Name: Answer Key

Problem #1: (10 pts)

A Relaxed, Linear, Time-Invariant system is described by the following differential equation:

$$y''(t) + 6y'(t) + 9y(t) = 9x(t-1) \qquad \textbf{Initial Conditions: } y(0)=y'(0)=0$$

a. Determine an analytical expression for the Frequency Response H(f) for this system: ***(You do not need to break up into magnitude / phase terms if it is complex.)*** **(10 pts)**

① Take Fourier Transform of the Diff. Equ., term-by-term using Derivative Property of F.T.:

$$(j2\pi f)^2 Y(f) + 6(j2\pi f)Y(f) + 9Y(f) = \underbrace{9X(f)\,e^{-j2\pi f(1)}}_{\text{Time Shift (Delay) Property}}$$

② Group Terms to find $H(f) = Y(f)/X(f)$

$$Y(f)\left[(j2\pi f)^2 + 6(j2\pi f) + 9\right] = 9X(f)\,e^{-j2\pi f}$$

$$\boxed{H(f) = \frac{9e^{-j2\pi f}}{(j2\pi f)^2 + 6(j2\pi f) + 9}}$$

or: $$H(f) = \frac{9e^{-j2\pi f}}{(j2\pi f + 3)^2}$$

LEARNING OBJECTIVE: Derive frequency response from impulse response or differential equation.

Put a BOX around your $\boxed{\textit{final answer}}$

Name: ANSWER KEY

Problem #2: **(15 pts)**

Determine an analytical expression for the Fourier Transform X(f) of the following function of time, using either the Fourier Transform definition (integral), or known transforms and properties of the Fourier Transform.

$$x(t) = 2t\left[\delta(t-2) + \delta(t+2)\right]$$

USING KNOWN TRANSFORMS / PROPERTIES

FT[$\delta(t)$]: $\quad \delta(t) \overset{FT}{\Rightarrow} 1$

Time Shift: $\quad x(t-t_0) \overset{FT}{\Rightarrow} X(f)\, e^{-j2\pi f t_0} \qquad \therefore [\delta(t-2) + \delta(t+2)] \rightarrow e^{-j2\pi f(2)} + e^{-j2\pi f(-2)}$

Times t: $\quad t\,x(t) \Rightarrow -\frac{1}{j2\pi}\frac{d}{df}X(f)$

$$\therefore FT[x(t)] = \frac{-2}{j2\pi}\left[\frac{d}{df}\left(e^{-j4\pi f}\right) + \frac{d}{df}\left(e^{+j4\pi f}\right)\right]$$

$$= \frac{-1}{j\pi}\left[-j4\pi\, e^{-j4\pi f} + j4\pi\, e^{+j4\pi f}\right]$$

$$= -4\left[\underbrace{e^{+j4\pi f} - e^{-j4\pi f}}_{2j\sin(4\pi f)}\right]$$

$$\boxed{X(f) = -8j\sin(4\pi f)}$$

USING DEFINITION

$$X(f) = \int_{-\infty}^{\infty} 2t\left[\delta(t-2) + \delta(t+2)\right] e^{-j2\pi f t}\, dt$$

$$= 2\int_{-\infty}^{\infty}\left[t\,e^{-j2\pi f t}\right]\delta(t-2)\,dt + 2\int_{-\infty}^{\infty}\left[t\,e^{-j2\pi f t}\right]\delta(t+2)\,dt$$

$$= 2\left[t\,e^{-j2\pi f t}\right]\Big|_{t=+2} + 2\left[t\,e^{-j2\pi f t}\right]\Big|_{t=-2} \quad \text{from Sifting Property}$$

$$= 4e^{-j4\pi f} - 4e^{+j4\pi f} = -4\left[\underbrace{e^{+j4\pi f} - e^{-j4\pi f}}_{2j\sin(4\pi f)}\right]$$

$$\boxed{X(f) = -8j\sin(4\pi f)}$$

Sifting Property of δ:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

(We used this many times in Lecture to derive Simple Fourier Transforms.)

