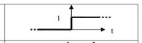
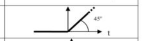
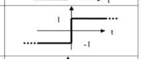
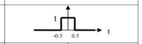
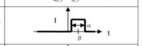
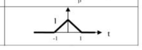
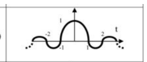
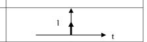
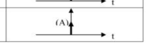
Signal Definition

EE228 – Common Signals

Unit Step	$u(t)=\begin{cases}0, t<0\\1, t>0\end{cases}$, undefined for t = 0. Heaviside unit step: $u(0)=0.5$	
Unit Ramp	$r(t)=\begin{cases}0, t<0\\t, t>=0\end{cases}$	
Signum (sign of argument)	$\mathrm{sgn}(t)=\begin{cases}+1, & t>0\\0, & t=0\\-1, & t<0\end{cases}$	
Rectangle or pulse	$rect(t)=\begin{cases}1, \lvert t\rvert<0.5\\0, \quad else\end{cases}$, width = 1	
Rectangle or pulse	$rect\left(\frac{t-\beta}{\alpha}\right)=\begin{cases}1, \beta-(\alpha/2)<t<\beta+(\alpha/2)\\0, \quad else\end{cases}$, width = α	
Triangle	$tri(t)=\begin{cases}1-\lvert t\rvert, & \lvert t\rvert<1\\0, & else\end{cases}$, width = 2	

Sinc	$\mathrm{sinc}(t)=\sin(\pi t)/\pi t$ $\mathrm{sinc}(t)=0,\ t=\pm1,\pm2,\pm3,\ldots$ (zero crossings) $\mathrm{sinc}(0)=1$	
Impulse	$\delta(t)=\begin{cases}0, & t\neq0\\\infty, & t=0\end{cases}$ Area = 1	
Weighted Impulse	$A\,\delta(t)=\begin{cases}0, & t\neq0\\\infty, & t=0\end{cases}$ Area = A	

$$e^{j\theta}=\cos\theta+j\sin\theta$$

$$\cos\theta=\mathcal{RE}\left\{e^{j\theta}\right\}=\frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta=\mathcal{IM}\left\{e^{j\theta}\right\}=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

...stration of Euler's Equation

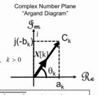
FS: Coefficients Relationship:

Fourier Series Coefficients Related

Same Complex # Value:

$$C_k e^{+j\theta_k}=C_k\angle+\theta_k=a_k-jb_k$$
$$=2X[k]=2|X[k]|\angle X[k],\quad k>0$$

Complex Number Plane "Argand Diagram"



$$Magnitude: C_k=|a_k-jb_k|=\sqrt{a_k^2+b_k^2}=2|X[k]|_{k>0}$$
$$Phase:\ \theta_k=\angle(a_k-jb_k)=\tan^{-1}\left(\frac{-b_k}{a_k}\right)=\angle X[k]_{k>0}$$
$$DC\ (k=0):\ C_0=a_0=X[0],\quad b_0=0,\quad \theta_0=0$$

Exponential Fourier Series Coefficient Symmetry

For $k>0$:

$$X[k]=\frac{1}{2}C_k e^{+j\theta_k}=\frac{1}{2}C\angle+\theta_k \qquad X[-k]=\frac{1}{2}C_k e^{-j\theta_k}=\frac{1}{2}C\angle-\theta_k$$

$$|X[k]|=|X[-k]|=\frac{1}{2}C_k$$ **|X[k]| is "Even Symmetric"**

$$\angle X[k]=-\angle X[-k]=\theta_k$$ **∠X[k] is "Odd Symmetric"**

$$X^*[k]=X[-k]$$ **X[k] has "Conjugate Symmetry"**

$$\Re e\{X[k]\}=\Re e\{X[-k]\}$$
$$Im\{X[k]\}=-Im\{X[-k]\}$$

FS: Finding Fourier Coefficients:

Finding Exponential Fourier Series Coefficients: a_k, b_k

By a similar derivations, it can be shown that:

