

Proof of Regularity of 2-Sum and 3-Sum of Matroids

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1 The 2-Sum of Regular Matroids Is Regular

Lemma 1. Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc} , the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic} \cdot a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \cdot \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column $a''_{\bullet k}$ with the pivot column $a_{\bullet c}$ without the pivot element a_{rc} .

By the cofactor expansion in A along row r , we have

$$\det A = \sum_{k=1}^n (-1)^{r+k} \cdot a_{rk} \cdot \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k . By swapping the order of columns in $B_{r,k}$ to match the form of B_k , we get

$$\det A = (-1)^{r+c} (a_{rc} \cdot \det A' - \sum_{k \in C} a_{rk} \cdot \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 2. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 3. Let B_l, B_r be matrices with $\{0, \pm 1\}$ entries expressed as $B_l = \begin{bmatrix} A_l \\ x \end{bmatrix}$ and $B_r = \begin{bmatrix} y & A_r \end{bmatrix}$, where x is a row vector, y is a column vector, and A_l, A_r are matrices of appropriate dimensions. Let D be the outer product of y and x . The 2-sum of B_l and B_r is defined as

$$B_l \oplus_{2,x,y} B_r = \begin{bmatrix} A_l & 0 \\ D & A_r \end{bmatrix}.$$

Definition 4. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k -TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 5. Note that a matrix is TU if and only if it is k -TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 6. Let B_l and B_r be TU matrices and let $B = B_l \oplus_{2,x,y} B_r$. Then B is 1-TU and 2-TU.

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B . If V is a submatrix of $\begin{bmatrix} A_l \\ D \end{bmatrix}$, $\begin{bmatrix} D & A_r \end{bmatrix}$, $\begin{bmatrix} A_l & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_l and A_r . Let i be the row shared by V and A_l and j be the column shared by V and A_r . Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij} . Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. \square

Lemma 7. Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_l and B_r their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is ℓ -TU for every $\ell < k$. Then for any TU matrices B_l and B_r their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is also k -TU.

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_l and B_r such that their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is not k -TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $\begin{bmatrix} A_l \\ D \end{bmatrix}$, $\begin{bmatrix} D & A_r \end{bmatrix}$, $\begin{bmatrix} A_l & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_l and A_r each.

Consider the row of V whose index appears in A_l . Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_l . Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc} . For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, \tilde{B} denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\begin{bmatrix} \tilde{A}_l \\ \tilde{D} \end{bmatrix}$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_r$, i.e., A_r remains unchanged. This holds because of the 0 block in B .
- \tilde{D} consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\begin{bmatrix} \tilde{D} & \tilde{A}_r \end{bmatrix}$ is TU, since this matrix consists of A_r and copies of y scaled by factors $\{0, \pm 1\}$.
- \tilde{D} can be represented as an outer product of a column vector \tilde{y} and a row vector \tilde{x} , and we can define $\tilde{B}_1 = \begin{bmatrix} \tilde{A}_l \\ \tilde{x} \end{bmatrix}$ and $\tilde{B}_2 = \begin{bmatrix} \tilde{y} & \tilde{A}_r \end{bmatrix}$ similar to B_l and B_r , respectively. Note that \tilde{B}_1 and \tilde{B}_2 have the same size as B_l and B_r , respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- \tilde{B} contains a square submatrix \tilde{V} of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 2 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k - 1) \times (k - 1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B , the submatrix V'' corresponds to a submatrix \tilde{V} of \tilde{B} with the same property.

To sum up, after pivoting we obtain a matrix \tilde{B} that represents a 2-sum of TU matrices \tilde{B}_1 and \tilde{B}_2 and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k - 1)$ -TUness of \tilde{B} , which proves the lemma. \square

Lemma 8. Let B_l and B_r be TU matrices. Then $B_l \oplus_{2,x,y} B_r$ is also TU.

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_l and B_r , their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is ℓ -TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 6.

Step: If the Proposition holds for some k , then it also holds for $k + 1$ by Lemma 7.

