Control of Singular Distributed Systems by Controllability: The Ill-Posed Backwards Heat Equation

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September 6, 2024

Abstract

This paper deals with the ill-posed backwards heat equation. To do this, we propose the controllability method. The point of view adopted, which consists in interpreting the state equation as an inverse problem, allows us to obtain a decoupled and strong singular optimality system for the optimal control-state pair, but also to propose an existence criteria for a regular solution of the backwards heat equation. It is important to note that this results are obtained without recourse to a Slater-type assumption, an assumption to which many analyses have had to recourse.

Keywords: Singular distributed systems, Optimal control, Ill-posed backwards heat equation, Controllability, Inverse Problem.

1 Introduction

Let us consider an open and bounded subset Ω of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, of boundary Γ, twice continuously differentiable, with Ω locally on one side only of Γ; that is to say that $\overline{\Omega}$ is a variety with a boundary of class $\mathscr{C}^2$.

For $T > 0$, we denote $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, and $\mathscr{U}_{ad}$ a closed and non-empty convex of $L^2(Q)$.

Given $v \in L^2(Q)$, we consider the following problem

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} - \Delta z & = & v & \text{in} \quad Q, \\ z & = & 0 & \text{on} \quad \Sigma, \\ z(\cdot, T) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{1}$$

being interested in the evolution, in Ω and at a time $t \in (0, T)$, of the temperature z, the value of which is kwnown at the final time T.

It is well known that problem (1), so called the ill-posed backwards heat equation, does not admits solution for any initial data $v \in L^2(Q)$.

So we consider, a priori, pairs $(v, z) \in \left(L^2(Q)\right)^2$ satisfying (1), saying that such pairs constitute the set of control-state pairs.

A control-state pair (v, z) for (1) will be said admissible if

$$v \in \mathscr{U}_{ad};$$

and we will denote $(v, z) \in \mathscr{A}$, designating by $\mathscr{A}$ the set of admissible control-state pairs.

Supposing that the set $\mathscr{A}$ is non-empty, we introduce the cost function

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)},$$

where $z_d \in L^2(Q)$ and $N > 0$.

Then, we are interested in the control problem which consists in finding the control-state pair

$$(u, y) \in \mathscr{A}_{ad}: \quad J(u, y) = \inf_{(v,z)\in\mathscr{A}_{ad}} J(v, z). \tag{2}$$

The following result in then immediate

Theorem 1. *The optimal control problem* (2) *admits a unique solution* (u, y)*, called the optimal control-state pair.*

Proof. Due to its structure, the cost functional J is clearly coercive, strictly convex and lower semi-continuous. From where, with the non-vacuity assumption of the closed convex set of admissible control-state pairs $\mathscr{A}$, we can conclude to existence and unicity of the optimal control-state pair (u, y). □

Hence, one can establish, (cf. [6]), using the first-order Euler-Lagrange optimality condition, that the optimal control-state pair (u, y) is characterized by

$$(y - z_d\,,\, z - y)_{L^2(Q)} + N(u\,,\, v - u)_{L^2(Q)} \;\geq\; 0, \qquad \forall\,(v, z) \in \mathscr{A}.$$

We now focus on the characterization of the optimal pair (u, y) through a strong and decoupled singular optimality system.

A classic method to do so is the penalization method introduced by J. L. Lions in [6]. This one consists of approaching the optimal control-state pair (u, y) by a penalized problem. More precisely, for $\varepsilon > 0$, we define the penalized cost function

$$J_\varepsilon(v, z) = J(v, z) + \frac{1}{\varepsilon}\left\|\frac{\partial z}{\partial t} - \Delta z - v\right\|^2_{L^2(Q)}.$$

We then establish that the optimal control-state pair $(u_\varepsilon, z_\varepsilon)$ corresponding to this last cost function converges towards the optimal control-state pair (u, y). We therefore have a theoretical approximation process for the optimal pair (u, y). Thanks to the first-order Euler-Lagrange optimality condition, we characterize the approached optimal control-state pair $(u_\varepsilon, y_\varepsilon)$ by a system of variational inequalities which we interpret as an approached optimality system after introducing the approached adjoint state

$$p_\varepsilon = -\frac{1}{\varepsilon}\left(\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon - u_\varepsilon\right).$$

The main step of this technique is the use of the *a priori* estimation method to obtain a strongly decoupled optimality system by passing to the limit in the approached optimality system. But this final step requires the Slater-type assumption that

"the set of admissible controls $\mathscr{U}_{ad}$ is of non empty interior in $L^2(Q)$". (3)

But in many situations, the Slater-type assumption does not hold; for example in the case

$$\mathscr{U}_{ad} = \left(L^2(Q)\right)^+ = \left\{v \in L^2(Q) : \; v \geq 0\right\}.$$

It therefore appears relevant to know how to do without this unrealistic assumption.

For this purpose, R. Dorville, O. Nakoulima and A. Omrane propose in [1], the notion of least regrets control, applied to a regularized elliptic state equation with missing data. An approach that allows them to characterize the least regrets control u^γ through a strongly decoupled singular optimality system. This, certainly without recourse to the Slater-type assumption (3), but to the detriment of the final condition

$$y^\gamma(\cdot, T) \quad = \quad 0 \quad \text{in} \quad \Omega.$$

That is to say, more precisely, that the state y^γ associated with the least regrets control u^γ is such that, in all generality,

$$y^\gamma(\cdot, T) \quad \neq \quad 0 \quad \text{in} \quad \Omega.$$

In response to this problem, the authors introduce the notion of zero-order corrector, which calls for the regularity assumption

$$y^\gamma(\cdot, T) \in H_0^1(\Omega) : \; \frac{\partial y^\gamma}{\partial t} \in L^2(Q)\,. \tag{4}$$

This last one allows them to obtain the characterization specified below of the initial optimal control-state pair.

