



Projective Geometry

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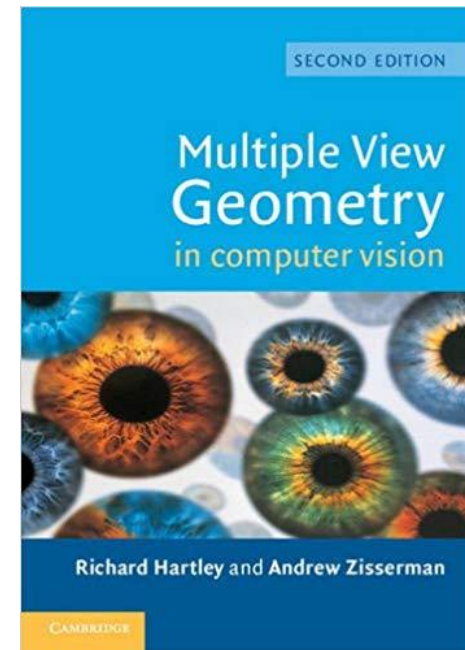
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Outline

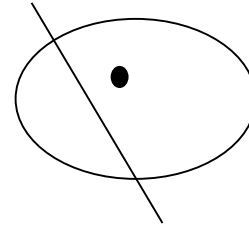
- Projective 2D geometry
- Projective 3D geometry



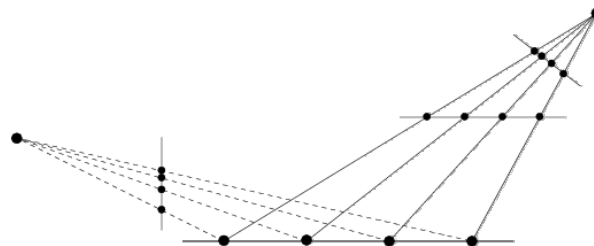
[Slides credit: Marc Pollefeys]

Projective 2D Geometry

- Points, lines & conics
- Transformations & invariants



- 1D projective geometry and the Cross-ratio



Homogeneous Coordinates

- Homogeneous representation of lines

$$ax + by + c = 0 \quad (a, b, c)^T$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \quad (a, b, c)^T \sim k(a, b, c)^T$$

equivalence class of vectors, any vector is representative

- Homogeneous representation of points

$$x = (x, y)^T \text{ on } l = (a, b, c)^T \text{ if and only if } ax + by + c = 0$$

$$(x, y, 1)(a, b, c)^T = (x, y, 1) \mathbf{L} = 0 \quad (x, y, 1)^T \sim k(x, y, 1)^T, \forall k \neq 0$$

The point x lies on the line l if and only if $x^T l = l^T x = 0$

Homogeneous coordinates $(x_1, x_2, x_3)^T$ but only 2DOF

Inhomogeneous coordinates $(x, y)^T$

The point $x = (x_1, x_2, x_3)^T$ represent the point $(x_1/x_3, x_2/x_3)^T$ in \mathbb{R}^2

Points and Lines

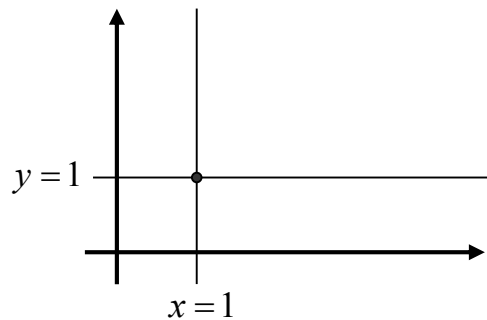
- Intersections of lines

The intersection of two lines l and l' is $x = l \times l'$

- Line joining two points

The line through two points x and x' is $l = x \times x'$

Example



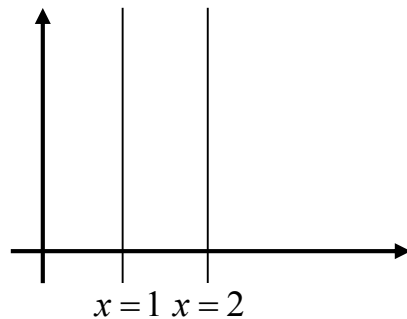
Ideal Points and the Line at Infinity

- Intersections of parallel lines

$$l = (a, b, c)^T \text{ and } l' = (a, b, c')^T \quad l \times l' = (b, -a, 0)^T$$



Example



$(b, -a)$ tangent vector
 (a, b) normal direction

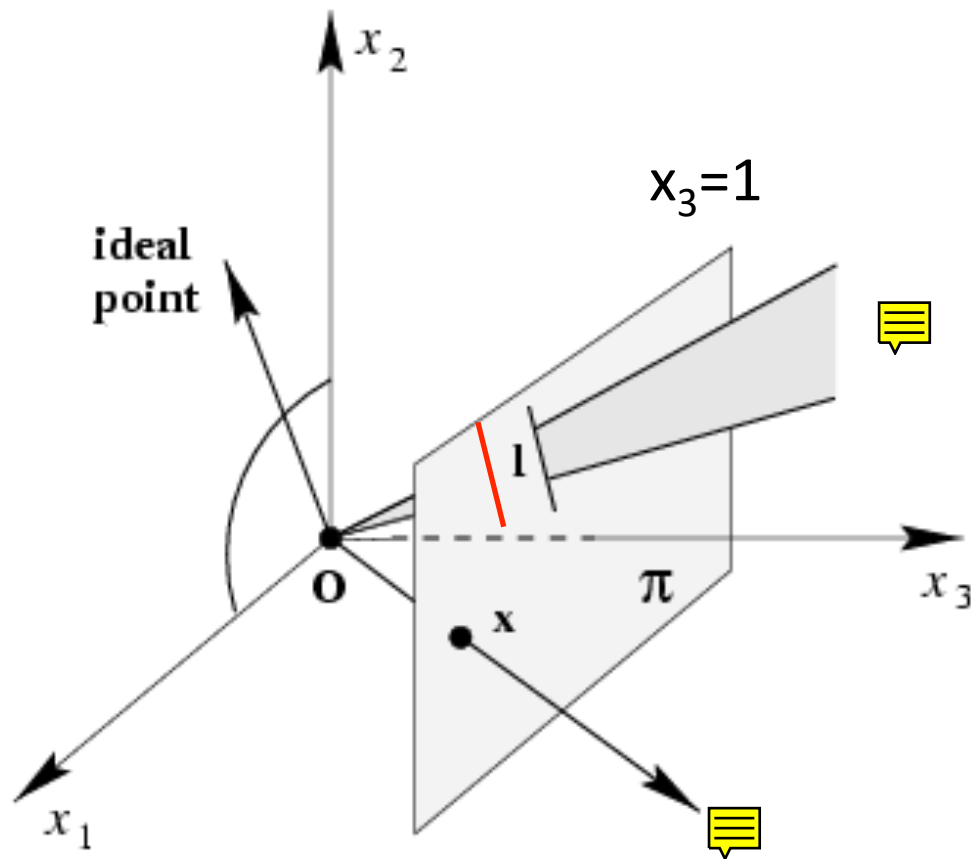
Ideal points $(x_1, x_2, 0)^T$

Line at infinity $l_\infty = (0, 0, 1)^T$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup l_\infty$$

Note that in \mathbf{P}^2 there is no distinction between ideal points and others

A Model for the Projective Plane



exactly one line through two points

exactly one point at intersection of two lines

Duality



$$\begin{array}{ccc} x & \longleftrightarrow & l \\ x^T l = 0 & \longleftrightarrow & l^T x = 0 \\ x = l \times l' & \longleftrightarrow & l = x \times x' \end{array}$$

- Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto \frac{x_1}{x_3}, y \mapsto \frac{x_2}{x_3}$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

symmetric

5DOF: $\{a:b:c:d:e:f\}$

Five Points Define a Conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

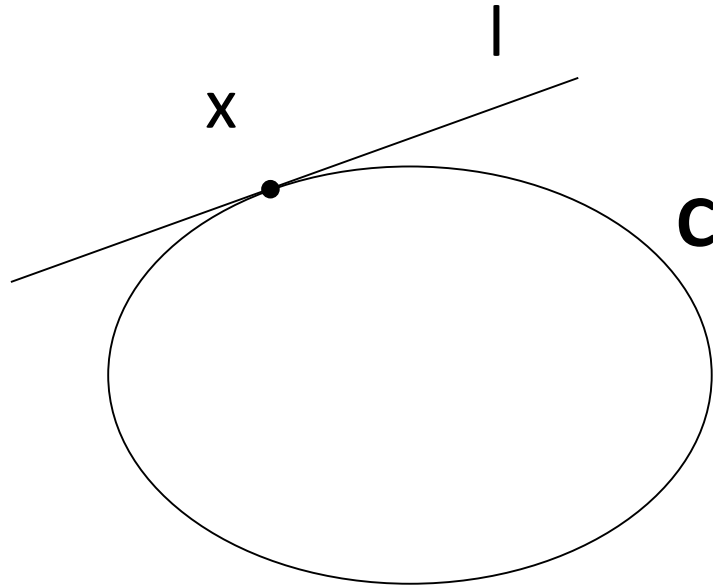
$$(x_i^2, x_iy_i, y_i^2, x_i, y_i, f)\mathbf{c} = 0 \quad \mathbf{c} = (a, b, c, d, e, f)^\top$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent Lines to Conics

The line l tangent to C at point x on C is given by $l=Cx$

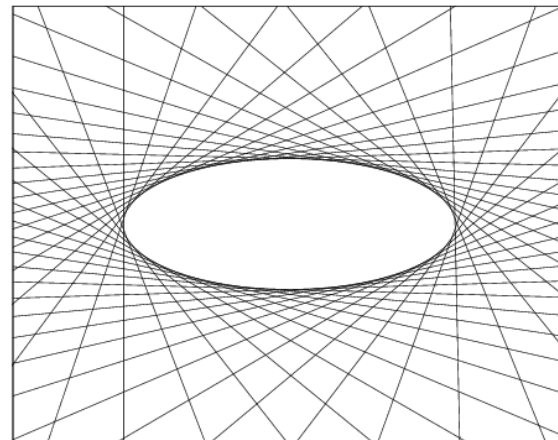
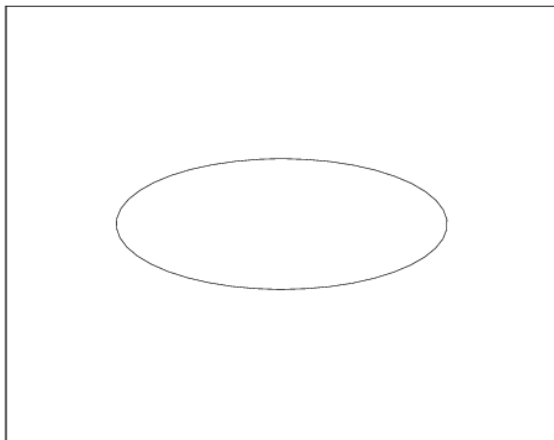


Dual Conics

A line tangent to the conic \mathbf{C} satisfies $\mathbf{1}^T \mathbf{C}^* \mathbf{1} = 0$

In general (\mathbf{C} full rank): $\mathbf{C}^* = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes



Projective Transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3×3 matrix \mathbf{H} such that for any point in P^2 represented by a vector \mathbf{x} it is true that $h(\mathbf{x}) = \mathbf{H}\mathbf{x}$

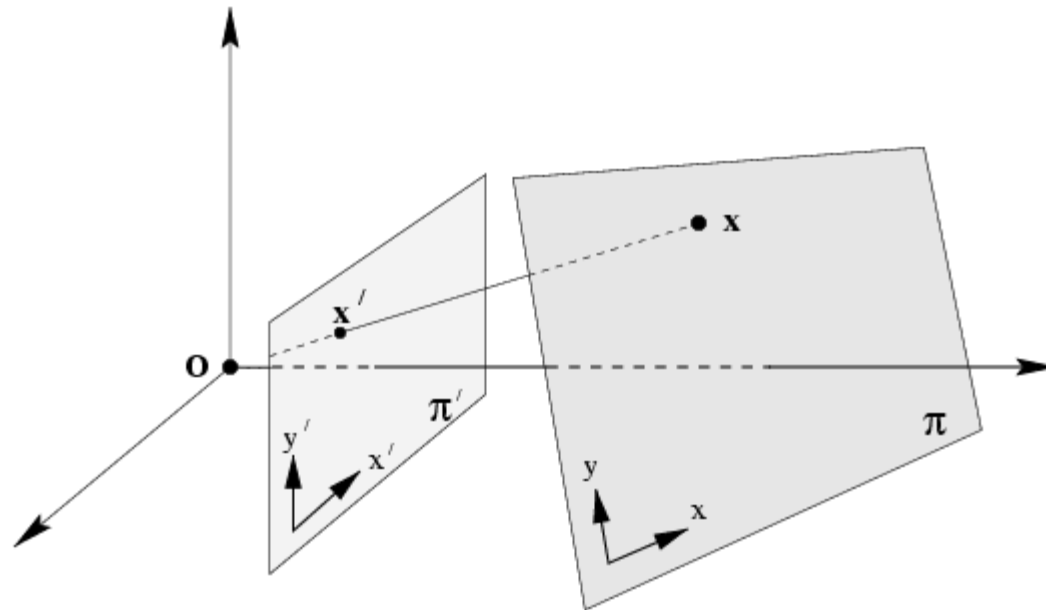
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H} \mathbf{x}$$

8DOF

projectivity=collineation=projective transformation=homography 14

Mapping between Planes



central projection may be expressed by $x' = Hx$
(application of theorem)

Removing Projective Distortion



select four points in a plane with know coordinates

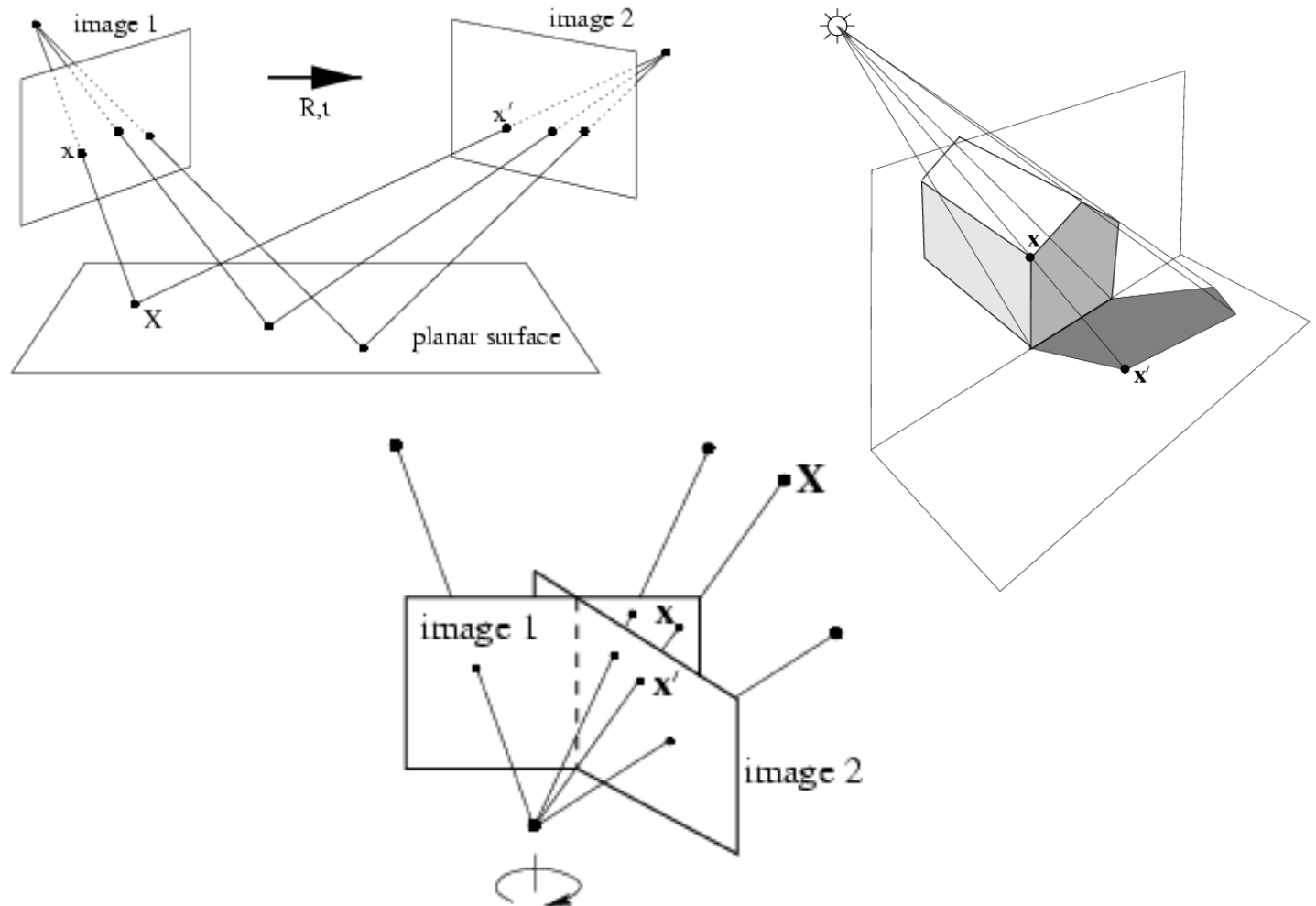
$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \quad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13} \quad (\text{linear in } h_{ij})$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

