# Camera Models

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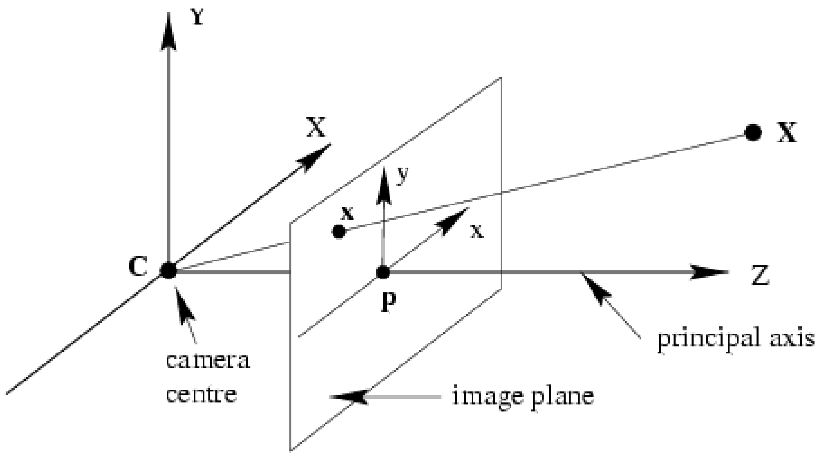
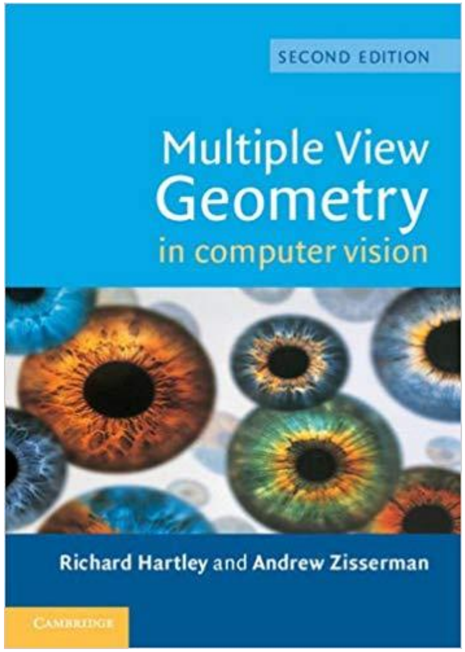
# Outline

- Camera models


[Slides credit: Marc Pollefeys]

# Camera Obscura: the Pre-Camera

- First idea: Mo-Ti, China (470BC to 390BC)
- First built: Alhazen, Iraq/Egypt (965 to 1039AD)

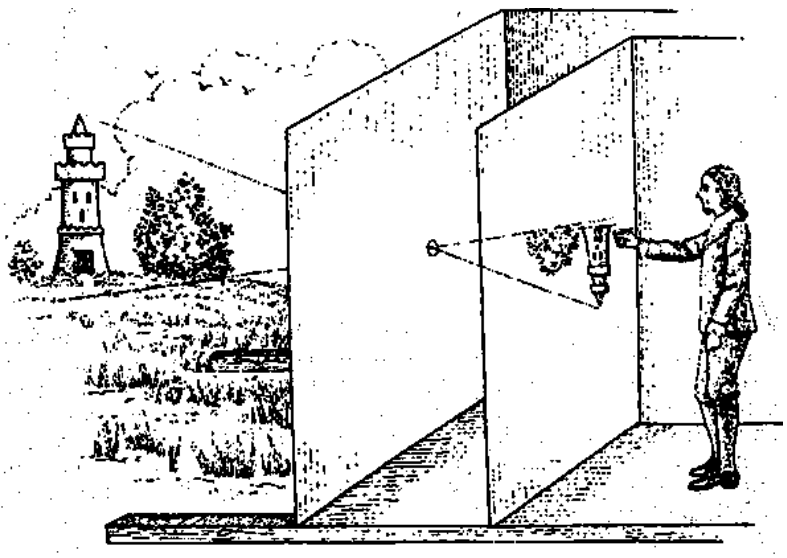


Illustration of Camera Obscura

Freestanding camera obscura at UNC Chapel Hill

Photo by Seth Ilys

# Camera Obscura

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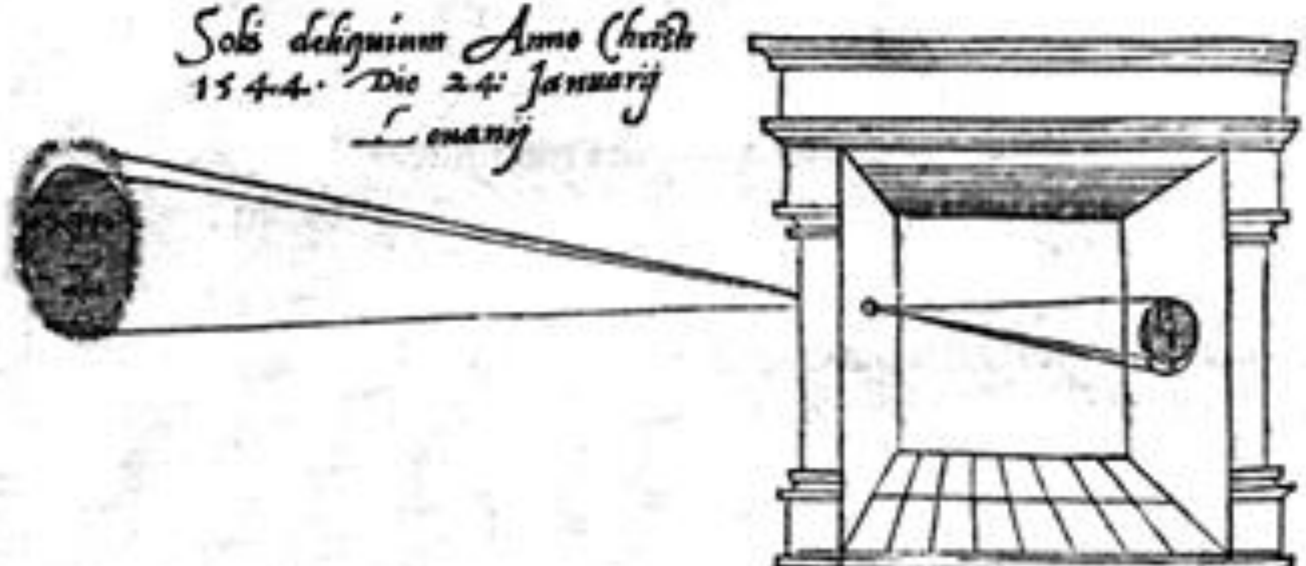