Theorem 2 (cf. [1]). *The no-regret control u of the problem* (1)(1)(2) *is characterized by the unique $\{u, y, \rho, p, \xi\}$ solution of the system:*

$$\left\{\begin{array}{rclrclrclrclcc} Ly &=& u, & L\rho &=& 0, & L^*p &=& y - z_d + \rho, & L^*\xi &=& y & in & Q, \\ y &=& 0, & \rho &=& 0, & p &=& 0, & \xi &=& 0, & on & \Sigma, \\ y(0) &=& 0, & \rho(0) &=& \lambda(0) & & & & & & & in & \Omega, \\ & & & & & & p(T) &=& 0 & \xi(T) &=& 0 & in & \Omega, \end{array}\right. \tag{5}$$

and we have the variational inequality

$$(p + Nu\,,\, v - u) \quad \geq \quad 0 \quad \forall v \in \mathscr{U}_{ad}, \tag{6}$$

with

$$\left|\begin{array}{l} L = \dfrac{\partial}{\partial t} - \Delta \quad \textit{and} \quad L^* = -\dfrac{\partial}{\partial t} - \Delta \quad \textit{its adjoint operator;} \\ u, y, p, \rho, \xi \in L^2(]0, T[; L^2(\Omega))\,, \quad \lambda(0) \in L^2(\Omega)\,. \end{array}\right.$$

Ultimately, the no-regret control thus obtained does not satisfy the final condition

$$y(\cdot, T) \quad = \quad 0 \quad \text{in} \quad \Omega,$$

so that, in all generality, the problem remains, to the best of our knowledge, globally open.

We propose in this paper the controllability method for the analysis, without recourse to the Slater-type assumption (3), of the control problem of the ill-posed backwards heat equation. Note that this method has made it possible to propose in [2], [4], [3] and [5], an answer to the same question of the control, without recourse to (3), of control problems of ill-posed Cauchy system for elliptic, parabolic and hyperbolic operators.

The rest of this paper is as follows. Section 2 is devoted to the announced interpretation of the initial problem as an inverse problem. We define there the so-called exact controllability problem which we approach, by density argument, by an equivalent problem (this one called the approached controllability problem). In Section 3, we return to the control problem (2), starting by regularized it based on the controllability results previously obtained. After established the convergence of the process in Section 3.1, then the approached optimality system in Section 3.2, we end in Section 3.3 with the optimality system for the initial control problem.

2 Controllability of the ill-posed backwards heat equation

In the present section, we introduce the controllability viewpoint here proposed. Which consists, starting from the notion of controllability introduced by J.-L. Lions in [7, p. 222], in interpreting the initial state equation as an inverse problem (we also say a controllability problem).

We establish that, when it exists, the solution of the ill-posed backwards heat equation (1) can be approximated by that of the approached controllability problem. Which implicitly allows us to propose a necessary and sufficient condition for the existence of a regular solution to the problem (1).

We therefore interpret (1) as an inverse problem whose observation objective consists in the final condition

$$z(\cdot,T) \;=\; 0 \quad \text{in} \quad \Omega.$$

More precisely, we consider the following problem

$$\left\{\begin{aligned} \frac{\partial y}{\partial t}-\Delta y &= v \quad \text{in} \quad Q,\\ y &= 0 \quad \text{on} \quad \Sigma,\end{aligned}\right.$$

posing the problem of finding $y_0 \in L^2(\Omega)$ such that if

$$\left\{\begin{aligned} \frac{\partial y}{\partial t}-\Delta y &= v \quad \text{in} \quad Q,\\ y &= 0 \quad \text{on} \quad \Sigma,\\ y(\cdot,0) &= y_0 \quad \text{in} \quad \Omega,\end{aligned}\right. \tag{7}$$

then

$$y(\cdot,T) \;=\; 0 \quad \text{in} \quad \Omega. \tag{8}$$

We say that (7)(8) constitutes an exact controllability problem associated with the ill-posed backwards heat equation (1).

Remark 1. *The controllability problem is well defined. Indeed, it is well known that for any $v \in L^2(Q)$ and $y_0 \in L^2(\Omega)$, (7) is well posed in the sense of Hadamard. That is to say that its admits a unique solution*

$$y(v,y_0) \in \mathbb{V} := L^2\big(0,T;H_0^1(\Omega)\big) \cap \mathscr{C}\big([0,T];L^2(\Omega)\big)$$

which depends continuously on the data. We deduce from this, after possible modification on a set of zero measure, that the solution $y(v,y_0)$ of (7) merges with a continuous function from $(0,T)$ to $L^2(\Omega)$.

Thus we can indeed speak of the final value $y(\cdot,T;v,y_0)$ of $y(v,y_0)$ in Ω.

Remark 2. *Let us denote*

- y^v *the unique solution of* (7);
- $y_0^v \in \mathbb{V} \subset L^2(Q)$ *that of*

$$\begin{cases} \dfrac{\partial y_0^v}{\partial t} - \Delta y_0^v = v & \text{in} \quad Q, \\ y_0^v = 0 & \text{on} \quad \Sigma, \\ y_0^v(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

- *and* $y^0 \in \mathbb{V} \subset L^2(Q)$ *that of*

$$\begin{cases} \dfrac{\partial y^0}{\partial t} - \Delta y^0 = 0 & \text{in} \quad Q, \\ y^0 = 0 & \text{on} \quad \Sigma, \\ y^0(\cdot,0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{9}$$

Then, the mapping

$$(v,y_0) \longmapsto y^v = y_0^v + y^0,$$

being linear and continuous from $L^2(Q) \times L^2(Q)$ to $\mathbb{V} \subset L^2(Q)$, we deduce that the exact controllability problem (7)(8) is equivalent to the following

$$\begin{cases} \text{find } y_0 \in L^2(\Omega) \text{ such that:} \\ \text{if } y^0 \text{ is solution of } (9), \text{ then} \\ y^0(\cdot,T) = -y_0^v(\cdot,T) \quad \text{in } \Omega. \end{cases} \tag{10}$$

We approach the exact controllability problem (10) by density argument as specified below.

Proposition 1. *When the initial data y_0 traverses $L^2(\Omega)$, the set*

$$E = \left\{ y^0(\cdot,T) \ ; \ y_0 \in L^2(\Omega) \right\},$$

described by the final values of the solution y^0 of (9), is dense in $L^2(\Omega)$.

Proof. It is clear that the set E constitute a vector subspace of $L^2(\Omega)$. From where, by the Hahn-Banach Theorem, E is dense in $L^2(\Omega)$ if and only if $E^\perp = \{0\}$.

