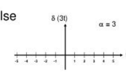
Area = 1



Time-Scaled Impulse

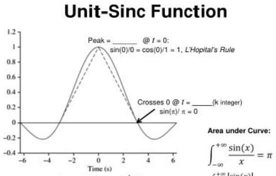
δ (αt) =( ) δ (t)

Area =



IMPULSE PRODUCT/Siefting PROP:

Unit-Sinc Function



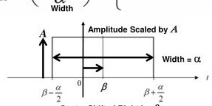
$$\int_{-\infty}^{+\infty}\frac{\sin(x)}{x}=\pi \qquad \int_{-\infty}^{+\infty}\left|\frac{\sin(x)}{x}\right|=+\infty$$

$$\text{sinc}(t)=\frac{\sin t}{t}$$

Scaled, Shifted Rectangular Pulse Function

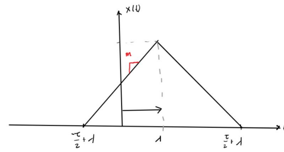
$$A\,\text{rect}\left(\frac{t-\beta}{\alpha}\right)=\begin{cases}A, & -\frac{1}{2}<\frac{t-\beta}{\alpha}<+\frac{1}{2}\\0, & \text{elsewhere}\end{cases}$$

Amplitude; Time Shift; Width



How to find is setting the $\beta-\alpha/2$ to end point of function.

Scaled, Shifted Triangluar Function



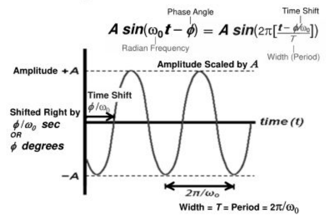
$$A\,tri\left[\frac{t-\lambda}{\tau}\right]=\begin{cases}A-2A\frac{|t-\lambda|}{\tau}, & -\frac{\tau}{2}+\lambda<t<+\frac{\tau}{2}+\lambda\\0, & \text{elsewhere}\end{cases}$$

① find A set $m=\frac{2A}{\tau}$ ⇒ $\boxed{A=\frac{m\tau}{2}}$

② set $-\frac{\tau}{2}+\lambda$ to left endpt and solve for τ.

Scaled, Shifted Sinusoids

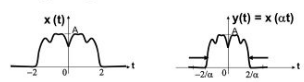
$$A\sin(\omega_0 t-\phi)=A\sin(2\pi[t-\phi/\omega_0]/T)$$



Operations on signals:

Operations on Signals: Time Scaled: $\mathcal{S}\{x(t)\}=x(\alpha\, t)$

Original Signal Time Scaled



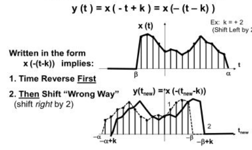
Compressed in time for |α| __ 1
Ex: y(t) = x( )

Expanded in time for |α| __ 1
Ex: y(t) = x( ) = x( )

$-2<t<2$ => $-2/\alpha<t<2/\alpha$

Operations on Signals: Time Shift With Reversal

y (t) = x (- t + k) = x (– (t – k))



Periodicity, Energy and Power:

Signal Measures

Signal Areas & Integrals:

$$Signal\ Area=\int_{-\infty}^{\infty}x(t)dt$$

$$Absolute\ Area=\int_{-\infty}^{\infty}|x(t)|dt$$

$Absolutely\ Integrable:\ \int_{-\infty}^{\infty}|x(t)|dt$ _____ (finite area)

$$Time\ Average=\frac{1}{-}\int\ \ x(t)dt$$

Signal Energy = Integral of Instantaneous Power over all time

$$Signal\ Energy=\int_{-\infty}^{\infty}P_i(t)dt=$$

Avg. Signal Power = Time average of Instant. Power over all time
= Total Signal Energy / Time Duration (*i.e.* energy per unit of time)

$Avg.\ Signal\ Power=\lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\left|x^2(t)\right|dt$ If nonperiodic signal

$=\qquad\qquad=\frac{\text{Energy in 1 Period}}{\text{Time in 1 Period}}$ If periodic signal with period "T"

Energy vs. Power Signal:

Signal is a power signal if: (example - periodic fns)

- If the signal is not time-limited (infinite time duration.
- Has Infinite total signal energy
- Has finite non zero Power (Energy/Period)

Signal is an Energy signal if: (example - non periodic fns, half cycle sinusoid)

- Has finite signal Energy
- Has zero signal power($E/\infty=0$)

Power of a Sinusoid: $P_{avg}=\frac{A^2}{2}$

Power of periodic signal made up of sinusoids ($X_0 is const$):

$$P_{avg}=P_0+\sum_{k=1}^{\infty}P_k=X_0^2+\sum_{k=1}^{\infty}\frac{A_k^2}{2}$$

Energy of half cycle sinusoid: $E=\frac{A^2b}{2}$

Signal Measures - Periodic Signals

Power Signals: Finite Power / Energy = ∞

$Average\ Value\quad \bar{x}=\frac{1}{T}\int_{t_0}^{t_0+T}x(t)dt$ (DC value)

$Effective\ Value\quad x_{eff}=\quad\int_{t_0}^{t_0+T}\quad dt=x_{rms}$ (rms value)

Effective Value = DC value having same average power as $x(t)$
= (square) ROOT of the MEAN (average) of $x(t)$ SQUARED
$=\sqrt{P}$

$$Signal\ Power=\frac{1}{T}\int\left|x^2(t)\right|dt$$

The "fundamental frequency" f_0 of the sum signal will be the "_____________________" of the individual frequencies.