(2 constraints/point, 8DOF \Rightarrow 4 points needed)

Remark: no calibration at all necessary

More Examples



Transformation of Lines and Conics

For a point transformation

$$\mathbf{x}' = \mathbf{H} \mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$$



Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$$

Transformation for dual conics

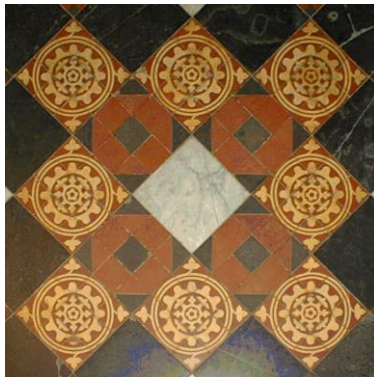
$$\mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$$

A Hierarchy of Transformations

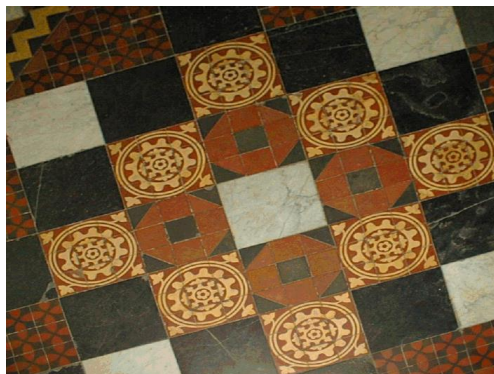
- Projective linear group
- Affine group (last row $(0,0,1)$)
- Euclidean group (upper left 2×2 orthogonal)
- Oriented Euclidean group (upper left 2×2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged



Similarity



Affine



Projective

Class I: Isometries

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon \cos \theta & -\sin \theta & t_x \\ \varepsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$ **(Euclidean transform)**

orientation reversing: $\varepsilon = -1$

$$\mathbf{x}' = \mathbf{H}_E \mathbf{x} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0^\top & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation), can be computed from 2 point correspondences

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}_s \mathbf{x} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ 0^\top & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)),

can be computed from 2 point correspondences

also know as *equi-form* (shape preserving)

metric structure = structure up to similarity (in literature)

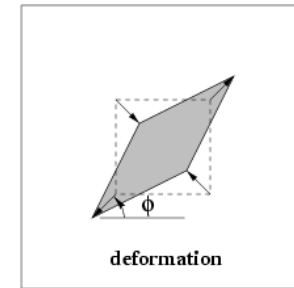
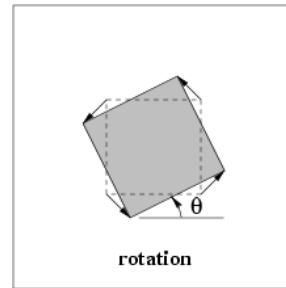
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine Transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}_A \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x}$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



6DOF (2 scale, 2 rotation, 2 translation),
can be computed from 3 point correspondences
non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Class VI: Projective Transformations

$$\mathbf{x}' = \mathbf{H}_P \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \mathbf{x} \quad \mathbf{v} = (v_1, v_2)^\top$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

can be computed from 4 point correspondences

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line, (ratio of ratio)

Action of Affinities and Projectivities on Line at Infinity

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0^\top & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon

Decomposition of Projective Transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{v}^\top & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$$

S: similarity

A: Affine

P: Projective

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^\top$$

decomposition unique (if chosen $s > 0$)

\mathbf{K} upper-triangular, $\det \mathbf{K} = 1$

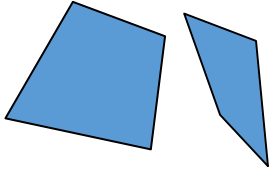
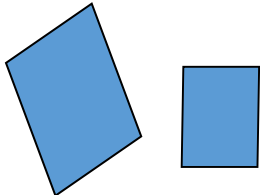
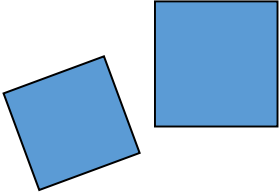
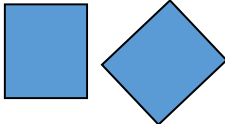
Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2 \cos 45^\circ & -2 \sin 45^\circ & 1.0 \\ 2 \sin 45^\circ & 2 \cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Summary of Transformations

Invariant Properties

Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		<p>Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio</p>
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). The line at infinity l_∞</p>
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>Ratios of lengths, angles. The circular points I,J</p>
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>lengths, areas.</p>

Number of Invariants?

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

e.g. configuration of 4 points in general position has 8 dof (2/pt)
and so 4 similarity, 2 affinity and zero projective invariants

Projective Geometry of 1D

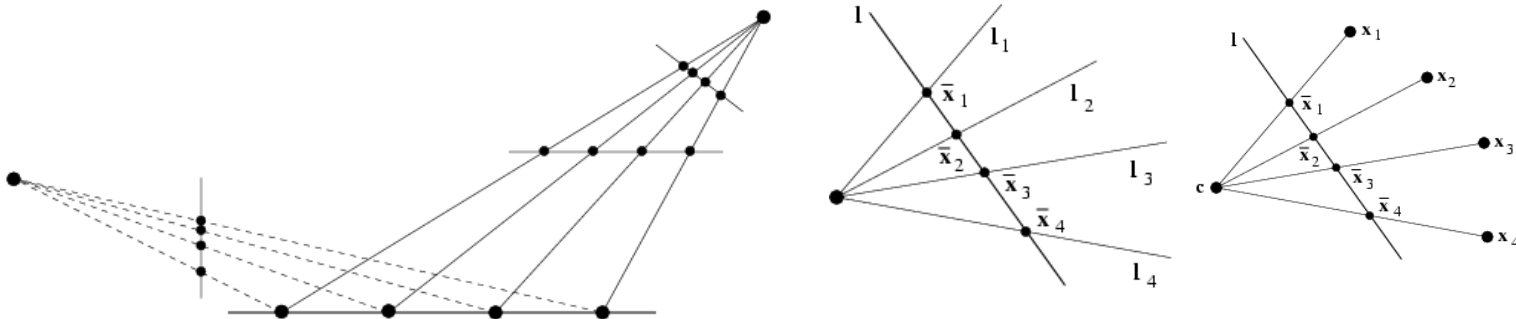
$$(x_1, x_2)^T \quad x_2 \neq 0$$

$$\bar{X}' = \mathbf{H}_{2 \times 2} \bar{X} \quad 3\text{DOF } (2 \times 2 - 1)$$

The cross ratio

$$\text{Cross}(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = \frac{|\bar{X}_1, \bar{X}_2| |\bar{X}_3, \bar{X}_4|}{|\bar{X}_1, \bar{X}_3| |\bar{X}_2, \bar{X}_4|} \quad |\bar{X}_i, \bar{X}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