Let us consider $k \in E^\perp$; so we have

$$\forall\, y_0 \in L^2(\Omega), \qquad \left(k\,,\, y^0(\cdot,T)\right)_{L^2(\Omega)} = 0.$$

But it comes from (9) that, for any test function $\varphi \in \mathscr{D}(Q)$, we have:

$$\begin{aligned} \left(\frac{\partial y^0}{\partial t} - \Delta y^0\,,\, \varphi\right)_{L^2(Q)} = 0 \iff & \left(\frac{\partial y^0}{\partial t}\,,\, \varphi\right)_{L^2(Q)} - \left(\Delta y^0\,,\, \varphi\right)_{L^2(Q)} = 0 \\ \iff & \left(y^0(\cdot,T)\,,\, \varphi(\cdot,T)\right)_{L^2(\Omega)} - \left(y_0\,,\, \varphi(\cdot,0)\right)_{L^2(\Omega)} \\ & - \left(y^0\,,\, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - \left(y^0\,,\, \Delta\varphi\right)_{L^2(Q)} + \left(y^0\,,\, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma)} \\ & - \left(\frac{\partial y^0}{\partial \nu}\,,\, \varphi\right)_{L^2(\Sigma)} = 0 \end{aligned}$$

i.e.

$$\begin{aligned}\big(y^0(\cdot,T)\ ,\ \varphi(\cdot,T)\big)_{L^2(\Omega)} - (y_0\ ,\ \varphi(\cdot,0))_{L^2(\Omega)} - \left(y^0\ ,\ \frac{\partial\varphi}{\partial t}\right)_{L^2(Q)} \\ - \big(y^0\ ,\ \Delta\varphi\big)_{L^2(Q)} - \left(\frac{\partial y^0}{\partial\nu}\ ,\ \varphi\right)_{L^2(\Sigma)} = 0.\end{aligned} \tag{11}$$

Choosing in the above, φ such that

$$\begin{cases} -\dfrac{\partial\varphi}{\partial t} - \Delta\varphi &= 0 \quad \text{in} \quad Q, \\ \varphi &= 0 \quad \text{on} \quad \Sigma, \\ \varphi(\cdot,T) &= k \quad \text{in} \quad \Omega, \end{cases} \tag{12}$$

it comes that (11) is equivalent to

$$\big(k\ ,\ y^0(\cdot,T)\big)_{L^2(\Omega)} - (y_0\ ,\ \varphi(\cdot,0))_{L^2(\Omega)} = 0, \tag{13}$$

where

$$k \in E^\perp \iff \big(k\ ,\ y^0(\cdot,T)\big)_{L^2(\Omega)} = 0.$$

Thus (13) becomes

$$\forall\, y_0 \in L^2(\Omega)\,, \qquad (y_0\ ,\ \varphi(\cdot,0))_{L^2(\Omega)} = 0. \tag{14}$$

But we can still choose, in (14), $y_0 = \varphi(\cdot,0)$ in Ω, and then it follows

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} = 0 \qquad i.e. \qquad \varphi(\cdot,0) \ = \ 0 \quad \text{in} \quad \Omega.$$

Which brings, with (12), that φ is solution of

$$\begin{cases} -\dfrac{\partial\varphi}{\partial t} - \Delta\varphi &= 0 \quad \text{in} \quad Q, \\ \varphi &= 0 \quad \text{on} \quad \Sigma, \\ \varphi(\cdot,0) &= 0 \quad \text{in} \quad \Omega, \end{cases}$$

that is to say that $\varphi \equiv 0$ and consequently that

$$\varphi(\cdot,T) \ = \ k \ = 0 \quad \text{in} \quad \Omega.$$

From where we deduce that $E^\perp = \{0\}$ and therefore that E is dense in $L^2(\Omega)$. $\square$

The following result is then immediate.

Corollary 1. *For any $\varepsilon > 0$ and $\kappa \in L^2(\Omega)$, there exists $y_{0\varepsilon} \in L^2(\Omega)$ such that the solution $y^0_\varepsilon \in \mathbb{V}$ of*

$$\begin{cases} \dfrac{\partial y^0_\varepsilon}{\partial t} - \Delta y^0_\varepsilon &= 0 \quad \textit{in} \quad Q, \\ y^0_\varepsilon &= 0 \quad \textit{on} \quad \Sigma, \\ y^0_\varepsilon(\cdot,0) &= y_{0\varepsilon} \quad \textit{in} \quad \Omega, \end{cases} \tag{15}$$

also satisfies

$$\left\|y^0_\varepsilon(\cdot,T) - \kappa\right\|_{L^2(\Omega)} < \varepsilon.$$

From which it also follows that

Corollary 2. *Given $v \in L^2(Q)$ and $\varepsilon > 0$, there exists $y_{0\varepsilon} \in L^2(\Omega)$ such that the solution $y^v_\varepsilon \in \mathbb{V}$ of*

$$\begin{cases} \dfrac{\partial y^v_\varepsilon}{\partial t} - \Delta y^v_\varepsilon &= v \quad in \quad Q, \\ \\ y^v_\varepsilon &= 0 \quad on \quad \Sigma, \\ \\ y^v_\varepsilon(\cdot, 0) &= y_{0\varepsilon} \quad in \quad \Omega, \end{cases} \tag{16}$$

satisfies

$$\|y^v_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

Proof. Let $v \in L^2(Q)$ and $\varepsilon > 0$. We know that there exists a unique $y^v_0 \in \mathbb{V}$, almost everywhere equal to a continuous mapping from $(0, T)$ to $L^2(\Omega)$, solution of

$$\begin{cases} \dfrac{\partial y^v_0}{\partial t} - \Delta y^v_0 &= v \quad \text{in} \quad Q, \\ \\ y^v_0 &= 0 \quad \text{on} \quad \Sigma, \\ \\ y^v_0(\cdot, 0) &= 0 \quad \text{in} \quad \Omega. \end{cases}$$

It follows that, with Corollary 1, there exists $y_{0\varepsilon} \in L^2(\Omega)$ such that the solution $y^0_\varepsilon \in \mathbb{V}$ of (15) satisfies

$$\left\|y^0_\varepsilon(\cdot, T) - (-y^v_0(\cdot, T))\right\|_{L^2(\Omega)} < \varepsilon.$$

Which allows us to conclude that $y^v_\varepsilon = \left(y^v_0 + y^0_\varepsilon\right) \in \mathbb{V}$ is unique solution of (16), with

$$\|y^v_\varepsilon(\cdot, T)\|_{L^2(\Omega)} = \|y^\varepsilon_0(\cdot, T) + y^v_0(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

$\square$

More over, we have the following theorem, characterizing the existence of a regular solution to the ill-posed backwards heat equation.