Conclusion: For any TU matrices B_l and B_r , their 2-sum $B_l \oplus_{2,x,y} B_r$ is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_l \oplus_{2,x,y} B_r$ is TU. \square

2 The 3-Sum of Regular Matroids Is Regular

2.1 Definition of 3-Sum

Definition 9. Let $B_l^{(0)} \in \mathbb{Z}_2^{(X_l \cup \{x_0, x_1\}) \times (Y_l \cup \{y_2\})}$, $B_r^{(0)} \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0, y_1\})}$ be matrices of the form

$$B_l^{(0)} = \begin{array}{|c|cc|c|} \hline & & & 0 \\ \hline & A_l^{(0)} & & \\ \hline & 1 & 1 & 0 \\ \hline D_l^{(0)} & D_0^{(0)} & & 1 \\ & & & 1 \\ \hline \end{array}, \quad B_r^{(0)} = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline D_0^{(0)} & 1 & & \\ & 1 & & A_r^{(0)} \\ \hline D_r^{(0)} & & & \\ \hline \end{array},$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{lr}^{(0)} = D_r^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_l^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_l^{(0)}$ and $B_r^{(0)}$ is

$$B^{(0)} = B_l^{(0)} \oplus_3 B_r^{(0)} = \begin{array}{|c|cc|c|} \hline & & & 0 \\ \hline & A_l^{(0)} & & \\ \hline & 1 & 1 & 0 \\ \hline D_l^{(0)} & D_0^{(0)} & 1 & \\ & & 1 & A_r^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} & & \\ \hline \end{array} \in \mathbb{Z}_2^{(X_l \cup X_r) \times (Y_l \cup Y_r)}.$$

Here $x_2 \in X_l$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_l$, $y_2 \in Y_r$, $A_l^{(0)} \in \mathbb{Z}_2^{X_l \times Y_l}$, $A_r^{(0)} \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_l^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_l \setminus \{y_0, y_1\})}$, $D_r^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{lr}^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times (Y_l \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_l^{(0)}$, $B_r^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{l,lr}^{(0)} = \begin{array}{|c|} \hline D_l^{(0)} \\ \hline D_{lr}^{(0)} \\ \hline \end{array}, \quad D_{0,r}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_r^{(0)} \\ \hline \end{array}, \quad D_{l,0}^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{lr,r}^{(0)} = \begin{array}{|c|c|} \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$.

can omit

Lemma 10. Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

Proof. Verify by complete enumeration.

\square

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2.2 Construction of Canonical Signing

Definition 11. We call B_l and B_r canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively, if they have the form

$$B_l = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_l & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_l & D_0 & 1 \\ \hline & & 1 \\ \hline \end{array}, \quad B_r = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline & D_0 & 1 & \\ \hline & & 1 & A_r \\ \hline D_r & & & \\ \hline \end{array}$$

where every block in B_l and B_r is a signing of the corresponding block in $B_l^{(0)}$ and $B_r^{(0)}$, and D_0 is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings B_l and B_r , the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_l & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_l & D_0 & 1 \\ \hline & & 1 & A_r \\ \hline D_{lr} & D_r & & \\ \hline \end{array}$$

where $D_{lr} = D_r \cdot (D_0)^{-1} \cdot D_l$ (calculated over \mathbb{Q}).

The following lemma helps construct canonical signings from arbitrary initial TU signings.

Lemma 12. Let Q' be a TU signing of the matrix

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0^{(0)} & 1 & \\ \hline & & 1 \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0, x_1, x_2) \times (y_0, y_1, y_2)}$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Define $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q as follows:

$$\begin{aligned} u(x_0) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0), \\ u(x_1) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), \\ u(x_2) &= 1, \\ v(y_0) &= Q'(x_2, y_0), \\ v(y_1) &= Q'(x_2, y_1), \\ v(y_2) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), \\ \forall i \in \{x_0, x_1, x_2\}, \forall j \in \{y_0, y_1, y_2\}, \quad Q(i, j) &= Q'(i, j) \cdot u(i) \cdot v(j). \end{aligned}$$

Then Q is a TU signing of T and $Q = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0 & 1 & \\ \hline & & 1 \\ \hline \end{array}$ where D_0 is the respective canonical signing of $D_0^{(0)}$.

Proof. Since Q' is a TU signing of T and Q is obtained from Q' by multiplying rows and columns by ± 1 factors, Q is also a TU signing of T . By construction, we have

$$\begin{aligned}
Q(x_2, y_0) &= Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1, \\
Q(x_2, y_1) &= Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1, \\
Q(x_2, y_2) &= 0, \\
Q(x_0, y_0) &= Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1, \\
Q(x_0, y_1) &= Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1), \\
Q(x_0, y_2) &= Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1, \\
Q(x_1, y_0) &= 0, \\
Q(x_1, y_1) &= Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)), \\
Q(x_1, y_2) &= Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1.
\end{aligned}$$

Thus, it remains to check that $Q(x_0, y_1)$ and $Q(x_1, y_1)$ are correct.

