IMO 2025 P5 Solution

ByteDance Seed AI4Math

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This is a compilation of ByteDance Seed Prover team's solutions to the 2025 IMO competition. All of the solutions are generated by Artificial Intelligence (AI) methods. However, to help human graders, ByteDance Seed Prover team translates the machine proof into natural language (English) solutions by human, with the original machine proof attached.

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Problem statement

Alice and Bazza are playing the *inekoalaty game*, a two-player game whose rules depend on a positive real number λ which is known to both players. On the n^{th} turn of the game (starting with $n = 1$) the following happens:

- If n is odd, Alice chooses a nonnegative real number x_n such that
$$x_1 + x_2 + \cdots + x_n \leqslant \lambda n.$$
- If n is even, Bazza chooses a nonnegative real number x_n such that
$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant n.$$

If a player cannot choose a suitable number x_n, the game ends and the other player wins. If the game goes on forever, neither player wins. All chosen numbers are known to both players.

Determine all values of λ for which Alice has a winning strategy and all those for which Bazza has a winning strategy.

Claim 1.1 — When $\lambda > \frac{\sqrt{2}}{2}$, Alice has a winning strategy.

We define the nth round means: Alice decides the number x_{2n+1}, and Bazza decides the x_{2n}. And we model the game state at the n-th round using the triple $(n, \text{sum1}, \text{sum2})$, where:

- $\text{sum1} = x_1 + x_2 + \cdots + x_{2n}$ is the sum of the first $2n$ numbers.
- $\text{sum2} = x_1^2 + x_2^2 + \cdots + x_{2n}^2$ is the sum of their squares.

Let $\mathrm{WinA}(n, \mathrm{sum1}, \mathrm{sum2})$ represent whether Alice has a winning strategy from the current state $(n, \mathrm{sum1}, \mathrm{sum2})$.

The *predicate* WinA satisfies the following properties:

- If $\mathrm{WinA}(n, \mathrm{sum1}, \mathrm{sum2})$ holds, then there exists a move for Alice:

$$\exists\ a \geq 0, s.t.\ \mathrm{sum1} + a \leq (2n+1)\lambda,$$

  and for any move that Bazza can make,

$$\forall\ b \geq 0, s.t.\ \mathrm{sum2} + a^2 + b^2 \leq (2n+2),$$

  the corresponding state satisfies:

$$\mathrm{WinA}(n+1, \mathrm{sum1} + a + b, \mathrm{sum2} + a^2 + b^2).$$

- Conversly, if there exists a move for Alice, such that for any move of Bazza,

$$\mathrm{WinA}(n+1, \mathrm{sum1} + a + b, \mathrm{sum2} + a^2 + b^2),$$

  then we have $\mathrm{WinA}(n, \mathrm{sum1}, \mathrm{sum2})$ holds.

Specifically, the initial state of the game is $(0, 0, 0)$. When $\lambda > \frac{\sqrt{2}}{2}$, we prove that for any *predicate* satisfying the properties outlined above, it must include the initial state. Since WinA satisfies these properties, we conclude that WinA contains the initial state $(0, 0, 0)$, meaning Alice has a winning state in the start of game.

Lemma 1.2 (Quadratic Growth Contradiction (round1_main))
For any $\lambda > \frac{\sqrt{2}}{2}$, there exists a natural number N such that

$$((2N+1)\lambda - \sqrt{2}N)^2 > 2N + 2$$

Proof. Since $\lambda \neq \sqrt{2}$, there exists N, such that

$$N > \frac{4}{(2\lambda - \sqrt{2})^2} + 2 > 2$$

We have

$$\begin{aligned}
((2N+1)\lambda - \sqrt{2}N)^2 &= ((2\lambda - \sqrt{2})N + \lambda)^2 \\
&> ((2\lambda - \sqrt{2})N)^2 \\
&= (2\lambda - \sqrt{2})^2 N^2 \\
&> (4 + 2(2\lambda - \sqrt{2})^2)N + 2 \\
&> 4N \\
&> 2N + 2
\end{aligned}$$

□

Lemma 1.3 (Property of large N (round8_quadratic_growth_contradiction))

For any $\lambda > \frac{\sqrt{2}}{2}$, there exists N such that for any sum1 and sum2, if $0 \le \text{sum1} \le \sqrt{2}N$ and $0 \le \text{sum2} \le 2N$ then

$$\text{sum2} + ((2N+1)\lambda - \text{sum1})^2 > 2N + 2$$

Proof. Use the N obtained from the previous lemma:

$$\begin{aligned}\text{sum2} + ((2N+1)\lambda - \text{sum1})^2 &\ge ((2N+1)\lambda - \text{sum1})^2 \\ &\ge ((2N+1)\lambda - \sqrt{2}N)^2 \\ &> 2N+2\end{aligned}$$