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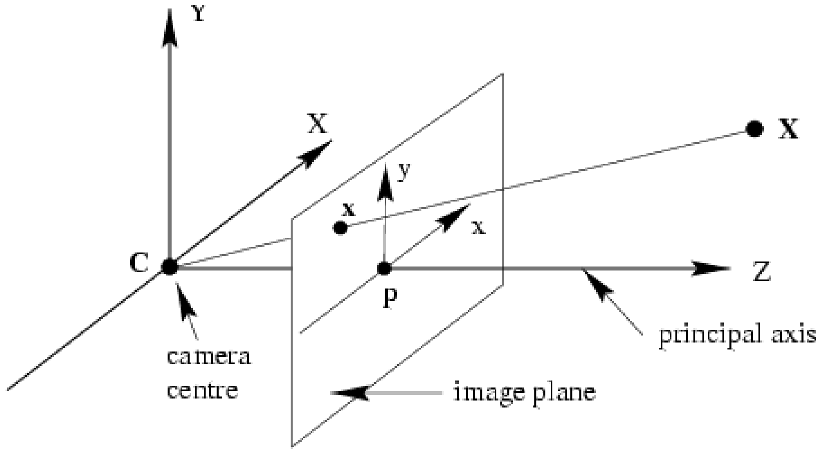
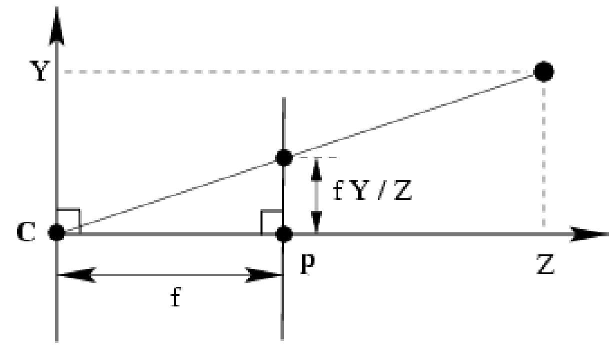
Camera Obscura, Reinerus Gemma-Frisius, 1544

# Pinhole Camera Model

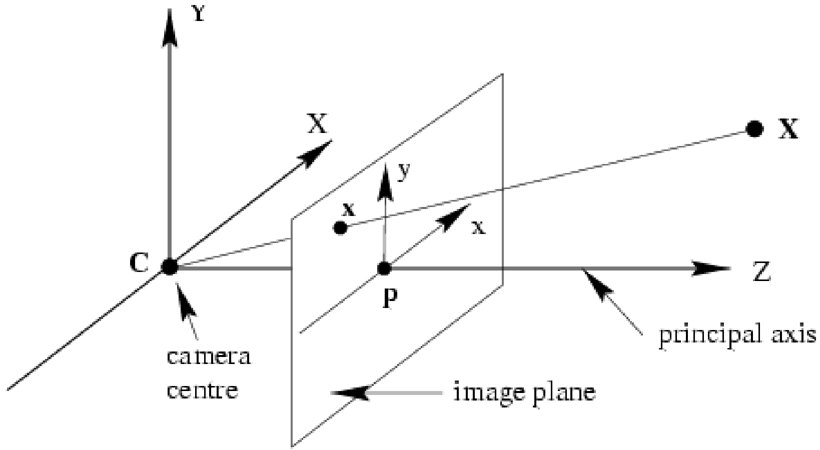


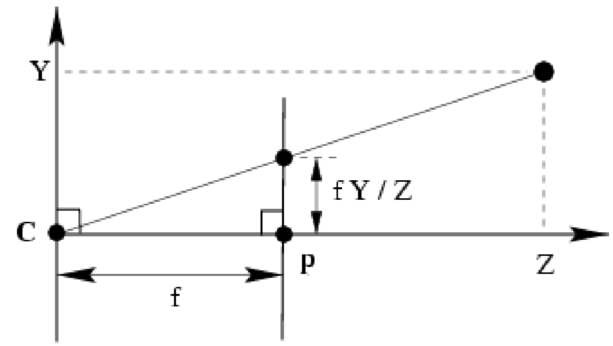




$$(X, Y, Z)^T \mapsto (fX / Z, fY / Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Pinhole Camera Model