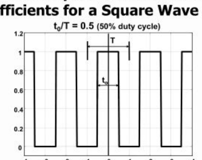
$a_k=$ $\qquad k>0$

$b_k=$ $\qquad k>0$

So that: $c_k=\sqrt{a_k^2+b_k^2}\qquad \theta_k=\tan^{-1}\left(\frac{-b_k}{a_k}\right)$

Find the Exponential Fourier Series Coefficients for a Square Wave Signal

$t_0/T=0.5$ (50% duty cycle)



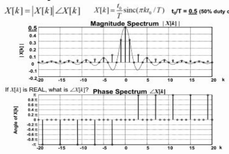
$$X[k]=\frac{1}{T}\int_T x(t)e^{-j2\pi kf_0t}dt$$

Exponential Fourier Coefficients

$$X[k]=\frac{1}{T}\int_T x(t)e^{-j2\pi kf_0t}dt=\frac{1}{T}\int_{-t_0/2}^{t_0/2}x(t)e^{-j2\pi kt/T}dt$$
$$=\frac{1}{T}\int_{-t_0/2}^{t_0/2}1e^{-j2\pi kt/T}dt=\frac{1}{T}\frac{e^{-j2\pi kt/T}}{-j2\pi k/T}\Bigg|_{t=-t_0/2}^{t=t_0/2}$$
$$=\frac{1}{-j2\pi k}\left[e^{-j2\pi kt_0/2T}-e^{-j2\pi k(-t_0/2T)}\right]$$
$$=\frac{1}{-j2\pi k}\left[e^{-j\pi kt_0/T}-e^{+j\pi kt_0/T}\right]=\frac{1}{\pi k}\left[\frac{e^{+j\pi kt_0/T}-e^{-j\pi kt_0/T}}{2j}\right]$$
$$=\frac{\sin(\pi kt_0/T)}{\pi k}=\frac{t_0}{T}\frac{\sin(\pi kt_0/T)}{\pi kt_0/T}=\boxed{}$$

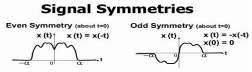
Exponential Fourier Series

$$X[k]=|X[k]|\angle X[k]\qquad X[k]=\frac{t_0}{T}\mathrm{sinc}(\pi kt_0/T)$$ $t_0/T=$ 0.5 (50% duty cycle)

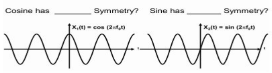


FS: Shortcut (Signal Symmetrys)

Signal Symmetries

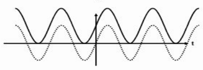


Cosine has ________ Symmetry? Sine has ________ Symmetry?



Odd Symmetry

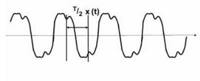
Sometimes Odd Symmetry can be **hidden** by a DC Offset



$X_1(t)=0.5+\sin(2\pi f_0t)$

Odd Symmetry Revealed When DC Removed

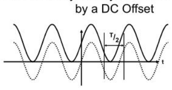
Half-Wave Symmetry



x (t) = ___________

Half-Wave Symmetry

Half-Wave Symmetry can also be **hidden** by a DC Offset



$X_1(t)=0.5+\sin(2\pi f_0t)$

Half-Wave Symmetry Revealed When DC Removed

Signal Symmetries

$$a_k=\frac{2}{T}\int_0^T x_p(t)\cos(2\pi kf_0t)dt\qquad b_k=\frac{2}{T}\int_0^T x_p(t)\sin(2\pi kf_0t)dt$$

- **$x_p(t)$ has Even Symmetry:**

$$a_k=\frac{}{T}\int_0\ x_p(t)\cos(2\pi kf_0t)dt\qquad b_k=$$
$k>0$

- **$x_p(t)$ has Odd Symmetry:**

$$b_k=\frac{}{T}\int_0\ x_p(t)\sin(2\pi kf_0t)dt\qquad a_k=$$
$k>0$

Signal Symmetries

- **$x_p(t)$ has Half-Wave Symmetry:**

$$a_k=\frac{4}{T}\int_0^{T/2}x_p(t)\cos(2\pi kf_0t)dt\qquad k=odd$$
$$b_k=\frac{4}{T}\int_0^{T/2}x_p(t)\sin(2\pi kf_0t)dt\qquad k=odd$$