Even and Odd Parts of f(t)

Any function $f(t)$ can be represented by the sum of an even symmetric function $f_e(t)$ and an odd symmetric function $f_o(t)$.

$$f(t)=f_e(t)+f_o(t)$$

determined by:

Even Part of $f(t)$: $f_e(t)=\frac{f(t)+f(-t)}{2}$

Odd Part of $f(t)$: $f_o(t)=\frac{f(t)-f(-t)}{2}$

System Classifications:

Checking Differential Equation For Linearity:

System is Nonlinear if the Diff. Equation describing it contains:

- **A constant term**
  $y(t)=x'(t)\ \underline{+\ 3}$
- **Products of inputs and/or output terms**
  $y'(t)=b_0\mathbf{x^2(t)}+b_1\mathbf{x'(t)y(t)}+\ldots$

Checking Differential Equation For Time-Invariance:

System is Time Varying if the Diff. Equ. describing its behavior contains:

- **Coefficients A_k or B_k that are functions of time t:**
  $y(t)=\underline{2t}x'(t)+\ldots$
- **Time-scaled inputs or output terms**
  $y(t)=b_0\mathbf{x}(\underline{\mathbf{2t}})+\ldots$

Static *vs.* Dynamic

Static Systems (instantaneous)

- Determining the **present output of the system** $y(t_0)$ at any particular time "t_0" **requires _____________________________ of the input $x(t_0)$**
- **All** x(t) and y(t) **terms** have the _________ arguments ( ).
- **Effects of previous inputs do not linger** to affect the output later – _____________________
- **No derivatives or integrals** in system equation.
  $y(t)=\mathcal{S}\{x(t)\}=Cx(t)+A$

Other Classes / Properties

Causal (non-anticipating)

- At any time "t_0" (the "present"), the output of the system $y(t_0)$ is determined completely by values of the input x(t) that arrived at the same or previous times … *i.e.* x(t) for t ___ t_0
- Causal systems do not use _________ values of the input $x(t+\tau)$ to determine the present output y(t)

Causal Systems (non-anticipating)

Determining Causality from the system differential equation:

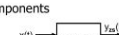
1. If the equation is in terms of $y(t), y'(t), y''(t),\ldots$ **there are no ___________terms in the equation**
2. **There are no ____________________ terms:**

SOLVING DIFFERENTIAL EQNS:

Alternate Solution Linear Diff. Equ.

(Method of Undetermined Coefficients)

Total Response $y_T(t)$ has 2 Alternate Components

- **Zero State Response $y_{ZS}(t)$**
  

  - Different from the Forced Response: $y_{ZS}(t)\neq y_F(t)$
  - Includes the Forced Response and part of the Natural Response:

    ____

- **Zero Input Response $y_{ZI}(t)$**
  

  - In general, different from the Natural Response:
    $y_{ZI}(t)\neq y_N(t)$ [unless x(t) = 0 for t >0, so that y_{ZS}=0]
  - Same form as $y_N(t)$, but with different coefficients

    ____

  - Includes the part of the Natural Response due to IC's:
    Total Natural Response: $y_N(t)=y_{N_1}(t)+y_{N_2}(t)$
  - If system is truly "relaxed" (all IC's=0), $y_{ZI}(t)=0$

Solution Method

Method of Undetermined Coefficients – **General Case**

Solution Steps:

1. Convert the Differential Equation to a **Single Input Case**
2. Find ZIR, the **zero-input response**, $y_{ZIR}(t) = y_{N_1}(t)$
3. Find the **forced response**, $y_F(t)$ of the Single Input Case
4. Find ZS1, the **zero-state response** of the Single Input Case

   $y_{ZS1}(t) = y_F(t) + y_{N_2}(t)$
5. Find ZSR, the **zero-state response** of the **General Case**
6. Find **General Case total response**, $y_T(t) = y_{ZSR}(t) + y_{ZIR}(t)$

Solution Method

Method of Undetermined Coefficients – **Single Input Case**

Solution Steps:

1. Find ZIR, the **zero-input response**, $y_{ZI}(t) = y_{N_1}(t)$

   [Solve with ________; using given I.C.'s.]

2. Find the **forced response**, $y_F(t)$

   [$y_F(t)$ is a solution of the Single Input Case Diff Equ.]

2. Find ZSR, the **zero-state response**, $y_{ZS}(t) = y_F(t) + y_{N_2}(t)$

   [Solve with actual x(t); using __________.]

Solution Method - General Case

When terms such as A x(t) + B dx(t)/dt appear on the right side of a generalized differential equation, a similar approach is used.

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = A\,x(t) + B\,dx(t)/dt$$

1. We first find the **zero state response** of the **single input case** $y_{zs1}(t)$, the response due to just the x(t) forcing function, for the modified (single input case) differential equation:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = x(t)$$

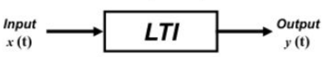
2. **Then linearity and superposition are used to find the overall ZSR** due to all the input terms [x(t) and its derivatives] on the right side of the equation, *i.e.:*

If input x(t) ➔ yields S.I.C. ZS response $y_{zs1}(t)$ from the system,

then input of [A x(t) + B *dx(t)/dt*] ➔ **yields Gen. Zero State Response [$Ay_{zs1}(t) + Bdy_{zs1}(t)/dt$]**

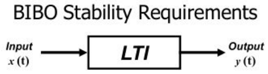
STABILITY FROM LOOKING AT DE:

BIBO Stability Requirements



- Every root of the characteristic equation must have a ____________________
  - Constrains natural response to be bounded

BIBO Stability Requirements



2. The **degree** of the **highest derivative of x(t)** in the differential equation (RHS) **must ______________the *order* of the differential equation** (the highest derivative of y(t) on Left Hand Side)

⇩ ⇩ UNSTABLE

$$y'(t) + a_0 y(t) = b_0\,x(t) + b_1\,x'(t) + b_2\,x''(t)$$

Impulse response and step response:

Methods for Determining h(t)

1. Find the Impulse Response h(t) as the derivative of the Step Response s(t)
   - First find the Step Response by solving the *relaxed* system differential equation [ICs=0] for: x(t) = u(t)

2. **Solve the *relaxed* system differential equation for: $x(t) = \delta(t)$**

Method for Finding Impulse Response $h(t)$

1. Set all SIC Initial Conditions to zero (relaxed)

   y(0)=0, y'(0)=0,...

