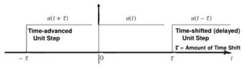
Signal Definitions:

UNIT - STEP:

Unit-Step Function

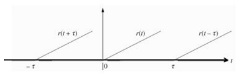
$$u(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$



$u(t+\tau) = \begin{cases} 1, & \text{for } t \geq \\ 0, & \text{for } t < \end{cases}$ $\quad u(t-\tau) = \begin{cases} 1, & \text{for } t \geq \\ 0, & \text{for } t < \end{cases}$

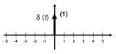
Unit-Ramp Function

$$r(t) = \begin{cases} t, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$



Unit Impulse (Delta Function)

$$\delta(t) = \begin{cases} \_\_, & t \neq 0 \\ \_\_, & t = 0 \end{cases}$$



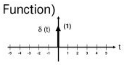
- Zero Duration (Infinitely narrow)
- Infinite Amplitude (height) @ $t = 0$
- Finite Area = ____ $\quad Area = \int_{-\infty}^{\infty} \delta(t)dt =$
- Infinite Energy $\quad Energy = \int_{-\infty}^{\infty} |\delta^2(t)| dt =$

OPERATIONS ON IMPULSES

a) Unit Impulse (Delta Function)

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

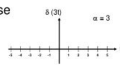
Area =1



Time-Scaled Impulse

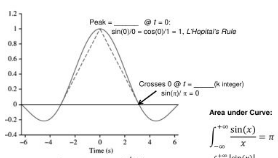
$\delta(\alpha t) = (\quad)\,\delta(t)$

Area =



IMPULSE PRODUCT/Siefting PROP:

Unit-Sinc Function



$\text{sinc}(t) = \frac{\sin t}{t}$

Area under Curve:

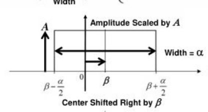
$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x} = \pi$

$\int_{-\infty}^{+\infty} \left|\frac{\sin(x)}{x}\right| = +\infty$

Scaled, Shifted Rectangular Pulse Function

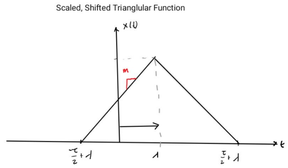
$$A\,\text{rect}\left(\frac{t-\beta}{\alpha}\right) = \begin{cases} A, & -\frac{1}{2} < \frac{t-\beta}{\alpha} < +\frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Amplitude · Time Shift · Width



How to find is setting the $\beta - \alpha/2$ to end point of function.

Scaled, Shifted Triangular Function



$A\,\text{tri}\left[\frac{t-\lambda}{\tau}\right] = \begin{cases} A - 2A\frac{|t-\lambda|}{\tau}, & -\frac{\tau}{2}+\lambda < t < \frac{\tau}{2}+\lambda \\ 0, & \text{elsewhere} \end{cases}$

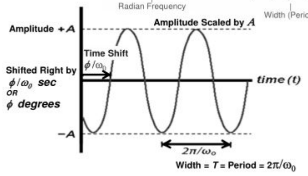
① find A set $m = \frac{2A}{\tau} \Rightarrow \boxed{A = \frac{m\tau}{2}}$

② set $-\frac{\tau}{2} + \lambda$ to left endpt and solve for τ.

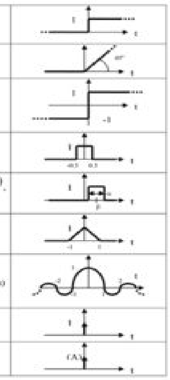
Scaled, Shifted Sinusoids

$$A \sin(\omega_0 t - \phi) = A \sin\left(2\pi\left[\frac{t - \frac{\phi}{\omega_0}}{T}\right]\right)$$

Phase Angle · Time Shift · Radian Frequency · Width (Period)



