1.3.5 Discrete Probability Models

The sample space is a countable (finite or infinite) set in discrete models.

Ex: An experiment involving a single coin toss. We say that the coin is "fair", equal probabilities are assigned to the possible outcomes. That is, $P(H) = P(T) = 1/2$.

In a discrete probability model,

- The probability of any event $\{s_1, s_2, \dots, s_k\}$ is the sum of the probabilities of its elements. (Recall "additivity".)

$$\begin{aligned} P(\{s_1, s_2, \dots, s_k\}) &= P(\{s_1\}) + P(\{s_2\}) + \ldots + P(\{s_k\}) \\ &= P(s_1) + P(s_2) + \ldots + P(s_k) \end{aligned}$$

  Ex: Throw a 6-sided die. Express the probability that the outcome is 1 or 6.

- Discrete uniform probability law: If the sample space consists of n possible outcomes which are equally likely, then

$$P(A) = \frac{|A|}{n}.$$

Ex: Throw a fair 6-sided die. Find the probability that the outcome is 1 or 6.

Ex: A fair coin is tossed until a tails is observed. Determine the probabilities of each outcome in the sample space.

Ex: A file contains 1Kbytes. The probability that there exists at least one corrupted byte is 0.01. The probability that at least two bytes are corrupted is 0.005. Let the outcome of the experiment be the number of bytes in error.

(a) Define the sample space.

(b) Find P(no errors).

(c) P(exactly one byte in error)=?

(d) P(at most one byte is in error)=?

1.3.6 Continuous Models

The sample space is an uncountable set in continuous models. We compute the probability by measuring the probability "weight" of the desired event relative to the sample space.

Ex: I start driving to work in the morning at some time uniformly chosen in the interval $[8:30, 9:00]$.

- What is the probability that I start driving before $8:45$?

- My favorite radio program comes on at 8 : 30, and may last anywhere between 5 to 15 minutes, with equal probability. What is the probability that I catch at least part of the program?

1.4 Conditional Probability

$P(A|B)$ = probability of A, given that B occurred

Definition 3 *Let A and B two events with $P(B) \neq 0$. The conditional probability $P(A|B)$ of A given B is defined as*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example: Consider two rolls of a tetrahedral die. Let B be the event that the minimum of the two rolls is 2. Let M be the maximum of the rolls.

- $\mathrm{P}(M = 1|B) =$
- $\mathrm{P}(M = 2|B) =$

1.4.1 Properties of Conditional Probability

Conditional probability is a probability law, where B is the new universe.

Theorem 1 *For a fixed event B with $P(B) > 0$, the conditional probabilities $P(A|B)$ form a probability law satisfying all three axioms.*

Proof 1

- If A and B are disjoint, $P(A|B) =$.
- If $B \subset A$, $P(A|B) =$.
- When all outcomes are equally likely,

$$P(A|B) = \frac{|\quad|}{|\;|}.$$

Ex: A girl I met told me she has 1 sibling. What is the probability that her sibling is a boy? (Assumption: each birth results in a boy or girl with equal probability.)

1.4.2 Chain (Multiplication) Rule

Assuming that all of the conditioning events have positive probability, the following expression holds

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_n|\bigcap_{i=1}^{n-1} A_i).$$

Ex: There are two balls in an urn numbered with 0 and 1. A ball is drawn. If it is 0, the ball is simply put back. If it is 1, another ball numbered with 1 is put in the urn along with the drawn ball. The same operation is performed once more. A ball is drawn in the third time. What is the probability that the drawn balls are all labeled 1?

1.4.3 Total Probability Theorem

- This is the "divide and conquer" idea. Very useful in modelling and solving problems.
- Partition set B into $A_1, A_2, \dots, A_n$. The A_i's should be disjoint inside B. That is, $A_i \cap A_j \cap B = \emptyset$ for all i, j.

- One way of computing $\mathrm{P}(B)$:

$$\mathrm{P}(B) = \mathrm{P}(A_1)P(B|A_1) + \mathrm{P}(A_2)\mathrm{P}(B|A_2) + \dots \mathrm{P}(A_n)\mathrm{P}(B|A_n)$$

- An often used partition is A and A^c, where A is any event in the sample space, not disjoint with B.

Ex: The "Monty Hall Problem" (Example 1.12 in textbook.) There is a prize behind one of three identical doors. You are told to pick a door. The game show host then opens one of the remaining doors with no prize behind it. At this point, you have the option to switch to the unopened door, or stick to your original choice. What is the better strategy- to stick or to switch? (Hint: Examine each strategy separately. In each, let B be the event of winning, and A the event that the initially chosen door has the prize behind it.)

1.4.4 Bayes's Rule

This is a rule for combining "evidence".

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \text{ Note how } A \text{ and } B \text{ changed places} \end{aligned}$$

Ex: Criminal X and Criminal Y are both 20 percent likely to commit a certain crime, and they are both 50 percent likely to be near the site of the crime at a given time. As a result of the investigation, it is revealed that Criminal X was near the site at the time of the crime, but Y was not. What are the posterior probabilities of committing the crime for X and Y?

Bayes's rule is often applied to events A_i that form a partition of the given event B.

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i \cap B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{P(B)} \\ &= \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)} \end{aligned}$$

Where, going from the second line to the third, we applied the Total Probability Theorem.

Ex: Let B be the event that the sum of the numbers obtained on two tosses of a die is seven. Given that B happened, find the probability that the first toss resulted in a 3.

1.5 Modelling using conditional probability

Ex: If an aircraft is present in a certain area, a radar detects it and generates and alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm with probability 0.10. We assume that an aircraft is present with probability 0.05.
Event A: Airplane is flying above
Event B: Something registers on the radar screen

(a) $P(A \cap B) =$

(b) $P(B) =$

(c) $P(A|B) =$? (Discuss how to improve this probability.)

Ex: The "false positive puzzle" (Example 1.18 in textbook.): A test for a certain disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person is not sick, the test results are negative with probability 0.95. A person randomly drawn from the population has the disease with probability 0.01. Given that a person is tested positive, what is the probability that the person is actually sick?