IMO 2025 Solution Notes

ByteDance Seed AI4Math

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This is a compilation of ByteDance Seed AI4Math team's solutions to the 2025 IMO competition. All of the solutions are generated by Artificial Intelligence (AI) methods. However, to help human graders, ByteDance Seed Prover team translates the machine proof into natural language (English) solutions by human, with the original machine proof attached.

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Problem statement

A **proper divisor** of a positive integer N is a positive divisor of N other than N itself. The infinite sequence $a_0, a_1, \ldots$ consists of positive integers, each of which has at least three proper divisors. For each $n \geqslant 0$, the integer a_{n+1} is the sum of the three **largest** proper divisors of a_n. Determine all possible values of a_0.

The set $\{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \land k \in \mathbb{N} \land n \text{ is coprime with } 10\}$ is exactly the set of all possible a_0.

First we prove that when $a_0 \in \{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \land k \in \mathbb{N} \land n \text{ is coprime with } 10\}$, infinite sequence $a_0, a_1, \ldots$ exists.

We'll show that: if a_m can be written as $6 \cdot 12^k \cdot n$, where n is coprime to 10, then a_{m+1} exists, and can also be written in this form.

Cases by k.

Lemma (1)

When $k = 0$, denote $12^k \cdot n$ as s, then $s, 2s, 3s$ should be exactly the three largest proper divisors of a_m.

Know that $(s, 2s, 3s \mid 6s = a_m) \land (s, 2s, 3s \leq 6s = a_m)$, these three numbers are truly three proper divisors of a_m.

Suppose e is a proper divisor of a_m, then $\exists t \in \mathbb{N}^*, e \cdot t = a_m = 6s$.

Cases by t.

$t = 1$ then $e = a_m$ contraction.
$t = 2$ then $e = 3s$.
$t = 3$ then $e = 2s$.
$t = 4$ then $4|a_m$ contraction.
$t = 5$ then $5|a_m$ contraction.
$t = 6$ then $e = s$.
$t > 6$ then $e < s, 2s, 3s$ so e cannot be one of the three largest proper divisors of a_m. $\square$

So $a_{m+1} = s + 2s + 3s = 6s = a_m$ by the definition of the infinite sequence, surely then a_{m+1} exists, and can also be written in this form.

Lemma (2)

When $k \geq 1$, denote $6 \cdot 12^{k-1} \cdot n$ as t, then $3t, 4t, 6t$ should be exactly the three largest proper divisors of a_m.

Know that $(3t, 4t, 6t \mid 12t = a_m) \wedge (3t, 4t, 6t \leq 12t = a_m)$, these three numbers are truly three proper divisors of a_m.

Suppose e is a proper divisor of a_m, then $\exists j \in \mathbb{N}^*, e \cdot j = a_m = 12t$.

Cases by j.

$j = 1$ then $e = a_m$ contraction.
$j = 2$ then $e = 6t$.
$j = 3$ then $e = 4t$.
$j = 4$ then $e = 3t$.
$j > 4$ then $e < 3t, 4t, 6t$ so e cannot be one of the three largest proper divisors of a_m. $\square$

So $a_{m+1} = 3t + 4t + 6t = 13t = 6 \cdot 12^{k-1} \cdot (13n)$, where $13n$ is coprime to 10, surely then a_{m+1} can also be written in this form. $\square$

Now we've shown that: if a_m can be written as $6 \cdot 12^k \cdot n$, where n is coprime to 10, then a_{m+1} exists, and can also be written in this form. By induction, easy to see that if a_0 can be written as $6 \cdot 12^k \cdot n$, where n is coprime to 10, infinite sequence $a_0, a_1, \ldots$ exists.

Then we prove that when infinite sequence $a_0, a_1, \ldots$ exists, $a_0 \in \{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \wedge k \in \mathbb{N} \wedge n \text{ is coprime with } 10\}$.

1. We'll show that: if infinite sequence $a_0, a_1, \ldots$ exists, $\forall n \in \mathbb{N}^*, 2|a_n$.

Lemma (3)

If $2 \nmid a_m$, then $a_{m+1} < a_m$.

Suppose $b < c < d$ are the three largest proper divisors of a_m.

Firstly, d should be odd, otherwise $2 \mid d \mid a_m$, contradiction.

Secondly, $\exists k \in \mathbb{N}^*, k \cdot d = a_m$, and then k should be odd, otherwise $2 \mid k \mid a_m$, contradiction. By the nature of integer, $k \geq 3$.

Finally, $a_{m+1} = b + c + d < d + d + d = 3 \cdot d \leq k \cdot d = a_m$. □

Lemma (4)
If $2 \nmid a_m$, then $2 \nmid a_{m+1}$.

