IMO 2025 P1 Solution

ByteDance Seed AI4Math

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This is a compilation of ByteDance Seed AI4Math team's solutions to the 2025 IMO competition. All of the solutions are generated by Artificial Intelligence (AI) methods. However, to help human graders, ByteDance Seed Prover team translates the machine proof into natural language (English) solutions by human, with the original machine proof attached. Some very minor proof details may be omitted and can be checked in the LEAN version. The proof of IMO 2025 P1 is running under LEAN v4.14.0.

§1 IMO 2025/1

Problem statement

A line in the plane is called *sunny* if it is **not** parallel to any of the x-axis, the y-axis, and the line $x + y = 0$. Let $n \geqslant 3$ be a given integer. Determine all nonnegative integers k such that there exist n distinct lines in the plane satisfying both of the following:

- for all positive integers a and b with $a + b \leqslant n + 1$, the point (a, b) is on at least one of the lines; and
- exactly k of the n lines are sunny.

Claim 1.1 — When $k = 0, 1, 3$, there exists a possible way to put k sunny lines in the plane.

Proof. See $\boxed{\text{theorem imo2025\_p1\_right}}$. We prove it by classification.

- When $k = 0$, we can place $i^{th}, 1 \leq i \leq n$ line to be $x = i$. Every point is covered by these lines.
- When $k = 1$, we can place $i^{th}, 1 \leq i < n$ line to be $x = i$, and n^{th} line to be $y = x + (1 - n)$. Only the n^{th} line is sunny. Every point is covered by these lines. Especially, $(n, 1)$ is on the n^{th} line.
- When $k = 3$, we can place $i^{th}, 1 \leq i \leq n - 3$ line to be $x = i$, and the $(n - 2)^{th}$ line to be $y = -\frac{1}{2}x + 1 + \frac{n}{2}$, the $(n - 1)^{th}$ line to be $y = x + 3 - n$, the n^{th} line to be $y = -2x + 2n - 1$. It is obvious there are 3 sunny lines. We only need to show every point is covered by our setting. Since for every point which $1 \leq x \leq n - 3$ is covered.

We need to show points $(n-2,1),(n-2,2),(n-2,3),(n-1,1),(n-1,2),(n,1)$ are covered. By calculation, $(n-2,1)$ is on $y=x+3-n$. $(n-2,2)$ is on $y=-\frac{1}{2}x+1+\frac{n}{2}$. $(n-2,3)$ is on $y=-2x+2n-1$. $(n-1,1)$ is on $y=-2x+2n-1$. $(n-1,2)$ is on $y=x+3-n$. $(n,1)$ is on $y=-\frac{1}{2}x+1+\frac{n}{2}$.

□

Definition 1.1

Let $P(n)$ denote the statement below.
Given

- a finite set $\mathcal{L}$ of lines in $\mathbb{R}^2$, each represented as a pair (m,c) (slope and y-intercept),
- a finite set $\mathcal{V}$ of vertical lines in $\mathbb{R}^2$, each given by a real number x,
- a finite set $\mathcal{P}$ of points in $\mathbb{N}\times\mathbb{N}$,

satisfying the following conditions:

1. $|\mathcal{L}|+|\mathcal{V}|=n$.
2. A point (a,b) lies in $\mathcal{P}$ if and only if $a\geq 1$, $b\geq 1$, and $a+b\leq n+1$.
3. Every point $(a,b)\in\mathcal{P}$ is either:
   - on some line $(m,c)\in\mathcal{L}$ (i.e., $b=m\cdot a+c$), or
   - on some vertical line $x\in\mathcal{V}$ (i.e., $a=x$).
4. The number of lines with slopes neither 0 nor -1 in $\mathcal{L}$ (i.e., the sunny line) is denoted by k.

Then $k=0$ or $k=1$ or $k=3$.

Claim 1.2 — There are at most 3 sunny lines when $3\leq n\leq 4$.

Proof. See theorem k_le_3_for_n_le_4.

The case $n=3$ is obvious since $k\leq n$. We only need to consider the case $n=4$. In fact, it is the same to Claim 1.8, but SeedProver proves it specifically by classification, so we describe the main process while overlook details.

We note that if $k=4$, then points $(1,1),(1,2),(1,3)$ and $(1,4)$ should be covered by 4 distinct sunny lines. Besides, there exists at least 1 sunny line that covers at least 3 points. This is impossible by calculating the slope of sunny lines through enumeration. □

Claim 1.3 — $P(n)$ holds for $n=3$.

Proof. See theorem imo2025_p1_prop_n_eq_3_k_eq_0_1_3. From Claim 1.1 and Claim 1.2, we only need to show that when $n=3$, $k\neq 2$. We prove it by classification and omit details.