Note: *You can not use the half-wave symmetry integrals above if the half-wave symmetry is **"hidden"** (i.e. if there is a DC offset).*
-The $k_{even}=0$ result for the Fourier Coefficients DOES apply for $k\geq2$ if there is "hidden" half-wave symmetry

Signal Symmetries

- **$x_p(t)$ has Even and Half-Wave Symmetry:**

$$a_k=\frac{8}{T}\int_0^{T/4}x_p(t)\cos(2\pi kf_0t)dt\qquad k=odd$$

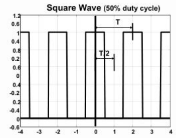
- **$x_p(t)$ has Odd and Half-Wave Symmetry:**

$$b_k=\frac{8}{T}\int_0^{T/4}x_p(t)\sin(2\pi kf_0t)dt\qquad k=odd$$

Note: *You can not use the half-wave symmetry integrals above if the half-wave symmetry is **"hidden"** (i.e. if there is a DC offset).*
-The $k_{even}=0$ result for the Fourier Coefficients DOES apply for $k\geq2$ if there is "hidden" half-wave symmetry

What Symmetries Does It Have?

Square Wave (50% duty cycle)

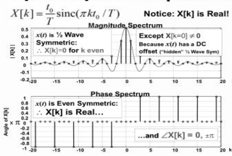


Even/Odd?

Half-Wave?

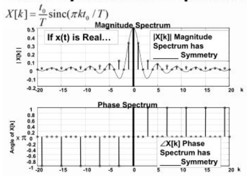
Symmetry Seen in Spectrum

$$X[k]=\frac{t_0}{T}\mathrm{sinc}(\pi kt_0/T)$$ **Notice: X[k] is Real!**

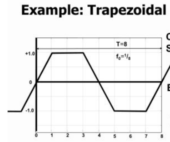


Other Symmetries in Spectrum

$$X[k]=\frac{t_0}{T}\mathrm{sinc}(\pi kt_0/T)$$



Example: Trapezoidal Signal



Check for Symmetries:

Even/Odd?

Half-Wave?

$$b_k=\frac{8}{T}\int_0^{T/4}x_p(t)\sin(2\pi kf_0t)dt\qquad k=odd$$

$$b_k=\frac{8}{8}\int_0^2 x_p(t)\sin(2\pi kt/8)dt=\int_0^1 t\sin(\pi kt/4)dt+\int_1^2 1\sin(\pi kt/4)dt$$
$$=\left\{\left[\sin(\pi kt/4)-\frac{\pi kt}{4}\cos(\pi kt/4)\right]\left(\frac{1}{\pi k/4}\right)^2\Bigg|_0^1-\left[\frac{\cos(\pi kt/4)}{\pi k/4}\right]\Bigg|_1^2\right\}$$
$$=\left[\frac{\sin(\pi k/4)-(\pi k/4)\cos(\pi k/4)-\sin(0)+0}{(\pi k/4)^2}\right]-\left[\frac{\cos(2\pi k/4)-\cos(\pi k/4)}{\pi k/4}\right]$$
$$=\frac{16\sin(\pi k/4)}{(\pi k)^2}-\frac{\cos(\pi k/4)+\cos(\pi k/2)-\cos(\pi k/4)}{\pi k/4}$$
$$=\frac{16\sin(\pi k/4)}{\pi^2k^2}-\frac{\cos(\pi k/2)}{\pi k/4}$$

$\cos(\pi k/2)=0\quad(k\ \text{odd})$
$\cos(\pi/2)=0,\quad\cos(3\pi/2)=0,\quad\cos(5\pi/2)=0$

$$\boxed{b_k=\frac{16\sin(\pi k/4)}{\pi^2k^2}\quad(k\ \text{odd})}\qquad \int t\sin(at)dt=\frac{\sin(at)-at\cos(at)}{a^2}$$