LEARNING OBJECTIVES: a) Evaluate Fourier Transform (analysis)
b) Apply properties of Fourier Transform

Put a BOX around your $\boxed{\textit{final answer}}$

Name: ______________________________

Problem #2: (15 pts)

Determine an analytical expression for the Fourier Transform X(f) of the following function of time, using either the Fourier Transform definition (integral), or known transforms and properties of the Fourier Transform.

$$x(t) = 2t\left[\delta(t-2) + \delta(t+2)\right]$$

ANOTHER OPTION USING PROPERTIES:

DUALITY: $x(t) \longleftrightarrow X(f)$ $\quad \cos(2\pi \alpha t) \longleftrightarrow \frac{1}{2}\left[\delta(f+\alpha) + \delta(f-\alpha)\right]$

$X(t) \longleftrightarrow x(-f)$ $\quad \frac{1}{2}\left[\delta(t-2) + \delta(t+2)\right] \longleftrightarrow \cos(2\pi(2)(-f)) = \cos(4\pi f)$

TIMES-t: $4t\left[\frac{1}{2}(\delta(t-2) + \delta(t+2))\right] \Rightarrow \frac{-4}{j2\pi}\frac{d}{df}\left[\cos(4\pi f)\right] = \frac{-2}{j\pi}\left[-4\pi \sin(4\pi f)\right]$

$x(f) = -8j\sin(4\pi f)$

USING MULTIPLICATION OF δ FUNCTIONS:

$2t\,\delta(t-2) = \underset{@t=+2}{\underline{2t}}\,\delta(t-2) = 4\,\delta(t-2)$ $\quad 2t\,\delta(t+2) = \underset{@t=-2}{\underline{2t}}\,\delta(t+2) = -4\,\delta(t+2)$

$x(t) = 4\left[\delta(t-2) - \delta(t+2)\right]$

$X(f) = 4\left[e^{-j2\pi(2)f} - e^{+j2\pi(2)f}\right] = -8j\sin(4\pi f)$

NOTE: $-8j\sin(4\pi f) = -8e^{j\frac{\pi}{2}}\sin(4\pi f) = -8\sin(4\pi f + \pi/2) = -8\cos(4\pi f)$

Problem #3: (40 pts)

LEARNING OBJECTIVE: Find Fourier Series coefficients for a periodic signal in all 3 forms.

$$x_p(t) \longrightarrow \boxed{H(f)} \longrightarrow y_p(t)$$

A periodic, real-valued, continuous-time signal $x_p(t)$ is input to a Linear, Time-Invariant system. This input signal can be represented by an Exponential Fourier Series with a fundamental frequency $\boxed{f_0 = 20\text{ Hz}}$ and the Fourier Series coefficients given in the table below:

a) Complete the Table below with the missing Coefficients for the alternative forms of the Fourier Series for this same input signal $x_p(t)$: (14 pts)

$X[-k] = X^*[k]$ for Real $x(t)$

Harmonic k	X[k] (complex)	|X[k]| (magnitude)	∠X[k] (phase)	c_k	θ_k	a_k	b_k
-1	$X^*[1]$ $\boxed{2-j2}$	$\lvert X[1]\rvert$ $\boxed{2\sqrt{2}}$	$-\angle X[1]$ $\boxed{-\pi/4}$				
0	1	$\boxed{1}$	$\boxed{0}$	$X[0]$ $\boxed{1}$	$\angle X[0]$ $\boxed{0}$	$X[0]$ $\boxed{1}$	0 $\boxed{0}$
1	2 + j2	$\sqrt{2^2+2^2}$ $\sqrt{8}$ $=\boxed{2\sqrt{2}}$	$\pi/4$	$2\lvert X[1]\rvert$ $2(\sqrt{8})$ $\boxed{4\sqrt{2}}$	$\angle X[1]$ $\boxed{\pi/4}$	$2\,Re(2+j2)$ $=2(2)$ $\boxed{4}$	$-2\,Im\{2+j2\}$ $=-2(2)$ $=\boxed{-4}$
2	j2	$\boxed{2}$	$\boxed{\pi/2}$	$2(2)$ $\boxed{4}$	$\boxed{\pi/2}$	$2(0)$ $\boxed{0}$	$-2(2)$ $\boxed{-4}$
3	-1 + j1	$\sqrt{(-1)^2+1^2}$ $\boxed{\sqrt{2}}$	$\boxed{3\pi/4}$	$\boxed{2\sqrt{2}}$	$\boxed{3\pi/4}$	$\boxed{-2}$	$\boxed{-2}$