Invariant under projective transformations



Recovering Metric and Affine Properties from Images

- Parallelism
- Parallel length ratios
- Angles
- Length ratios

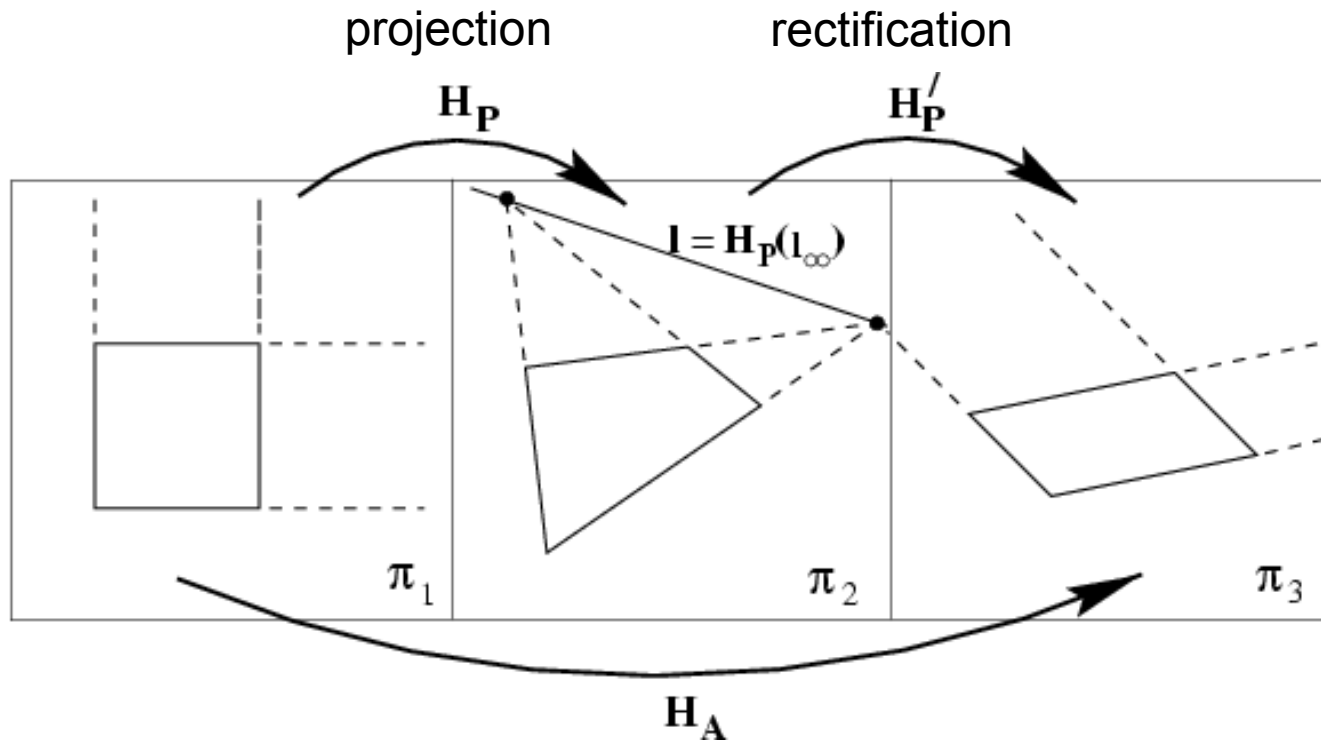
The Line at Infinity

$$l'_\infty = \mathbf{H}_A^{-T} l_\infty = \begin{bmatrix} \mathbf{A}^{-T} & 0 \\ -\mathbf{A} \mathbf{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

Affine Properties from Images



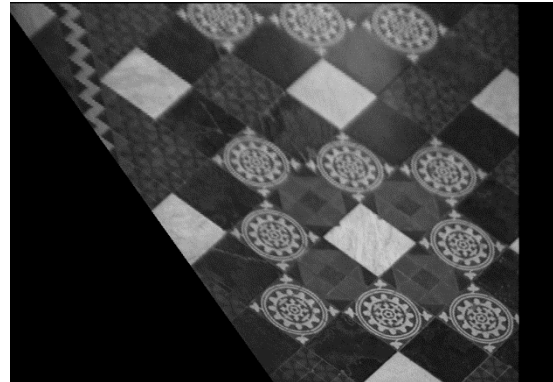
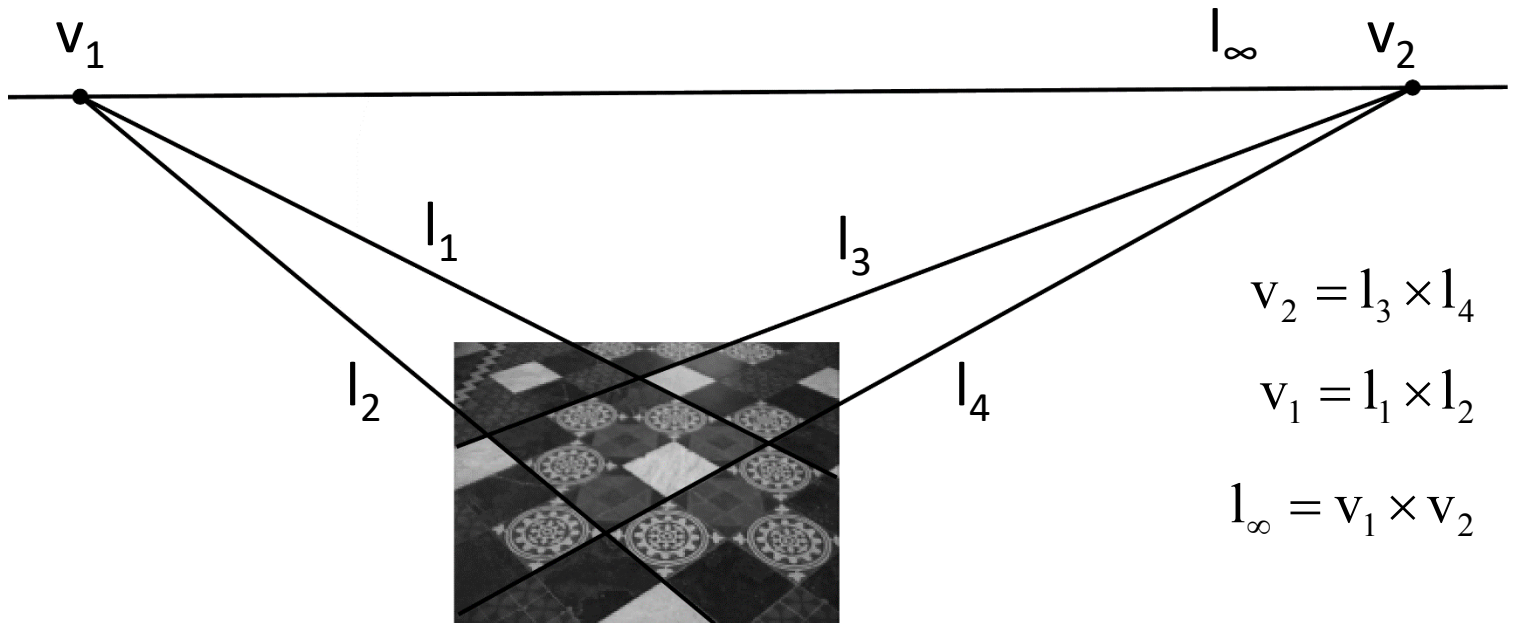
$$\mathbf{H}_{PA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \mathbf{H}_A$$