FS: properties

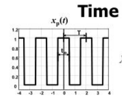
Properties of Fourier Series

Symmetries:

If $x(t)$ real:	$x(t)=x^*(t)$	$\Longleftrightarrow X_n=X^*_{-n}$	**Conjugate Symmetric**
If $x(t)$ even sym:	$x(t)=x(-t)$	$\Longleftrightarrow X_n=X_{-n}$	**Real**
If $x(t)$ odd sym:	$x(t)=-x(-t)$	$\Longleftrightarrow X_n=-X_{-n}$	**Imaginary**
If ½ wave sym:	$x(t)=-x(t+T/2)$	$\Longleftrightarrow X_n=0$ for n even	
Even part of $x(t)$:	$\frac{x(t)+x(-t)}{2}$	$\Longleftrightarrow \mathcal{RE}\{X_n\}$	
Odd part of $x(t)$:	$\frac{x(t)-x(-t)}{2}$	$\Longleftrightarrow j\mathcal{IM}\{X_n\}$	
Linearity:	$ax(t)+by(t)$	$\Longleftrightarrow aX_n+bY_n$	
Time Reversal:	$x(-t)$	$\Longleftrightarrow X^*_n$	
Time Shift:	$x(t-t_0)$	$\Longleftrightarrow X_ne^{-j2\pi nt_0/T}$	

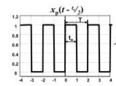
Time Delay

$x_p(t)$



$$X[k]=\frac{t_0}{T}\mathrm{sinc}(\pi kt_0/T)$$

$x_p(t-t_0/2)$



Time Shifting $x_p(t)$ by $(t_0/2)$

$$X_{shift}[k]=\frac{t_0}{T}\mathrm{sinc}(\pi kt_0/T)e^{\underline{-j2\pi kf_0t_0/2}}$$

Leads to a ***Phase Shift*** in $X[k]$ by

$\theta=$ ______________ $(t_0/2)$ Time Shift

FS: POWER

Power in Harmonic Signals

"___________ Theorem"

So we can compute the power in a harmonic signal from the **time-domain** waveform itself:

$$Signal\ Power=\frac{1}{T}\int_T|x_p^2(t)|dt=$$

OR from its frequency spectrum & **Fourier Series** coef:

And we should get the same answer!!

$|X[k]|^2$ – "Power Spectral Density"

Power in Harmonic Signals

$$P=\left[c_0^2+0.5\sum_{k=1}^{\infty}c_k^2\right]\qquad \text{Polar}$$
$$=\left[a_0^2+0.5\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right]\qquad \text{Trigonometric}$$
$$=\sum_{k=-\infty}^{\infty}|X[k]|^2=|X[0]|^2+2\sum_{k=1}^{\infty}|X[k]|^2$$

Exponential (2-sided)

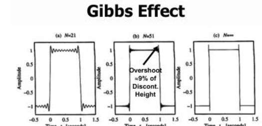
Power in Few Harmonics

The Power in a subset of the harmonic frequencies in a signal can also be found using the **Exponential Fourier Coefficients** of those harmonics:

$$P_{(\text{in } k=k_1\to k_2)}=0.5\sum_{k=k_1}^{k_2}c_k^2$$
$$=0.5\sum_{k=k_1}^{k_2}\left[2|X[k]|\right]^2=0.5\sum_{k=k_1}^{k_2}4|X[k]|^2$$
$$P_{(\text{in } k=k_1\to k_2)}=\boxed{}$$

FS: GIBBS EFFECT

Gibbs Effect

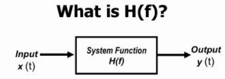


Even with an infinite number of harmonics, a signal with discontinuities can not be exactly reconstructed from its Fourier Series

FS: System Function

$$x(t)=\sum_k c_k\cos(2\pi kf_0t+\theta_k)$$

What is H(f)?