$C_0 = a_0 = |X[0]|$

$C_k = 2|X[k]|$ $(k>0)$ $\qquad \Theta_k = \angle X[k]$

Put a BOX around your $\boxed{\textit{final answer}}$

$X[k] = \frac{1}{2}[a_k - j b_k]$

$\therefore a_k = 2\,Re\{X[k]\}$

$b_k = -2\,Im\{X[k]\}$

b) Give complete analytical expressions for the DC and first 3 harmonic terms (k=0-3) of the Fourier Series of the <u>input</u> signal $x_p(t)$, using the <u>Polar Form</u> (c_k, θ_k) of the Fourier Series expansion. (8 pts)

Fourier Series (Polar Form):

$$x_p(t) = C_0 + \sum_{k=1}^{3 \leftarrow 3 \text{ harmonics}} C_k \cos(2\pi k f_0 t + \Theta_k) \qquad f_0 = 20\text{ Hz}$$

$$= C_0 + C_1\cos(2\pi f_0 t + \Theta_1) + C_2\cos(2\pi(2)f_0 t + \Theta_2) + C_3\cos(2\pi(3)f_0 t + \Theta_3)$$

$$\boxed{x_p(t) = 1 + 4\sqrt{2}\cos(40\pi t + \pi/4) + 4\cos(80\pi t + \pi/2) + 2\sqrt{2}\cos(120\pi t + 3\pi/4)}$$

Using answers from Part (a)

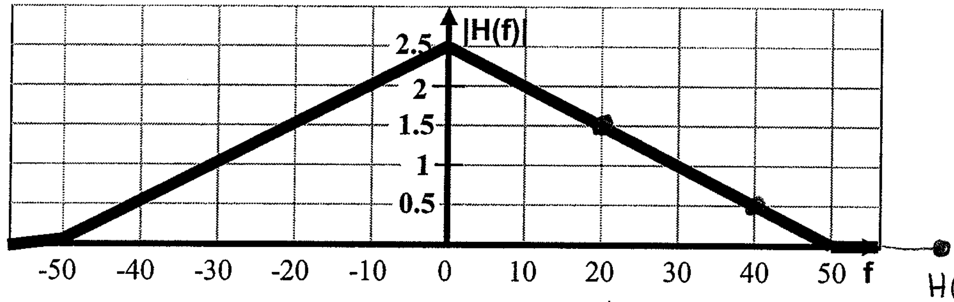
Put a BOX around your $\boxed{\textit{final answer}}$

LEARNING OBJECTIVE: Find response of relaxed System to harmonic Signals.

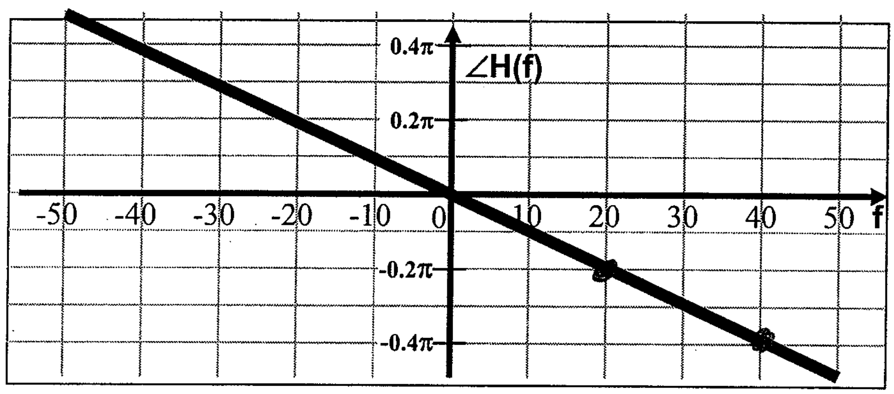
The signal $x_p(t)$ is input to a linear, time-invariant system that has the following frequency response:

$$|H(f)| = 2.5\left[1 - |f/50|\right] \qquad \angle H(f) = -\pi f/100 \qquad \text{for } -50 < f < 50 \text{ Hz}$$

$|H(f)| = 0$, elsewhere ($|f| > 50$ Hz) $H(60) = 0$



H(60) = 0



c) Give a complete expression for the steady-state <u>output</u> of the system $y_p(t)$, for the given Frequency Response and the same input signal $x_p(t)$ from steps (a) + (b), using the <u>Polar Form</u> (c_k, θ_k) of the Fourier Series expansion of $y_p(t)$. **(12 pts)**

For each term in Fourier Series of $x_p(t)$:

$$y_p(t)_k = |H(kf_0)|\, C_k \cos\left[2\pi k f_0 t + \theta_k + \angle H(kf_0)\right]$$

k	kf_0	$\lvert H(kf_0)\rvert$	$\angle H(kf_0)$
0	0	2.5	0
1	20	1.5	-0.2π
2	40	0.5	-0.4π
3	60	0	(don't care)
$4\to\infty$	-	0	-

$$y_p(t) = H(0)C_0 + |H(20)|C_1 \cos\left[40\pi t + \theta_1 + \angle H(20)\right] + |H(40)|C_2 \cos\left[80\pi t + \theta_2 + \angle H(40)\right] + |H(60)|C_3 \cos\left[120\pi t + \theta_3 + \angle H(60)\right]$$

$$y_p(t) = (2.5)(1) + (1.5)(4\sqrt{2})\cos(40\pi t + .25\pi - .2\pi) + (0.5)(4)\cos(80\pi t + .5\pi - .4\pi) + (\underline{\underline{0}})(2\sqrt{2})\cos(120\pi t + \ldots)$$

Put a BOX around your $\boxed{\textit{final answer}}$

$$\boxed{y_p(t) = 2.5 + 6\sqrt{2}\cos(40\pi t + 0.05\pi) + 2\cos(80\pi t + 0.1\pi)}$$

d) Determine the Power in the periodic <u>output signal</u> $y_p(t)$. (6 pts)

FOR FOURIER SERIES:

$$P = C_0^2 + \frac{1}{2}\sum_{k=1}^{N} C_k^2$$

For the output signal $y_p(t)$, the Fourier Series Coefficients (c_{k_y}, θ_{k_y}) are the magnitude and phase angle of each harmonic term in the Fourier Series Expansion [like Problem 8.18 d from Homework #4]

$$P = (2.5)^2 + \frac{1}{2}\left[(6\sqrt{2})^2 + (2)^2\right]$$

$$= 6.25 + \frac{1}{2}\left[72 + 4\right]$$

$$\boxed{P = 44.25 \text{ Watts}}$$

LEARNING OBJECTIVE: Use Parseval's relationship to find signal power in harmonics

Put a BOX around your *final answer*

Problem #4: (20 pts) A Relaxed, Linear, Time-Invariant system has the following impulse response:

$$h(t) = e^{-2t}\left[\cos(2t) - \sin(2t)\right]u(t)$$

a. Determine an expression for the Frequency Response H(f) of this system. ***(You do not need to break it up into magnitude / phase terms.)*** **(6 pts)**

FOURIER TRANSFORM USING TABLES: $e^{-\alpha t}\sin(2\pi\beta t)u(t)$ $\quad \alpha = 2$