First, consider the entry $Q(x_0, y_1)$. If $D_0^{(0)}(x_0, y_1) = 0$, then $Q(x_0, y_1) = 0$, as needed. Otherwise, if $D_0^{(0)}(x_0, y_1) = 1$, then $Q(x_0, y_1) \in \{\pm 1\}$, as Q is a signing of T . Our goal is to show that $Q(x_0, y_1) = 1$. For the sake of deriving a contradiction suppose that $Q(x_0, y_1) = -1$. Then the determinant of the submatrix of Q indexed by $\{x_0, x_2\} \times \{y_0, y_1\}$ is

$$\det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q . Thus, $Q(x_0, y_1) = 1$, as needed.

Consider the entry $Q(x_1, y_1)$. Since Q is a signing of T , we have $Q(x_1, y_1) \in \{\pm 1\}$. Note that we know all the other entries of Q , so we can determine the sign of $Q(x_1, y_1)$ using TUness of Q . Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $Q(x_1, y_1) = 1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q . Thus, $Q(x_1, y_1) = -1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $Q(x_1, y_1) = -1$, then $\det Q_{\{x_0, x_1\}, \{y_1, y_2\}} = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q . Thus, $Q(x_1, y_1) = 1$, as needed.

□

Definition 13. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q' \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q^{(0)} \in \mathbb{Z}_2^{X \times Y}$. Let $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q be constructed as follows:

$$\begin{aligned}
u(i) &= \begin{cases} Q'(x_2, y_0) \cdot Q'(x_0, y_0), & i = x_0, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases} \\
v(j) &= \begin{cases} Q'(x_2, y_0), & j = y_0, \\ Q'(x_2, y_1), & j = y_1, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases} \\
&\quad \forall i \in X, \forall j \in Y, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j).
\end{aligned}$$

We call Q a canonical resigning of Q' .

Lemma 14. Let B'_l be a TU signing of $B_l^{(0)}$. Let B_l be the canonical resigining (constructed following Definition 13) of B'_l . Then B_l is a canonical signing of $B_l^{(0)}$ (in the sense of Definition 11) and B_l is TU. Going forward, we refer to B_l as a TU canonical signing for short of $B_l^{(0)}$. A TU canonical signing B_r of $B_r^{(0)}$ is defined similarly (up to replacing subscripts 1 by 2).

Proof. This follows directly from Lemma 12. \square

Lemma 15. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = (D_{0,r})_{\bullet, y_0}$ and $c_1 = (D_{0,r})_{\bullet, y_1}$. Then the following matrices are TU:

$$B_r^{(a)} = \begin{bmatrix} c_0 - c_1 & c_0 & A_r \end{bmatrix}, \quad B_r^{(b)} = \begin{bmatrix} c_0 - c_1 & c_1 & A_r \end{bmatrix}.$$

Proof. Pivoting in B_r on (x_2, y_0) and (x_2, y_1) yields:

$$\begin{array}{ccc} B_r = \begin{bmatrix} \textcircled{1} & 1 & 0 \\ c_0 & c_1 & A_r \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c_1 - c_0 & A_r \end{bmatrix} \\ \\ B_r = \begin{bmatrix} 1 & \textcircled{1} & 0 \\ c_0 & c_1 & A_r \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 1 & 0 \\ c_0 - c_1 & -c_1 & A_r \end{bmatrix} \end{array}$$

By removing row x_2 from the resulting matrices and then multiplying columns y_0 and y_1 by $\{\pm 1\}$ factors, we obtain $B_r^{(a)}$ and $B_r^{(b)}$. By Lemma 14, B_r is TU. Since TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $B_r^{(a)}$ and $B_r^{(b)}$ are TU. \square

Lemma 16. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.

1. For every $i \in X_r$, we have $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$.
2. $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$ is TU.

Proof. 1. Let $i \in X_r$. If $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r (which holds by Lemma 14). Similarly, if $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r .

2. Let V be a square submatrix of $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$. We will show that $\det V \in \{0, \pm 1\}$.