Unit Step	$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$, undefined for $t = 0$. Heaviside unit step: $u(0) = 0.5$	
Unit Ramp	$r(t) = \begin{cases} 0, t < 0 \\ t, t >= 0 \end{cases}$	
Signum (sign of argument)	$\text{sgn}(t) = \begin{cases} = 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$	
Rectangle or pulse	$rect(t) = \begin{cases} 1, \lvert t \rvert < 0.5 \\ 0, \quad else \end{cases}$, width = 1	
Rectangle or pulse	$rect\left(\frac{t-\beta}{\alpha}\right) = \begin{cases} 1, \beta - (\alpha/2) < t < \beta + (\alpha/2) \\ 0, \quad else \end{cases}$, width = α	
Triangle	$tri(t) = \begin{cases} 1 - \lvert t \rvert, & \lvert t \rvert < 1 \\ 0, & else \end{cases}$, width = 2	
Sinc	$\text{sinc}(t) = \sin(\pi t)/\pi t$; $\text{sinc}(t) = 0$, $t = \pm1, \pm2, \pm3, \ldots$ (zero-crossings); $\text{sinc}(0) = 1$	
Impulse	$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$ Area = 1	
Weighted Impulse	$A\,\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$ Area = A	

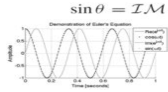


Complex Exponential Representation of Sinusoids

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \mathcal{RE}\left\{e^{j\theta}\right\} = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

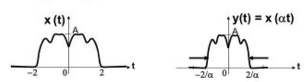
$$\sin\theta = \mathcal{IM}\left\{e^{j\theta}\right\} = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$



"Euler Identities"

Operations on signals:

Operations on Signals: Time Scaled: $\mathcal{S}\{x(t)\} = x(\alpha t)$



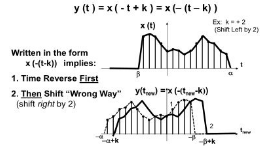
Compressed in time for |α| __ 1
Ex: y(t) = x( )

Expanded in time for |α| __ 1
Ex: y(t) = x( ) = x( )

$-2 < t < 2$ => $-2/\alpha < t < 2/\alpha$

Operations on Signals: Time Shift With Reversal

y (t) = x (- t + k) = x (– (t – k))



Periodicity, Energy and Power:

Signal Measures

Signal Areas & Integrals:

$Signal\ Area = \int_{-\infty}^{\infty} x(t)dt$

$Absolute\ Area = \int_{-\infty}^{\infty} |x(t)|dt$

$Absolutely\ Integrable: \int_{-\infty}^{\infty} |x(t)|dt \quad \_\_\_\_\_$ (finite area)

$Time\ Average = \frac{1}{-}\int \quad x(t)dt$

Signal Energy = Integral of Instantaneous Power over all time

$Signal\ Energy = \int_{-\infty}^{\infty} P_i(t)dt =$

Avg. Signal Power = Time average of Instant. Power over all time
= Total Signal Energy / Time Duration (*i.e.* energy per unit of time)

$Avg.\ Signal\ Power = \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |x^2(t)|dt$ — If nonperiodic signal

$= \qquad = \frac{\text{Energy in 1 Period}}{\text{Time in 1 Period}}$ — If periodic signal with period "T"

Energy vs. Power Signal:

Signal is a power signal if: (example - periodic fns)

- If the signal is not time-limited (infinite time duration.
- Has Infinite total signal energy
- Has finite non zero Power (Energy/Period)

Signal is an Energy signal if: (example - non periodic fns, half cycle sinusoid)

- Has finite signal Energy
- Has zero signal power($E/\infty = 0$)

Power of a Sinusoid: $P_{avg} = \frac{A^2}{2}$

Power of periodic signal made up of sinusoids ($X_0 is const$):

$P_{avg} = P_0 + \sum_{k=1}^{\infty} P_k = X_0^2 + \sum_{k=1}^{\infty} \frac{A_k^2}{2}$

Energy of half cycle sinusoid: $E = \frac{A^2 b}{2}$

Signal Measures - Periodic Signals

Power Signals: Finite Power / Energy = ∞

$Average\ Value \quad \bar{x} = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$ (DC value)

$Effective\ Value \quad x_{eff} = \int_{t_0}^{t_0+T} \quad dt = x_{rms}$ (rms value)

Effective Value = DC value having same average power as $x(t)$
= (square) **R**OOT of the **M**EAN (average) of $x(t)$ **S**QUARED
= $\sqrt{P}$

$Signal\ Power = \frac{1}{T}\int_T |x^2(t)|dt$

The "fundamental frequency" f_0 of the sum signal will be the "_______________________" of the individual frequencies.