Theorem 3. *Let $v \in L^2(Q)$. The ill-posed backwards heat equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z &= v \quad in \quad Q, \\ \\ z &= 0 \quad on \quad \Sigma, \\ \\ z(\cdot, T) &= 0 \quad in \quad \Omega, \end{cases} \tag{17}$$

admits a regular solution $z \in \mathbb{V}$ if and only if the sequence $\left(y_{0\varepsilon}\right)_\varepsilon$ is bounded in $L^2(\Omega)$.

Proof. 1. Let $\varepsilon > 0$. From Corollary 1, there exists $y_{0\varepsilon} \in L^2(\Omega)$ such as $y^v_\varepsilon \in \mathbb{V}$ is solution of

$$\left\{\begin{array}{rcll} \dfrac{\partial y^v_\varepsilon}{\partial t} - \Delta y^v_\varepsilon &=& v & \text{in} \quad Q, \\ \\ y^v_\varepsilon &=& 0 & \text{on} \quad \Sigma, \\ \\ y^v_\varepsilon(\cdot, 0) &=& y_{0\varepsilon} & \text{in} \quad \Omega, \end{array}\right. \tag{18}$$

with the estimate

$$\|y^v_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

Then, we generate sequences

$$(y_{0\varepsilon})_\varepsilon \subset L^2(\Omega) \qquad \text{and} \qquad (y^v_\varepsilon)_\varepsilon \subset \mathbb{V}.$$

Let suppose that the sequence $(y_{0\varepsilon})_\varepsilon$ is bounded in $L^2(\Omega)$. It therefore follows, the homogeneous Dirichlet problem (18) for the heat equation being well-posed, that the sequence $(y^v_\varepsilon)_\varepsilon$ is bounded in $\mathbb{V}$, and also in $L^2(Q)$.

So we can extract from $(y_{0\varepsilon})_\varepsilon$ and $(y^v_\varepsilon)_\varepsilon$ subsequences, still denoted in the same way, which converge weakly in $L^2(\Omega)$ and $\mathbb{V}$, respectively.

So that, there exist $y_0 \in L^2(\Omega)$ and $y^v \in \mathbb{V}$, such as

$$\left\{\begin{array}{rcll} y_{0\varepsilon} & \longrightarrow & y_0 & \text{weakly in} \quad L^2(\Omega), \\ \\ y^v_\varepsilon & \longrightarrow & y^v & \text{weakly in} \quad \mathbb{V}. \end{array}\right.$$

Thus we have on the one hand, that

$$\|y^v_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon \quad \text{and} \quad y^v_\varepsilon \quad \longrightarrow \quad y^v \quad \text{weakly in} \quad \mathbb{V}$$

imply, y^v_ε being almost everywhere equal to a continuous mapping from $(0, T)$ to $L^2(\Omega)$, that

$$y^v(\cdot, T) \quad = \quad 0 \quad \text{in} \quad \Omega. \tag{19}$$

On the other hand, we have for any $\varphi \in \mathscr{D}(Q)$ that:

$$\begin{aligned} \frac{\partial y^v_\varepsilon}{\partial t} - \Delta y^v_\varepsilon = v \text{ in } Q \iff & \left(\frac{\partial y^v_\varepsilon}{\partial t} - \Delta y^v_\varepsilon \,,\, \varphi\right)_{L^2(Q)} = (v \,,\, \varphi)_{L^2(Q)} \\ \iff & (y^v_\varepsilon(\cdot, T) \,,\, \varphi(\cdot, T))_{L^2(\Omega)} - (y_{0\varepsilon} \,,\, \varphi(\cdot, 0))_{L^2(\Omega)} \\ & - \left(y^v_\varepsilon \,,\, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (y^v_\varepsilon \,,\, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial y^v_\varepsilon}{\partial \nu} \,,\, \varphi\right)_{L^2(\Sigma)} \\ & = (v \,,\, \varphi)_{L^2(Q)}, \end{aligned}$$

so, by passing to the limit,

$$\begin{aligned} & (y^v(\cdot, T) \,,\, \varphi(\cdot, T))_{L^2(\Omega)} - (y_0 \,,\, \varphi(\cdot, 0))_{L^2(\Omega)} - \left(y^v \,,\, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} \\ & \quad - (y^v \,,\, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial y^v}{\partial \nu} \,,\, \varphi\right)_{L^2(\Sigma)} = (v \,,\, \varphi)_{L^2(Q)} \\ \iff & (y^v(\cdot, 0) - y_0 \,,\, \varphi(\cdot, 0))_{L^2(\Omega)} + \left(\frac{\partial y^v}{\partial t} - \Delta y^v \,,\, \varphi\right)_{L^2(Q)} \\ & \quad - \left(y^v \,,\, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma)} = (v \,,\, \varphi)_{L^2(Q)}. \end{aligned}$$

This last equality being valid for any $\varphi \in \mathscr{D}(Q)$, it follows that

$$\begin{cases} \dfrac{\partial y^v}{\partial t} - \Delta y^v &= v \quad \text{in} \quad Q, \\ \\ y^v &= 0 \quad \text{on} \quad \Sigma, \\ \\ y^v(\cdot, 0) &= y_0 \quad \text{in} \quad \Omega, \end{cases}$$

which implies, with (19), that

$$\begin{cases} \dfrac{\partial y^v}{\partial t} - \Delta y^v &= v \quad \text{in} \quad Q, \\ \\ y^v &= 0 \quad \text{on} \quad \Sigma, \\ \\ y^v(\cdot, T) &= 0 \quad \text{in} \quad \Omega, \end{cases}$$

that is to say that $y^v \in \mathbb{V}$ is solution of the ill-posed backwards heat equation (17).

2. Now, let assume that the ill-posed backwards heat equation (17) admits a regular solution $z \in \mathbb{V}$.

   Then, since $z \in \mathbb{V}$ implies that z is almost everywhere equal to a continuous function from $(0, T)$ to $L^2(\Omega)$, we can, after possible modification on a set of zero measure, speak of the initial value $z(\cdot, 0) \in L^2(\Omega)$.

   Then, by choosing, for any $\varepsilon > 0$,

$$y_{0\varepsilon} \quad = \quad z(\cdot, 0) \quad \text{in} \quad \Omega,$$

   we obtain that the sequence $(y_{0\varepsilon})_\varepsilon$, since constant, is bounded in $L^2(\Omega)$.