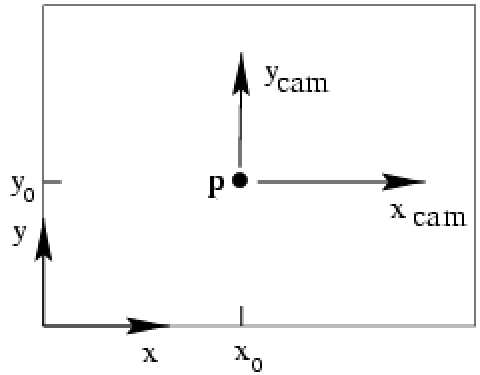




$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{PX}$$

$$\mathrm{P} = \mathrm{diag}(f, f, 1)\left[\mathrm{I} \mid 0\right]$$

# Principal Point Offset

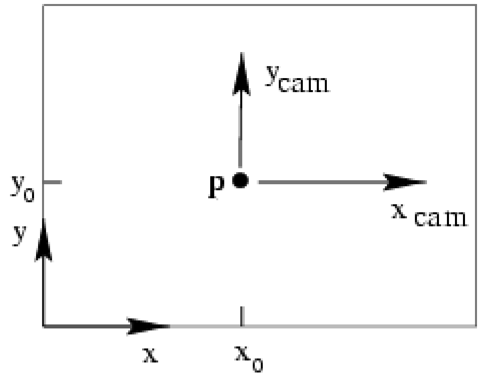



$$(X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Principal Point Offset

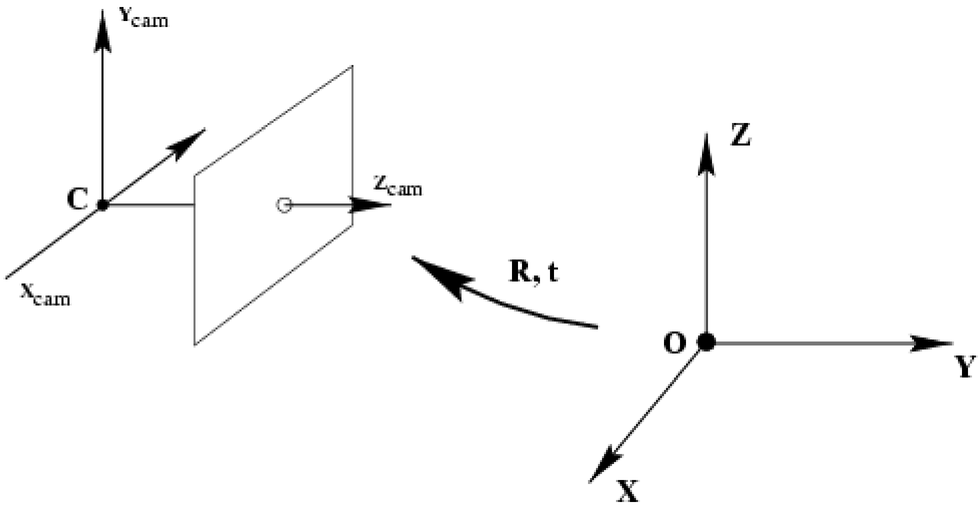



$$\mathrm{x} = \mathrm{K}\left[\mathrm{I} \mid 0\right]\mathrm{X}_{\mathrm{cam}}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

Calibration Matrix

# Camera Rotation and Translation

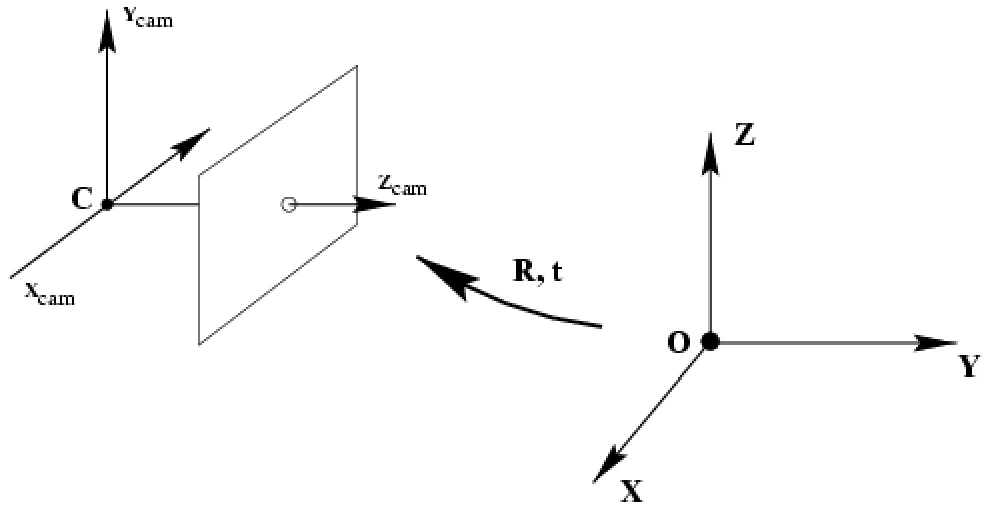



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -RC \\ 0 & 1 \end{bmatrix} X$$

$$x = K\left[I \mid 0\right] X_{cam}$$

# Camera Rotation and Translation




$$x = KR\left[I \mid -\tilde{C}\right]X$$

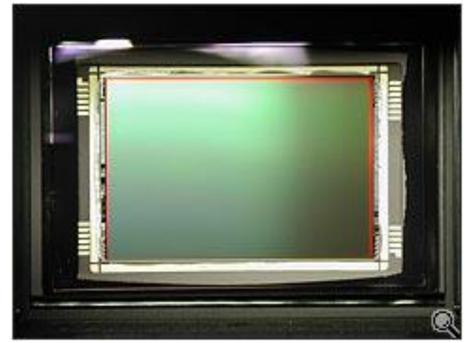
$$x = PX$$

$$P = K\left[R \mid t\right] \qquad t = -R\tilde{C}$$

Internal Camera Parameter
Internal Orientation
Intrinsic Matrix

External Camera Parameter
Exterior Orientation
Extrinsic Matrix

# CCD Camera

Non-square pixels →

$$K = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

# Finite Projective Camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

s: skew parameter,
=0 for most normal cameras

$$\mathrm{P} = \underbrace{\mathrm{KR}}\left[\mathrm{I} \mid -\tilde{\mathrm{C}}\right]$$

11 dof (5+3+3)

non-singular

decompose P in K,R,C?

$$\mathrm{P} = \left[\mathrm{M} \mid \mathrm{p}_4\right] \qquad \left[\mathrm{K}, \mathrm{R}\right] = RQ(\mathrm{M}) \quad \tilde{\mathrm{C}} = -\mathrm{M}^{-1}\mathrm{p}_4$$