Suppose $b < c < d$ are the three largest proper divisors of a_m.

b should be odd, otherwise $2 \mid b \mid a_m$, contradiction.
c should be odd, otherwise $2 \mid c \mid a_m$, contradiction.
d should be odd, otherwise $2 \mid d \mid a_m$, contradiction.

Finally, $a_{m+1} = b + c + d$ should be odd. □

Now use contradiction, we prove that $2|a_0$, otherwise if $2 \nmid a_0$, by induction we have $\forall n \in \mathbb{N}^*, 2 \nmid a_n$, then by induction and Lemma 4, we have $\forall n \in \mathbb{N}^*, a_{n+1} < a_n$, so infinite sequence $a_0, a_1, \ldots$ is a strictly decreasing infinite sequence of positive integers, which is contradiction.

Similarly, if $\exists n \in \mathbb{N}^*, 2 \nmid a_n$, using the same reasoning process, we know infinite sequence $a_n, a_{n+1}, \ldots$ is a strictly decreasing infinite sequence of positive integers, which is again contradiction.

So now we have proven that : if infinite sequence $a_0, a_1, \ldots$ exists, $\forall n \in \mathbb{N}^*, 2|a_n$.

2. We'll show that: if infinite sequence $a_0, a_1, \ldots$ exists, $\forall n \in \mathbb{N}^*, 3|a_n$.

Lemma (5)
There exists $j \in \mathbb{N}^*, 3|a_j$.

By contradiction. Suppose $\forall n \in \mathbb{N}^*, 3 \nmid a_n$, then we'll prove that $\forall n \in \mathbb{N}^*, a_{n+1} < a_n$.

Suppose $b < c < d$ be the three largest proper divisors of a_n, then $\exists i > j > k \in \mathbb{N}^*, b \cdot i = c \cdot j = d \cdot k = a_n$.

Cases by k, j, i.

$k = 2$, then$j > 2$. If $j = 3$ then $3 \mid a_m$, contradiction.
$k = 2$, then$j > 2$. If $j \geq 4$ then $i > 4$, $a_{n+1} = b + c + d < 1/i + 1/j + 1/k < a_n$.
$k > 2$, then$j, i > 3$, then $a_{n+1} = b + c + d < 1/i + 1/j + 1/k < a_n$.

Now that $\forall n \in \mathbb{N}^*, a_{n+1} < a_n$, by induction we know that infinite sequence $a_0, a_1, \ldots$ is a strictly decreasing infinite sequence of positive integers, which is again contradiction. □

Above we can see that the model prefer to use a special way to handle proper divisor.

After presenting the original taste of the model, from now on we'll translate the answer into more human-alike style.

Suppose not all a_n is divisible by 3, choose the least index m so that $3|a_m$, then $3 \nmid a_{m-1}$.

Denote $b < c < d$ as the three largest proper divisor of a_{m-1}.

Then $3 \nmid b, c, d \wedge 3|(b+c+d) = a_m$, This means $b \equiv c \equiv d \mod 3$, claim $i > j > k$ that $b \cdot i = c \cdot j = d \cdot k = a_m$. Since now we have $2|a_{m-1}$, k should only be 2.

If $4|a_{m-1}$, then j must be 4 but as we know $2 \not\equiv 4 \mod 3$, contradiction.

So $4 \nmid a_{m-1}$, then $d = a_{m-1}/2$ is odd. So $a_m = b + c + d$ is even means that there should be exactly one odd and one even number in b, c, then there should be exactly one odd and one even number in i, j. Suppose the even number in $\{i, j\}$ is in the form of $2l$, then l is a divisor of a_{j-1}. Since $l < 2l$, then $l \in \{c, d\}$. In the shape of $\{l, 2l\}$, We know $i = l, j = 2l$, so $c = 2b$, but $c \equiv b \mod 3$ as we've shown, $3 \mid b, c$ is guaranteed, which is contradicted to $3 \nmid a_{j-1}$. This contradiction show that "not all a_n is divisible by 3" is not possible. □

3. We'll show that: if infinite sequence $a_0, a_1, \ldots$ exists, $\forall n \in \mathbb{N}^*, 5 \nmid a_n$.

Lemma (6)

If $12|a_m$ for some m, then $v_2(a_{m+1}) = v_2(a_m) - 2 \wedge v_3(a_{m+1}) = v_2(a_m) - 1$.

The largest three proper divisors of a_m must be $a_m/2, a_m/3, a_m/4$.
Then $a_{m+1} = (13/12) \cdot a_m$, hence the proof. □

Lemma (7)

If $12|a_m$ for some m, then $v_2(a_{m+1}) - v_3(a_{m+1}) = v_2(a_m) - v_3(a_m) - 1$.

Just simply use Lemma 6. □

Lemma (8)

If $\forall i < k, 12|a_i$, then $v_2(a_k) = v_2(a_0) - 2k \wedge v_3(a_k) = v_3(a_0) - k$.