□

3 The optimal control problem

Let us start by recalling that we are here interested in the control problem of the ill-posed backwards heat equation. That is to say, more precisely, that for $v, z \in L^2(Q)$ satisfying

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z &= v \quad \text{in} \quad Q, \\ \\ z &= 0 \quad \text{on} \quad \Sigma, \\ \\ z(\cdot, T) &= 0 \quad \text{on} \quad \Omega, \end{cases} \tag{20}$$

we introduce the cost function

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)}, \tag{21}$$

being interested in the control problem

$$\inf\{J(v, z);\ (v, z) \in \mathscr{A}\}\,. \tag{22}$$

As underlined in the introduction, the optimal control problem (20)(21)(22) admits a unique solution (u, y) whose characterization, via a strong and decoupled singular optimality system, and this without using the Slater-type assumption (3), is the main objective.

To do so, starting from the non-vacuity assumption of the set of admissible control-state pairs, and using the results previously obtained, we have for any $v \in L^2(Q)$ and $\varepsilon > 0$ that there exist

$$y_{0_\varepsilon} \in L^2(\Omega) \qquad \text{and} \qquad y_\varepsilon^v \in \mathbb{V}$$

such that

$$\begin{cases} \dfrac{\partial y_\varepsilon^v}{\partial t} - \Delta y_\varepsilon^v = v & \text{in} \quad Q, \\ y_\varepsilon^v = 0 & \text{on} \quad \Sigma, \\ y_\varepsilon^v(\cdot, 0) = y_{0_\varepsilon} & \text{in} \quad \Omega, \end{cases}$$

with the estimate

$$\|y_\varepsilon^v(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

Assuming that the optimal solution (u, y) for (20)(21)(22) is such that $y(\cdot, 0) \in L^2(\Omega)$, we introduce the functional

$$J_\varepsilon(v) = \frac{1}{2}\|y_\varepsilon^v - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)} + \frac{1}{2\varepsilon}\|y_{0_\varepsilon} - y(\cdot, 0)\|^2_{L^2(\Omega)},$$

being interested in the control problem

$$\inf\{J_\varepsilon(v);\ v \in \mathscr{U}_{ad}\}\,. \tag{23}$$

The following result is then immediate.

Proposition 2. *For any $\varepsilon > 0$, the control problem* (23) *admits a unique solution: the approached optimal control-state pair $\overline{u}_\varepsilon$.*

3.1 Convergence of the method

Let $\varepsilon > 0$. For the approached optimal control, the results previously obtained lead to the existence of

$$\overline{y}_{0_\varepsilon} \in L^2(\Omega) \qquad \text{and} \qquad \overline{y}_\varepsilon \in \mathbb{V}$$

satisfying

$$\begin{cases} \dfrac{\partial \overline{y}_\varepsilon}{\partial t} - \Delta \overline{y}_\varepsilon = \overline{u}_\varepsilon & \text{in} \quad Q, \\ \overline{y}_\varepsilon = 0 & \text{on} \quad \Sigma, \\ \overline{y}_\varepsilon(\cdot, 0) = \overline{y}_{0_\varepsilon} & \text{in} \quad \Omega, \end{cases} \tag{24}$$

and the estimate

$$\|\overline{y}_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon,$$

with

$$J_\varepsilon(\overline{u}_\varepsilon) \le J_\varepsilon(v), \quad \forall\, v \in \mathscr{U}_{ad}.$$

In particular
$$J_\varepsilon(\overline{u}_\varepsilon) \leq J_\varepsilon(u), \tag{25}$$
where u is the optimal control for (20)(21)(22).

We have in fact that $J_\varepsilon(u)$ is independant of ε. Indeed, since the optimal state y associated with the optimal control u satisfies $y(\cdot,0) \in L^2(\Omega)$, we can take $y^*_{0\varepsilon} = y(\cdot,0)$ to obtain that y satisfies
$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y &=& u & \text{in} \quad Q, \\ \\ y &=& 0 & \text{on} \quad \Sigma, \\ \\ y(\cdot,0) &=& y^*_{0\varepsilon} & \text{in} \quad \Omega \end{array}\right.$$
with
$$y(\cdot,T) \;=\; 0 \;\text{ in }\; \Omega, \qquad \textit{a fortiori} \qquad \|y(\cdot,T)\|_{L^2(\Omega)} < \varepsilon.$$
It follows that $J_\varepsilon(u)$ is defined, with
$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|u\|^2_{L^2(Q)} + \frac{1}{2\varepsilon}\|y^*_{0\varepsilon} - y(\cdot,0)\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|u\|^2_{L^2(Q)} = J(u,y). \end{aligned}$$
Hence (25) becomes
$$J_\varepsilon(\overline{u}_\varepsilon) \leq J_\varepsilon(u) = J(u,y), \tag{26}$$
and it follows that there exist constants $C_i \in \mathbb{R}^*$ independant of ε such as
$$\|\overline{y}_\varepsilon\|_{L^2(Q)} \leq C_1, \qquad \|\overline{u}_\varepsilon\|_{L^2(Q)} \leq C_2 \qquad \text{and} \qquad \|\overline{y}_{0\varepsilon}\|_{L^2(\Omega)} \leq C_3.$$
Then, we immediately deduce that there exists $\hat{u} \in L^2(Q)$, $\hat{y} \in L^2(Q)$ and $\hat{y}_0 \in L^2(\Omega)$ such as
$$\left\{\begin{array}{rcll} \overline{u}_\varepsilon & \longrightarrow & \hat{u} & \text{weakly in} \quad L^2(Q)\,, \\ \overline{y}_\varepsilon & \longrightarrow & \hat{y} & \text{weakly in} \quad L^2(Q)\,, \\ \overline{y}_{0\varepsilon} & \longrightarrow & \hat{y}_0 & \text{weakly in} \quad L^2(\Omega)\,. \end{array}\right.$$
But much more, (26) also implies
$$\|\overline{y}_{0\varepsilon} - y(\cdot,0)\|_{L^2(\Omega)} \leq 2\varepsilon\, C_4, \tag{27}$$
which leads, since
$$\overline{y}_{0\varepsilon} \;\longrightarrow\; \hat{y}_0 \;\text{ weakly in }\; L^2(\Omega)\,, \tag{28}$$
to
$$\hat{y}_0 \;=\; y(\cdot,0) \;\text{ in }\; \Omega. \tag{29}$$
Then it comes, with (24), that for any $\varphi \in \mathscr{D}(Q)$,
$$\begin{aligned} &\left(\frac{\partial \overline{y}_\varepsilon}{\partial t}\,,\,\varphi\right)_{L^2(Q)} - (\Delta\overline{y}_\varepsilon\,,\,\varphi)_{L^2(Q)} = (\overline{u}_\varepsilon\,,\,\varphi)_{L^2(Q)} \\ &\qquad\iff (\overline{y}_\varepsilon(\cdot,T)\;,\;\varphi(\cdot,T))_{L^2(\Omega)} - (\overline{y}_{0\varepsilon}\,,\,\varphi(\cdot,0))_{L^2(\Omega)} - \left(\overline{y}_\varepsilon\,,\,\frac{\partial\varphi}{\partial t}\right)_{L^2(Q)} \\ &\qquad\quad - (\overline{y}_\varepsilon\,,\,\Delta\varphi)_{L^2(Q)} - \left(\frac{\partial\overline{y}_\varepsilon}{\partial\nu}\,,\,\varphi\right)_{L^2(\Sigma)} = (\overline{u}_\varepsilon\,,\,\varphi)_{L^2(Q)}, \end{aligned}$$