2. Set the highest order initial condition to 1

   $y^{n-1}(0)=\mathbf{1}$

   Ex: n=2: y(0)=0, y'(0)=1

3. Solve Single-Input Diff Equ. for x(t) = 0
   - Solve for the natural response only ($y_F(t)=0$)
   - Use the above IC's to find unknown coef's

4. Determine generalized $h(t)$ same way as generalizing the ZSR.

Other System Properties from h(t)

Stability

- **For a stable system**:

  h(t) must be "________________________"

Causality

- For a system to be causal:

  $h(t) =$ _____, for t < 0

Convolution:

Properties of Convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$

Time Shift: $x(t) * h(t) = y(t)$

$x(t-\tau) * h(t) = y(t-\tau)$

$x(t-\tau) * h(t-\alpha) =$

Derivative: $x(t) * h(t) = y(t)$

$x'(t) * h(t) =$

Properties of Convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$

Convolution with Impulse

Convolution: $\delta(t) * h(t) =$

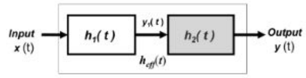
Multiplication: $\delta(t)h(t) =$

Convolution with a Shifted Impulse

Convolution: $\delta(t-\tau) * h(t) =$

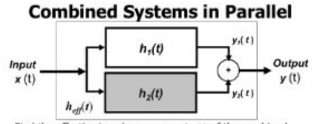
Multiplication: $\delta(t-\tau)h(t) =$

Combined Systems in Cascade



Find the effective impulse response $h_{eff}(t)$ of the combined **CASCADED** Filters:

Combined Systems in Parallel



Find the effective impulse response $h_{eff}(t)$ of the combined **PARALLEL** Filter:

Evaluation of Convolution Integral

Two Methods:

1. Analytical Evaluation

1. Use step functions u(t) to define the beginning and end of x() and h()
2. Break up the integral based on similar pairings of u(t)'s in x() and h() functions

- Best suited for convolution of two causal functions that are infinite duration
  - Then you're still doing "Convolution By Ranges", but there is only one solution "range"

2. Graphical Evaluation (Convolution By Ranges)

1. Break up problem into different regions based on where signal transitions and overlaps occur
2. In each region, find a unique solution using the appropriate analytical functions for x() and h()

Convolution by Ranges:

1.) First determine both x(t) and h(t) Range End Point.

2.) Then find pairwise sum

$$REPT(y(t)) = SORT(set(PWS(REPT(h(t)), REPT(x(t)))))$$

3.) Then write y(t) as a piecewise where y_1,.., y_k is the value between the endpoints.

GRAPHICAL CONVOLUTION

For each range of "t" determined above:

- Draw x(t-λ) relative to h(λ) {or h(t-λ) relative to x(λ)} on a single set of λ axes.
- Observe the endpoints of the regions where the two functions overlap. The beginning and endpoint of the overlapping regions (in terms of λ) determine the lower and upper limits of integration for λ in the convolution integral in this range.
- Solve the integral and evaluate over the upper and lower limits.
- **REPEAT Steps above for each range of t**

CHECKing answers for solution:

$$Lenth_y = Lenth_x + Lenth_h$$
$$T_{start,y} = T_{start,x} + T_{start,h}$$
$$T_{end,y} = T_{end,x} + T_{end,h}$$

Correlation:

Correlation

$$r_{xh}(t) = x(t) ** h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)dt = x(t) * h(-t)$$

- **Measure of similarity** between x, h
- Like **Convolution**, But Shift h(t) **without Flipping** it
- Not Commutative: $r_{xh}(t) = r_{hx}(-t)$
- "**Cross-Correlation**" if $x(t) \neq h(t)$
- "**Auto-Correlation**" if $x(t) = h(t)$

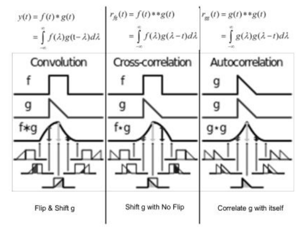
Cross-Correlation

- For **non-periodic** functions:

$$r_{12}(\tau) = \int_{-\infty}^{+\infty} x_1(t)x_2(t-\tau)dt, \quad -\infty < \tau < +\infty$$

- For **periodic** functions, with period T, the correlation function is given by:

$$r_{12}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} v_1(t)v_2(t-\tau)dt$$



Summary - Correlation

- Correlation is computed similarly to Convolution, but without the signal flip

$$r_{xh}(\tau) = x(t) ** h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)dt = x(t) * h(-t)$$

- **Cross-Correlation** $r_{xh}(\tau)$ reveals the **degree of similarity** between two different signals, and how the similarity **changes with time alignment** of the signals

- **Auto-Correlation** $r_{xx}(\tau)$ (correlating a signal with itself) reveals the **randomness or periodicity** of a signal

$$r_{xx}(\tau) = x(t) ** x(t) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = x(t) * x(-t)$$

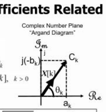
- Filtering (convolution) a noisy signal with a time-reversed clean version of a signal implements ***correlation detection*** (also called "**matched filtering**") by cross correlating the noisy and clean signals to **find if the signal is detected and at what time position.**

FS: Coefficients Relationship:

Fourier Series Coefficients Related

Same Complex # Value:

$$C_k e^{+j\theta_k} = C_k\angle+\theta_k = a_k - jb_k$$
$$= 2X[k] = 2|X[k]|\angle X[k], \quad k > 0$$



$$Magnitude: C_k = |a_k - jb_k| = \sqrt{a_k^2 + b_k^2} = 2|X[k]|_{k>0}$$
$$Phase: \theta_k = \angle(a_k - jb_k) = \tan^{-1}\left(\frac{-b_k}{a_k}\right) = \angle X[k]_{k>0}$$
$$DC\ (k=0): \quad C_0 = a_0 = X[0], \quad b_0 = 0, \quad \theta_0 = 0$$

Exponential Fourier Series Coefficient Symmetry

For $k > 0$:

$$X[k] = \frac{1}{2}C_k e^{+j\theta_k} = \frac{1}{2}C\angle+\theta_k \qquad X[-k] = \frac{1}{2}C_k e^{-j\theta_k} = \frac{1}{2}C\angle-\theta_k$$

$|X[k]| = |X[-k]| = \frac{1}{2}C_k$ **|X[k]| is "Even Symmetric"**

$\angle X[k] = -\angle X[-k] = \theta_k$ **∠X[k] is "Odd Symmetric"**

$X^*[k] = X[-k]$ **X[k]** has **"Conjugate Symmetry"**

$$\Re e\{X[k]\} = \Re e\{X[-k]\}$$
$$Im\{X[k]\} = -Im\{X[-k]\}$$

FS: Finding Fourier Coefficients:

Finding Exponential Fourier Series Coefficients: a_k, b_k

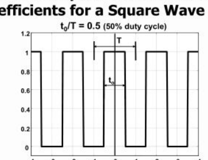
By a similar derivations, it can be shown that:

$a_k =$ $k > 0$

$b_k =$ $k > 0$

So that: $c_k = \sqrt{a_k^2 + b_k^2}$ $\qquad \theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$

Find the Exponential Fourier Series Coefficients for a Square Wave Signal



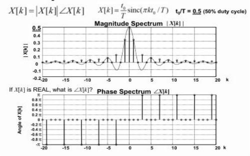
$$X[k] = \frac{1}{T}\int_T x(t)e^{-j2\pi k f_0 t}dt$$

Exponential Fourier Coefficients

$$X[k] = \frac{1}{T}\int_T x(t)e^{-j2\pi k f_0 t}dt = \frac{1}{T}\int_{-t_0/2}^{t_0/2} x(t)e^{-j2\pi kt/T}dt$$
$$= \frac{1}{T}\int_{-t_0/2}^{t_0/2} 1e^{-j2\pi kt/T}dt = \frac{1}{T}\left.\frac{e^{-j2\pi kt/T}}{-j2\pi k/T}\right|_{t=-t_0/2}^{t=t_0/2}$$
$$= \frac{1}{-j2\pi k}\left[e^{-j2\pi kt_0/2T} - e^{-j2\pi k(-t_0/2T)}\right]$$
$$= \frac{1}{-j2\pi k}\left[e^{-j\pi kt_0/T} - e^{+j\pi kt_0/T}\right] = \frac{1}{\pi k}\left[\frac{e^{+j\pi kt_0/T} - e^{-j\pi kt_0/T}}{2j}\right]$$
$$= \frac{\sin(\pi kt_0/T)}{\pi k} = \frac{t_0}{T}\frac{\sin(\pi kt_0/T)}{\pi kt_0/T} = \boxed{\quad}$$

Exponential Fourier Series

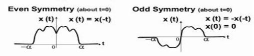
$X[k] = |X[k]| \angle X[k]$ $\quad X[k] = \frac{t_0}{T}\mathrm{sinc}(\pi k t_0/T)$ $\quad t_0/T = 0.5$ (50% duty cycle)



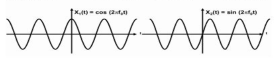
FS: Shortcut (Signal Symmetrys)

Signal Symmetries

Even Symmetry (about t=0) $\quad$ Odd Symmetry (about t=0)

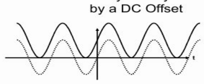


Cosine has _______ Symmetry? $\quad$ Sine has _______ Symmetry?



Odd Symmetry

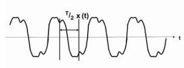
Sometimes Odd Symmetry can be **hidden** by a DC Offset



Odd Symmetry Revealed When DC Removed

$X_1(t) = 0.5 + \sin(2\pi f_0 t)$

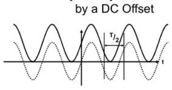
Half-Wave Symmetry



x (t) = ________

Half-Wave Symmetry

Half-Wave Symmetry can also be **hidden** by a DC Offset



Half-Wave Symmetry Revealed When DC Removed

$X_1(t) = 0.5 + \sin(2\pi f_0 t)$

Signal Symmetries

$$a_k = \frac{2}{T}\int_0^T x_p(t)\cos(2\pi k f_0 t)dt \qquad b_k = \frac{2}{T}\int_0^T x_p(t)\sin(2\pi k f_0 t)dt$$

- **$x_p(t)$ has Even Symmetry:**

$$a_k = \frac{-}{T}\int_0 x_p(t)\cos(2\pi k f_0 t)dt \qquad b_k =$$

$k > 0$

- **$x_p(t)$ has Odd Symmetry:**

$$b_k = \frac{-}{T}\int_0 x_p(t)\sin(2\pi k f_0 t)dt \qquad a_k =$$

$k > 0$

Signal Symmetries

- **$x_p(t)$ has Half-Wave Symmetry:**

$$a_k = \frac{4}{T}\int_0^{T/2} x_p(t)\cos(2\pi k f_0 t)dt \qquad k = odd$$

$$b_k = \frac{4}{T}\int_0^{T/2} x_p(t)\sin(2\pi k f_0 t)dt \qquad k = odd$$

Note: You can not use the half-wave symmetry integrals above if the half-wave symmetry is "hidden" (i.e. if there is a DC offset). -The k_{even} = 0 result for the Fourier Coefficients DOES apply for k ≥ 2 if there is "hidden" half-wave symmetry

Signal Symmetries

- **$x_p(t)$ has Even and Half-Wave Symmetry:**

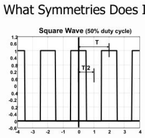
$$a_k = \frac{8}{T}\int_0^{T/4} x_p(t)\cos(2\pi k f_0 t)dt \qquad k = odd$$

- **$x_p(t)$ has Odd and Half-Wave Symmetry:**

$$b_k = \frac{8}{T}\int_0^{T/4} x_p(t)\sin(2\pi k f_0 t)dt \qquad k = odd$$

Note: You can not use the half-wave symmetry integrals above if the half-wave symmetry is "hidden" (i.e. if there is a DC offset). -The k_{even} = 0 result for the Fourier Coefficients DOES apply for k ≥ 2 if there is "hidden" half-wave symmetry

What Symmetries Does It Have?

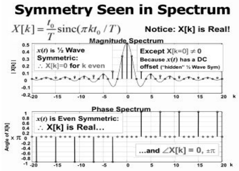


Even/Odd?

Half-Wave?