Even and Odd Parts of f(t)

Any function $f(t)$ can be represented by the sum of an even symmetric function $f_e(t)$ and an odd symmetric function $f_o(t)$,

$$f(t) = f_e(t) + f_o(t)$$

determined by:

Even Part of $f(t)$: $f_e(t) = \frac{f(t) + f(-t)}{2}$

Odd Part of $f(t)$: $f_o(t) = \frac{f(t) - f(-t)}{2}$

System Classifications:

Checking Differential Equation For Linearity:

System is Nonlinear if the Diff. Equation describing it contains:

- **A constant term**
  $y(t) = x'(t) \underline{+\ 3}$
- **Products of inputs and/or output terms**
  $y'(t) = b_0 \mathbf{x^2(t)} + b_1 \mathbf{x'(t)y(t)} + \ldots$

Checking Differential Equation For Time-Invariance:

System is Time Varying if the Diff. Equ. describing its behavior contains:

- **Coefficients A_k or B_k that are functions of time t:**
  $y(t) = \underline{2t}x'(t) + \ldots$
- **Time-scaled inputs or output terms**
  $y(t) = b_0 \mathbf{x(\underline{2}t)} + \ldots$

Static *vs.* Dynamic

Static Systems (instantaneous)

- Determining the **present output of the system $y(t_0)$** at any particular time "t_0" **requires ______ ______________________ of the input $x(t_0)$**
- **All** x(t) and y(t) **terms** have the _________ arguments ( ).
- **Effects of previous inputs do not linger** to affect the output later – _______________
- **No derivatives or integrals** in system equation.
  $y(t) = \mathcal{S}\{x(t)\} = \mathbf{C}x(t) + \mathbf{A}$

Other Classes / Properties

Causal (non-anticipating)

- At any time "t_0" (the "present"), the output of the system $y(t_0)$ is determined completely by values of the input x(t) that arrived at the same or previous times … *i.e.* x(t) for t ___ t_0
- Causal systems do not use _________ values of the input $x(t+\tau)$ to determine the present output y(t)

Causal Systems (non-anticipating)

Determining Causality from the system differential equation:

1. If the equation is in terms of $y(t), y'(t), y''(t), \ldots$ **there are no __________terms in the equation**
2. **There are no _________________ terms:**

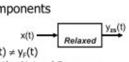
SOLVING DIFFERENTIAL EQNS:

Alternate Solution Linear Diff. Equ.

(Method of Undetermined Coefficients)

Total Response $y_T(t)$ has 2 Alternate Components

- **Zero State Response $y_{ZS}(t)$**
  - Different from the Forced Response: $y_{ZS}(t) \neq y_F(t)$
  - Includes the Forced Response and part of the Natural Response:



- **Zero Input Response $y_{ZI}(t)$**
  - In general, different from the Natural Response: $y_{ZI}(t) \neq y_N(t)$ [unless x(t) = 0 for t >0, so that y_{ZS}=0]
  - Same form as $y_N(t)$, but with different coefficients
  - Includes the part of the Natural Response due to IC's: Total Natural Response: $y_N(t) = y_{N_1}(t) + y_{N_2}(t)$
  - If system is truly "relaxed" (all IC's=0), $y_{ZI}(t) = 0$

Solution Method

Method of Undetermined Coefficients – **General Case**

Solution Steps:

1. Convert the Differential Equation to a **Single Input Case**
2. Find ZIR, the **zero-input response**, $y_{ZIR}(t) = y_{N_1}(t)$
3. Find the **forced response**, $y_F(t)$ of the Single Input Case
4. Find ZS1, the **zero-state response** of the Single Input Case

   $y_{ZS1}(t) = y_F(t) + y_{N_2}(t)$
5. Find ZSR, the **zero-state response** of the **General Case**
6. Find **General Case total response**, $y_T(t) = y_{ZSR}(t) + y_{ZIR}(t)$

Solution Method

Method of Undetermined Coefficients – **Single Input Case**

Solution Steps:

1. Find ZIR, the **zero-input response**, $y_{ZI}(t) = y_{N_1}(t)$

   [Solve with ________; using given I.C.'s.]

2. Find the **forced response**, $y_F(t)$

   [$y_F(t)$ is a solution of the Single Input Case Diff Equ.]

2. Find ZSR, the **zero-state response**, $y_{ZS}(t) = y_F(t) + y_{N_2}(t)$

   [Solve with actual x(t); using __________.]