System (Transfer) Function:
- **Relation of output signal $y(t)$ to input signal $x(t)$**
- **Frequency-dependent (changes with f) scale factor**
- **Complex value:**

H(f) = |H(f)| ∠H(f)
Magnitude Scale Factor (Gain)   Phase Shift (Time Shift)

$$y(t)=\sum_k|H(kf_0)|\ c_k\cos(2\pi kf_0t+\theta_k+\angle H(kf_0))$$

Fourier Series Concepts

- Any (almost) periodic signal can be represented by a combination of harmonically-related ($f=kf_0$, k integer) sinusoids (sin / cos / $e^{j2\pi ft}$)
  - Requirements: x(t) absolutely integrable, finite # of finite discontinuities, finite # max/mins
- Reconstruction from Fourier Series will be imperfect at discontinuities (overshoot)
  - "Gibbs Effect"

FT: Integral Definition

Fourier Series → Fourier Transform

Exponential Form:

$$x_p(t)=\sum_{k=-\infty}^{+\infty}X[k]e^{j2\pi kf_0t}\qquad X[k]=\frac{1}{T}\int_{-T/2}^{T/2}x_p(t)e^{-j2\pi kf_0t}dt$$

Period T ⇒ ∞, f_0 ⇒ 0
kf_0 ⇒ Continuous f

$$X[k]=\frac{1}{\infty}\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt$$ ***Uh-Oh!!***

$$\sum_{k=-\infty}^{+\infty}\Rightarrow\int_{-\infty}^{+\infty}\qquad \lim_{T\to\infty}\{TX[k]\}=\frac{T}{T}\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt$$

X[k] **⇒ Continuous $X(f)$**

Inverse Fourier Transform: **Fourier Transform:**

What IS the Fourier Transform?

Like the Fourier Series:
- It is a ____________________description of a signal
  - Tells us how much of what frequencies are present in a signal
- It is a ___________ valued function

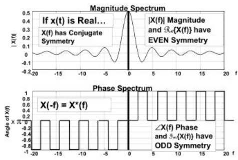
$$X(f)=|X(f)|\angle X(f)=|X(f)|\,e^{+j\angle X(f)}$$
$$=\Re e\{X(f)\}+j\,Im\{X(f)\}$$

What IS the Fourier Transform?

Unlike the Fourier Series:

- It is a _______________of frequency X(f)
  - Not a collection of particular, discrete, harmonically related frequencies X[kf_0]
  - Uses ALL frequencies to recreate signal
- Represents _______________ signals

Symmetries in Fourier Spectrum



Signal Symmetries

x(t) is Real and has Even Symmetry:
- X(f) is _______ and has _______ Symmetry
- ∠X(f) = ______ or ______

x(t) is Real and has Odd Symmetry:
- X(f) is _______ and has ________ Symmetry
- ∠X(f) = ___________

Existence of Fourier Transforms

A non-periodic signal x(t) can be represented by a unique Fourier Transform IF:

1. x(t) is "absolutely integrable" $\int_{-\infty}^{\infty}|x(t)|dt < \infty$ Integral over ALL time t
2. x(t) has a finite number of maxima & minima.
3. x(t) has a finite number of finite discontinuities.

Existence of Fourier Transforms

However, some signals that don't meet these strict requirements still have valid Fourier Transforms.

(Ex: x(t)=u(t) is not abs. integrable $\int \to \infty$)

- F.T. of an "absolutely integrable" signal _________________
- F.T. of a "not absolutely integrable" signal _________________, if it does not grow exponentially (e^{+at}).
- F.T. of a signal that grows exponentially or faster _________________.

