Multi-domain Bivariate Spectral Local Linearisation method for solving non-similar boundary layer partial differential equations

MAGAGULA VUSI MPENDULO

Supervisors: Prof. S.S. Motsa & Prof. P. Sibanda

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

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Overview

Aim & Objectives

Aim

To present an extension of the **Bivariate Spectral Local Linearisation Method** (BSLLM)[1] for solving non-similar nonlinear PDEs over **large time intervals**.

- BSLLM uses Chebyshev-Gauss-Lobbatto points (see [3, 4])
- Drawback of BSLLM - accuracy deteriorates over **large time intervals**.
- New approach termed - **Multi-domain Bivariate Spectral Local Linearisation Method** (MD-BSLLM)

Objectives

- Solve non-linear non-similar boundary layer equations over a large time domain using the MD-BSLLM.
- Validate the results using a series solution approach.

Solution Method

The solution approach involves

- Domain decomposition
- linearisation and decoupling
- bivariate interpolation.
- pseudo-spectral approximation

Numerical Experiment

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{4}(n+3)f\frac{\partial^2 f}{\partial \eta^2} - \frac{1}{2}(n+1)\left(\frac{\partial f}{\partial \eta}\right)^2 + \zeta\frac{\partial^2 f}{\partial \eta^2} + (1-w)g + wh$$
$$= \frac{1}{4}(1-n)\zeta\left[\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \zeta \partial \eta} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \zeta}\right], \quad (1)$$

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial \eta^2} + \frac{1}{4}(n+3)f\frac{\partial g}{\partial \eta} + \zeta\frac{\partial g}{\partial \eta} = \frac{1}{4}(1-n)\zeta\left[\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \zeta} - \frac{\partial g}{\partial \eta}\frac{\partial f}{\partial \zeta}\right], \quad (2)$$

$$\frac{1}{Sc}\frac{\partial^2 h}{\partial \eta^2} + \frac{1}{4}(n+3)f\frac{\partial h}{\partial \eta} + \zeta\frac{\partial h}{\partial \eta} = \frac{1}{4}(1-n)\zeta\left[\frac{\partial f}{\partial \eta}\frac{\partial h}{\partial \zeta} - \frac{\partial h}{\partial \eta}\frac{\partial f}{\partial \zeta}\right], \quad (3)$$

subject to

$$f(\zeta,0)=0, \quad \frac{\partial f}{\partial \eta}(\zeta,0)=0, \quad g(\zeta,0)=h(\zeta,0)=1, \quad \frac{\partial f}{\partial \eta}(\zeta,\infty)=g(\zeta,\infty)=h(\zeta,\infty)=0.$$

- **Formulated** and **Solved** by **Hussain** et.al. [2] using finite-difference based **Keller-box technique**
- Results were validated using **Series Solutions**, for small and large ζ

Domain Decomposition

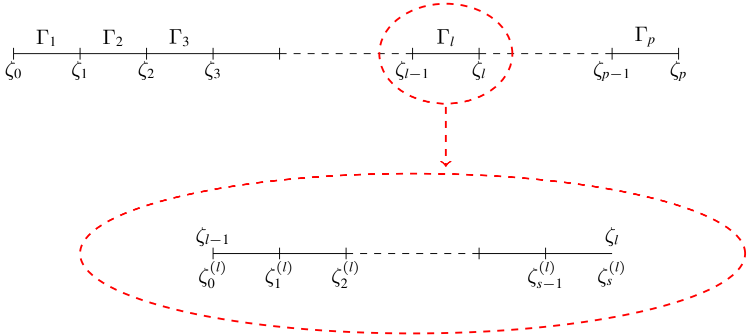

Figure: Multi-domain Grid

- The patching condition requires that

$$f^{(l)}(\eta, \zeta_{l-1}) = f^{(l-1)}(\eta, \zeta_{l-1}), \quad \eta \in [a, b], \tag{4}$$

where $f^{(l)}(\eta, \zeta)$ denotes the solution of equation (2) at each sub-interval Γ_l

Linearisation and Decoupling

- The PDEs are linearised using the **quasi-linearisation method**.
    - an iterative procedure based on the **Taylor-series expansion** about the previous estimates of the solution.
- Applying the **quasi-linearisation method** on one function at a time, results in three **decoupled linear PDEs**.
- In each sub-interval $[\zeta_{l-1}, \zeta_l]$, these decoupled linear PDEs can be solved iteratively in a sequential manner until the desired solution is obtained.