$e^{-\alpha t}\cos(2\pi\beta t)u(t)$ $\quad 2\pi\beta = 2$

$$H(f) = \frac{2 + j2\pi f}{(2+j2\pi f)^2 + (2)^2} - \frac{2}{(2+j2\pi f)^2 + (2)^2}$$

Same denominator. Combine terms

$$\boxed{H(f) = \frac{j2\pi f}{(2+j2\pi f)^2 + 4}}$$

LEARNING OBJECTIVE: Derive Frequency Response from Impulse Response.

b. What is the gain or attenuation scale factor that this system would apply to the DC part of an input signal in its steady-state response? **(2 pts)**

For DC, $f = \emptyset$.

$$H(f=0) = \frac{0}{(2)^2 + 4} = 0$$

$\boxed{H(0) = 0 \quad \text{Blocks (filters out) any DC}}$

Put a BOX around your final answer

c. Determine an analytical expression for the zero-state output response $\boxed{y_{ZSR}(t)}$ of this same system using Transform methods, for the following input signal:

(12 pts)

$$x(t) = 4u(t)$$

$$Y(f) = X(f) \cdot H(f)$$

$$X(f) = FT[4u(t)] = 4\left[0.5\,\delta(f) + \frac{1}{j2\pi f}\right]$$

$$X(f) = 2\delta(f) + \frac{4}{j2\pi f}$$

$$Y(f) = X(f)H(f) = \left[2\delta(f) + \frac{4}{j2\pi f}\right]\left[\frac{j2\pi f}{(2+j2\pi f)^2+4}\right]$$

$$= \underline{\left[\frac{j4\pi f}{(2+j2\pi f)^2+4}\right]\delta(f)} + \frac{4(\cancel{j2\pi f})}{(\cancel{j2\pi f})[(2+j2\pi f)^2+4]}$$

$G(f)\delta(f) = G(0)\delta(f)$

"fires" at $f=\emptyset$

$$= \left[\frac{j4\pi f}{(2+j2\pi f)^2+4}\right]\Bigg|_{f=\emptyset} \delta(f) + \frac{4}{(2+j2\pi f)^2+4}$$

$$= 0 + \frac{4}{(2+j2\pi f)^2+4}$$

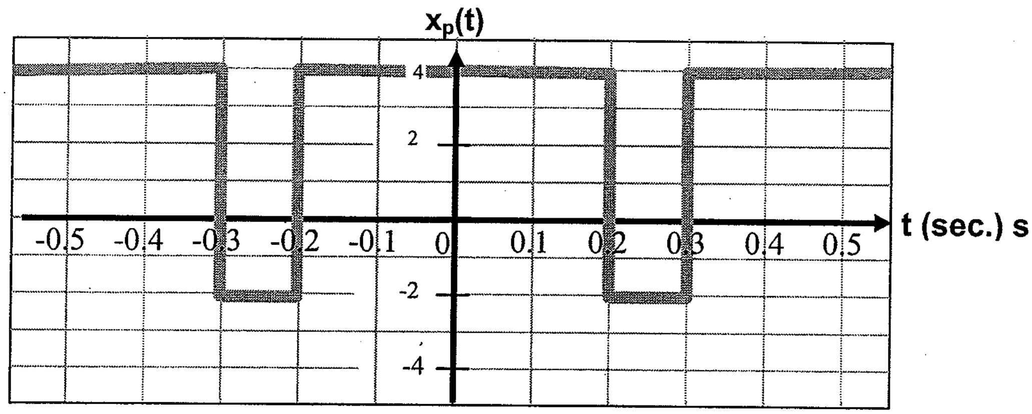
$$y_{ZSR}(t) = \text{Inv FT}\left[\frac{4}{(2+j2\pi f)^2+4}\right]$$ Using Table

$$\boxed{y_{ZSR}(t) = 2e^{-2t}\sin(2t)u(t)}$$

LEARNING OBJECTIVE: Determine Zero State Response of System Using Fourier Transforms