Let z denote the index of the appended column c_2 . Suppose that column z is not in V . Then V is a submatrix of B_r , which is TU by Lemma 14. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V .

Suppose that columns y_0 and y_1 are both in V . Then V contains columns z , y_0 , and y_1 , which are linearly dependent by construction of c_2 . Thus, $\det V = 0$. Going forward we assume that at most one of the columns y_0 and y_1 is in V .

Suppose that column y_0 is in V . Then V is a submatrix of $B_r^{(b)}$ from Lemma 15, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $B_r^{(a)}$ from Lemma 15, and so $\det V \in \{0, \pm 1\}$.

Thus, every square submatrix V of \tilde{T} has $\det V \in \{0, \pm 1\}$, and hence \tilde{T} is TU. \square

Remark 17. Vectors c_0 , c_1 , and c_2 can be defined directly in terms of entries of B_r , e.g., c_2 consists of entries of B_r indexed by $(X_r \setminus \{x_2\}) \times \{y_0\}$.

Lemma 18. Let B_l be a TU canonical signing of $B_l^{(0)}$. Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$. Then the following properties hold.

1. For every $j \in Y_r$, we have $\begin{bmatrix} d_0(j) \\ d_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

2. $\begin{bmatrix} A_l \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$ is TU.

Proof. Apply Lemma 16 to B_l^\top , or repeat the same argument up to interchanging rows and columns. □

Lemma 19. Let B_l and B_r be TU canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively.

- Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$.
- Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$.
- If $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.
- Let D be the bottom-left block in the canonical signing B of $B^{(0)}$ corresponding to B_l and B_r .

Then the following properties hold.

1. $D = c_0 \cdot r_0 + c_1 \cdot r_1$.
2. Rows of D are in $\begin{bmatrix} \pm r_0 \\ \pm r_1 \\ \pm r_2 \\ 0 \end{bmatrix}$.
3. Columns of D are in $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm c_2 & 0 \end{bmatrix}$.
4. $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$ is TU.
5. $\begin{bmatrix} A_r & D \end{bmatrix}$ is TU.
6. $\begin{bmatrix} A_l \\ r_0 \\ r_1 \\ r_2 \end{bmatrix}$ is TU.
7. $\begin{bmatrix} A_l \\ D \end{bmatrix}$ is TU.
8. $\begin{bmatrix} c_0 & c_1 \end{bmatrix}$ contains D_0 (the canonical signing of $D_0^{(0)}$) as a submatrix.

Proof. 1. Follows via a direct calculation.

need details?

2. By item 1, for every $i \in X_r$ we have $D(i, \bullet) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$. By Lemma 15.1, we know that $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$. Therefore, $D(i, \bullet)$ is equal to either 0 , $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.

3. Holds by the same argument as item 2 up to interchanging rows and columns.

4. Holds by Lemma 16.2.

5. By item 3, columns of $\begin{bmatrix} A_r & D \end{bmatrix}$ are in $\begin{bmatrix} A_r & \pm c_0 & \pm c_1 & \pm c_2 & 0 \end{bmatrix}$. Since $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$ is TU and since adding zero columns and copies of columns multiplied by ± 1 factors preserves TUness, $\begin{bmatrix} A_r & D \end{bmatrix}$ is also TU.

6. By Lemma 18.2 (or by the same argument as item 4 up to interchanging rows and columns),

$\begin{bmatrix} A_l \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$ is TU. Since TUness is preserved under multiplication of rows by ± 1 and exchanging rows, $\begin{bmatrix} A_l \\ r_0 \\ r_1 \\ r_2 \end{bmatrix}$ is also TU.

7. Holds by the same argument as item 5 up to interchanging rows and columns.

8. Holds by construction. □

Definition 20. Let $A_l \in \mathbb{Q}^{X_l \cup Y_l}$, $A_r \in \mathbb{Q}^{X_r \cup Y_r}$, $c_0, c_1 \in \mathbb{Q}^{X_r}$, $r_0, r_1 \in \mathbb{Q}^{Y_l}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–8 from the statement of Lemma 19 are satisfied for $A_l, A_r, c_0, c_1, r_0, r_1$. Given $k \in \mathbb{Z}_{\geq 1}$,

define $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, k)$ to mean “ $C = \begin{bmatrix} A_l & 0 \\ D & A_r \end{bmatrix}$ is k -TU”.