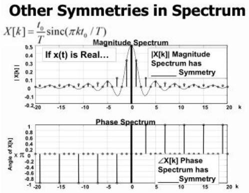
Symmetry Seen in Spectrum

$X[k] = \frac{t_0}{T}\mathrm{sinc}(\pi k t_0/T)$ $\quad$ **Notice: X[k] is Real!**

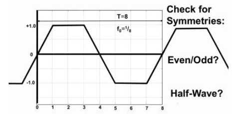


Other Symmetries in Spectrum

$X[k] = \frac{t_0}{T}\mathrm{sinc}(\pi k t_0/T)$



Example: Trapezoidal Signal



$$b_k = \frac{8}{T}\int_0^{T/4} x_p(t)\sin(2\pi k f_0 t)dt \qquad k = odd$$

$$b_k = \frac{8}{8}\int_0^2 x_p(t)\sin(2\pi k t/8)dt = \int_0^1 t\sin(\pi k t/4)dt + \int_1^2 1\sin(\pi k t/4)dt$$

$$= \left\{\left[\sin(\pi k t/4) - \frac{\pi k t}{4}\cos(\pi k t/4)\right]\left(\frac{1}{\pi k/4}\right)^2\Bigg|_0^1 - \left[\frac{\cos(\pi k t/4)}{\pi k/4}\right]_1^2\right\}$$

$$= \left[\frac{\sin(\pi k/4) - (\pi k/4)\cos(\pi k/4) - \sin(0) + 0}{(\pi k/4)^2}\right] - \left[\frac{\cos(2\pi k/4) - \cos(\pi k/4)}{\pi k/4}\right]$$

$$= \frac{16\sin(\pi k/4)}{(\pi k)^2} - \frac{\cos(\pi k/4) + \cos(\pi k/2) - \cos(\pi k/4)}{\pi k/4}$$

$$= \frac{16\sin(\pi k/4)}{\pi^2 k^2} - \frac{\cos(\pi k/2)}{\pi k/4} \qquad \cos(\pi k/2) = 0 \;\; (k \text{ odd}) \quad \cos(\pi/2) = 0,\; \cos(3\pi/2) = 0,\; \cos(5\pi/2) = 0$$

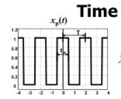
$$\boxed{b_k = \frac{16\sin(\pi k/4)}{\pi^2 k^2} \quad (k \text{ odd})} \qquad \int t\sin(at)dt = \frac{\sin(at) - at\cos(at)}{a^2}$$

FS: properties

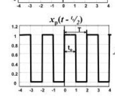
Properties of Fourier Series

Symmetries:			
If $x(t)$ real:	$x(t) = x^*(t)$	$\iff X_n = X^*_{-n}$	Conjugate Symmetric
If $x(t)$ even sym:	$x(t) = x(-t)$	$\iff X_n = X_{-n}$	Real
If $x(t)$ odd sym:	$x(t) = -x(-t)$	$\iff X_n = -X_{-n}$	Imaginary
If ½ wave sym:	$x(t) = -x(t+T/2)$	$\iff X_n = 0$ for n even	
Even part of $x(t)$:	$\frac{x(t)+x(-t)}{2}$	$\iff \mathcal{RE}\{X_n\}$	
Odd part of $x(t)$:	$\frac{x(t)-x(-t)}{2}$	$\iff j\mathcal{IM}\{X_n\}$	
Linearity:	$ax(t) + by(t)$	$\iff aX_n + bY_n$	
Time Reversal:	$x(-t)$	$\iff X_n^*$	
Time Shift:	$x(t - t_0)$	$\iff X_n e^{-j2\pi n t_0/T}$	

Time Delay



$$X[k] = \frac{t_0}{T}\mathrm{sinc}(\pi k t_0/T)$$



Time Shifting $x_p(t)$ by (t_0/ 2)

$$X_{shift}[k] = \frac{t_0}{T}\mathrm{sinc}(\pi k t_0/T)e^{-j2\pi k f_0 t_0/2}$$

Leads to a ***Phase Shift*** in $X[k]$ by

θ = ____________(t_0/ 2) Time Shift

FS: POWER

Power in Harmonic Signals

____________ Theorem"

So we can compute the power in a harmonic signal from the **time-domain** waveform itself:

$$Signal\ Power = \frac{1}{T}\int_T |x_p^2(t)|dt =$$

OR from its frequency spectrum & **Fourier Series** coef:

And we should get the same answer!!

$|X[k]|^2$ ~ "Power Spectral Density"

Power in Harmonic Signals

$$P = \left[c_0^2 + 0.5\sum_{k=1}^{\infty} c_k^2\right] \qquad \text{Polar}$$

$$= \left[a_0^2 + 0.5\sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right] \qquad \text{Trigonometric}$$

$$= \sum_{k=-\infty}^{\infty}|X[k]|^2 = |X[0]|^2 + 2\sum_{k=1}^{\infty}|X[k]|^2$$

Exponential (2-sided)

Power in Few Harmonics

The Power in a subset of the harmonic frequencies in a signal can also be found using the **Exponential Fourier Coefficients** of those harmonics:

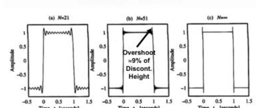
$$P_{(\text{in } k = k_1 \to k_2)} = 0.5\sum_{k=k_1}^{k_2} c_k^2$$

$$= 0.5\sum_{k=k_1}^{k_2}\left[2|X[k]|\right]^2 = 0.5\sum_{k=k_1}^{k_2} 4|X[k]|^2$$

$$P_{(\text{in } k = k_1 \to k_2)} = \boxed{\quad}$$

FS: GIBBS EFFECT

Gibbs Effect

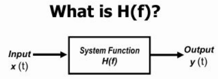


Even with an infinite number of harmonics, a signal with discontinuities can not be exactly reconstructed from its Fourier Series

FS: System Function

$$x(t) = \sum_{\pm k} c_k \cos(2\pi k f_0 t + \theta_k)$$

What is H(f)?



System (Transfer) Function:

- **Relation of output signal $y(t)$ to input signal $x(t)$**
- **Frequency-dependent (changes with f) scale factor**
- **Complex value:**

H(f) = |H(f)| ∠H(f)

Magnitude Scale Factor (Gain) $\quad$ ***Phase Shift (Time Shift)***

$$y(t) = \sum_{\pm k} |H(kf_0)|\, c_k \cos(2\pi k f_0 t + \theta_k + \angle H(kf_0))$$

Fourier Series Concepts

- Any (almost) periodic signal can be represented by a combination of harmonically-related ($f = kf_0$, k integer) sinusoids (sin / cos / $e^{j2\pi ft}$)
  - Requirements: x(t) absolutely integrable, finite # of finite discontinuities, finite # max/mins
- Reconstruction from Fourier Series will be imperfect at discontinuities (overshoot)
  - "Gibbs Effect"

FT: Integral Definiti

What IS the Fourier Transform?