FT AND FS Relationship:

Methods of Computing Fourier Transforms and Inverse Transforms

1. Converting known exponential Fourier Series coefficients of a periodic signal version $x_p(t)$

$$X_p[k] \Rightarrow X(f) = T\,X_p[k]_{kf_0 \to f}$$

Fourier Series of A Periodic Signal $x_p(t)$ — **Fourier Transform of A Non-periodic Signal $x(t)$ made up of 1 Period of the periodic signal $x_p(t)$**

FT: Properties

Duality: Similarity Theorem

Fourier Transform: $X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$

Inverse Fourier Transform: $x(t) = \int_{-\infty}^{+\infty} X(f)e^{+j2\pi ft}df$

If FT{ $x(t)$ } = X(f)

X( ) has different functional form than x( )

Then FT{ X(t) } = x(-f)

For functions with Even Symmetry in x( ):

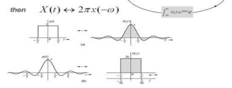
$x(+t) = x(-t)$ X(+f) = X(-f)

So FT{ X(t) } = x(f)

Properties of the Fourier Transform

- Duality (Similarity) :
- Let $x(t) \Leftrightarrow X(\omega)$

then $X(t) \leftrightarrow 2\pi x(-\omega)$



Time / Frequency Scaling

- There is a coupled and inverse relationship ("duality") between the width of a signal representation in the Time Domain vs. the FT Frequency Domain

Time Scaling Property $x(\alpha t) \xrightleftharpoons[InvFT]{FT} \frac{1}{|\alpha|}X(f/\alpha)$

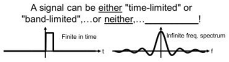
$\xrightleftharpoons[InvFT]{FT} X(\beta f)$ Frequency Scaling Property

Time / Frequency Scaling

An extension of this concept is the Time/Frequency Limit Theorem:

A signal can not have BOTH a finite time duration ("time-limited") AND a finite length frequency spectrum ("band-limited").

A signal can be either "time-limited" or "band-limited",...or neither,...............!



FT: ENERGY

Energy in Non-Periodic Signals

We can compute the energy in an aperiodic signal from the time-domain waveform itself:

$$Total\ Energy = \int_{-\infty}^{\infty} x^2(t)dt = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

= _______________

OR from its Fourier Transform frequency spectrum:

And we should get the same answer!!

"Parseval's Theorem"

(again)

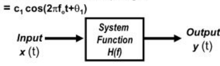
---FT: input a sinusoid

System Response to Periodic Input Signals

From the Fourier Series, we saw that:

One Term of the Fourier Series of Input Signal

$x(t) = c_1\cos(2\pi f_o t+\theta_1)$



System Response to the Fourier Series Term of Input Signal

$y(t) = |H(f_0)|c_1\cos(2\pi f_o t+\theta_1+\angle H(f_0))$

LT: Definition

Limitations of the Fourier Transform

1. Fourier Transform is _______________ for a number of interesting signals
   - Fourier Transform does not converge for them
2. Fourier System Analysis: Y(f)=X(f)H(f) only allows us to determine the ___________ __________of _________________,
   - Only solve Differential Equations with no IC's
   - Can not determine Zero Input Responses

Two Forms of the Laplace Transform

Unilateral Laplace transform:

$$\mathcal{L}[x(t)] = X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

For *Causal* Systems....and *Causal* Input Signals

Bilateral Laplace transform:

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

For *Non-Causal* Systems....or *Non-Causal* Input Signals

Region of Convergence (ROC)

For the Laplace Transform to exist, the Transform *integral* must converge, limiting the values of s (or σ) that we can use to only those for which:

$$\int_{-\infty}^{+\infty}|x(t)e^{-\sigma t}u(t)|dt < \infty \quad \text{(absolutely integrable)}$$

- Choosing σ *too small* could make it too weak to force convergence
- Choosing negative values for σ would take a converging function and make it go to $+\infty$ (!)

How do we choose and specify the allowed range of σ??

Methods of Computing Laplace Transforms

1. Using the integral Transform definitions
$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$
2. Converting known Fourier Transforms for causal signals: X(f) ⇒ X(s)
   a) Drop any δ(f±f0)'s in the Fourier Transform
   b) Change $j2\pi f \Rightarrow s$ ($j\omega \Rightarrow s$)
3. Using known Laplace Transform pairs (from Tables) and modifications from L.T. Properties

Region of Convergence (ROC)

The ROC of a unilateral Laplace transform (causal signal) is a right side half-plane.

