MATH 165 - Sections 09–14 **Spring 2015**

TEST 1 – ANSWERS

RULES

- No books or notes or calculators allowed.
- No bathroom breaks until after you have completed and turned in your exam.
- Out of consideration for your classmates, do not make disturbing noises during the exam.
- **Phones must be turned off.**

Cheating will not be tolerated. If there are any indications that a student may have given or received unauthorized aid on this exam, the case will be brought to the ISU Office of Academic Integrity.
When you finish the exam, please sign the following statement acknowledging that you understand this policy:

"On my honor as a student I, ________________________, have neither given nor received unauthorized aid on this exam." (Print Name)

Signature: ____________________________________ Date: _______________

1. Consider the following function of a real variable: $f(x) = \sqrt{6 - 9x}$.

   i. What is the *domain* of $f(x)$?

   (a) $(-\infty, -2/3)$ (b) $(-\infty, 0]$ (c) $(-\infty, 1/3]$

   (d) $(-\infty, 2/3]$ (e) $[0, \infty)$ (f) $(-\infty, \infty)$

   Answer: (d)

   ii. What is the *range* of $f(x)$?

   (a) $(-\infty, -2/3)$ (b) $(-\infty, 0]$ (c) $(-\infty, 1/3]$

   (d) $(-\infty, 2/3]$ (e) $[0, \infty)$ (f) $(-\infty, \infty)$

   Answer: (e)

   iii. Find a formula for the inverse function of $f(x)$.

Solution: The inverse function $f^{-1}(x)$ satisfies $f(f^{-1}(x)) = x$. In this case, we have

$$\begin{array}{rrcl} & f(f^{-1}(x)) & = & x \\ \Leftrightarrow & \sqrt{6 - 9(f^{-1}(x))} & = & x \\ \Leftrightarrow & 6 - 9(f^{-1}(x)) & = & x^2 \\ \Leftrightarrow & f^{-1}(x) & = & \frac{x^2 - 6}{-9} \end{array}$$

Answer: $f^{-1}(x) = \frac{6 - x^2}{9}$

Also Acceptable: $f^{-1}(x) = -\frac{1}{9}x^2 + \frac{2}{3}$

2. Evaluate each limit. If the limit does not exist, indicate why. That is, write "∞", or "$-\infty$", or "oscillating", or... Do NOT simply write "dne." (No justification required here. No partial credit for incorrect answers.)

   i.

$$\lim_{x \to -2^+} \frac{3x}{4 - x^2}$$

Answer: $-\infty$

   ii.

$$\lim_{x \to \infty} \frac{x(1-4x)}{2x^2 - 7x + 1}$$

Answer: -2

   iii.

$$\lim_{x \to -\infty} \frac{8x^5 + 3}{7x^3 - x}$$

Answer: ∞

3. Let f be the function defined by

$$f(x) = \frac{16 - x^2}{x + 4}$$

   i. Compute the limit of $f(x)$ as x approaches -4. **Answer:** 8

   ii. Write down a function $g(x)$ with domain $(-\infty, \infty)$ that is continuous everywhere and satisfies $g(x) = f(x)$ for all x in the domain $(-\infty, -4) \cup (-4, \infty)$ of f.

Answer: $g(x) = 4 - x$

Also Acceptable: $g(x) = \begin{cases} f(x), & x \neq -4 \\ 8, & x = -4 \end{cases}$

Also Acceptable: $g(x) = \begin{cases} \frac{16-x^2}{x+4}, & x \neq -4 \\ 8, & x = -4 \end{cases}$

4. Evaluate each limit. If the limit does not exist, explain why. Justify all answers. (Answers without justification will receive no credit.)

i.

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3}$$

Solution:

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3} = \lim_{x \to 3} \frac{(x+5)(x-3)}{x-3} = \lim_{x \to 3} (x+5) = 8.$$

Answer: 8

ii.

$$\lim_{\theta \to 0} \frac{\tan(7\theta)}{\cos(2\theta)}$$

Solution:

$$\lim_{\theta \to 0} \frac{\tan(7\theta)}{\cos(2\theta)} = \lim_{\theta \to 0} \frac{\sin(7\theta)}{\cos(7\theta)} \frac{1}{\cos(2\theta)} = \frac{0}{1} \frac{1}{1} = 0.$$

Answer: 0

iii.

$$\lim_{x \to \infty} (x - \sqrt{x^2 + 6x})$$

Solution:

$$\begin{aligned}
\lim_{x \to \infty} (x - \sqrt{x^2 + 6x}) &= \lim_{x \to \infty} \left(\frac{x - \sqrt{x^2 + 6x}}{1} \frac{x + \sqrt{x^2 + 6x}}{x + \sqrt{x^2 + 6x}} \right) \\
&= \lim_{x \to \infty} \left(\frac{x^2 - (x^2 + 6x)}{x + \sqrt{x^2 + 6x}} \right) \\
&= \lim_{x \to \infty} \left(\frac{-6x}{x + \sqrt{x^2(1 + 6/x)}} \right) \\
&= \lim_{x \to \infty} \left(\frac{-6x}{x + x\sqrt{1 + 6/x}} \right) \\
&= \lim_{x \to \infty} \left(\frac{-6}{1 + \sqrt{1 + 6/x}} \right) = \frac{-6}{2} = -3.
\end{aligned}$$

Answer: -3

5. *i.* Evaluate the limit, if it exists.