1. If there exists one vertical line $x=c$, then $c=1,2,3$ (the other cases make no progress). Without loss of generality, we assume $c=1$, then there are 3 points

left. However, we cannot cover them by 2 sunny lines by classification. Thus it is impossible.

2. If there exists one horizontal line $y = c$, then $c = 1, 2, 3$ (the other cases make no progress). Without loss of generality, we assume $c = 1$, then there are 3 points left. However, we cannot cover them by 2 sunny lines by classification. Thus it is impossible.

3. If there exists one anti-diagonal line $x + y = c$, then $c = 1, 2, 3$ (the other cases make no progress). Without loss of generality, we assume $c = 3$, then there are 3 points left. However, we cannot cover them by 2 sunny lines by classification. Thus it is impossible.

Thus $P(n)$ holds for $n = 3$. □

Definition 1.2

For positive natural number n, let $\mathcal{D}_n$ denote the points $(x, y) \in \mathbb{N} \times \mathbb{N}$ satisfying

- $x \geq 1$ and $y \geq 1$ and $x + y \leq n + 1$
- $x = 1$ or $y = 1$ or $x + y = n + 1$

Claim 1.4 — $\operatorname{card}(\mathcal{D}_n) = 3n - 2$ if $n = 1$ and $3n - 3$ otherwise.

Proof. See $\boxed{\text{theorem num\_points\_on\_boundary}}$. The cases $n = 1$ and $n = 2$ are obvious, so we start from $n \geq 3$. We define three sets

$$A = \{(1, y) \mid y \in \mathbb{N}, y \in [1, n]\},$$

$$B = \{(x, 1) \mid x \in \mathbb{N}, x \in [2, n]\},$$

$$C = \{(x, n + 1 - x) \mid x \in \mathbb{N}, x \in [2, n - 1]\}.$$

Then we have

$$A \cup B \cup C = \mathcal{D}_n,$$

$$A \cap B = A \cap C = B \cap C = \varnothing.$$

Thus

$$\begin{aligned}
\operatorname{card}(\mathcal{D}_n) &= \operatorname{card}(A) + \operatorname{card}(B) + \operatorname{card}(C) \\
&= n + (n - 1) + (n - 2) \\
&= 3n - 3.
\end{aligned}$$

□

Claim 1.5 — For natural number $n \geq 3$ and any pair (m, c) in $\mathbb{R} \times \mathbb{R}$ such that $(m, c) \neq (0, 1)$ and $(m, c) \neq (-1, n + 1)$, then $\operatorname{card}(\{(x, mx + c) : x \in \mathbb{R}\} \cap \mathcal{D}_n) \leq 2$.

Proof. See $\boxed{\text{theorem line\_not\_boundary\_covers\_at\_most\_2\_boundary\_points}}$.

If $m = 0$, then for $S := \{(x, mx + c) : x \in \mathbb{R}\} \cap \mathcal{D}_n$, we have $S \subseteq A \cup B$ where $A := \{(x, y) \in S \mid x = 1\}$ and $B := \{(x, y) \in S \mid x + y = n + 1\}$, since the lower boundary is not covered by the line as $c \neq 1$. But if $p, q \in A$ then by expanding the defining equations, $p = q$, so $\operatorname{card}(A) \leq 1$. Similarly $\operatorname{card}(B) \leq 1$. So $\operatorname{card}(S) \leq 2$.

If $m = -1$, then for $S := \{(x, mx + c) : x \in \mathbb{R}\} \cap \mathcal{D}_n$, we have $S \subseteq A \cup B$ where $A := \{(x, y) \in S \mid x = 1\}$ and $B := \{(x, y) \in S \mid y = 1\}$, since the boundary $x+y = n+1$ is not covered by the line as $c \neq n+1$. Similarly one can derive that $\text{card}(A) \leq 1$ and $\text{card}(B) \leq 1$ so $\text{card}(S) \leq 2$.

Now suppose $m \neq 0$ and $m \neq -1$, i.e. the line defined by $y = mx + c$ is not sunny. We assume there are three points $p, q, r \in \{(x, mx + c) : x \in \mathbb{R}\} \cap \mathcal{D}_n$ and show at least two of them are equal. The core idea here is to conduct a classified discussion on each point.