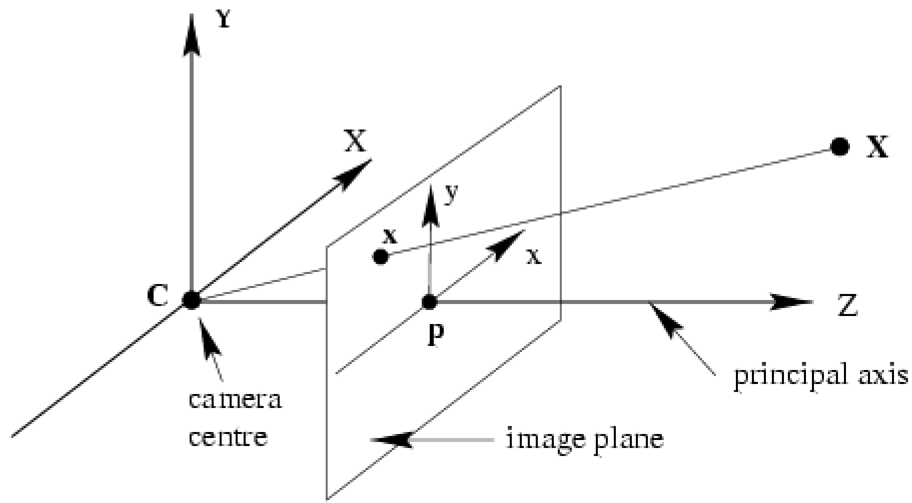
$P_4$: Last column of P

{finite cameras}={$P_{4x3}$ | det M≠0}

If rank P=3, but rank M<3, then cam at infinity

# Camera Anatomy

- Camera center
- Column points
- Principal plane
- Axis plane
- Principal point
- Principal ray



# Camera Center

Camera Center (C): Null-space of camera projection matrix

$$\mathrm{PC} = 0$$

Proof:

$$\mathrm{X} = \lambda \mathrm{A} + (1-\lambda)\mathrm{C}$$

$$\mathrm{x} = \mathrm{PX} = \lambda \mathrm{PA} + (1-\lambda)\mathrm{PC}$$

For all A all points on AC project on image of A,

therefore C is camera center

Image of camera center is $(0,0,0)^{\mathrm{T}}$, i.e. undefined

Finite cameras: $C = \begin{pmatrix} -\mathrm{M}^{-1}\mathrm{p}_4 \\ 1 \end{pmatrix}$

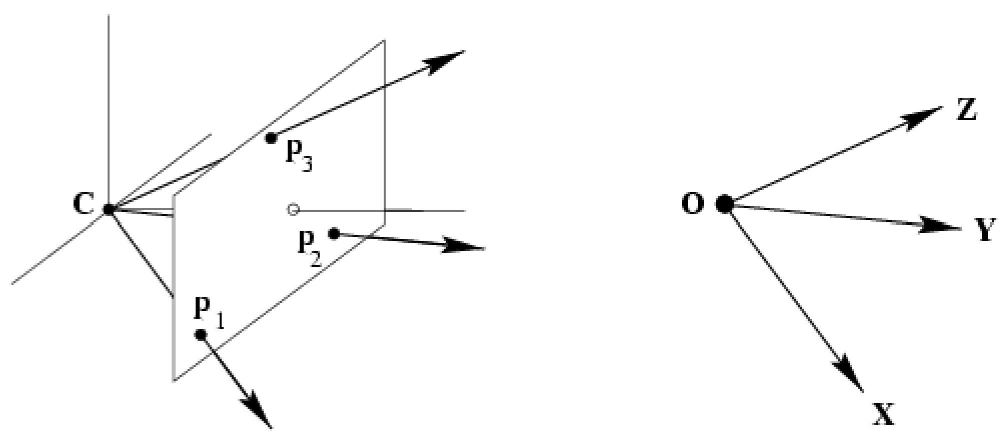
Infinite cameras: $C = \begin{pmatrix} \mathrm{d} \\ 0 \end{pmatrix}, \mathrm{Md} = 0$

The camera center is a point at infinity

# Column Vectors

$$[p_2] = [p_1 p_2 p_3 p_4]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$p_1$, $p_2$, $p_3$: vanishing points of the world coordinate X, Y, and Z axes
Image points corresponding to X,Y,Z directions and origin

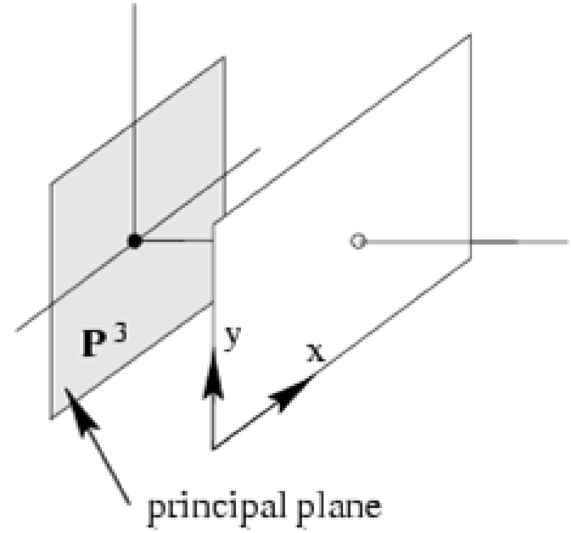


# Row Vectors

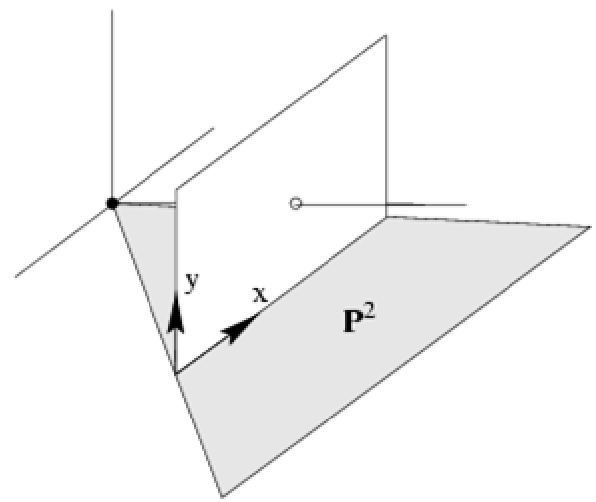
$$\mathtt{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{1\mathsf{T}} \\ \mathbf{P}^{2\mathsf{T}} \\ \mathbf{P}^{3\mathsf{T}} \end{bmatrix}$$