$$\beta_{0,r}\frac{\partial^3 f_{r+1}^{(l)}}{\partial\eta^3} + \beta_{1,r}\frac{\partial^2 f_{r+1}^{(l)}}{\partial\eta^2} + \beta_{2,r}\frac{\partial f_{r+1}^{(l)}}{\partial\eta} + \beta_{3,r} f_{r+1}^{(l)} + \beta_{4,r}\frac{\partial f_{r+1}^{(l)}}{\partial\xi} + \beta_{5,r}\frac{\partial f_{r+1}'^{(l)}}{\partial\xi} = R_{f,r}^{(l)}, \tag{5}$$

$$\sigma_{1,r}\frac{\partial^2 g_{r+1}^{(l)}}{\partial\eta^2} + \sigma_{2,r}\frac{\partial g_{r+1}^{(l)}}{\partial\eta} + \sigma_{3,r} g_{r+1}^{(l)} + \sigma_{4,r}\frac{\partial g_{r+1}^{(l)}}{\partial\xi} = R_{g,r}^{(l)}, \tag{6}$$

$$\omega_{1,r}\frac{\partial^2 h_{r+1}^{(l)}}{\partial\eta^2} + \omega_{2,r}\frac{\partial h_{r+1}^{(l)}}{\partial\eta} + \omega_{3,r} h_{r+1}^{(l)} + \omega_{4,r}\frac{\partial h_{r+1}^{(l)}}{\partial\xi} = R_{h,r}^{(l)}, \tag{7}$$

Pseudo-spectral Approximation

- Assume that the solution at each sub-interval Γ_l, denoted by $f^{(l)}(\eta,\zeta)$, can be approximated by a **bivariate Lagrange interpolation** polynomial of the form

$$f^{(l)}(\eta,\zeta) \approx \sum_{i=0}^{N_x}\sum_{j=0}^{N_t} f^{(l)}(\eta_i,\zeta_j)\mathcal{L}_i(\eta)\mathcal{L}_j(\zeta). \tag{8}$$

- Transpiration parameter derivative values are computed at the grid points (η_i,ζ_j):

$$\left.\frac{\partial f^{(l)}}{\partial \zeta}\right|_{(\eta_i,\zeta_j)} = \left(\frac{2}{\zeta_l-\zeta_{l-1}}\right)\sum_{v=0}^{N_t} d_{jv} f^{(l)}(\eta_j,\zeta_v) = \left(\frac{2}{\zeta_l-\zeta_{l-1}}\right)\sum_{v=0}^{N_t} d_{jv}\mathbf{F}_v^{(l)} \tag{9}$$

- The nth order space derivative is defined as

$$\left.\frac{\partial^n f^{(l)}}{\partial \eta^n}\right|_{(\eta_i,\zeta_j)} = \left(\frac{2}{\eta_\infty}\right)^n \sum_{\rho=0}^{N_x} D^n_{i\rho} f^{(l)}(\eta_\rho,\zeta_j) = \left(\frac{2}{\eta_\infty}\right)^n \mathbf{D}^n\mathbf{F}_j^{(l)}, \quad j=0,1,2,\ldots,N_t, \tag{10}$$

where the vector $\mathbf{F}_j^{(l)}$ is defined as

$$\mathbf{F}_j^{(l)} = [f^{(l)}(\eta_0,\zeta_j), f^{(l)}(\eta_1,\zeta_j),\ldots, f^{(l)}(\eta_{N_x},\zeta_j)]^T. \tag{11}$$

Pseudo-spectral Approximation

- Substituting equations (9) and (10) into equation (5), we get

$$\left[\beta_{0,r}\mathbf{D}^3+\beta_{1,r}\mathbf{D}^2+\beta_{2,r}\mathbf{D}+\beta_{3,r}\right]\mathbf{F}^{(l)}_{r+1,j}+\beta_{4,r}\sum_{v=0}^{N_t}d_{jv}\mathbf{F}^{(l)}_{r+1,v}+\beta_{5,r}\sum_{v=0}^{N_t}d_{jv}\mathbf{D}\mathbf{F}^{(l)}_{r+1,v}=\mathbf{R}^{(l)}_{f,r}, \quad (12)$$

  for $j=0,1,2,\ldots,N_t$.
- The patching condition requires that

$$f^{(l)}_{r+1}(\eta_i,\zeta_{(l-1,j)})=f^{(l-1)}_{r+1}(\eta_i,\zeta_{(l-1,j)}), \quad \eta\in[a,b], \quad (13)$$