H_P

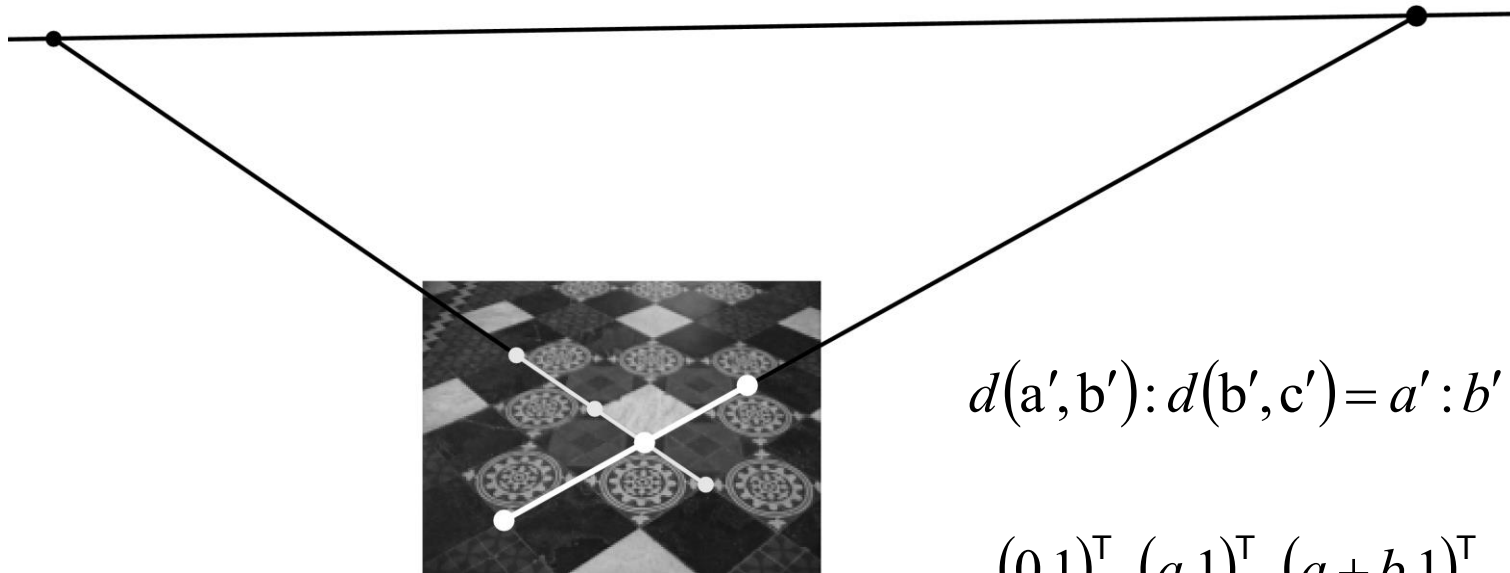
$$l_\infty = [l_1 \quad l_2 \quad l_3]^T, l_3 \neq 0$$

$$H_P^{-T}(l_1, l_2, l_3)^T = (0, 0, 1)^T = l_\infty$$

Affine Rectification



Distance Ratios



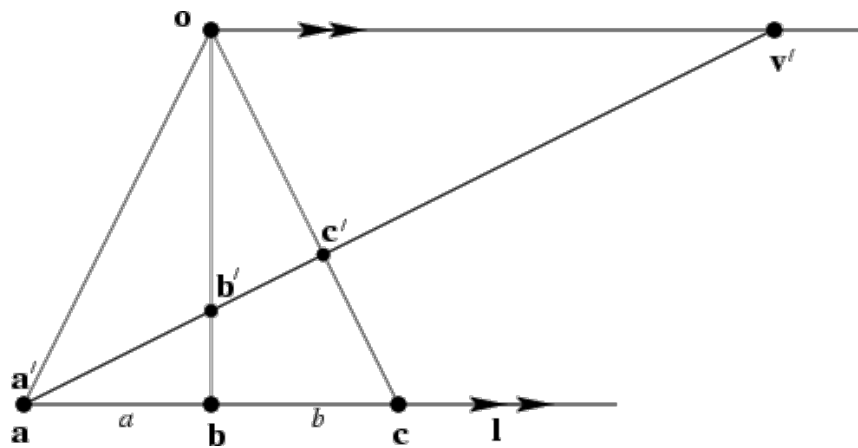
$$d(a', b') : d(b', c') = a' : b'$$

$$(0, 1)^T, (a, 1)^T, (a + b, 1)^T$$

$$\downarrow \mathbf{H}$$

$$a', b', c'$$

$$v' = \mathbf{H}(1, 0)^T$$



The Circular Points

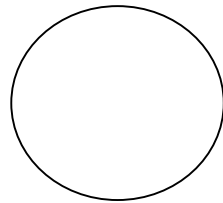
$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$I' = \mathbf{H}_S I = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = s e^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = I$$

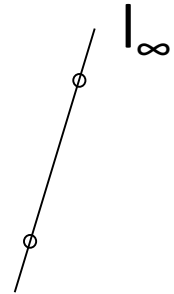
The circular points I, J are fixed points under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

The Circular Points

“circular points”



$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$



$$x_1^2 + x_2^2 = 0$$

$$I = (1, i, 0)^\top$$

$$J = (1, -i, 0)^\top$$

Algebraically, encodes orthogonal directions

$$I = (1, 0, 0)^\top + i(0, 1, 0)^\top$$

Conic dual to the Circular Points

$$\mathbf{C}_{\infty}^* = \mathbf{I}\mathbf{J}^{\top} + \mathbf{J}\mathbf{I}^{\top} \quad \mathbf{C}_{\infty}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_{\infty}^* = \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^{\top}$$

The dual conic \mathbf{C}_{∞}^* is fixed conic under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

Angles

Euclidean: $\mathbf{l} = (l_1, l_2, l_3)^\top$ $\mathbf{m} = (m_1, m_2, m_3)^\top$

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Projective: $\cos \theta = \frac{\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^\top \mathbf{C}_\infty^* \mathbf{m})}}$

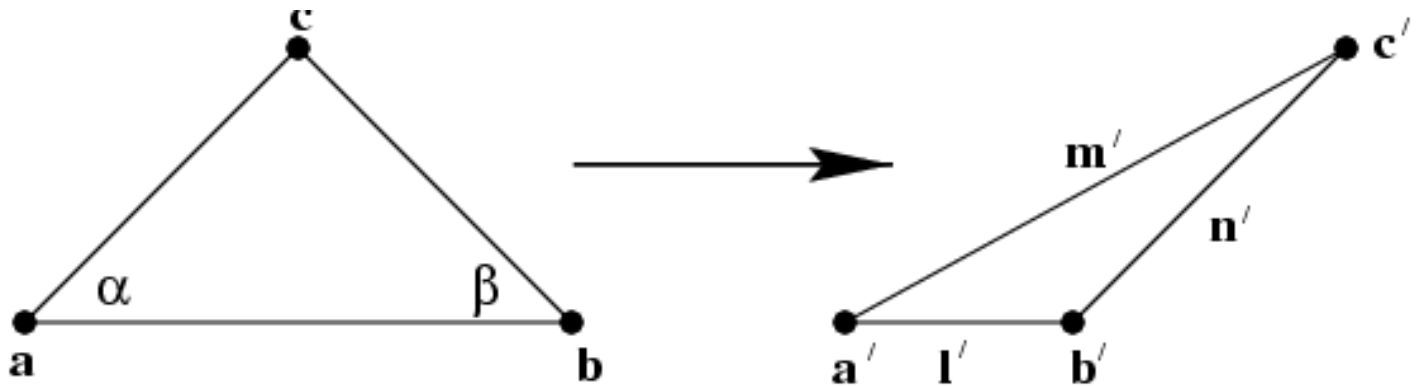
(This equation is Invariant to projective transform)

$$\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m} = 0 \quad \text{If orthogonal}$$

Length Ratios

$$\frac{d(b,c)}{d(a,c)} = \frac{\sin \alpha}{\sin \beta}$$

$\cos \alpha$ and $\cos \beta$ can be derived with the equations in the previous page



Metric Properties from Images

$$\begin{aligned}\mathbf{C}_{\infty}^{*'} &= (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S)^{\top} \\ &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^{\top} (\mathbf{H}_P \mathbf{H}_A)^{\top} \\ &= (\mathbf{H}_P \mathbf{H}_A) \mathbf{C}_{\infty}^* (\mathbf{H}_P \mathbf{H}_A)^{\top} \\ &= \begin{bmatrix} \mathbf{K} \mathbf{K}^{\top} & \mathbf{K}^{\top} \mathbf{v} \\ \mathbf{v}^{\top} \mathbf{K} & \mathbf{v}^{\top} \mathbf{v} \end{bmatrix}\end{aligned}$$