Put a BOX around your $\boxed{\text{final answer}}$

Problem #5: **(15 pts) For the following periodic signal $x_p(t)$:**



a) **What is the fundamental frequency of this signal?** Period = 0.5 sec

$f_0 = 1/\text{Period} = 2\ Hz$

b) **Determine the value of the DC term in a Fourier Series expansion of this signal.** $C_0 = \frac{1}{T}\int_T x_p(t)\,dt = \frac{\text{Area under } x_p(t)}{\text{Period}} = \frac{1}{0.5}\left[(0.4)(+4)+(0.1)(-2)\right] = 2.8$

c) **Does this time-domain signal have any of the following symmetries?** ***(Circle all that apply to this signal.)***

i. **Even symmetry**

ii. **Odd symmetry**

iii. **Half-Wave symmetry**

iv. **No symmetry**

d) **Will the Exponential Fourier Series coefficients (X[k]'s) be:**

(Circle ONE): **Real** **Imaginary** **Complex**

Zero **Can't Be Determined**

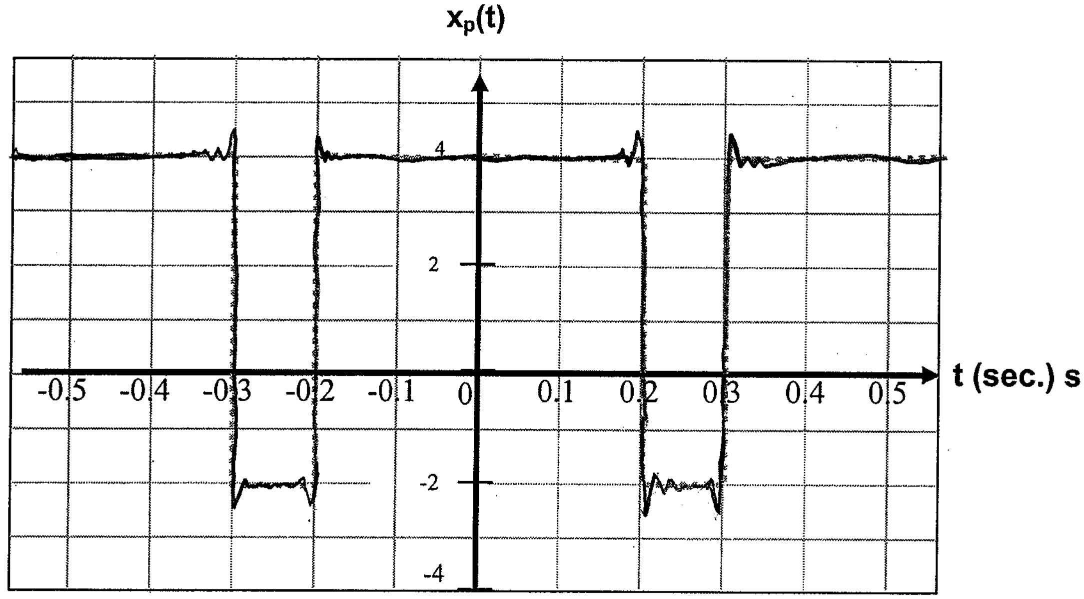
e) **Will any of the following Fourier Series Coefficients be zero?**
(Circle all that apply to this signal.)

$a_k = 0$, **for all k** $a_k = 0$, **only for even k** $a_k = 0$, **for k=0**

$b_k = 0$, **for all k** $b_k = 0$, **only for even k**

LEARNING OBJECTIVE: Identify symmetry to simplify Fourier analysis

f) **Sketch on the axes below what the signal would look like if it is reconstructed from the Fourier Series representation of this $x_p(t)$ signal; assuming that a large number (like 50) of harmonics (terms in the Fourier Series) were used in the reconstruction?**



Due to Gibbs Effect, the reconstructed signal will display overshoot (≈ 9%) and ringing at any vertical discontinuities. Even with many harmonics used in the reconstruction, the overshoot never goes away,

(Just show a little overshoot above on the plot was all I was looking for.)

LEARNING OBJECTIVE: Knowledge of Gibb's Effect