- The initial unsteady solution of equation (5) when $\zeta=0$ corresponds to $t=t_{N_t}=-1$.
- Equation (12) is evaluated for $j=0,1,\cdots,N_t-1$

$$\left[\beta_{0,r}\mathbf{D}^3+\beta_{1,r}\mathbf{D}^2+\beta_{2,r}\mathbf{D}+\beta_{3,r}\right]\mathbf{F}^{(l)}_{r+1,j}+\beta_{4,r}\sum_{v=0}^{N_t-1}d_{jv}\mathbf{F}^{(l)}_{r+1,v}+\beta_{5,r}\sum_{v=0}^{N_t-1}d_{jv}\mathbf{D}\mathbf{F}^{(l)}_{r+1,v}=\boldsymbol{\mathcal{R}}^{(l)}_{1,j}, \quad (14)$$

  and

$$\boldsymbol{\mathcal{R}}^{(l)}_{1,j}=\mathbf{R}^{(l)}_{f,r}-\beta_{4,r}d_{jN_t}\mathbf{F}^{(l)}_{N_t}-\beta_{5,r}d_{jN_t}\mathbf{D}\mathbf{F}^{(l)}_{N_t}.$$

- Imposing boundary conditions for $j = 0, 1, \cdots, N_t - 1$, equation (14) can be expressed as the following

$$N_t(N_x + 1) \times N_t(N_x + 1)$$

matrix system

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,N_t-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,N_t-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N_t-1,0} & A_{N_t-1,1} & \cdots & A_{N_t-1,N_t-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_0^{(l)} \\ \mathbf{F}_1^{(l)} \\ \vdots \\ \mathbf{F}_{N_t-1}^{(l)} \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{1,0}^{(l)} \\ \mathcal{R}_{1,1}^{(l)} \\ \vdots \\ \mathcal{R}_{1,N_t-1}^{(l)} \end{bmatrix}, \tag{15}$$

where

$$A_{i,i} = \beta_{0,r}\mathbf{D}^3 + \beta_{1,r}\mathbf{D}^2 + \beta_{2,r}\mathbf{D} + \beta_{3,r}\mathbf{I} + \beta_{4,r}d_{ii}\mathbf{I} + \beta_{5,r}d_{ii}\mathbf{D} \tag{16}$$

$$A_{i,j} = \beta_{4,r}d_{ij}\mathbf{I} + \beta_{5,r}d_{ij}\mathbf{D}, \quad \text{when } i \neq j, \tag{17}$$

Results

- Comparison of Multi-domain bivariate spectral local linearisation solution for the **skin friction** against the series solution for large ζ.

	MD-BSLLM	Series Solution for large ζ
ζ	$f''(0,\zeta)$	$f''(0,\zeta)$
5	0.3088214	0.3088214
10	0.1547399	0.1547399
15	0.1031717	0.1031717
20	0.0773803	0.0773803
25	0.0619045	0.0619045
30	0.0515872	0.0515872
35	0.0442176	0.0442176
40	0.0386905	0.0386905

Table: $N_x = 60, N_t = 5, p = 20$

Results

- Comparison of Multi-domain bivariate spectral local linearisation solution for the **Sherwood number** against the series solution for large ζ

	MD-BSLLM	Series Solution for large ζ
ζ	$-h'(0,\zeta)$	$-h'(0,\zeta)$
5	3.0018658	3.0018658
10	6.0002332	6.0002332
15	9.0000691	9.0000691
20	12.0000292	12.0000292
25	15.0000149	15.0000149
30	18.0000086	18.0000086
35	21.0000054	21.0000054
40	24.0000036	24.0000036

Table: $N_x = 60$, $N_t = 5$, $p = 20$

Results

- Comparison of Multi-domain bivariate spectral local linearisation solution for the **Nusselt number** against the series solution for large ζ

	MD-BSLLM	Series Solution for large ζ
ζ	$-g'(0,\zeta)$	$-g'(0,\zeta)$
5	3.5018961	3.5018961
10	7.0002370	7.0002370
15	10.5000702	10.5000702
20	14.0000296	14.0000296
25	17.5000152	17.5000152
30	21.0000088	21.0000088
35	24.5000055	24.5000055
40	28.0000037	28.0000037

Table: $N_x = 60$, $N_t = 5$, $P = 20$

Conclusion

- The MD-BSLLM method can be used to solve non-linear non-similar boundary layer equations over a large time domain.
- We were able to validate the results using a series solution approach.

Future Research Direction

- Solve different **types** of NPDEs arising from Physics, Mathematical Biology, etc.

References

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Thank you ...