# Row Vectors

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\mathsf{T}} \\ \mathrm{p}^{2\mathsf{T}} \\ \mathrm{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

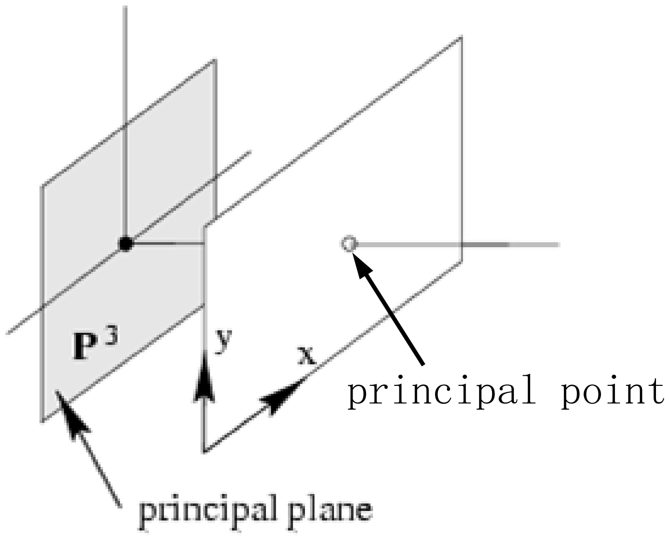


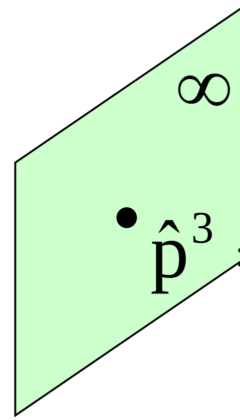

$$\begin{bmatrix} 0 \\ y \\ w \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\mathsf{T}} \\ \mathrm{p}^{2\mathsf{T}} \\ \mathrm{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$




note: $\mathrm{p}^1$,$\mathrm{p}^2$ dependent on image reparametrization

# The Principal Point



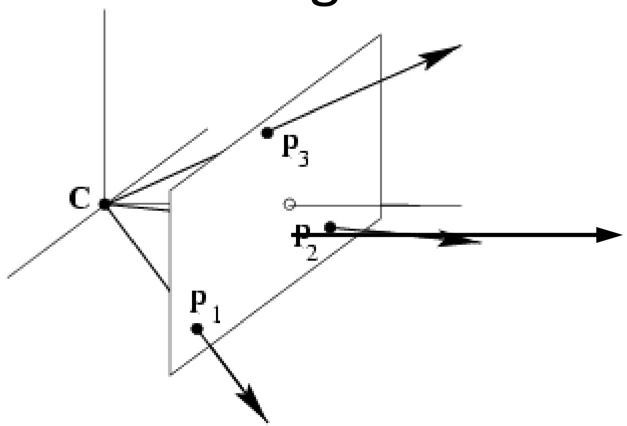




$$\hat{\mathrm{p}}^3 = \underbrace{\left(p_{31}, p_{32}, p_{33}\right.}_{m^3}\left.,0\right)$$

$$\mathrm{x}_0 = \mathrm{P}\hat{\mathrm{p}}^3 = \mathrm{M}\mathrm{m}^3$$

# The Principal Axis Vector

vector defining front side of camera




$m^3$

$$x = P_{cam} X_{cam} = K[I \mid 0] X_{cam} \qquad v = \det(M) m^3 = (0,0,1)^T$$

$$P_{cam} \mapsto k P_{cam} \qquad v \mapsto k^4 v$$

(direction unaffected)

$$P = kKR\left[I \mid -\tilde{C}\right] = [M \mid p_4]$$

Direction unaffected because

$$\det(R) > 0$$

The principal axis vector ***v***=*det(M)m*$^3$ is directed towards the front of the camera

# Action of Projective Camera on Point

**Forward projection**

$$x = PX$$

$$x = PD = [M \mid p_4]D = Md$$

**Back-projection**

$$PC = 0$$

$$X = P^{+}x \qquad P^{+} = P^{\mathsf{T}}\left(PP^{\mathsf{T}}\right)^{-1} \qquad PP^{+} = I$$

(pseudo-inverse)

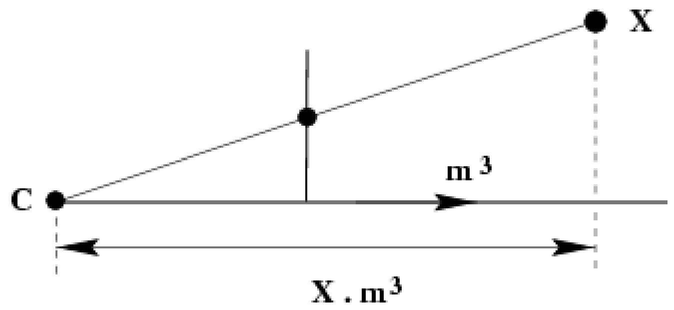
$$X(\lambda) = P^{+}x + \lambda C$$

For finite camera

$$d = M^{-1}x$$

$$X(\lambda) = \mu\underbrace{\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}}_{C} = \begin{pmatrix} M^{-1}(\mu x - p_4) \\ 1 \end{pmatrix}$$