LT: Properties

Time Scaling

$$LT_1\{x(\alpha t)\} = \int_{0^-}^{\infty} x(\alpha t)e^{-st}dt$$

Change variables: Let λ=αt t=λ/α dλ=αdt dt=dλ/α Limits of Integration: As t→∞ λ→∞, α>0

$$LT_1\{x(\alpha t)\} = \int_{0^-}^{\infty} x(\lambda)e^{-s\lambda/\alpha}d\lambda/\alpha, \quad \alpha>0$$

$$= \frac{1}{\alpha}\underline{\int_{0^-}^{\infty} x(\lambda)e^{-(s/\alpha)\lambda}d\lambda} = X(s/\alpha)$$

$$LT_1\{x(\alpha t)\} = \frac{1}{\alpha}X(s/\alpha)$$

Time Reversal

$$LT_1\{x(-t)\} = \int_0^{\infty} x(-t)u(-t)e^{-st}dt \quad \text{If } x(t) \text{ is causal}$$

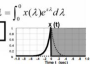
Change variables: Let λ=–t t=-λ dλ=–dt dt=-dλ Limits of Integration: As t→∞ λ→ –∞

$$LT_1\{x(-t)\} = \int_0^{-\infty} x(\lambda)u(\lambda)e^{s\lambda}(-d\lambda)$$

$$= \int_{-\infty}^{0} x(\lambda)u(\lambda)e^{s\lambda}d\lambda = \int_0^0 x(\lambda)e^{s\lambda}d\lambda$$

$$\boxed{LT_1\{x(-t)\} = 0 \text{ for causal } x(\lambda)}$$

Can't do time reversal with 1-sided Laplace Transform



Time Shift Property

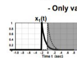
Fourier Transform:

$$X_1(f) = FT\{x(t-t_0)\} = e^{-j2\pi ft_0}X(f)$$

Unilateral Laplace Transform:

$$X_1(s) = LT_1\{x(t-t_0)\underline{u(t-t_0)}\} = e^{-st_0}X(s),$$

- Need the explicit unit step function
- Only valid for _______________ $\boxed{t_0 > 0}$

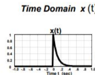


(Part of signal not transformed if time advanced)

$$X(s) = \int_{0^-}^{\infty} x(t-t_0)e^{-st}dt$$

S Domain Shift: The S Domain "Dual" of the Time-Shift Property

Time Domain x (t) $\xleftrightarrow{\text{Laplace Transform}}$ X (s) *Laplace Transform*



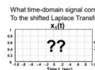
IF x(t) ⇨ X(s)

Shift in S Domain

$$x_1(t) = ILT\{X(s-a)\} = e^{+at}x(t)$$

Multiplies Signal by e^{+at}

What time-domain signal corresponds To the shifted Laplace Transform??



THEN ???⇦X(s-a)

Multiply Signal x(t) by e^{+at}

$$LT\{e^{+at}x(t)\} = X(s-a)$$

Shifts X(s) by a

Convolution Property

- The most important property of the LAPLACE TRANSFORM for System Analysis:

$$x(t) * h(t) \xrightleftharpoons[InvLT]{LT}$$

The CONVOLUTION of two *causal* Functions in the Time Domain... — **Results in MULTIPLICATION of the Laplace Transforms of the individual functions (Product of Complex Functions)**

Property holds for BOTH the Unilateral AND Bilateral Laplace Transforms

Can use the *1-sided LTs* Only IF both x(t) and h(t) are _____________

LT: Transfer Function - poles and zeros

$H(s)$ is a "rational function"

- ratio of functions of the same variable

$$\boxed{H(s) = \frac{b_0 + b_1s^1 + b_2s^2 + ... + b_Ms^M}{s^N + a_{N-1}s^{N-1} + ... + a_2s^2 + a_1s + a_0} = \frac{Num(s)}{Denom(s)}}$$

It is a

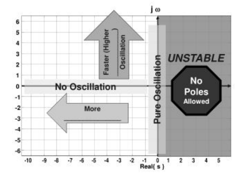
- "_________ ***rational function"*** if the order of the denominator polynomial (N) is equal to or greater than the order of the numerator polynomial (M)