There are two types of scenarios (see the proof of the lemma: non_sunny_lines_cover_at_most_2_points_on_boundary_h_main):

1. If two points, say p and q, lie on the same boundary (both in $x = 1$ or both in $y = 1$ or both in $x + y = n + 1$), then:
   a) If $x_p = x_q = 1$, then since they both lie on $y = mx + c$ we can derive $y_p = y_q$, so $p = q$.
   b) If $y_p = y_q = 1$, then since they both lie on $y = mx + c$ and $m \neq 0$, $x_p = x_q$, so $p = q$.
   c) If $x_p + y_p = x_q + y_q = n + 1$, then since they both lie on $y = mx + c$ and $m \neq -1$, we can derive $x_p = x_q$ and $y_p = y_q$, so $p = q$.
2. Otherwise, all three points are on different boundaries, say $x_p = 1$, $y_q = 1$, and $x_r + y_r = n + 1$. Suppose p, q, r are pairwise distinct. Similarly to above, we can derive that $y_p \neq 1$ and $x_q \neq 1$ and $x_r \neq 1$. Also $y_p \neq y_r$; otherwise $y_p = y_r = 1$ so $p = r$. Similarly $y_q \neq y_r$. Then:
   a) If $m < -1$, then by linear arithmetic, as q, r are on $y = mx + c$, $y_p > n$, a contradiction to $p \in \mathcal{D}_n$.
   b) If $m > -1$, then by linear arithmetic, as p, r are on $y = mx + c$, $x_q > n$, a contradiction to $q \in \mathcal{D}_n$.

□

Claim 1.6 — For natural number $n \geq 3$ and any real number c such that $c \neq 1$, $\text{card}(\{(c, y) : y \in \mathbb{R}\} \cap \mathcal{D}_n) \leq 2$.

Proof. See lemma vert_line_covers_at_most_two_boundary_points.

Clearly, the vertical boundary is disjoint from $\{(c, y) : y \in \mathbb{R}\}$, so points p in $\{(c, y) : y \in \mathbb{R}\} \cap \mathcal{D}_n$ have either $y_p = 1$ or $x_p + y_p = n + 1$. In the two cases we derive $p = (c, 1)$ or $p = (c, n + 1 - c)$ respectively. Hence $\{(c, y) : y \in \mathbb{R}\} \cap \mathcal{D}_n$ is a subset of $\{(c, 1), (c, n + 1 - c)\}$, which has cardinality 2. □

Claim 1.7 — Given

- a finite set $\mathcal{L}$ of lines in $\mathbb{R}^2$, each represented as a pair (m, c) (slope and y-intercept),
- a finite set $\mathcal{V}$ of vertical lines in $\mathbb{R}^2$, each given by a real number x,
- a finite set $\mathcal{P}$ of points in $\mathbb{N} \times \mathbb{N}$,

satisfying the following conditions:

1. $|\mathcal{L}| + |\mathcal{V}| = n$.
2. A point (a, b) lies in $\mathcal{P}$ if and only if $a \geq 1$, $b \geq 1$, and $a + b \leq n + 1$.
3. Every point $(a, b) \in \mathcal{P}$ is either:
   - on some line $(m, c) \in \mathcal{L}$ (i.e., $b = m \cdot a + c$), or
   - on some vertical line $x \in \mathcal{V}$ (i.e., $a = x$).

if $1 \notin \mathcal{V}$ and $(-1, n+1) \notin \mathcal{L}$ and $(0, 1) \notin \mathcal{L}$, then $\mathrm{card}(\mathcal{D}_n) \leq 2n$.

Proof. See theorem total_boundary_points_covered_by_at_most_2n.

By Claim 1.5 and Claim 1.6, if $1 \notin \mathcal{V}$ and $(-1, n+1) \notin \mathcal{L}$ and $(0, 1) \notin \mathcal{L}$, then each line covers at most 2 distinct boundary points, i.e. $|l \cap \mathcal{D}_n| \leq 2$ for $l \in \mathcal{L}$ and $|v \cap \mathcal{D}_n| \leq 2$ for $v \in \mathcal{V}$. $\mathcal{L}$ and $\mathcal{V}$ cover $\mathcal{D}_n$, so $\mathcal{D}_n = \bigcup_{l \in \mathcal{L}} (l \cap \mathcal{D}_n) \cup \bigcup_{v \in \mathcal{V}} (v \cap \mathcal{D}_n)$. Since $|\mathcal{L}| + |\mathcal{V}| = n$, $\mathrm{card}(\mathcal{D}_n) \leq 2|\mathcal{L}| + 2|\mathcal{V}| = 2n$ by subadditivity of cardinality. □

Claim 1.8 — Given

- a finite set $\mathcal{L}$ of lines in $\mathbb{R}^2$, each represented as a pair (m, c) (slope and y-intercept),
- a finite set $\mathcal{V}$ of vertical lines in $\mathbb{R}^2$, each given by a real number x,
- a finite set $\mathcal{P}$ of points in $\mathbb{N} \times \mathbb{N}$,

satisfying the following conditions:

1. $|\mathcal{L}| + |\mathcal{V}| = n \geq 4$.
2. A point (a, b) lies in $\mathcal{P}$ if and only if $a \geq 1$, $b \geq 1$, and $a + b \leq n + 1$.
3. Every point $(a, b) \in \mathcal{P}$ is either:
   - on some line $(m, c) \in \mathcal{L}$ (i.e., $b = m \cdot a + c$), or
   - on some vertical line $x \in \mathcal{V}$ (i.e., $a = x$).