which gives, passing to the limit,

$$
\begin{aligned}
(\hat y(\cdot,T)\,,\,\varphi(\cdot,T))_{L^2(\Omega)} - (\hat y_0\,,\,\varphi(\cdot,0))_{L^2(\Omega)} - \left(\hat y\,,\,\frac{\partial\varphi}{\partial t}\right)_{L^2(Q)} &- (\hat y\,,\,\Delta\varphi)_{L^2(Q)} \\
- \left(\frac{\partial \hat y}{\partial\nu}\,,\,\varphi\right)_{L^2(\Sigma)} &= (\hat u\,,\,\varphi)_{L^2(Q)} \\
\Longleftrightarrow\ (\hat y(\cdot,0) - \hat y_0\,,\,\varphi(\cdot,0))_{L^2(\Omega)} + \left(\frac{\partial\hat y}{\partial t} - \Delta\hat y\,,\,\varphi\right)_{L^2(Q)} & \\
- \left(\hat y\,,\,\frac{\partial\varphi}{\partial\nu}\right)_{L^2(\Sigma)} &= (\hat u\,,\,\varphi)_{L^2(Q)},
\end{aligned}
$$

i.e.

$$
\left\{
\begin{array}{rcll}
\dfrac{\partial\hat y}{\partial t} - \Delta\hat y & = & \hat u & \text{in} \quad Q, \\[2ex]
\hat y & = & 0 & \text{on} \quad \Sigma, \\[2ex]
\hat y(\cdot,0) & = & \hat y_0 & \text{in} \quad \Omega.
\end{array}
\right.
\tag{30}
$$

Moreover, the norm $\|\cdot\|_{L^2(\Omega)}$ being continuous, *a fortiori* weakly continuous,

$$
\overline{y}_\varepsilon \quad\longrightarrow\quad \hat y \quad \text{weakly in} \quad L^2(Q) \qquad \text{and} \qquad \|\overline{y}_\varepsilon(\cdot,T)\|_{L^2(\Omega)} < \varepsilon
$$

bring

$$
\hat y(\cdot,T) \quad = \quad 0 \quad \text{in} \quad \Omega. \tag{31}
$$

Thus (30) and (31) allow to conclude that $\hat y \in L^2(Q)$ satisfies

$$
\left\{
\begin{array}{rcll}
\dfrac{\partial\hat y}{\partial t} - \Delta\hat y & = & \hat u & \text{in} \quad Q, \\[2ex]
\hat y & = & 0 & \text{on} \quad \Sigma, \\[2ex]
\hat y(\cdot,T) & = & 0 & \text{in} \quad \Omega.
\end{array}
\right.
$$

Then, noting that $\hat u \in \mathscr{U}_{ad}$, since $\overline{u}_\varepsilon \in \mathscr{U}_{ad}$ and $\mathscr{U}_{ad}$ is closed and therefore weakly closed, we get that the control-state pair $(\hat u, \hat y)$ is admissible for (20)(21)(22), so that

$$
J(u,y) \le J(\hat u, \hat y)\,. \tag{32}
$$

On the other hand, passing (26) to the limit when $\varepsilon \to 0$, it comes $J(\hat u,\hat y) \le J(u,y)$; that is to say, with (32), that $J(u,y) = J(\hat u,\hat y)$.

Then we conclude, by uniqueness of the optimal control-state pair (u,y), that $(\hat u,\hat y) = (u,y)$, which ends up proving the following result.

Proposition 3. *For any $\varepsilon > 0$, the approach optimal control $\overline{u}_\varepsilon$ and the associated state $\overline{y}_\varepsilon$ satisfy*

$$
\left\{
\begin{array}{llll}
\overline{u}_\varepsilon & \longrightarrow & u & \textit{weakly in} \quad L^2(Q)\,, \\
\overline{y}_\varepsilon & \longrightarrow & y & \textit{weakly in} \quad L^2(Q)\,,
\end{array}
\right.
$$

where (u,y) is the optimal control-state pair for (20)(21)(22).

We establish below that we actually have more: the strong convergence.

Theorem 4. *For any $\varepsilon > 0$, the approach optimal control $\overline{u}_\varepsilon$ and the associated state $\overline{y}_\varepsilon$ are such that, when $\varepsilon \to 0$,*

$$\begin{cases} \overline{u}_\varepsilon & \longrightarrow & u & \textit{strongly in} & L^2(Q)\,, \\ \overline{y}_\varepsilon & \longrightarrow & y & \textit{strongly in} & L^2(Q)\,, \end{cases}$$

where (u, y) is the optimal control-state pair for (20)(21)(22).