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$$

Answer: The limit does not exist.

ii. Complete the following statement of the "Sandwich Theorem:"

Suppose f, g, and h are functions satisfying $g(x) \leq f(x) \leq h(x)$ for all x. If $\lim\limits_{x \to c} g(x) = L$ and $\lim\limits_{x \to c} h(x) = L$, then...

Answer: $\lim\limits_{x \to c} f(x) = L$.

iii. Suppose

$$f(x) = \sqrt{x} \sin\left(\frac{1}{x}\right).$$

and consider the limit of $f(x)$ as x approaches 0. Can you apply the Sandwich Theorem to compute this limit? If so, then write down the functions $g(x)$ and $h(x)$ that satisfy the assumptions of the theorem, and then compute the limit:

Answer: Yes, we can apply the Sandwich Theorem to compute the limit. Two functions which satisfy the hypotheses of the theorem are

$$g(x) = -\sqrt{x} \quad \text{and} \quad h(x) = \sqrt{x}.$$

Note that the functions $g(x) = -x$ and $h(x) = x$ do not work since $x < \sqrt{x}$ for x near 0.

The limit is $\lim\limits_{x \to 0} f(x) = 0$.

6. By following the steps below, find the equation of the line tangent to the graph of the function $f(x) = \sqrt{x}$ at the point where $x = 16$. Present your answer in $y = mx + b$ form. (For full credit, follow the steps given and show your work.)

(i) Find the instantaneous rate of change of the function $f(x) = \sqrt{x}$ at an arbitrary point $x = x_0$ by computing the following limit:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Solution:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \\
&= \lim_{h \to 0} \frac{\left(\sqrt{x_0 + h} - \sqrt{x_0}\right)}{h} \frac{\left(\sqrt{x_0 + h} + \sqrt{x_0}\right)}{\left(\sqrt{x_0 + h} + \sqrt{x_0}\right)} \\
&= \lim_{h \to 0} \frac{x_0 + h - x_0}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} \\
&= \frac{1}{2\sqrt{x_0}}
\end{aligned}$$

(ii) Find the instantaneous rate of change of the function at the point $x_0 = 16$ by plugging 16 into the result you obtained in part (i).

Answer: 1/8

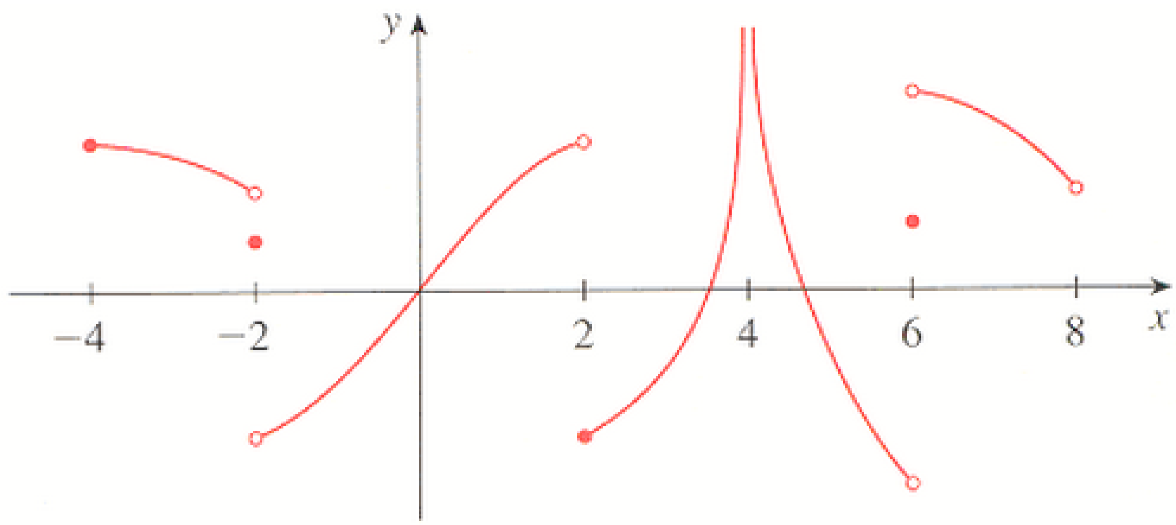
(iii) Recall the relationship between the instantaneous rate of change of a function and the line tangent to the graph of the function. Using this and your answer to part (ii), write down the equation of the line tangent to the graph of $\sqrt{x}$ at the point where $x = 16$. Use the form $y = mx + b$.

Solution: A point on the line is $(x_0, f(x_0)) = (16, \sqrt{16}) = (16, 4)$. We plug this point into the equation for the line, $y = (1/8)x + b$, to arrive at $4 = 16/8 + b$, which yields $b = 2$.

Answer: $y = \left(\frac{1}{8}\right) x + 2$

Also Acceptable: $y = \frac{x+16}{8}$

7. Let $f(x)$ be a function with domain $[-4, 8)$ and with graph shown below.



(a) Write down the set of points where $f(x)$ is continuous. (Use interval notation, with the union symbol $\cup$ if necessary).

Answer: $[-4, -2) \cup (-2, 2) \cup (2, 4) \cup (4, 6) \cup (6, 8)$

(b) At how many points in the domain is the function discontinuous?

Answer: 4

(c) How many points in the domain are removable discontinuities?

Answer: 0