(noting that $\text{sum1} \le \sqrt{2}N \le (2N+1)\lambda$.) □

Lemma 1.4 (Inductive step (round15_lemma1))

For any predicate WinA satisfying the above characterization, and $\lambda > \sqrt{2}/2$, and any $k \in \mathbb{N}$, $0 \le \text{sum1}$, $0 \le \text{sum2} \le 2k$, if $\text{sum1}^2 \le k\text{sum2}$, and if Alice would cannot win from the state $(k, \text{sum1}, \text{sum2})$, then there is a move $b \ge 0$ from Bazza, such that in the state $(k+1, \text{sum1}+b, \text{sum2}+b^2)$, Alice cannot win, and the move satisfies: $\text{sum2} + b^2 \le 2(k+1)$ and $(\text{sum1}+b)^2 \le (k+1)(\text{sum2}+b^2)$.

Proof. Since Alice cannot win, by the contrapositive of the characterization, for any move Alice makes, there is a valid Bazza move b.

Now Alice makes the move $a = \boxed{0}$ (see line `have h6 := h5 0 (by linarith)` in the original Lean proof). This move is valid: $a \ge 0$ and $\text{sum1} + a \le \sqrt{2k} \le (2k+1)\lambda$.

Hence there is a valid Bazza move $b \ge 0$ such that $\text{sum2} + b^2 \le 2k + 2$ and Alice cannot win from the new position $(k+1, \text{sum1}+b, \text{sum2}+b^2)$.

Finally it remains to show $(\text{sum1}+b)^2 \le (k+1)(\text{sum2}+b^2)$. But this is obvious because by Cauchy–Schwarz,

$$\begin{aligned}(\text{sum1}+b)^2 &\le (\sqrt{k}\sqrt{\text{sum2}} + 1 \cdot b)^2 \\ &\le (k+1)(\text{sum2}+b^2).\end{aligned}$$

(Human annotator's note: the model proved the Cauchy–Schwarz inequality directly by considering the nonnegativity of suitable squared terms. This Cauchy–Schwarz inequality is also a part in the inductive step of the overall induction argument, and the model, in effect, used this to show AM–QM by induction (we note that Lean's mathematical library lacks a direct formulation of AM–QM), and the AM–QM inequality is not explicit in the argument.) □

Lemma 1.5 (Induction (round15_h_main))

Given the previous lemma, if Alice cannot win from the initial state $(0, 0, 0)$, then For any step k, there are states $\text{sum1} \ge 0$ and $\text{sum2} \ge 0$ such that $\text{sum1}^2 \le k\text{sum2}$ and $\text{sum2} \le 2k$, such that Alice cannot win from the state $(k, \text{sum1}, \text{sum2})$.

Proof. We proceed by induction on k. The case 0 is obvious: take $\mathrm{sum1} = \mathrm{sum2} = 0$. The inductive step is immediate from the previous lemma: if Alice cannot win from $(k, \mathrm{sum1}, \mathrm{sum2})$, then there exists a suitable b, such that by taking $\mathrm{sum1}' = \mathrm{sum1} + b$ and $\mathrm{sum2} = \mathrm{sum2} + b^2$, Alice cannot win from $(k+1, \mathrm{sum1}', \mathrm{sum2}')$. $\square$

Theorem 1.6 (Alice's Winning Strategy)

For any $\lambda > \frac{\sqrt{2}}{2}$, Alice has a winning strategy in the inekoalaty game.

Proof. This is simply by combining previous lemmas: suppose Alice cannot win from the initial state, then for any step N, there are states $\mathrm{sum1} \geq 0$ and $\mathrm{sum2} \geq 0$ such that $\mathrm{sum1}^2 \leq N\mathrm{sum2}$ and $\mathrm{sum2} \leq 2N$, such that Alice cannot win from the state $(N, \mathrm{sum1}, \mathrm{sum2})$.