# Depth of Points




$$w = \mathrm{P}^{3^T}\mathrm{X} = \mathrm{P}^{3^T}\left(\mathrm{X} - \mathrm{C}\right) = \mathrm{m}^{3^T}\left(\tilde{\mathrm{X}} - \tilde{\mathrm{C}}\right)$$

(PC=0) (dot product)

If $\det M > 0; \left\|\mathrm{m}^3\right\|, = 1$

then m³ unit vector in positive direction

$$\boxed{\mathrm{depth}\left(\mathrm{X}; \mathrm{P}\right) = \frac{\mathrm{sign}(\mathrm{detM})w}{T\left\|\mathrm{m}^3\right\|}}$$

$$\mathrm{X} = \left(X, Y, Z, T\right)^{\mathrm{T}}$$

# Camera Matrix Decomposition

Finding the camera center

$$\mathrm{PC} = 0$$ (use SVD to find null-space)

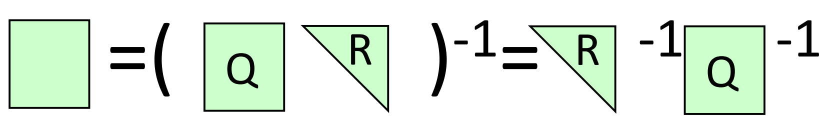
$$X = \det([\mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4]) \quad Y = -\det([\mathrm{p}_1, \mathrm{p}_3, \mathrm{p}_4])$$

$$Z = \det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_4]) \quad T = -\det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3])$$

Finding the camera orientation and internal parameters

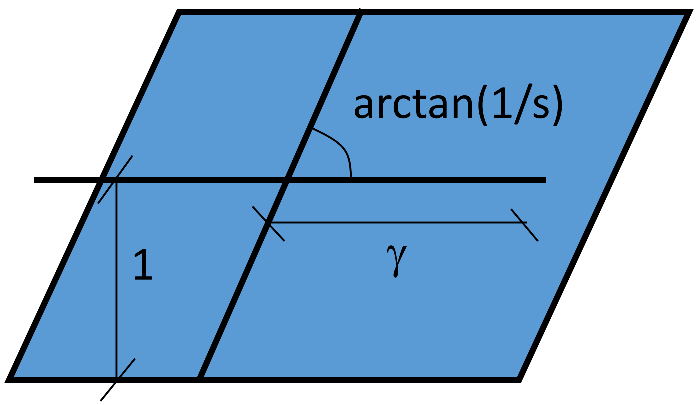
$$\mathrm{M} = \mathrm{KR}$$ (use RQ decomposition ~QR)

(if only QR, invert)



# When is Skew Non-zero?

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$




for CCD/CMOS, always s=0

Image from image, s≠0 possible
(non coinciding principal axis)

resulting camera: $HP$

# Euclidean vs. Projective

general projective interpretation

$$P = \left[ 3 \times 3\,\text{homography} \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left[ 4 \times 4\,\text{homography} \right]$$