Then $(0, 1) \in \mathcal{L}$ or $(-1, n+1) \in \mathcal{L}$ or $1 \in \mathcal{V}$.

Proof. See theorem boundary_line_exists_at_any_cover_simplified.

We prove it by contradiction. When $n \geq 4$, if $(0, 1) \notin \mathcal{L}$ and $(-1, n+1) \notin \mathcal{L}$ and $1 \notin \mathcal{V}$, then by Claim 1.4 and Claim 1.7, we have

$$3n - 3 = \mathcal{D}_n \leq 2n,$$

which is a contradiction. Thus $(0, 1) \in \mathcal{L}$ or $(-1, n+1) \in \mathcal{L}$ or $1 \in \mathcal{V}$. □

Claim 1.9 — If $n \geq 4$ and $P(n-1)$ holds and the horizontal line $y = 1$ lies in the finite set of lines $\mathcal{P}$ with respect to statement $P(n)$, then $P(n)$ holds.

Proof. See theorem inductive_step_if_contains_horizontal_line.

Removing $y = 1$ from $\mathcal{L}$, $|\mathcal{L}| + |\mathcal{V}| = n - 1$, covering all points in $a \geq 1$, $b \geq 2$, and $a + b \leq n + 1$. Since $y = 1$ is not sunny, we wish to show there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$. We translate $\mathcal{L}$ by the injection $(m, c) \mapsto (m, c-1)$ to obtain $\mathcal{L}'$. Then $\mathcal{L}'$, $\mathcal{V}$ cover all points in $a \geq 1$, $b \geq 1$, and $a + b \leq n$, so there are 0, 1, or 3 sunny lines among $\mathcal{L}'$ and $\mathcal{V}$ by $P(n-1)$. Since translation preserves sunny lines, there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$. □

Claim 1.10 — If $n \geq 4$ and $P(n-1)$ holds and the vertical line $x = 1$ lies in the finite set of vertical lines $\mathcal{V}$ with respect to statement $P(n)$, then $P(n)$ holds.

Proof. See $\boxed{\text{theorem inductive\_step\_if\_contains\_vertical\_line}}$.

Removing $x = 1$ from $\mathcal{V}$, $|\mathcal{L}| + |\mathcal{V}| = n - 1$, covering all points in $a \geq 2$, $b \geq 1$, and $a + b \leq n + 1$. Since $x = 1$ is not sunny, we wish to show there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$. We translate $\mathcal{V}$ by the injection $c \mapsto c - 1$ to obtain $\mathcal{V}'$, and $\mathcal{L}$ by the injection $(m, c) \mapsto (m, m + c)$ to obtain $\mathcal{L}'$. Then $\mathcal{L}'$, $\mathcal{V}'$ cover all points in $a \geq 1$, $b \geq 1$, and $a + b \leq n$, so there are 0, 1, or 3 sunny lines among $\mathcal{L}'$ and $\mathcal{V}'$ by $P(n-1)$. Since translation preserves sunny lines, there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$. □

Claim 1.11 — If $n \geq 4$ and $P(n-1)$ holds and $x + y = n + 1$ lies in the finite set of lines $\mathcal{P}$ with respect to statement $P(n)$, then $P(n)$ holds.

Proof. See $\boxed{\text{theorem inductive\_step\_if\_contains\_rainy\_diagonal\_line\_refined}}$.

Removing $x + y = n + 1$ from $\mathcal{L}$, $|\mathcal{L}| + |\mathcal{V}| = n - 1$, covering all points in $a \geq 2$, $b \geq 1$, and $a + b \leq n + 1$. Since $x + y = n + 1$ is not sunny, we wish to show there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$. Then $\mathcal{L}$, $\mathcal{V}$ cover all points in $a \geq 1$, $b \geq 1$, and $a + b \leq n$, so there are 0, 1, or 3 sunny lines among $\mathcal{L}$ and $\mathcal{V}$ by $P(n-1)$. □

Claim 1.12 — $P(n)$ holds for all $n \geq 3$.

Proof. See $\boxed{\text{theorem imo2025\_p1\_left}}$. We proceed by induction on $n \geq 3$.

The base case is by Claim 1.3. For the inductive step, suppose $P(n-1)$ holds for $n \geq 4$. By the three scenarios in Claim 1.8, one of the three boundary lines exist, and in each case we may apply Claim 1.9, 1.10, or 1.11 respectively to show $P(n)$ holds. □

By combining Claims 1.1 and 1.12, the statement is thus proved.