However, if we consider N in the lemma round8_quadratic_growth_contradiction, since $\mathrm{sum2} \leq 2N$ and $\mathrm{sum1} \leq \sqrt{N}\sqrt{\mathrm{sum2}} \leq \sqrt{2}N$, we must have $\mathrm{sum2} + ((2N+1)\lambda - \mathrm{sum1})^2 > 2N + 2$. But then Alice can win in the state $(N, \mathrm{sum1}, \mathrm{sum2})$: let's consider making the move $a = \boxed{(2N+1)\lambda - \mathrm{sum1}}$. a is valid because $\mathrm{sum1} \leq \sqrt{2}N \leq (2N+1)\lambda$. (see `have h11 := h10 ((2 * (N : R) + 1) * l - s1) h9` in the proof). However, by the given $\mathrm{sum2} + a^2 > 2N + 2 = \mathrm{sum2} + ((2N+1)\lambda - \mathrm{sum1})^2 > 2N + 2$, Bazza can no longer make a move, so Alice wins. This is a contradiction with the previous result that Alice cannot win. $\square$

Claim 1.7 — When $\lambda < \frac{\sqrt{2}}{2}$, Bazza has a winning strategy.

The proof is by a reverse induction from a large N where Bazza will win, to show that Bazza will win at step 0.

Let WinB represent that when played optimally, Bazza can win from the current state. The predicate WinB satisfies the following property, similar to WinA: $\mathrm{WinB}(n, \mathrm{sum1}, \mathrm{sum2})$ holds, iff for any valid move from Alice, there is a valid move from Bazza, such that $\mathrm{WinB}(n+1, \mathrm{sum1}+a+b, \mathrm{sum2}+a^2+b^2)$.

Lemma 1.8 (Base case (round3_P_holds_for_large_n))

There is $N \in \mathbb{N}$ such that for any $n \geq N$ and any $\mathrm{sum1} \geq n\sqrt{2}$ and $\mathrm{sum2} = 2n$, Bazza wins in the state $(n, \mathrm{sum1}, \mathrm{sum2})$.

Proof. Take $N > \lambda/(\sqrt{2} - 2\lambda) + 1$. Let $n \geq N$ and $\mathrm{sum1} \geq n\sqrt{2}$ and $\mathrm{sum2} = 2n$. In the next state, suppose Alice makes move $0 \leq a \leq (2n+1)\lambda - \mathrm{sum1}$; we show this is invalid. This is invalid because $n(\sqrt{2} - 2\lambda) > \lambda$, so $n\sqrt{2} > (2n+1)\lambda$, but $\mathrm{sum1} \geq n\sqrt{2}$. □

Lemma 1.9 (Inductive step (round3_P_inductive_step_backward))

Let $P(k)$ be the predicate: for all $\mathrm{sum1} \geq k\sqrt{2}$ and $\mathrm{sum2} = 2k$, Bazza wins from state $(k, \mathrm{sum1}, \mathrm{sum2})$. For any k, if $P(k+1)$ holds, then $P(k)$ holds.

Proof. We assume $P(k+1)$. Let current state $\mathrm{sum1} \geq k\sqrt{2}$ and $\mathrm{sum2} = 2k$. We want to show Bazza wins from state $(k, \mathrm{sum1}, \mathrm{sum2})$. Therefore consider any Alice move $a \geq 0$ s.t. $\mathrm{sum1}+a \leq (2k+1)\lambda$. Since $\mathrm{sum1} \geq k\sqrt{2}$, $\mathrm{sum1}+a \leq (2k+1)\lambda$, and $\lambda < \sqrt{2}/2$, we have $a < \sqrt{2}/2$. Let $b = \boxed{\sqrt{2 - a^2}}$. (This is `set b := Real.sqrt (2 - a ^ 2) with hb_def` in the original proof). Clearly b is well defined and $b \geq 0$. The move b is valid because $\mathrm{sum2} + a^2 + b^2 = \mathrm{sum2} + 2 = 2k + 2$. Also $a + b \geq \sqrt{2}$, so $\mathrm{sum1} + a + b \geq (k+1)\sqrt{2}$. Applying IH to the next state $(k+1, \mathrm{sum1}+a+b, \mathrm{sum2}+a^2+b^2 = 2k)$, Bazza wins in this next state. So Bazza wins in the current state. □

Theorem 1.10 (Bazza's Winning Strategy)

For any $\lambda < \frac{\sqrt{2}}{2}$, Bazza has a winning strategy in the inekoalaty game.

Proof. This is by reverse induction with the previous 2 lemmas: since $P(N)$ holds for a large N (base case), and if $P(k+1)$ holds, then $P(k)$ holds, then $P(0)$ holds. But in this case since $\mathrm{sum1} = 0 \geq 0\sqrt{2}$ and $\mathrm{sum2} = 0 = 2 \cdot 0$, $P(0)$ simply states that Bazza wins from state $(0, 0, 0)$, i.e. that Bazza wins the game. □

Claim 1.11 — When $\lambda = \frac{\sqrt{2}}{2}$, the game will end in a draw.