Meaningfull decomposition in K,R,t requires Euclidean image and space

Camera center is still valid in projective space

Principal plane requires affine image and space

Principal ray requires affine image and Euclidean space

# Cameras at Infinity

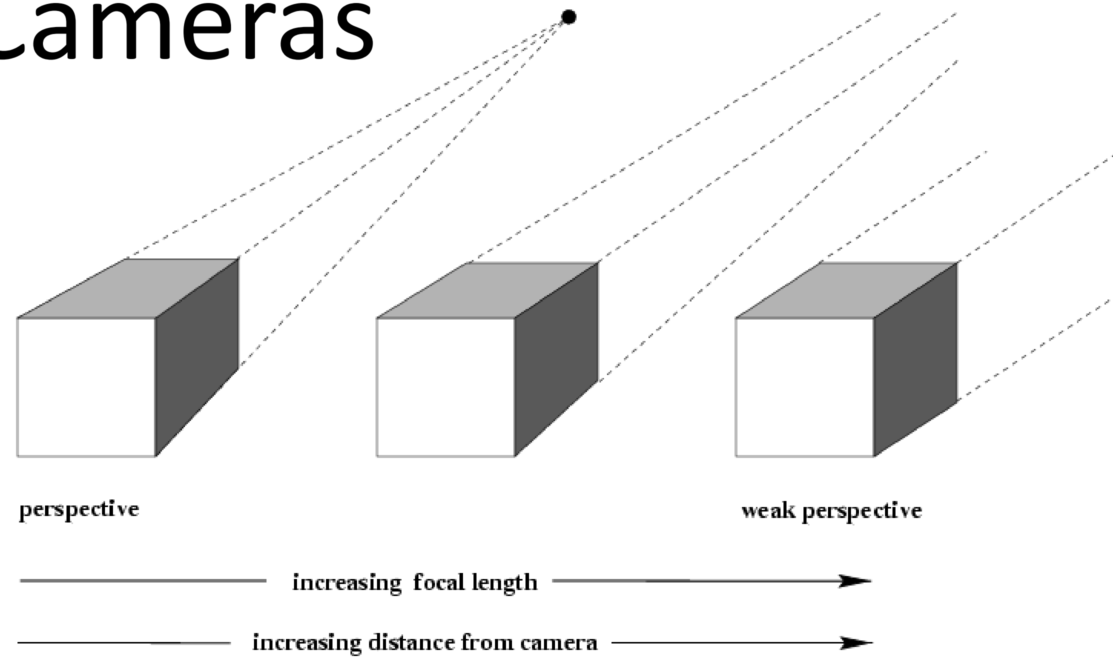
Camera center at infinity

$$P\begin{bmatrix} d \\ 0 \end{bmatrix} = 0 \Rightarrow \det M = 0$$

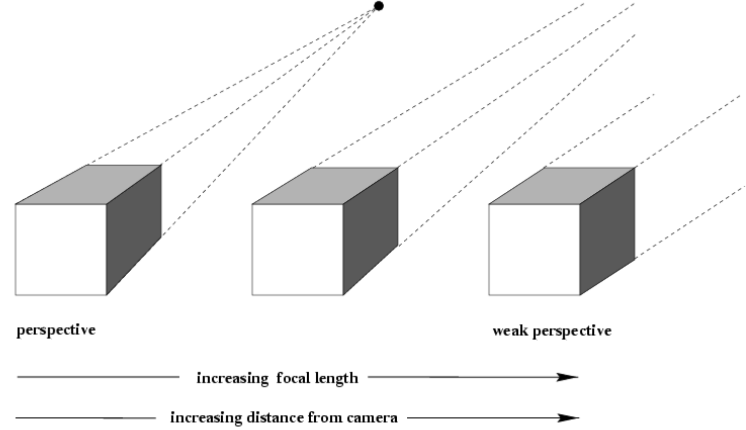
Affine and non-affine cameras

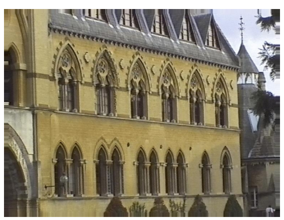
<u>Definition</u>: affine camera has $P^{3T}=(0,0,0,1)$

# Affine Cameras




# Affine Cameras



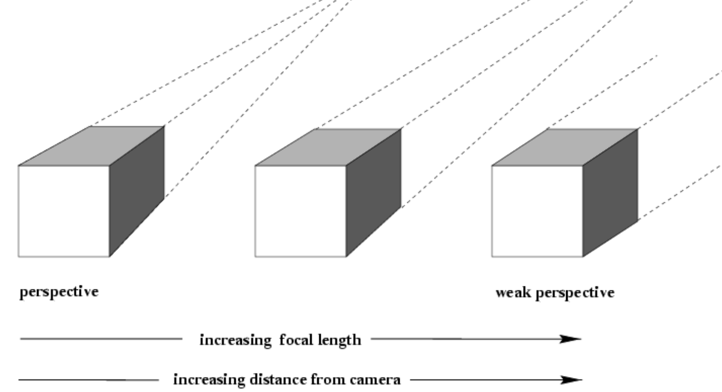
$$P_0 = KR\left[I \mid -\tilde{C}\right] = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & -r^{3T}\tilde{C} \end{bmatrix}$$

$$d_0 = -r^{3T}\tilde{C}$$

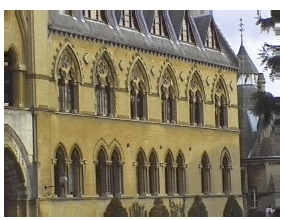
$$P_t = K\begin{bmatrix} r^{1T} & -r^{1T}\left(\tilde{C} - t r^3\right) \\ r^{2T} & -r^{2T}\left(\tilde{C} - t r^3\right) \\ r^{3T} & -r^{3T}\left(\tilde{C} - t r^3\right) \end{bmatrix} = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

modifying $p_{34}$ corresponds to moving along principal ray

# Affine Cameras




now adjust zoom to compensate



$$P_t = K\begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix}\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

$$= \frac{d_t}{d_0} K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} d_t/d_0 & d_0 \end{bmatrix}$$

$$P_\infty = \lim_{t\to\infty} P_t = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ 0 & d_0 \end{bmatrix}$$

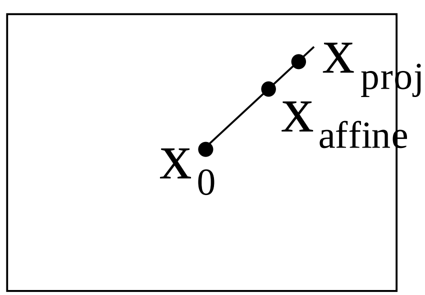
# Error in Employing Affine Cameras

$$X = \begin{pmatrix} \alpha r^1 + \beta r^2 \\ 1 \end{pmatrix}$$ point on plane parallel with principal plane and through origin, then

$$P_0 X = P_t X = P_\infty X$$

$$X = \begin{pmatrix} \alpha r^1 + \beta r^2 + \Delta r^3 \\ 1 \end{pmatrix}$$ general points

$$x_{\text{proj}} = P_0 X = K \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 + \Delta \end{pmatrix} \qquad x_{\text{affine}} = P_\infty X = K \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 \end{pmatrix}$$



# Error in Employing Affine Cameras

$$\mathrm{x}_{\mathrm{affine}} - \mathrm{x}_{\mathrm{proj}} = \frac{\Delta}{d_0}\left(\mathrm{x}_{\mathrm{proj}} - \mathrm{x}_0\right)$$

Approximation should only cause small error

1. $\Delta$ much smaller than $d_0$
2. Points close to principal point
   (i.e. small field of view)

# Decomposition of $P_\infty$

$$P_\infty = \begin{bmatrix} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & d_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix}$$

absorb $d_0$ in $K_{2x2}$

$$= \begin{bmatrix} K_{2x2}\tilde{R} & K_{2x2}\tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2x2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} + K_{2x2}^{-1}\tilde{x}_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{2x2} & K_{2x2}\tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

Prefer this one

$$P_\infty = \boxed{\begin{bmatrix} K_{2x2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

alternatives, because 8dof (3+3+2), not more

# Summary Parallel Projection

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

canonical representation

$$K = \begin{bmatrix} K_{2\times 2} & 0 \\ 0 & 1 \end{bmatrix}$$

calibration matrix

principal point is not defined

# A Hierarchy of Affine Cameras

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1 \end{bmatrix}$$ (5dof)