Rectifying transformation from SVD

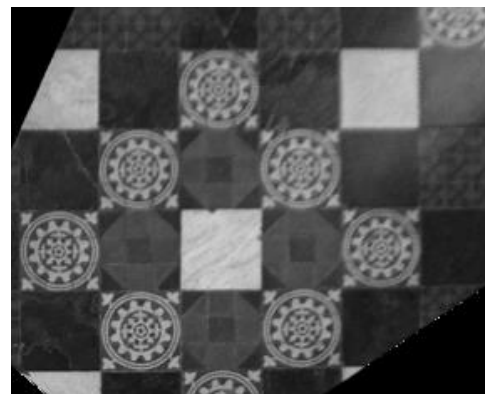
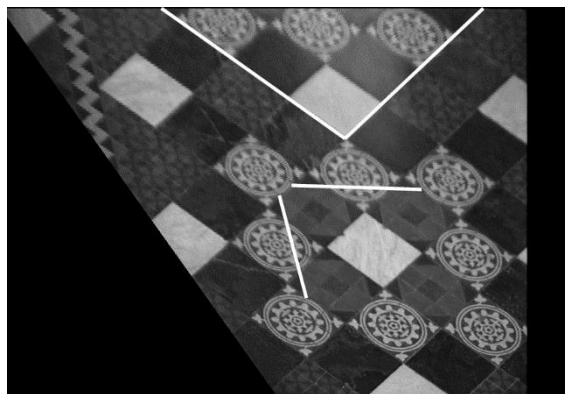
$$\mathbf{C}_{\infty}^{*'} = \mathbf{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\top} \quad \mathbf{H} = \mathbf{U}$$

Metric from Affine

Suppose an image has been affinely rectified ($\mathbf{v}=0$)

$$(l'_1 \quad l'_2 \quad l'_3) \begin{bmatrix} \mathbf{K}\mathbf{K}^\top & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2) (k_{11}^2 + k_{12}^2, k_{11} k_{12}, k_{22}^2)^\top = 0$$



Metric from Projective

$$\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m} = 0 \quad \begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{K}\mathbf{K}^\top & \mathbf{K}^\top \mathbf{v} \\ \mathbf{v}^\top \mathbf{K} & \mathbf{v}^\top \mathbf{v} \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

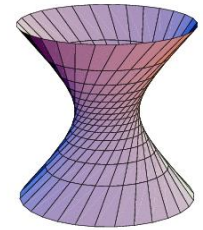
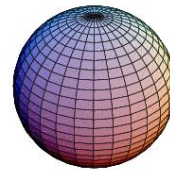
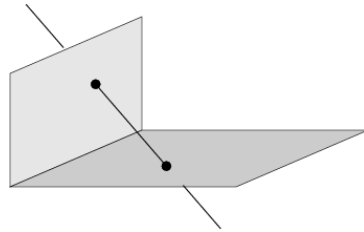
$$(l'_1 m'_1, 0.5(l'_1 m'_2 + l'_2 m'_1), l'_2 m'_2, 0.5(l'_1 m'_3 + l'_3 m'_1), 0.5(l'_2 m'_3 + l'_3 m'_2), l'_3 m'_3) \mathbf{c} = 0$$

$$\mathbf{c} = (a, b, c, d, e, f)^\top$$

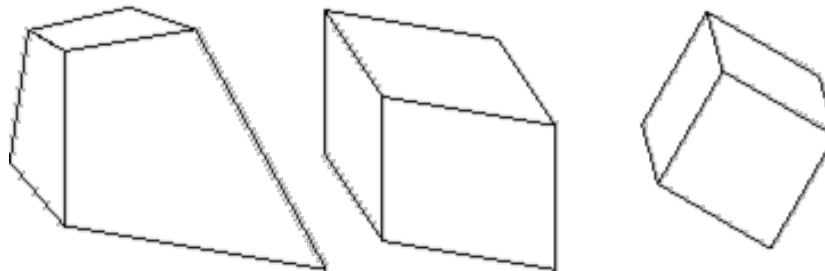


Projective 3D Geometry

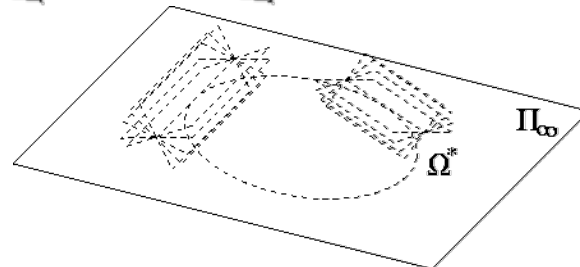
- Points, lines, planes and quadrics



- Transformations



- Π_∞ , ω_∞ and Ω_∞



3D Points

3D point

$$(X, Y, Z)^T \text{ in } \mathbf{R}^3$$

$$\mathbf{X} = (X_1, X_2, X_3, X_4)^T \text{ in } \mathbf{P}^3$$

$$\mathbf{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1 \right)^T = (X, Y, Z, 1)^T \quad (X_4 \neq 0)$$

projective transformation

$$\mathbf{X}' = \mathbf{H} \mathbf{X} \quad (4 \times 4 - 1 = 15 \text{ dof})$$

Dual: points \leftrightarrow planes, lines \leftrightarrow lines

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^\top X = 0$$

Transformation

$$X' = H X$$

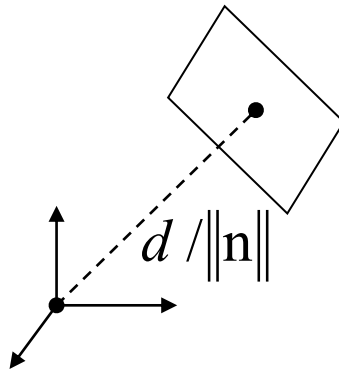
$$\pi' = H^{-\top} \pi$$

Euclidean representation

$$n \cdot \tilde{X} + d = 0 \quad n = (\pi_1, \pi_2, \pi_3)^\top \quad \tilde{X} = (X, Y, Z)^\top$$

$$\pi_4 = d$$

$$X_4 = 1$$



Planes from Points

Solve π from $X_1^\top \pi = 0$, $X_2^\top \pi = 0$ and $X_3^\top \pi = 0$

$$\begin{bmatrix} X_1^\top \\ X_2^\top \\ X_3^\top \end{bmatrix} \pi = 0 \quad \left(\text{solve } \boldsymbol{\pi} \text{ as right nullspace of } \begin{bmatrix} X_1^\top \\ X_2^\top \\ X_3^\top \end{bmatrix} \right)$$

Or implicitly from coplanarity condition

$$\det \begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^\top$$

Points from Planes

Solve X from $\pi_1^\top X = 0$, $\pi_2^\top X = 0$ and $\pi_3^\top X = 0$

$$\begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix} X = 0 \quad (\text{solve } X \text{ as right nullspace of } \begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix})$$

Points and Planes

- Projective transformation

Under the point transformation $\mathbf{X}' = H\mathbf{X}$, a plane transforms as $\boldsymbol{\pi}' = H^{-T}\boldsymbol{\pi}$

- Parametrized points on a plane

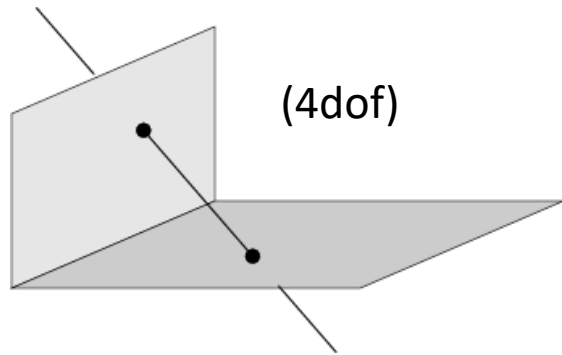
Representing a plane $\boldsymbol{\pi} = (a, b, c, d)^T$ by its span

$\mathbf{X} = \mathbf{M}\mathbf{x}$ \mathbf{x} is a 3-vector parameter (a point on the projective plane)

$$\boldsymbol{\pi}^T \mathbf{M} = 0$$

$$\text{M is not unique} \quad \mathbf{M} = \begin{bmatrix} \mathbf{p} \\ \mathbf{I} \end{bmatrix} \quad \mathbf{p} = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a} \right)^T$$

Lines



Defined as the join of two points A, B

$$W = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \quad \lambda A + \mu B$$

(Dual) Defined as the intersection of two planes P, Q

$$W^* = \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix} \quad \lambda P + \mu Q$$

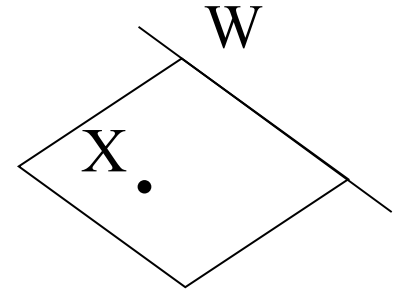
$$W^* W^\top = W W^{*\top} = 0_{2 \times 2}$$