Proof. From the results previously obtained, we have that

$$\overline{u}_\varepsilon \quad \longrightarrow \quad u \quad \text{weakly in} \quad L^2(Q)\,, \tag{33}$$

$$\overline{y}_\varepsilon \quad \longrightarrow \quad y \quad \text{weakly in} \quad L^2(Q)\,, \tag{34}$$

and

$$J(u, y) = \lim_{\varepsilon \to 0} J_\varepsilon(\overline{u}_\varepsilon)\,.$$

Where, from (27)(28) and (29), the last equality above can still be written

$$\|y - z_d\|^2_{L^2(Q)} + N\|u\|^2_{L^2(Q)} = \lim_{\varepsilon \to 0} \left(\|\overline{y}_\varepsilon - z_d\|^2_{L^2(Q)} + N\|\overline{u}_\varepsilon\|^2_{L^2(Q)} \right). \tag{35}$$

But then, the norm $\|\cdot\|_{L^2(Q)}$ being continous, *a fortiori* weakly lower semi-continous, it comes, with (33) and (34), that

$$\begin{cases} \|y - z_d\|^2_{L^2(Q)} & \leq & \liminf_{\varepsilon \to 0} \|\overline{y}_\varepsilon - z_d\|^2_{L^2(Q)}, \\ \\ \|u\|^2_{L^2(Q)} & \leq & \liminf_{\varepsilon \to 0} \|\overline{u}_\varepsilon\|^2_{L^2(Q)}. \end{cases}$$

From where it follows, with (35), that

$$\|y - z_d\|^2_{L^2(Q)} = \lim_{\varepsilon \to 0} \|\overline{y}_\varepsilon - z_d\|^2_{L^2(Q)}, \tag{36}$$

and

$$\|u\|^2_{L^2(Q)} = \lim_{\varepsilon \to 0} \|\overline{u}_\varepsilon\|^2_{L^2(Q)}. \tag{37}$$

Hence, since

$$\|\overline{y}_\varepsilon - y\|^2_{L^2(Q)} = \|\overline{y}_\varepsilon - z_d\|^2_{L^2(Q)} + \|y - z_d\|_{L^2(Q)} - 2(\overline{y}_\varepsilon - z_d\,,\, y - z_d)_{L^2(Q)},$$

we conclude with (34) and (36) that

$$\lim_{\varepsilon \to 0} \|\overline{y}_\varepsilon - y\|^2_{L^2(Q)} = 0 \qquad i.e. \qquad \overline{y}_\varepsilon \quad \longrightarrow \quad y \quad \text{strongly in} \quad L^2(Q)\,.$$

In a similar way, (33) and (37) lead to

$$\overline{u}_\varepsilon \quad \longrightarrow \quad u \quad \text{weakly in} \quad L^2(Q)\,,$$

which ends up proving the announced result. ◻

3.2 Approached optimality system

Let us start by recalling that, for any $\varepsilon > 0$ and for the optimal control $\overline{u}_\varepsilon \in \mathscr{U}_{ad}$, we have the existence of

$$\overline{y}_{0\varepsilon} \in L^2(\Omega) \qquad \text{and} \qquad \overline{y}_\varepsilon \in L^2(Q)\,,$$

verifying

$$\begin{cases} \dfrac{\partial \overline{y}_\varepsilon}{\partial t} - \Delta \overline{y}_\varepsilon &= \overline{u}_\varepsilon \quad \text{in} \quad Q, \\ \\ \overline{y}_\varepsilon &= 0 \quad \text{on} \quad \Sigma, \\ \\ \overline{y}_\varepsilon(\cdot, 0) &= \overline{y}_{0\varepsilon} \quad \text{in} \quad \Omega, \end{cases}$$

and the estimate

$$\|\overline{y}_\varepsilon(\cdot, T)\|_{L^2(\Omega)} < \varepsilon.$$

So, given $v \in \mathscr{U}_{ad}$ and $\lambda \in \mathbb{R}^*$; we have:

$$\begin{aligned} J_\varepsilon(\overline{u}_\varepsilon + \lambda\,(v - \overline{u}_\varepsilon)) &= \frac{1}{2}\|y(\overline{u}_\varepsilon + \lambda\,(v - \overline{u}_\varepsilon)\,, \overline{y}_{0\varepsilon}) - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|\overline{u}_\varepsilon + \lambda\,(v - \overline{u}_\varepsilon)\|^2_{L^2(Q)} \\ &\quad + \frac{1}{2\varepsilon}\|\overline{y}_{0\varepsilon} - y(\cdot, 0)\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\|\overline{y}_\varepsilon - z_d + \lambda\phi_\varepsilon\|^2_{L^2(Q)} + \frac{N}{2}\|\overline{u}_\varepsilon + \lambda\,(v - \overline{u}_\varepsilon)\|^2_{L^2(Q)} \\ &\quad + \frac{1}{2\varepsilon}\|\overline{y}_{0\varepsilon} - y(\cdot, 0)\|^2_{L^2(\Omega)} \\ &= J_\varepsilon(\overline{u}_\varepsilon) + \frac{\lambda^2}{2}\left(\|\phi_\varepsilon\|^2_{L^2(Q)} + N\|v - \overline{u}_\varepsilon\|^2_{L^2(Q)}\right) \\ &\quad + \lambda(\overline{y}_\varepsilon - z_d\,,\, \phi_\varepsilon)_{L^2(Q)} + \lambda N(\overline{u}_\varepsilon\,,\, v - \overline{u}_\varepsilon)_{L^2(Q)}, \end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J_\varepsilon(\overline{u}_\varepsilon + \lambda\,(v - \overline{u}_\varepsilon))\bigg|_{\lambda=0} = (\overline{y}_\varepsilon - z_d\,,\, \phi_\varepsilon)_{L^2(Q)} + N(\overline{u}_\varepsilon\,,\, v - \overline{u}_\varepsilon)_{L^2(Q)},$$

where $\phi_\varepsilon = y(v - \overline{u}_\varepsilon, \overline{y}_{0\varepsilon}) - y(0, \overline{y}_{0\varepsilon})$ is defined by

$$\begin{cases} \dfrac{\partial \phi_\varepsilon}{\partial t} - \Delta \phi_\varepsilon &= v - \overline{u}_\varepsilon \quad \text{in} \quad Q, \\ \\ \phi_\varepsilon &= 0 \qquad \text{on} \quad \Sigma, \\ \\ \phi_\varepsilon(\cdot, 0) &= 0 \qquad \text{in} \quad \Omega. \end{cases} \tag{38}$$

Hence, with the first-order optimality condition of Euler-Lagrange, we obtain that the approached optimal control $\overline{u}_\varepsilon$ is the unique element of $\mathscr{U}_{ad}$ satisfying

$$(\overline{y}_\varepsilon - z_d\,,\, \phi_\varepsilon)_{L^2(Q)} + N(\overline{u}_\varepsilon\,,\, v - \overline{u}_\varepsilon)_{L^2(Q)} \geq 0, \qquad \forall\, v \in \mathscr{U}_{ad}. \tag{39}$$

Let introduce here the adjoint state $p_\varepsilon \in L^2(Q)$ by

$$\left\{\begin{array}{rcll} -\dfrac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon &=& \overline{y}_\varepsilon - z_d & \text{in} \quad Q, \\ p_\varepsilon &=& 0 & \text{on} \quad \Sigma, \\ p_\varepsilon(\cdot, T) &=& 0 & \text{in} \quad \Omega. \end{array}\right.$$