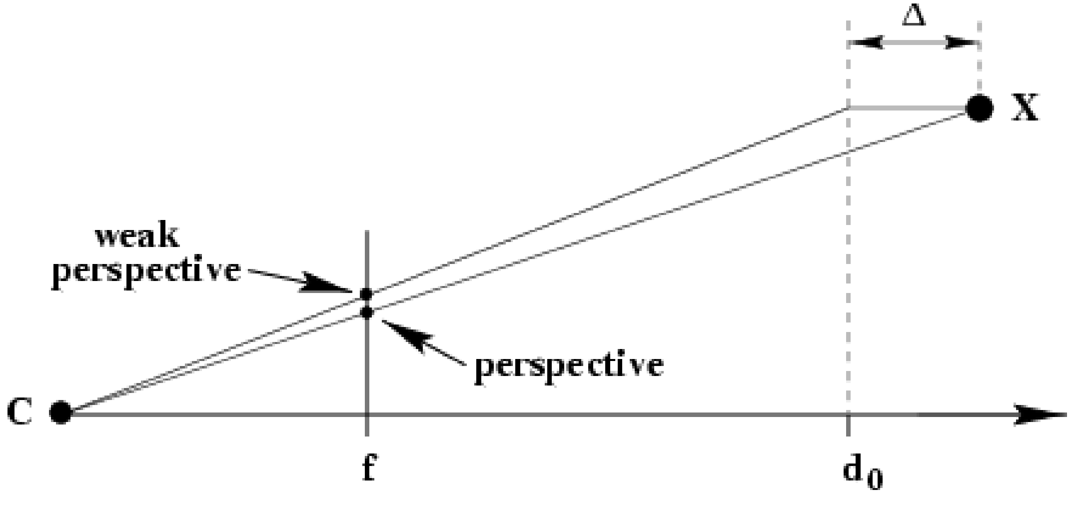
Scaled orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1/k \end{bmatrix}$$ (6dof)

# A Hierarchy of Affine Cameras

Weak perspective projection

$$\mathrm{P}_\infty = \begin{bmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathrm{r}^{1\mathrm{T}} & t_1 \\ \mathrm{r}^{1\mathrm{T}} & t_2 \\ 0 & 1/k \end{bmatrix} \quad \text{(7dof)}$$



# A Hierarchy of Affine Cameras

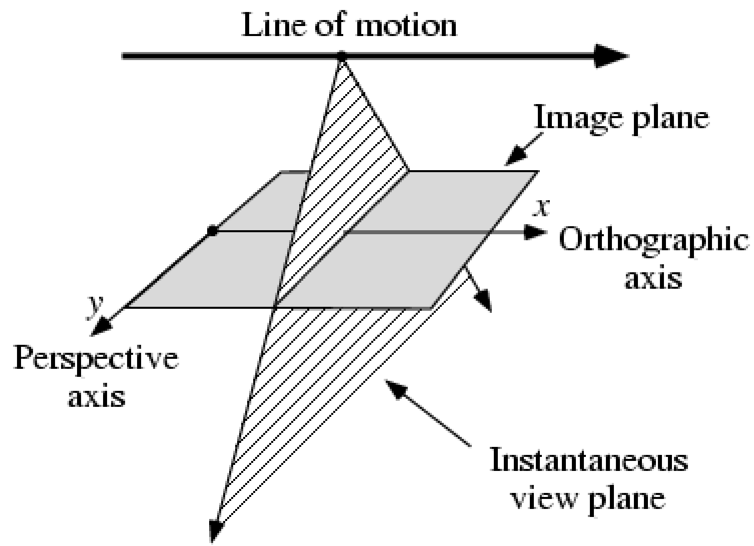
Affine camera (8dof)

$$P_A = \begin{bmatrix} \alpha_x & s & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathrm{r}^{1T} & t_1 \\ \mathrm{r}^{1T} & t_2 \\ 0 & 1/k \end{bmatrix} \quad P_A = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_A = \left[3\times 3\,\mathrm{affine}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[4\times 4\,\mathrm{affine}\right]$$

1. Affine camera=camera with principal plane coinciding with $\Pi_\infty$
2. Affine camera maps parallel lines to parallel lines
3. No center of projection, but direction of projection $P_A D=0$ (point on $\Pi_\infty$)

# Pushbroom Cameras




(11dof)

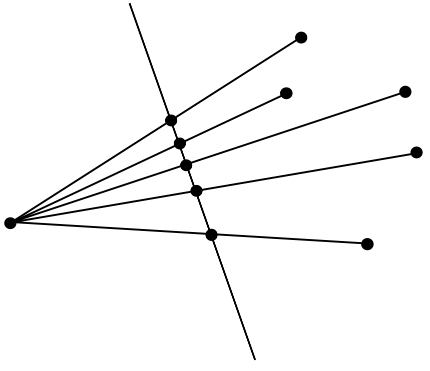
$$\mathrm{X} = (X, Y, X, T)^{\mathrm{T}} \qquad \mathrm{PX} = (x, y, w)^{\mathrm{T}} \qquad (x, y/w)^{\mathrm{T}}$$

$$\tilde{x} = x = \mathrm{P}^{1\mathsf{T}}\mathbf{X} \qquad \tilde{y} = y/z = \frac{\mathrm{P}^{2\mathsf{T}}\mathbf{X}}{\mathrm{P}^{3\mathsf{T}}\mathbf{X}}$$

Straight lines are not mapped to straight lines!
(otherwise it would be a projective camera)

# Line Cameras

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad \text{(5dof)}$$



Null-space PC=0 yields camera center

Also decomposition $\quad \mathrm{P}_{2\times3} = \mathrm{K}_{2\times2}\mathrm{R}_{2\times2}\left[\mathrm{I}_{2\times2} \mid -\tilde{\mathrm{c}}\right]$