Example: X -axis

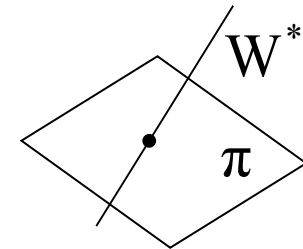
$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, Lines and Planes

$$\mathbf{M} = \begin{bmatrix} \mathbf{W} \\ \mathbf{X}^\top \end{bmatrix} \quad \mathbf{M} \pi = 0$$



$$\mathbf{M} = \begin{bmatrix} \mathbf{W}^* \\ \pi^\top \end{bmatrix} \quad \mathbf{M} X = 0$$



Plücker Matrices

Plücker matrix (4x4 skew-symmetric homogeneous matrix)

$$l_{ij} = A_i B_j - B_i A_j$$

$$L = AB^T - BA^T$$

1. L has rank 2 $LW^{*T} = 0_{4 \times 2}$
2. 4dof
3. generalization of $l = x \times y$
4. L independent of choice A and B
5. Transformation $L' = HLH^T$

Example: X-axis

$$L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Plücker matrices

Dual Plücker matrix L^*

$$L^* = PQ^T - QP^T$$

$$L^{*'} = H^{-T} L H^{-1}$$

Correspondence

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l_{34}^* : l_{42}^* : l_{23}^* : l_{14}^* : l_{13}^* : l_{12}^*$$

Join and incidence

$$\pi = L^* X \quad (\text{plane through point and line})$$

$$L^* X = 0 \quad (\text{point on line})$$

$$X = L\pi \quad (\text{intersection point of plane and line})$$

$$L\pi = 0 \quad (\text{line in plane})$$

$$[L_1, L_2, \dots]\pi = 0 \quad (\text{coplanar lines})$$

Quadrics and dual quadrics

$$X^T Q X = 0 \quad (Q : 4 \times 4 \text{ symmetric matrix})$$

1. 9 d.o.f.
2. in general 9 points define quadric
3. $\det Q = 0 \leftrightarrow$ degenerate quadric
4. Polar plane $\pi = QX$
5. (plane \cap quadric)=conic $C = M^T Q M \quad \pi : X = Mx$
6. transformation $Q' = H^{-T} Q H^{-1}$

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

Q^* : dual quadric, equations on planes

$$\pi^T Q^* \pi = 0$$

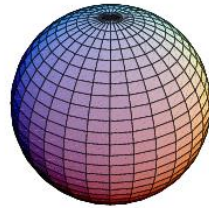
1. relation to quadric $Q^* = Q^{-1}$ (non-degenerate)
2. transformation $Q'^* = H Q^* H^T$

Quadric Classification

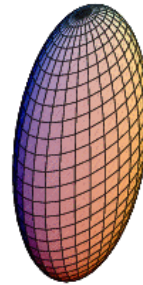
Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2 + Y^2 + Z^2 + 1 = 0$	No real points
	2	(1,1,1,-1)	$X^2 + Y^2 + Z^2 = 1$	Sphere
	0	(1,1,-1,-1)	$X^2 + Y^2 = Z^2 + 1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2 + Y^2 + Z^2 = 0$	Single point
	1	(1,1,-1,0)	$X^2 + Y^2 = Z^2$	Cone
2	2	(1,1,0,0)	$X^2 + Y^2 = 0$	Single line
	0	(1,-1,0,0)	$X^2 = Y^2$	Two planes
1	1	(1,0,0,0)	$X^2 = 0$	Single plane

Quadric Classification

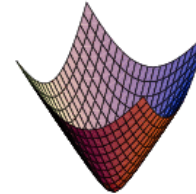
Projectively equivalent to *sphere*:



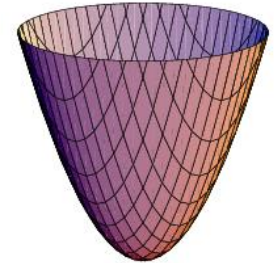
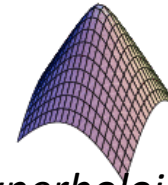
sphere



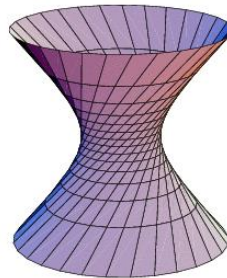
ellipsoid



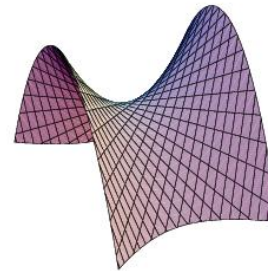
hyperboloid of two sheets



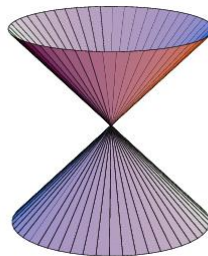
Ruled quadrics: (contain straight line)



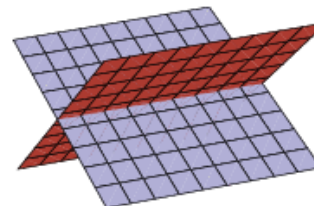
hyperboloids of one sheet



Degenerate ruled quadrics:



cone



two planes

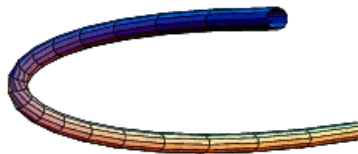
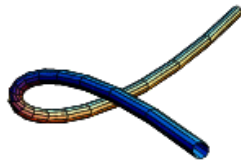
Twisted Cubic

conic

twisted cubic

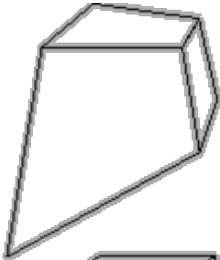
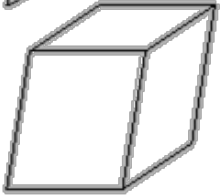
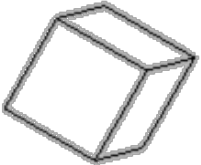
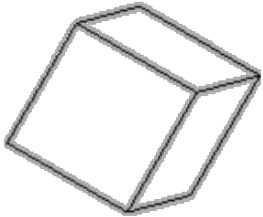
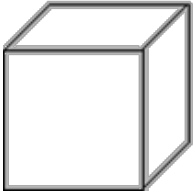
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 + a_{14}\theta^3 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 + a_{24}\theta^3 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 + a_{34}\theta^3 \\ a_{41} + a_{42}\theta + a_{43}\theta^2 + a_{44}\theta^3 \end{pmatrix}$$



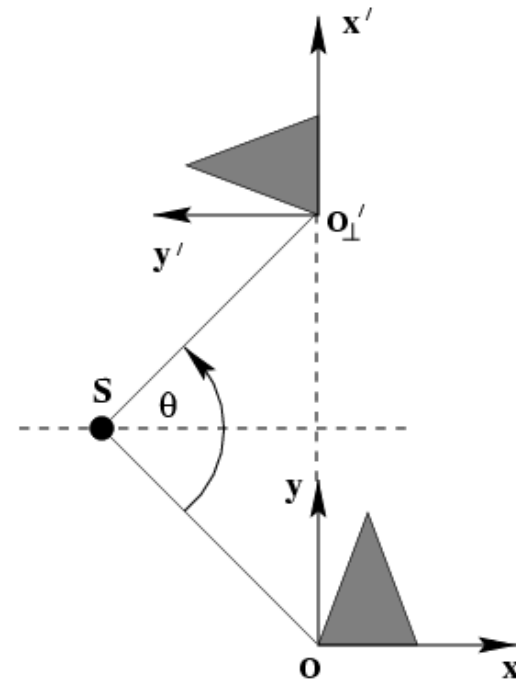
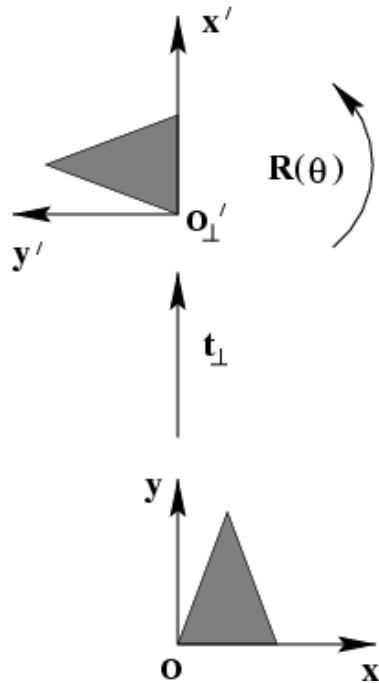
1. 3 intersection with plane (in general)
2. 12 dof (15 for A – 3 for reparametrisation (1 θ θ^2 θ^3))
3. 2 constraints per point on cubic, defined by 6 points
4. projectively equivalent to (1 θ θ^2 θ^3)
5. Horopter & degenerate case for reconstruction