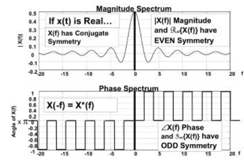
Like the Fourier Series:

- It is a ________________description of a signal
  - Tells us how much of what frequencies are present in a signal
- It is a __________ valued function

$X(f) = |X(f)| \angle X(f) = |X(f)|\, e^{+j\angle X(f)}$

$= \mathcal{R}e\{X(f)\} + j\, \mathcal{I}m\{X(f)\}$

Symmetries in Fourier Spectrum



Signal Symmetries

x(t) is Real and has Even Symmetry:

- X(f) is ________ and has ________ Symmetry
- ∠X(f) = _______ or ________

x(t) is Real and has Odd Symmetry:

- X(f) is ________ and has _________ Symmetry
- ∠X(f) = ____________

Existence of Fourier Transforms

A non-periodic signal x(t) can be represented by a unique Fourier Transform IF:

1. x(t) is "absolutely integrable"

   $\int_{-\infty}^{\infty}|x(t)|dt < \infty$ $\quad$ Integral over ALL time t

2. x(t) has a finite number of maxima & minima.
3. x(t) has a finite number of finite discontinuities.

Existence of Fourier Transforms

However, some signals that don't meet these strict requirements still have valid Fourier Transforms. (Ex: x(t)=u(t) is not abs. integrable $\int \to \infty$)

- F.T. of an "absolutely integrable" signal ______________
- F.T. of a "not absolutely integrable" signal ___________________, if it does not grow exponentially (e^{+at}).
- F.T. of a signal that grows exponentially or faster ____________________.

FT AND FS Relationship:

Methods of Computing Fourier Transforms and Inverse Transforms

1. Converting known exponential Fourier Series coefficients of a periodic signal version $x_p(t)$

$$X_p[k] \Rightarrow X(f) = T\, X_p[k]_{kf_0 \to f}$$

Fourier Series of A Periodic Signal $x_p(t)$ $\quad$ Fourier Transform of A Non-periodic Signal $x(t)$ made up of 1 Period of the periodic signal $x_p(t)$

FT: Properties

Duality: Similarity Theorem

Fourier Transform: $X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$ $\quad$ **Inverse Fourier Transform:** $x(t) = \int_{-\infty}^{+\infty} X(f)e^{+j2\pi ft}df$

If FT{ $x(t)$ } = $X(f)$

$X(\,)$ has different functional form than $x(\,)$

Then FT{ $X(t)$ } = $x(-f)$

For functions with Even Symmetry in $x(\,)$:

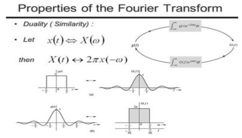
$x(+t) = x(-t) \qquad X(+f) = X(-f)$

So FT{ $X(t)$ } = $x(f)$

Properties of the Fourier Transform

- *Duality (Similarity)* :
- Let $x(t) \Leftrightarrow X(\omega)$

then $X(t) \leftrightarrow 2\pi x(-\omega)$



Time / Frequency Scaling

- There is a coupled and inverse relationship ("duality") between the width of a signal representation in the Time Domain vs. the FT Frequency Domain

Time Scaling Property $\quad x(\alpha t) \underset{InvFT}{\overset{FT}{\rightleftarrows}} \frac{1}{|\alpha|}X(f/\alpha)$

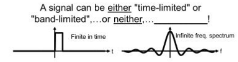
$\underset{InvFT}{\overset{FT}{\rightleftarrows}} X(\beta f)$ $\quad$ Frequency Scaling Property

Time / Frequency Scaling

An extension of this concept is the Time/Frequency Limit Theorem:

A signal can not have BOTH a finite time duration ("time-limited") AND a finite length frequency spectrum ("band-limited").

A signal can be either "time-limited" or "band-limited",...or neither,.........!



FT: ENERGY

Energy in Non-Periodic Signals

We can compute the energy in an aperiodic signal from the time-domain waveform itself:

$$Total\ Energy = \int_{-\infty}^{\infty} x^2(t)dt = \_\_\_\_\_\_\_\_\_\_\_\_$$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

OR from its Fourier Transform frequency spectrum:

And we should get the same answer!!

"Parseval's Theorem" (again)

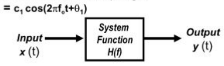
FT: input a sinusoid

System Response to Periodic Input Signals

From the Fourier Series, we saw that:

One Term of the Fourier Series of Input Signal

$x(t) = c_1\cos(2\pi f_o t+\theta_1)$



System Response to the Fourier Series Term of Input Signal

$y(t) = |H(f_o)|c_1\cos(2\pi f_o t+\theta_1+\angle H(f_o))$

LT: Definition

Limitations of the Fourier Transform

1. Fourier Transform is ______________ for a number of interesting signals
   – Fourier Transform does not converge for them
2. Fourier System Analysis: $Y(f)=X(f)H(f)$ only allows us to determine the ______________ of ______________
   – Only solve Differential Equations with no IC's
   – Can not determine Zero Input Responses

Region of Convergence (ROC)

For the Laplace Transform to exist, the Transform *integral* must converge, limiting the values of s (*or* σ) that we can use to only those for which:

$$\int_{-\infty}^{+\infty}\left|x(t)e^{-\sigma t}u(t)\right|dt<\infty \quad \text{(absolutely integrable)}$$

- Choosing σ *too small* could make it too weak to force convergence
- Choosing negative values for σ would take a converging function and make it go to $+\infty$ (!)

How do we choose and specify the allowed range of σ??

Methods of Computing Laplace Transforms

1. Using the integral Transform definitions $X(s)=\int_{0^-}^{\infty}x(t)e^{-st}dt$
2. Converting known Fourier Transforms for causal signals: $X(f)\Rightarrow X(s)$
   a) Drop any $\delta(f\pm f_0)$'s in the Fourier Transform
   b) Change $j2\pi f\Rightarrow s$ $(j\omega\Rightarrow s)$
3. Using known Laplace Transform pairs (from Tables) and modifications from L.T. Properties

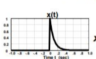
Region of Convergence (ROC)

The ROC of a unilateral Laplace transform (causal signal) is a right side half-plane.

LT: Properties

S Domain Shift: The S Domain "Dual" of the Time-Shift Property

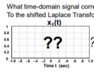
Time Domain $x(t)$ ⟷ $X(s)$ *Laplace Transform*



IF $x(t)\Rightarrow X(s)$

Shift in S Domain
$x_1(t)=ILT\{X(s-a)\}$
$=e^{+at}x(t)$
Multiplies Signal by e^{+at}

What time-domain signal corresponds To the shifted Laplace Transform??