It comes with (38) that

$$-\frac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon \;=\; \overline{y}_\varepsilon - z_d \quad \text{in} \quad Q$$

leads to

$$\begin{aligned} (\overline{y}_\varepsilon - z_d \,,\, \phi_\varepsilon)_{L^2(Q)} &= -\left(\frac{\partial p_\varepsilon}{\partial t} \,,\, \phi_\varepsilon\right)_{L^2(Q)} - (\Delta p_\varepsilon \,,\, \phi_\varepsilon)_{L^2(Q)} \\ &= (p_\varepsilon(\cdot, 0) \,,\, \phi_\varepsilon(\cdot, 0))_{L^2(\Omega)} - (p_\varepsilon(\cdot, T) \,,\, \phi_\varepsilon(\cdot, T))_{L^2(\Omega)} + \left(p_\varepsilon \,,\, \frac{\partial \phi_\varepsilon}{\partial t}\right)_{L^2(Q)} \\ &\quad - (p_\varepsilon \,,\, \Delta\phi_\varepsilon)_{L^2(Q)} - \left(\frac{\partial p_\varepsilon}{\partial \nu} \,,\, \phi_\varepsilon\right)_{L^2(\Sigma)} + \left(p_\varepsilon \,,\, \frac{\partial \phi_\varepsilon}{\partial \nu}\right)_{L^2(\Sigma)} \\ &= \left(p_\varepsilon \,,\, \frac{\partial \phi_\varepsilon}{\partial t} - \Delta\phi_\varepsilon\right)_{L^2(Q)} = (p_\varepsilon \,,\, v - \overline{u}_\varepsilon)_{L^2(Q)}, \end{aligned}$$

so that the optimality condition (39) reduces to

$$(p_\varepsilon + N\overline{u}_\varepsilon \,,\, v - \overline{u}_\varepsilon)_{L^2(Q)} \;\geq\; 0, \qquad \forall\, v \in \mathscr{U}_{ad}.$$

We thus obtain the following result.

Theorem 5. *Let $\varepsilon > 0$. The control $\overline{u}_\varepsilon$ is unique solution of (23) if and only if the quadruplet*

$$\{\overline{y}_{0\varepsilon}, \overline{u}_\varepsilon, \overline{y}_\varepsilon, p_\varepsilon\} \in L^2(\Omega) \times \left(L^2(Q)\right)^3$$

is solution of the approached optimality system defined by the partial differential equations systems

$$\left\{\begin{array}{rcll} \dfrac{\partial \overline{y}_\varepsilon}{\partial t} - \Delta \overline{y}_\varepsilon &=& \overline{u}_\varepsilon & \textit{in} \quad Q, \\ \overline{y}_\varepsilon &=& 0 & \textit{on} \quad \Sigma, \\ \overline{y}_\varepsilon(\cdot, 0) &=& \overline{y}_{0\varepsilon} & \textit{in} \quad \Omega, \end{array}\right. \tag{40}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon &=& \overline{y}_\varepsilon - z_d & \textit{in} \quad Q, \\ p_\varepsilon &=& 0 & \textit{on} \quad \Sigma, \\ p_\varepsilon(\cdot, T) &=& 0 & \textit{in} \quad \Omega, \end{array}\right. \tag{41}$$

the estimate

$$\|\overline{y}_\varepsilon(\cdot, T)\|_{L^2(\Omega)} \;<\; \varepsilon, \tag{42}$$

and the variational inequality

$$(p_\varepsilon + N\overline{u}_\varepsilon \,,\, v - \overline{u}_\varepsilon)_{L^2(Q)} \;\geq\; 0, \qquad \forall\, v \in \mathscr{U}_{ad}. \tag{43}$$

3.3 Singular optimality system

From the results obtained in Section 3.1, we have:

$$\overline{u}_\varepsilon \longrightarrow u \quad \text{strongly in} \quad L^2(Q)$$

$$\overline{y}_\varepsilon \longrightarrow y \quad \text{strongly in} \quad L^2(Q)\,,$$

where (u, y) is the optimal control-state pair of (20)(21)(22).

So that, (41) being well-posed in the sense of Hadamard, it follows that there exists $p \in L^2(Q)$ such that

$$p_\varepsilon \longrightarrow p \quad \text{strongly in} \quad L^2(Q)\,.$$

Then, we easily pass the results of Theorem 5 to the limit, when $\varepsilon \to 0$, to obtain that the strong singular optimaly system characterizing the optimal control-state pair of (20)(21)(22) is as specified below.

Theorem 6. *The control-state pair (u, y) is unique solution of the control problem* (20)(21)(22) *if and only if the triple*

$$\{u, y, p\} \in \left(L^2(Q)\right)^3$$

is solution of the singular optimality system defined by the partial differential equation systems

$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y &=& u & in \quad Q, \\[2ex] y &=& 0 & on \quad \Sigma, \\[2ex] y(\cdot, T) &=& 0 & in \quad \Omega, \end{array}\right. \tag{44}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial p}{\partial t} - \Delta p &=& y - z_d & in \quad Q, \\[2ex] p &=& 0 & on \quad \Sigma, \\[2ex] p(\cdot, T) &=& 0 & in \quad \Omega, \end{array}\right. \tag{45}$$

and the variational inequality

$$(p + Nu\,,\, v - u)_{L^2(Q)} \;\geq\; 0, \qquad \forall\, v \in \mathscr{U}_{ad}. \tag{46}$$

4 Conclusion

In this work, we succeed in characterizing the optimal control-state pair of the control problem for the ill-posed backwards heat equation, using the controllability concept. The method consists in interpreting the initial problem as an inverse problem, we are also saying a controllability problem. An approach that allows us to obtain a strong and decoupled singular optimality system. As expected, the approach here proposed does well without using an additional assumption of the class of the regularity assumption (4) used in [1] and the Slater-type one (3). All that is required is the following

$$(u, y): \quad y(\cdot, 0) \in L^2(\Omega)\,. \tag{47}$$

Finally, in view of the similar results obtained for the ill-posed Cauchy system for elliptic (see [2] and [4]), parabolic (see [3]) and hyperbolic (see [5]) operators, the controllability method here proposed seems relevant for control problems (of singular distributed systems) which require recourse to Slater-type assumptions such as (3).

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