Hierarchy of Transformations

				<u>Invariant Properties</u>
	Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Intersection and tangency
5 for affine scaling	Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, The plane at infinity π_∞
3 for rotation 3 for translation 1 for isotropic scaling	Similarity 7dof	$\begin{bmatrix} s R & t \\ 0^T & 1 \end{bmatrix}$		The absolute conic Ω_∞
	Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume 

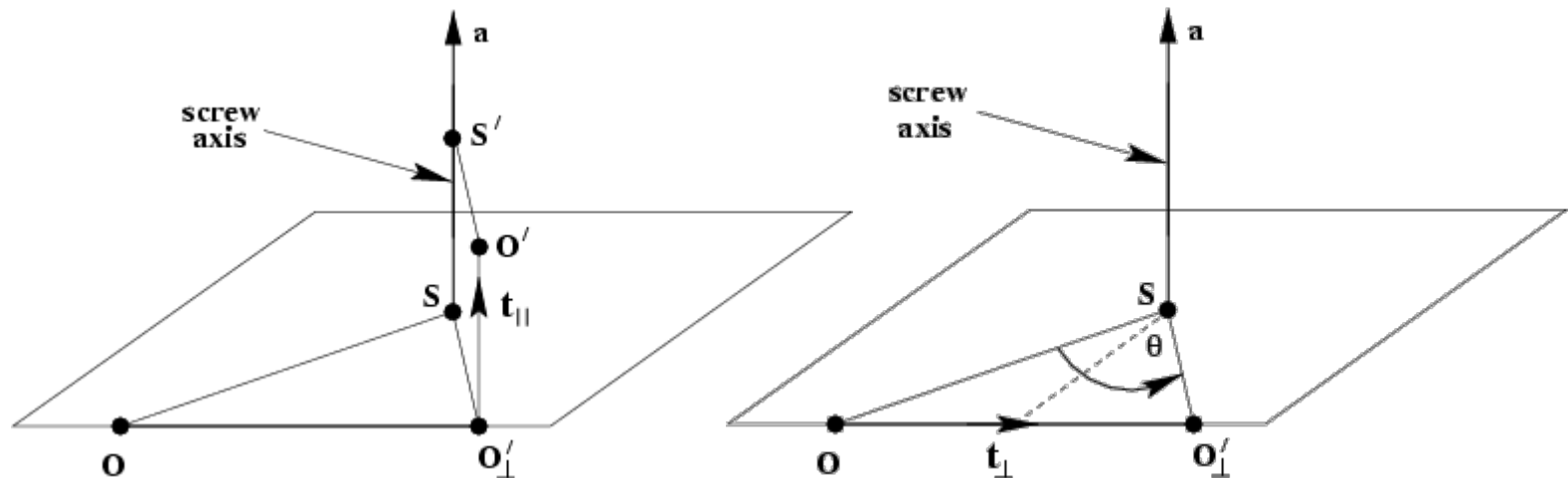
Screw Decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



Screw Decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



screw axis // rotation axis

$$\mathbf{t} = \mathbf{t}_{||} + \mathbf{t}_{\perp}$$

The Plane at Infinity

$$\pi'_\infty = \mathbf{H}_A^{-T} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-T} & 0 \\ -\mathbf{A} \mathbf{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canonical position $\pi_\infty = (0,0,0,1)^T$
2. contains directions $D = (X_1, X_2, X_3, 0)^T$
3. two planes are parallel \Leftrightarrow line of intersection in π_∞
4. line // line (or plane) \Leftrightarrow point of intersection in π_∞

The Absolute Conic

The absolute conic Ω_∞ is a (point) conic on π_∞ .

In a metric frame:

$$\left. \begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array} \right\} = 0$$

or conic for directions: $(X_1, X_2, X_3) \mathbf{I} (X_1, X_2, X_3)^\top$
(with no real points)

The absolute conic Ω_∞ is a fixed conic under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two points
3. Spheres intersect π_∞ in Ω_∞

The Absolute Conic

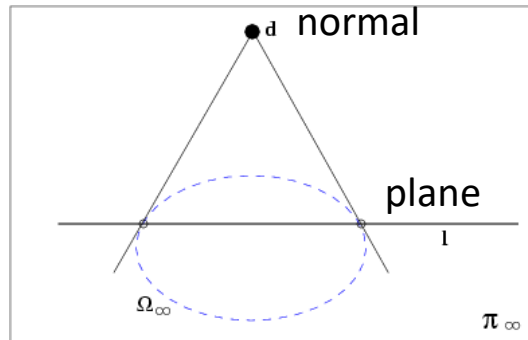
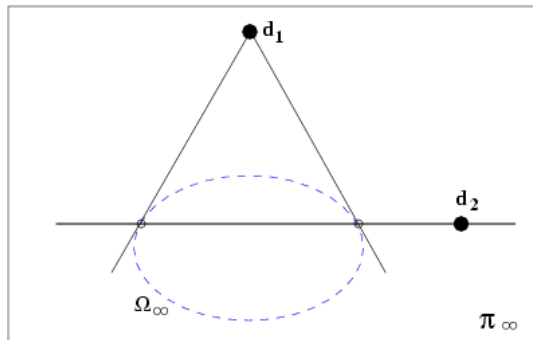
Euclidean:

$$\cos \theta = \frac{(d_1^T d_2)}{\sqrt{(d_1^T d_1)(d_2^T d_2)}}$$

Projective:

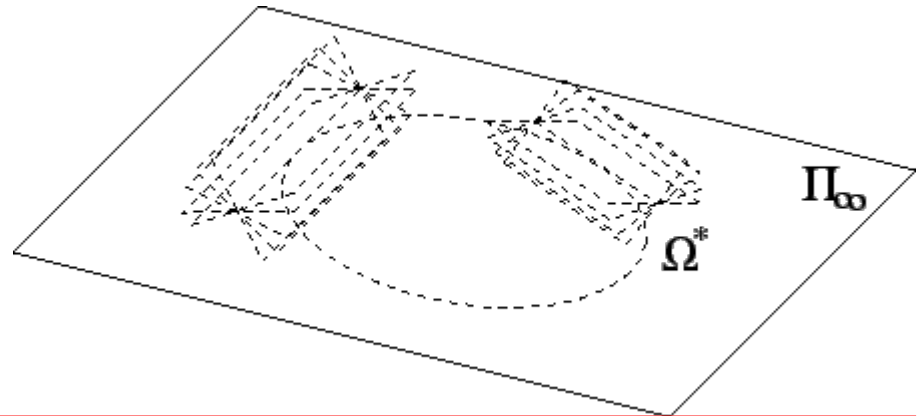
$$\cos \theta = \frac{(d_1^T \Omega_\infty d_2)}{\sqrt{(d_1^T \Omega_\infty d_1)(d_2^T \Omega_\infty d_2)}}$$

$$d_1^T \Omega_\infty d_2 = 0 \quad (\text{orthogonality=conjugacy})$$



The Absolute Dual Quadric

$$\Omega_{\infty}^* = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$



The absolute conic Ω_{∞}^* is a fixed conic under the projective transformation \mathbf{H} iff \mathbf{H} is a similarity

1. 8 dof
2. plane at infinity π_{∞} is the nullvector of Ω_{∞}
3. Angles:

$$\cos \theta = \frac{\pi_1^T \Omega_{\infty}^* \pi_2}{\sqrt{(\pi_1^T \Omega_{\infty}^* \pi_1)(\pi_2^T \Omega_{\infty}^* \pi_2)}}$$