THEN ???⇔X(s-a)

Multiply Signal x(t) by e^{+at}
$LT\{e^{+at}x(t)\}=X(s-a)$
Shifts X(s) by a

Convolution Property

- The most important property of the LAPLACE TRANSFORM for System Analysis:

$x(t)*h(t) \underset{InvLT}{\overset{LT}{\rightleftarrows}}$

The CONVOLUTION of two *causal* Functions in the Time Domain... | Results in MULTIPLICATION of the Laplace Transforms of the individual functions (Product of Complex Functions)

Property holds for BOTH the Unilateral AND Bilateral Laplace Transforms

Can use the *1-sided LTs* **Only IF** both $x(t)$ and $h(t)$ are ______________

LT: Transfer Function - poles and zeros

H(s) is a "rational function"

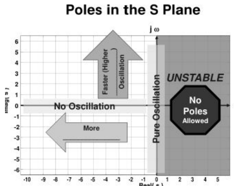
- ratio of functions of the same variable

$$H(s)=\frac{b_0+b_1s^1+b_2s^2+\dots+b_Ms^M}{s^N+a_{N-1}s^{N-1}+\dots+a_2s^2+a_1s+a_0}=\frac{Num(s)}{Denom(s)}$$

It is a

- "__________ *rational function"* if the order of the denominator polynomial (*N*) is equal to or greater than the order of the numerator polynomial (*M*) $N\geq M$
- "______________ *rational function"* if the order of the denominator polynomial (*N*) is greater than the order of the numerator polynomial (*M*) $N>M$

Poles in the S Plane



Stability Requirements

TIME DOMAIN - Diff Equation	LAPLACE S-PLANE - H(s)
(1) Highest y derivative order N ≥ Highest x deriv. order M; N ≥ M	(1) H(s) must be a ______________ ______________ N ≥ M; # of ______ ≥ # of ______
(2) All roots of the Characteristic Equation must have negative real parts	(2) All __________ of H(s) must have negative real parts (σ < 0) - All Poles in the ______________ (left of the jω axis)

Stability and the ROC of H(s)

- The Region of Convergence for a System Function H(s) **can not include any** ____________.
- The ROC of a right-sided time function (incl. Causal) takes the form of $\mathcal{R}e\{s\}=\sigma>\sigma_0$
  - ROC is to the right of some σ_0
- The ROC of a ***Causal System*** [causal h(t)] is $\mathcal{R}e\{s\}=\sigma>\mathcal{R}e\{p_r\}$ where p_r is the ______________________
  - ROC *is to the right of the rightmost pole*
- **For a Stable, Causal System, the ROC must include** ______________

LLT: Partial Fraction Expansion

Partial Fraction Expansion:

Special Cases:

2. **Repeated Roots (Poles)**
   - The coefficients of these pole terms are determined by multiplication of H(s) only by the highest power of the repeated pole term, and evaluating at the pole value.

$$H(s)(s+p_2)^2=\frac{N(s)(s+p_2)^2}{(s+p_1)(s+p_2)^2}=\frac{K_1(s+p_2)^2}{(s+p_1)}+\frac{A_0(s+p_2)^2}{(s+p_2)^2}+\frac{A_1(s+p_2)^2}{(s+p_2)}$$

$$H(s)(s+p_2)^2\Big|_{s=-p_2}=\frac{N(s)}{(s+p_1)}\Big|_{s=-p_2}=\frac{K_1\cancel{(s+p_2)^2}}{(s+p_1)}\Big|_{s=-p_2}+A_0+A_1\cancel{(s+p_2)}\Big|_{s=-p_2}$$

$$H(s)(s+p_2)^2\Big|_{s=-p_2}=A_0$$

Partial Fraction Expansion:

Special Cases:

2. **Repeated Roots (Poles)**
   - The coefficients for decreasing powers of the repeated pole term are determined from derivatives of the previous function, again evaluated at the pole value.

$$\frac{d}{ds}\left[H(s)(s+p_2)^2\right]_{s=-p_2}=\frac{d}{ds}\left[\frac{N(s)}{(s+p_1)}\right]_{s=-p_2}$$

$$=\frac{d}{ds}\left[\frac{K_1(s+p_2)^2}{(s+p_1)}\right]_{s=-p_2}+\frac{d}{ds}A_0+\frac{d}{ds}\left[A_1(s+p_2)\right]_{s=-p_2}$$

$$=\left[\frac{2K_1\cancel{(s+p_2)}}{(s+p_1)}-\frac{K_1\cancel{(s+p_2)^2}}{(s+p_1)^2}\right]_{s=-p_2}+0+A_1$$

$$\frac{d}{ds}\left[H(s)(s+p_2)^2\right]_{s=-p_2}=A_1$$

Partial Fraction Expansion:

Special Cases:

2. **Repeated Roots (Poles)**
   - If the root appears "*k*" times, the general expression for each coefficient of the descending powers of the pole terms are given by:

$$A_n=\frac{1}{n!}\frac{d^n}{ds^n}\left[H(s)(s+p_2)^k\right]_{s=-p_2}$$

where: $k-n$ = the power (exponent) of the pole in the partial fraction term that goes with A_n

Inverse Laplace Transform

Non-"Strictly Proper" Rational Functions:

Inverse Laplace Transform

Functions with Time Shifts (e^{-as}):

LT: Initial Value and Final Value theorems

Initial Value Theorem

Method:

1) Convert $X(s)$ to Strictly Proper $X_p(s)$ (if needed)
2) Multiply $X_p(s)$ by s.
3) Divide both numerator & denominator polynomials by the highest power of "s" in the denominator.
4) Evaluate $x(0^+)=\lim_{s\to\infty}\left[sX_p(s)\right]$

Final Value Theorem

Predicts the "final" value of the signal: $x(t)|_{t\to\infty}$

from its Laplace Transform $X(s)$:

$x(\infty)=$

IF: All Poles w/ σs<0 (any ω) $x(t)=Ae^{-\sigma t}\cos(\omega t+\theta)+..$ when common factors (s-a) in numerator & denominator cancelled (in case zero cancels bad pole). [OK: 1 pole @ s=0 (σ=0, ω=0)] $x(t)=Bu(t)+..$