Proof. We will prove that both Alice and Bazza will not lose when $\lambda = \frac{\sqrt{2}}{2}$. Here are some notations. We define the kth round means: Alice decides the kth value, and Bazza decides the kth value. For the kth round, Alice's value is a_k and Bazza's value is b_k.

Firstly, Alice will not lose by taking $a_k = 0, \forall k \geq 1$. If the game not end in the nth round, then for sequence $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$ satisfy

$$\sum_{k=1}^n (a_k + b_k) \leq 2\lambda n = \sqrt{2}n,$$

$$\sum_{k=1}^n (a_k^2 + b_k^2) \leq 2n.$$

By Cauchy-Schwarz inequality, we have

$$2n \geq \sum_{k=1}^n (a_k^2 + b_k^2) = \sum_{k=1}^n b_k^2 \geq \frac{(\sum_{k=1}^n b_k)^2}{n}.$$

Thus $\sqrt{2}n \geq \sum_{k=1}^n b_k$. Then

$$\sum_{k=1}^n (a_k + b_k) + a_{k+1} = \sum_{k=1}^n b_k \leq \sqrt{2}n < \frac{\sqrt{2}}{2}(2n+1) = (2n+1)\lambda.$$

Secondly, Bazza will not lose by taking

$$b_k = \begin{cases} \boxed{\sqrt{2} - a_k} & \text{if } a_k \leq \sqrt{2}, \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ k \geq 1.$$

We will show that $a_k \leq \frac{\sqrt{2}}{2}, \forall\ k \geq 1$ by induction. If $k = 1$, then $a_1 \leq \lambda = \frac{\sqrt{2}}{2}$. If $a_k \leq \frac{\sqrt{2}}{2}, \forall k \leq n$, this means that the game will not end before the nth round, then for sequence $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^{n-1}$ which satisfies the above condition, we have

$$\sum_{k=1}^{n-1} (a_k + b_k) + a_n \leq \frac{\sqrt{2}}{2}(2n-1),$$

$$\sum_{k=1}^{n-1} (a_k^2 + b_k^2) \leq 2n - 2.$$

Note that $a_k + b_k = \sqrt{2}, \forall\ k \leq n-1$, then $a_n \leq \frac{\sqrt{2}}{2}$. This indicates that b_n can take $\sqrt{2} - a_n$ and satisfies

$$\begin{aligned}
\sum_{k=1}^n (a_k^2 + b_k^2) &= \sum_{k=1}^{n-1} (a_k^2 + b_k^2) + (a_n^2 + b_n^2) \\
&\leq 2n - 2 + a_n^2 + (\sqrt{2} - a_n)^2 \\
&= 2n - 2 + 2(a_n - \frac{\sqrt{2}}{2})^2 + 1 \\
&\leq 2n.
\end{aligned}$$

If Alice wants to continue the game, then a_{n+1} satisfies

$$\sum_{k=1}^{n}(a_k + b_k) + a_{n+1} \leq \frac{\sqrt{2}}{2}(2n + 1).$$

This is the same to the above case. Thus Bazza can keep taking $b_{n+1} = \sqrt{2} - a_{n+1}$ and remain undefeated.

To sum up, the game will end in a draw when $\lambda = \frac{\sqrt{2}}{2}$.

$\square$

Remark 1.12. Our model proposed two different strategies to keep Bazza from losing. One to deal with the case of Bazza winning $\boxed{\sqrt{2 - a_n^2}}$, and the other to deal with the case of Bazza not losing $\boxed{\sqrt{2} - a_n}$.

Remark 1.13. Human translator's note : our strategy for any conjunction problem is to split the problem into subproblems and let the prover model solve each subproblem. This problem is split to 4 subproblems: Alice wins when $\lambda > \sqrt{2}/2$, Bazza wins when $\lambda < \sqrt{2}/2$, Alice cannot win when $\lambda = \sqrt{2}/2$, and Bazza cannot win when $\lambda = \sqrt{2}/2$, where the value $\sqrt{2}/2$ is determined by our LLM. We note that the proof of winning conditions of Alice and Bazza use the WinA/WinB predicate formulation (below), but the proof of the drawing condition uses a more direct formulation. This is because we (the human translators) when translating from the natural language problem statement to Lean, first came up with the predicate formulation, and the model solved the winning condition cases in this formulation, but we struggled to formulate the more direct formulation in Lean on the day the problem is released. We also could not formulate a predicate-style formulation for the drawing condition. Eventually we did successfully formulate the direct formulation in Lean (see the statement of the drawing condition), and the model proved the drawing condition in the direct formulation, but due to time constraints we did not let our model re-prove the winning conditions with the new direct formulation.