Lemma 21. Assume the notation of Definition 20. Then $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, 1)$ holds.

Proof. Every entry of C is in one of four blocks: 0, A_l , D , A_r . By the assumptions of Definition 20, all of these blocks are TU. Thus, C is 1-TU. □

Lemma 22. Assume the notation of Definition 20. Let $i \in X_l$, let $T = \begin{bmatrix} A_l(i, \bullet) \\ D \end{bmatrix}$. Suppose we pivot on entry

$T(i, j) \in \{\pm 1\}$ in T and obtain matrix $T' = \begin{bmatrix} a' \\ D' \end{bmatrix}$. Then columns of D' are in $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \end{bmatrix}$.

Proof. Since T is a submatrix of $\begin{bmatrix} A_l \\ D \end{bmatrix}$, which is TU by assumptions of Definition 20, we have that T is TU.

Since pivoting preserves TUness, T' is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column p in T could be and what another column q in T could be. This uniquely determines the resulting columns p' and q' in T' by the pivot formula. In every case, either $\begin{bmatrix} p' & q' \end{bmatrix}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of T' , or the restriction of p' and q' to X_r is in $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \end{bmatrix}$. □

need details?

Lemma 23. Assume the notation of Definition 20. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k-1)$ holds for all A'_l, r'_0 , and r'_1 satisfying the assumptions of Definition 20 (together with A_r, c_0 , and c_1). Then $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, k)$ holds.

Proof. Let V be a $k \times k$ submatrix of C . For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that V is a submatrix of $\begin{bmatrix} A_l \\ D \end{bmatrix}$, $\begin{bmatrix} A_l & 0 \end{bmatrix}$, $\begin{bmatrix} D & A_r \end{bmatrix}$, or $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$. Since all of those four matrices are TU by the assumptions of Definition 20, we have $\det V \in \{0, \pm 1\}$. Thus, V shares at least one row and one column index with A_l and A_r each.

Consider the row index shared by V and A_l . Note that this row in V cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a ± 1 entry shared by V and A_l . Let i and j denote the row and the column index of this entry, respectively.

Perform a pivot in C on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, B' denotes the entire matrix after the pivot. Note that the following statements hold.

- C' contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix V' of C' with the same row and column index sets as V minus the pivot row i and pivot column j .
- $C' = \begin{bmatrix} A'_l & 0 \\ D' & A_r \end{bmatrix}$, i.e., the 0 and the A_r blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.
- $\begin{bmatrix} A'_l \\ D' \end{bmatrix}$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.
- The columns of D' are in $\begin{bmatrix} 0 & \pm c_0 & \pm c_1 & \pm(c_0 - c_1) \end{bmatrix}$. This holds by Lemma 22.
- There exist r'_0 and r'_1 such that $D' = c_0 \cdot r'_0 + c_1 \cdot r'_1$ and the assumptions of Definition 20 are satisfied for $A'_l, A_r, c_0, c_1, r'_0, r'_1$. This follows from the previous bullet point by carefully checking assumptions. need details?
- C' is $(k-1)$ -TU. This follows from the hypothesis: $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k-1)$ holds.

To sum up, after pivoting we obtain a matrix C' (which can be obtained in the manner of Definition 20) that is $(k-1)$ -TU and contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. \square

Lemma 24. Let B_l and B_r be TU canonical signings. Then the corresponding canonical signing B is TU.

Proof. Define $A_l, A_r, c_0, c_1, r_0, r_1$ as in Lemma 19. Note that canonical signing B has the form of C in the notation of Definition 20.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$ holds for all A'_l, r'_0 , and r'_1 satisfying the assumptions of Definition 20.

Base: The Proposition holds for $k = 1$ by Lemma 21.

Step: If the Proposition holds for some k , then it also holds for $k+1$ by Lemma 23.

Conclusion: $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$.

Specializing the conclusion to $A_l, A_r, c_0, c_1, r_0, r_1$ (obtained from B_l and B_r as described in the statement of Lemma 19) shows that canonical signing B is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, B is TU. \square

Corollary 25. Suppose that $B_l^{(0)}$ and $B_r^{(0)}$ have TU signings. Then $B_l \oplus_3 B_r$ has a TU signing.

Proof sketch. Start with some TU signings, obtain canonical signings, apply Lemma 24. \square