In final value theorem don't worry about if X(s) Is strictly proper or not

Method for Finding Output of Non-Relaxed Systems (ZIR, Total Resp)

1. $x(t)\Rightarrow Laplace\ Transform\Rightarrow X(s)$
2. Laplace Transform Differential Equation Using Deriv. Property WITH Initial Conditions
   $Y(s)P(s)-y(0^-)Q(s)-y'(0^-)R(s)-..=X(s)Z(s)$
3. Isolate Y(s) and group terms on RHS
   $$Y(s)=\underbrace{X(s)\frac{Z(s)}{P(s)}}_{ZSR}+\underbrace{\frac{y(0^-)Q(s)+y'(0^-)R(s)+\dots}{P(s)}}_{ZIR}$$
4. Insert X(s) expression and y(0), y'(0) values
5. $y_{Total}(t)=InverseLaplaceTr\{Y(s)\}$

LT: Circuit Analysis

Non-Relaxed Laplace Circuit Models

Current Source (Norton) IC Models

Circuit Model	Time-Domain	Laplace Transform
R	$v(t)=Ri(t)$	$V(s)=RI(s)$
sL, $i_L(0)/s$	$v(t)=L\frac{d}{dt}i(t)$	With Initial Conditions: $V(s)=L[sI(s)-i_L(0)]$; $I(s)=\frac{V(s)}{sL}+\frac{i_L(0)}{s}$
$Cv_C(0)$, 1/sc	$i(t)=C\frac{d}{dt}v(t)$	$I(s)=C[sV(s)-v_c(0)]$; $I(s)=CsV(s)-Cv_c(0)$

Non-Relaxed Laplace Circuit Models

Voltage Source (Thevenin) IC Models

Circuit Model	Time-Domain	Laplace Transform
R	$v(t)=Ri(t)$	$V(s)=RI(s)$
sL, $Li_L(0)$	$v(t)=L\frac{d}{dt}i(t)$	With Initial Conditions: $V(s)=L[sI(s)-i_L(0)]$; $V(s)=sLI(s)-Li_L(0)$
1/sc, $v_C(0)/s$	$i(t)=C\frac{d}{dt}v(t)$	$I(s)=C[sV(s)-v_c(0)]$; $V(s)=I(s)\frac{1}{sC}+\frac{v_c(0)}{s}$

LT: Graphical Rep of Freq response

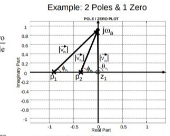
Graphical Interpretation of Frequency Response Values

$$|H(\omega_o)|=K\frac{|j\omega_o-z_1|}{|j\omega_o-p_1||j\omega_o-p_2|}=K\frac{\text{Product of distances to each zero}}{\text{Product of distances to each pole}}$$

$$\angle H(\omega_o)=\angle(\omega_o-z_1)-[\angle(\omega_o-p_1)\angle(\omega_o-p_2)]$$

= Sum of angles from each zero − Sum of angles from each pole $=(\theta_z)-(\theta_{p1}+\theta_{p2})$

Example: 2 Poles & 1 Zero



Approach is Useful for:
1) Finding response @ 1 freq. [or just evaluate complex $H(\omega_o)$]
2) Qualitative feel for $H(\omega)$ as move along $j\omega$ axis

Graphical Interpretation of Frequency Response Values

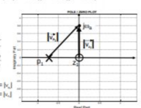
$$H(s)=K\frac{(s-z_1)(s-z_2)(s-z_3)\dots}{(s-p_1)(s-p_2)(s-p_3)\dots}$$

$$|H(s)|=K\frac{|(s-z_1)(s-z_2)(s-z_3)\dots|}{|(s-p_1)(s-p_2)(s-p_3)\dots|}=K\frac{|s-z_1||s-z_2||s-z_3|\dots}{|s-p_1||s-p_2||s-p_3|\dots}$$

At a particular frequency ω_o: $|H(\omega_o)|=|H(s)|$ at $s=j\omega_o$ on $j\omega$ axis

$$|H(\omega_o)|=K\frac{|j\omega_o-z_1||j\omega_o-z_2||j\omega_o-z_3|\dots}{|j\omega_o-p_1||j\omega_o-p_2||j\omega_o-p_3|\dots}=K\frac{\text{Product of distances from each zero}}{\text{Product of distances from each pole}}$$

Where
$|(\omega_o-p_1)|$ = length of vector from pole p_1 to $j\omega_o$
$|(\omega_o-z_1)|$ = length of vector from zero z_1 to $j\omega_o$



BODE PLOTS:

Pole or zero at s=0 (jw)^n:

Where n > 0 for zero
Where n < 0 for pole

$H(\omega)=(j\omega)^n$
$H(\omega)_{dB}=20nlog|j\omega|$
$\angle H(\omega)=(\pi/2)*n$

General Real zero or pole at s=-w_c:

Where n > 0 for zero
Where n < 0 for pole

$H(\omega)=(j\omega/\omega_c+1)^n$
$H(\omega)_{dB}=20*n*log|j\omega/\omega_c|$

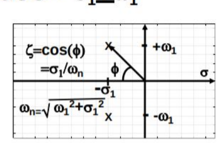
$$\angle H_3(\omega)=n\angle\left(1+j\frac{\omega}{\omega_c}\right)=\begin{cases}0, & \omega<<\omega_c\\ n\pi/4, & \omega=\omega_c\\ n\pi/2, & \omega>>\omega_c\end{cases}$$

$H(\omega)=(j\omega/\omega_c-1)^n$

Above shows a positive zero the only thing that as negative zero, difference is the phase is the phase is neg.

Complex Poles at $s=-\sigma_1\pm\omega_1$

$$H_5(\omega)\Rightarrow\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}$$



$$|H_{5dB}(\omega)|\approx\begin{cases}0, & \omega<<\omega_n\\ -20\log|2\zeta|, & \omega=\omega_n\\ -40\log\left|\frac{\omega}{\omega_n}\right|, & \omega>>\omega_n\end{cases}$$

$$\angle H_5(\omega)\approx\begin{cases}0, & \omega<<\omega_n\\ -\pi/4, & \omega=\omega_c\\ -\pi/2, & \omega>>\omega_n\end{cases}$$

Bode Plot is like a repeated pole, but with large deviation at ω_n that depends on ζ