Solution Method - General Case

When terms such as A x(t) + B dx(t)/dt appear on the right side of a generalized differential equation, a similar approach is used.

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_1y'(t) + a_0y(t) = A\,x(t) + B\,dx(t)/dt$$

1. We first find the **zero state response** of the **single input case** $y_{zs1}(t)$, the response due to just the x(t) forcing function, for the modified (single input case) differential equation:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_1y'(t) + a_0y(t) = x(t)$$

2. **Then linearity and superposition are used to find the overall ZSR** due to all the input terms [x(t) and its derivatives] on the right side of the equation, *i.e.*:

   If input **x(t)** → **yields S.I.C. ZS response $y_{zs1}(t)$** from the system,

   then input of [**A x(t) + B** *d*x(t)/*dt*] → **yields Gen. Zero State Response** [$Ay_{zs1}(t) + B\,dy_{zs1}(t)/dt$]

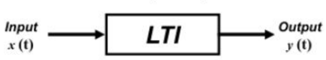
Special Case:
Forcing Function Matches Nat. Response Term

Forcing Function (RHS)		Form of Forced Response
C_0		C_1
$e^{\alpha t}$	*	$C\ e^{\alpha t}$
$\cos(\omega t + \theta_1)$		$C_1 \cos(\omega t) + C_2 \sin(\omega t)$ *or* $C \cos(\omega t + \theta_2)$
$e^{\alpha t}\cos(\omega t + \theta_1)$	*	$e^{\alpha t}\,[\,C_1 \cos(\omega t) + C_2 \sin(\omega t)\,]$
t^p		$C_0 + C_1t + C_2t^2 + \ldots + C_pt^p$
$t^p\,e^{\alpha t}$	*	$e^{\alpha t}\,[C_0 + C_1t + C_2t^2 + \ldots + C_pt^p\,]$
$t\cos(\omega t + \theta_1)$		$(C_1 + C_2\,t)\cos(\omega t) + (C_3 + C_4\,t)\sin(\omega t)$

Note: For entries including an $e^{\alpha t}$ forcing function (*), if 'α' is also a *root of the characteristic equation*, repeated r times, then the forced response should be multiplied by t^r.

STABILITY FROM LOOKING AT DE:

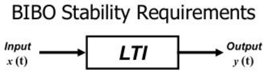
BIBO Stability Requirements



. Every root of the characteristic equation must have a ____________________

– Constrains natural response to be bounded

BIBO Stability Requirements



2. The **degree** of the **highest derivative of x(t)** in the differential equation (RHS) **must ______________ the *order* of the differential equation** (the highest derivative of y(t) on Left Hand Side)

   ⇩ ⇩ UNSTABLE

   $y'(t) + a_0y(t) = b_0\,x(t) + b_1\,x'(t) + b_2\,x''(t)$

Impulse response and step response:

Methods for Determining h(t)

1. Find the Impulse Response h(t) as the derivative of the Step Response s(t)
   – First find the Step Response by solving the *relaxed* system differential equation [ICs=0] for: x(t) = u(t)

2. **Solve the *relaxed* system differential equation for:** $x(t) = \delta(t)$

Method for Finding Impulse Response $h(t)$

1. Set all SIC Initial Conditions to zero (relaxed)

   y(0)=0, y´(0)=0,...

2. Set the highest order initial condition to 1

   $y^{n-1}(0)=$**1**

   Ex: n=2: y(0)=0, y´(0)=1

3. Solve Single-Input Diff Equ. for x(t) = 0
   – Solve for the natural response only ($y_F(t)=0$)
   – Use the above IC's to find unknown coef's

4. Determine generalized $h(t)$ same way as generalizing the ZSR.

Other System Properties from h(t)

Stability

- **For a stable system**:
  h(t) must be "_______________________"

Causality

- For a system to be causal:

  $h(t) =$ _____, for t < 0

Convolution:

Properties of Convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$

Time Shift: $x(t) * h(t) = y(t)$

$x(t-\tau) * h(t) = y(t-\tau)$

$x(t-\tau) * h(t-\alpha) =$

Derivative: $x(t) * h(t) = y(t)$

$x'(t) * h(t) =$

Properties of Convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$

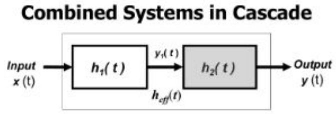
Convolution with Impulse

Convolution: $\delta(t) * h(t) =$ Multiplication: $\delta(t)h(t) =$

Convolution with a Shifted Impulse

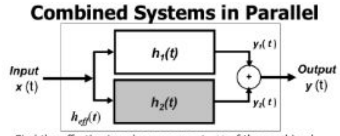
Convolution: $\delta(t-\tau) * h(t) =$ Multiplication: $\delta(t-\tau)h(t) =$

Combined Systems in Cascade



Find the effective impulse response $h_{eff}(t)$ of the combined **CASCADED** Filters:

Combined Systems in Parallel



Find the effective impulse response $h_{eff}(t)$ of the combined **PARALLEL** Filter:

Evaluation of Convolution Integral

Two Methods:

1. Analytical Evaluation

1. Use step functions u(t) to define the beginning and end of x() and h()
2. Break up the integral based on similar pairings of u(t)'s in x() and h() functions

– Best suited for convolution of two causal functions that are infinite duration
  - Then you're still doing "Convolution By Ranges", but there is only one solution "range"

2. Graphical Evaluation (Convolution By Ranges)

1. Break up problem into different regions based on where signal transitions and overlaps occur
2. In each region, find a unique solution using the appropriate analytical functions for x() and h()

Convolution by Ranges:

1.) First determine both x(t) and h(t) Range End Point.

2.) Then find pairwise sum

$REPT(y(t)) = SORT(set(PWS(REPT(h(t)), REPT(x(t)))))$

3.) Then write y(t) as a piecewise where y_1,.., y_k is the value between the endpoints.

GRAPHICAL CONVOLUTION

For each range of "t" determined above:

- Draw x(t-λ) relative to h(λ) {or h(t-λ) relative to x(λ)} on a single set of λ axes.
- Observe the endpoints of the regions where the two functions overlap. The beginning and endpoint of the overlapping regions (in terms of λ) determine the lower and upper limits of integration for λ in the convolution integral in this range.
- Solve the integral and evaluate over the upper and lower limits.
- **REPEAT Steps above for each range of t**

CHECKing answers for solution:

$$Lenth_y = Lenth_x + Lenth_h$$
$$T_{start,y} = T_{start,x} + T_{start,h}$$
$$T_{end,y} = T_{end,x} + T_{end,h}$$

Correlation:

Correlation

$$r_{xh}(t) = x(t) ** h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)dt = x(t) * h(-t)$$

- **Measure of similarity** between x, h
- Like **Convolution**, But Shift h(t) **without Flipping** it
- Not Commutative: $r_{xh}(t) = r_{hx}(-t)$
- "**Cross-Correlation**" if $x(t) \neq h(t)$
- "**Auto-Correlation**" if $x(t) = h(t)$

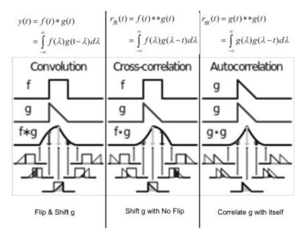
Cross-Correlation

- For **non-periodic** functions:

$$r_{12}(\tau) = \int_{-\infty}^{+\infty} x_1(t)x_2(t-\tau)dt, \qquad -\infty < \tau < +\infty$$

- For **periodic** functions, with period T, the correlation function is given by:

$$r_{12}(\tau) = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} v_1(t)v_2(t-\tau)dt$$



Summary - Correlation

- Correlation is computed similarly to Convolution, but without the signal flip

  $$r_{xh}(\tau) = x(t) ** h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)dt = x(t) * h(-t)$$

- **Cross-Correlation** $r_{xh}(\tau)$ reveals the **degree of similarity** between two different signals, and how the similarity **changes with time alignment** of the signals

- **Auto-Correlation** $r_{xx}(\tau)$ (correlating a signal with itself) reveals the **randomness or periodicity** of a signal

  $$r_{xx}(\tau) = x(t) ** x(t) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = x(t) * x(-t)$$

- Filtering (convolution) a noisy signal with a time-reversed clean version of a signal implements ***correlation detection*** (also called "**matched filtering**") by cross correlating the noisy and clean signals to **find if the signal is detected and at what time position.**