Recursively using Lemma 6, and we can easily get the result. □

Lemma (9)

There exists k, such that $12 \nmid a_k$.

Using contradiction, suppose $\forall n \in \mathbb{N}^*, 12|a_n$, then apply Lemma 6, we know that $v_2(a_0), v_2(a_1), \ldots$ should be strictly decreasing infinite sequence, obviously wrong. □

Lemma (A)

$\forall m, 12|a_m \to v_5(a_{m+1}) = v_5(a_m)$.

Just like the result under Lemma 6, $a_{m+1} = (13/12) \cdot a_m$, so $v_5(a_{n+1}) = v_5(a_n)$. □

Lemma (B)

If $30|a_m$ and the third largest proper divisor of a_m is 5, then $v_2(a_{m+1}) = v_2(a_m) - 1$.

Since $30|a_m$, the three largest proper divisors of a_m are $2, 3, 5$.

Then $a_{m+1} = (1/2 + 1/3 + 1/5) \cdot a_m = (31/30) \cdot a_m$. Easy to see that $v_2(a_{m+1}) = v_2(a_m) - 1$. □

Lemma (C)

There exists k, $5 \mid a_0 \to 12 \nmid a_k, 5 \mid a_k$

If not, for any k, $12 \mid a_k \vee 5 \nmid a_k$. Due to $5 \mid a_0$ and Lemma A, we have $5 \mid a_k$. Since $12 \mid a_k \vee 5 \nmid a_k$, we have $12 \mid a_k$. This contradicts Lemma 9. □

Lemma (D)

If $a_n = 30k, 2 \nmid k$, then $2 \nmid a_{n+1}$

Since 4 is not the factor of a_n, then the three largest factors of a_n are $\frac{a_n}{2}, \frac{a_n}{3}, \frac{a_n}{5}$. And $a_{n+1} = \frac{31}{30} a_n$, which is odd due to $v_2(a_n) = 1$. □

Lemma (E)

$v_5(a_n) = 0$

If not, we set $5|a_n$ for one n. Based on Lemma C, we have some $k > n$ (view a_n is the 0^{th} term in the sequence), s.t. $12 \nmid a_k, 5 \mid a_k$. Since we always have $6 \mid a_k$. We can use Lemma D to know $2 \nmid a_{k+1}$, this contradicts for any item in the sequence is even number. □

We have prove that for any n, $5 \nmid a_n$.

Lemma (10)

If $\exists k, 12 \nmid a_k \wedge \forall i < k, 12|a_i$, then $v_2(a_0)$ is odd $\wedge$ $v_2(a_0) < v_3(a_0) \wedge v_5(a_0) = 0$

Cases by k.

If $k = 0$, it's obvious.

If $k > 0$, then since $\forall j < k, 12|a_j$, we know that $a_j = (12/13) \cdot a_{j+1}$, which is just another way to say the fact under Lemma 6. Then by induction we get $a_0 = (12/13)^k \cdot a_k$.

At the same time since $2, 3, 6 \mid a_k \wedge 4, 5 \nmid a_k$, the three largest proper divisors of a_k must be $a_k/2, a_k/3, a_k/6$. The sum of the three is a_k, so after a_k the sequence doesn't change.

Since $12 \nmid a_k$, we know that $v_2(a_k) = 1 \wedge v_3(a_k) \geq 1 \wedge v_5(a_k) = 0$. Using Lemma 8, $v_2(a_0) = (1 + 2k) \wedge v_3(a_k) \geq (1 + k) \wedge v_5(a_0) = 0$. Hence the proof. □

Finally, we can prove the total statement.

If $12 \nmid a_0$, since $5 \nmid a_k, \forall k$, we can know that $v_2(a_0) = 1 \wedge v_3(a_0) \geq 1 \wedge v_5(a_0) = 0$, which is obviously an element of the set $\{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \wedge k \in \mathbb{N} \wedge n \text{ is coprime with } 10\}$.

Else $12|a_0$, then apply Lemma 9, we must have some $k, 12 \nmid a_k$, choose the least k_0 satisfying this condition. Then using Lemma 10 and k_0. We know $v_2(a_0)$ is odd $\wedge\ v_2(a_0) < v_3(a_0) \wedge v_5(a_0) = 0$. Easy to see that numbers satisfying this must be an element of the set $\{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \wedge k \in \mathbb{N} \wedge n \text{ is coprime with } 10\}$ as well.

So we've proven that if infinite sequence $a_0, a_1, \ldots$ exists, $a_0 \in \{6 \cdot 12^k \cdot n \mid n \in \mathbb{N}^* \wedge k \in \mathbb{N} \wedge n \text{ is coprime with } 10\}$.

Both sides have been proved. Problem solved. □