$N \geq M$

"_____________***rational function"*** if the order of the denominator polynomial (N) is greater than the order of the numerator polynomial (M)

$N > M$

Poles in the S Plane



Stability Requirements

TIME DOMAIN - Diff Equation	*LAPLACE S-PLANE - H(s)*
)Highest y derivative order N ≥ Highest x deriv. order M N ≥ M	(1) H(s) must be a ____________ ____________ N ≥ M # of _______ ≥ # of _________
2) All roots of the Characteristic Equation must have negative real parts	(2) All __________ of H(s) must have negative real parts (σ < 0) - All Poles in the ____________ (left of the jω axis)

Stability and the ROC of H(s)

The **Region of Convergence** for a System Function H(s) **can not include any** _____________.

The ROC of a right-sided time function (incl. Causal) takes the form of $\mathcal{R}e\{s\} = \sigma > \sigma_0$
- ROC is to the right of some σ_0

The ROC of a ***Causal System*** [causal h(t)] is $\mathcal{R}e\{s\} = \sigma > \mathcal{R}e\{p_r\}$

where p_r is the _____________________________
- ROC is *to the right of the rightmost pole*

For a Stable, Causal System, the ROC must include _______________

LT: Partial Fraction Expansion

Partial Fraction Expansion:

Special Cases:

2. **Repeated Roots (Poles)**
   - The coefficients of these pole terms are determined by multiplication of H(s) only by the highest power of the repeated pole term, and evaluating at the pole value.

$$H(s)(s+p_2)^2 = \frac{N(s)(s+p_2)^2}{(s+p_1)(s+p_2)^2} = \frac{K_1(s+p_2)^2}{(s+p_1)} + \frac{A_0(s+p_2)^2}{(s+p_2)^2} + \frac{A_1(s+p_2)^2}{(s+p_2)}$$

$$H(s)(s+p_2)^2\Big|_{s=-p_2} = \frac{N(s)}{(s+p_1)}\Big|_{s=-p_2} = \frac{K_1(\overset{0}{\cancel{s+p_2}})^2}{(s+p_1)}\Big|_{s=-p_2} + A_0 + A_1(\overset{0}{\cancel{s+p_2}})\Big|_{s=-p_2}$$

$$H(s)(s+p_2)^2\Big|_{s=-p_2} = A_0$$

Partial Fraction Expansion:

Special Cases:

2. **Repeated Roots (Poles)**
   - The coefficients for decreasing powers of the repeated pole term are determined from derivatives of the previous function, again evaluated at the pole value.

$$\frac{d}{ds}\left[H(s)(s+p_2)^2\right]_{s=-p_2} = \frac{d}{ds}\left[\frac{N(s)}{(s+p_1)}\right]_{s=-p_2}$$

$$= \frac{d}{ds}\left[\frac{K_1(s+p_2)^2}{(s+p_1)}\right]_{s=-p_2} + \frac{d}{ds}A_0 + \frac{d}{ds}\left[A_1(s+p_2)\right]_{s=-p_2}$$

$$= \left[\frac{2K_1(\overset{0}{\cancel{s+p_2}})^1}{(s+p_1)} - \frac{K_1(\overset{0}{\cancel{s+p_2}})^2}{(s+p_1)^2}\right]_{s=-p_2} + 0 + A_1$$

$$\frac{d}{ds}\left[H(s)(s+p_2)^2\right]_{s=-p_2} = A_1$$

al Fraction Expansion:

ases:

ated Roots (Poles)

e root appears "k" times, the general expression for coefficient of the descending powers of the pole s are given by:

$$\frac{1}{n!}\frac{d^n}{ds^n}\left[H(s)(s+p_2)^k\right]_{s=-p_2}$$

: k-n = the power (exponent) of the pole in the partial fraction term that goes with A_n

Inverse Laplace Transform

Non-"Strictly Proper" Rational Functions:

Inverse Laplace Transform

Functions with Time Shifts (e^{-as}):