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Regularity of 1-, 2-, and 3-Sums of Matroids

Ivan Sergeev

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Chapter 1

Preliminaries

1.1 Total Unimodularity

Definition 1. Matrix is a function that takes a row index and returns a vector, which is a function that takes a column index and returns a value. The former aforementioned identity is definitional, the latter is syntactical. By abuse of notation $(R^Y)^X \equiv R^{X \times Y}$ we do not curry functions in this text. When a matrix happens to be finite (that is, both X and Y are finite) and its entries are numeric, we like to represent it by a table of numbers.

Definition 2. Matrix Let A be a square matrix over a commutative ring. Determinant of A is the sum over all permutations, sign of the permutation times the product of (Todo: complete definition).

Definition 3. Matrix.det Let R be a commutative ring. We say that a matrix $A \in R^{X \times Y}$ is totally unimodular, or TU for short, if for every $k \in \mathbb{N}$, every (not necessarily contiguous) $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 4. Matrix.IsTotallyUnimodular Let A be a TU matrix. Suppose rows of A are multiplied by $\{0, \pm 1\}$ factors. Then the resulting matrix A' is also TU.

Proof. Matrix.IsTotallyUnimodular We prove that A' is TU by Definition ???. To this end, let T' be a square submatrix of A' . Our goal is to show that $\det T' \in \{0, \pm 1\}$. Let T be the submatrix of A that represents T' before pivoting. If some of the rows of T were multiplied by zeros, then T' contains zero rows, and hence $\det T' = 0$. Otherwise, T' was obtained from T by multiplying certain rows by -1 . Since T' has finitely many rows, the number of such multiplications is also finite. Since multiplying a row by -1 results in the determinant getting multiplied by -1 , we get $\det T' = \pm \det T \in \{0, \pm 1\}$ as desired. \square

Lemma 5. Matrix.IsTotallyUnimodular Let A be a TU matrix. Suppose columns of A are multiplied by $\{0, \pm 1\}$ factors. Then the resulting matrix A' is also TU.

Proof. `Matrix.IsTotallyUnimodular, Matrix.IsTotallyUnimodular.mul_rowsApplyLemma` to A^\top . \square

Definition 6. `Matrix.det` Given $k \in \mathbb{N}$, we say that a matrix A is k -partially unimodular, or k -PU for short, if every (not necessarily contiguous, not necessarily injective) $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 7. `Matrix.IsTotallyUnimodular, Matrix.IsPartiallyUnimodular` A matrix A is TU if and only if A is k -PU for every $k \in \mathbb{N}$.

Proof. `Matrix.IsTotallyUnimodular, Matrix.IsPartiallyUnimodular` This follows from Definitions ?? and ??. \square

Definition 8. `Matrix` Matrix made of 4 blocks (2x2).

1.2 Pivoting

Definition 9. `Matrix` Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. A long tableau pivot in A on (x, y) is the operation that maps A to the matrix A' where

$$\forall i \in X, \forall j \in Y, A'(i, j) = \begin{cases} \frac{A(i, j)}{A(x, y)}, & \text{if } i = x, \\ A(i, j) - \frac{A(i, y) \cdot A(x, j)}{A(x, y)}, & \text{if } i \neq x. \end{cases}$$

Lemma 10. `Matrix.IsTotallyUnimodular, Matrix.longTableauPivot` Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the long tableau pivot in A on (x, y) yields a TU matrix.

Proof. See implementation in Lean. \square

Definition 11. `Matrix.longTableauPivot` Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Perform the following sequence of operations.

1. Adjoin the identity matrix $1 \in R^{X \times X}$ to A , resulting in the matrix $B = \begin{bmatrix} 1 & A \end{bmatrix} \in R^{X \times (X \oplus Y)}$.
2. Perform a long tableau pivot in B on (x, y) , and let C denote the result.
3. Swap columns x and y in C , and let D be the resulting matrix.
4. Finally, remove columns indexed by X from D , and let A' be the resulting matrix.

A short tableau pivot in A on (x, y) is the operation that maps A to the matrix A' defined above.

Lemma 12. `Matrix.shortTableauPivot` Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then the short tableau pivot in A on (x, y) maps A to A' with

$$\forall i \in X, \forall j \in Y, A'(i, j) = \begin{cases} \frac{1}{A(x, y)}, & \text{if } i = x \text{ and } j = y, \\ \frac{A(x, j)}{A(x, y)}, & \text{if } i = x \text{ and } j \neq y, \\ -\frac{A(i, j)}{A(x, y)}, & \text{if } i \neq x \text{ and } j = y, \\ A(i, j) - \frac{A(i, y) \cdot A(x, j)}{A(x, y)}, & \text{if } i \neq x \text{ and } j \neq y. \end{cases}$$

Proof. Follows by direct calculation. \square

Lemma 13. `Matrix.shortTableauPivot` Let $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{Q}^{\{X_1 \cup X_2\} \times \{Y_1 \times Y_2\}}$.

Let $B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$ be the result of performing a short tableau pivot on $(x, y) \in X_1 \times Y_1$ in B . Then $B'_{12} = 0$, $B'_{22} = B_{22}$, and $\begin{bmatrix} B'_{11} \\ B'_{21} \end{bmatrix}$ is the matrix resulting from performing a short tableau pivot on (x, y) in $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$.

Proof. This follows by a direct calculation. Indeed, because of the 0 block in B , B_{12} and B_{22} remain unchanged, and since $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is a submatrix of B containing the pivot element, performing a short tableau pivot in it is equivalent to performing a short tableau pivot in B and then taking the corresponding submatrix. \square

Lemma 14. `Matrix.shortTableauPivot` Let $k \in \mathbb{N}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of performing a short tableau pivot in A on (x, y) with $x, y \in \{1, \dots, k\}$ such that $A(x, y) \neq 0$. Then A' contains a submatrix A'' of size $(k-1) \times (k-1)$ with $|\det A''| = |\det A|/|A(x, y)|$.

Proof. Let $X = \{1, \dots, k\} \setminus \{x\}$ and $Y = \{1, \dots, k\} \setminus \{y\}$, and let $A'' = A'(X, Y)$. Since A'' does not contain the pivot row or the pivot column, $\forall (i, j) \in X \times Y$ we have $A''(i, j) = A(i, j) - \frac{A(i, y) \cdot A(x, j)}{A(x, y)}$. For $\forall j \in Y$, let B_j be the matrix obtained from A by removing row x and column j , and let B'_j be the matrix obtained from A'' by replacing column j with $A(X, y)$ (i.e., the pivot column without the pivot element). The cofactor expansion along row x in A yields

$$\det A = \sum_{j=1}^k (-1)^{y+j} \cdot A(x, j) \cdot \det B_j.$$

By reordering columns of every B_j to match their order in B'_j , we get

$$\det A = (-1)^{x+y} \cdot \left(A(x, y) \cdot \det A' - \sum_{j \in Y} A(x, j) \cdot \det B'_j \right).$$

By linearity of the determinant applied to $\det A''$, we have

$$\det A'' = \det A' - \sum_{j \in Y} \frac{A(x, j)}{A(x, y)} \cdot \det B_j''$$

Therefore, $|\det A''| = |\det A|/|A(x, y)|$. \square

Lemma 15. `Matrix.IsTotallyUnimodular, Matrix.shortTableauPivot` Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the short tableau pivot in A on (x, y) yields a TU matrix.

Proof. `Matrix.IsTotallyUnimodular.longTableauPivot` See implementation in Lean. \square

1.3 Vector Matroids

Definition 16. (Todo: Add definition of matroids)

Definition 17. `Matrix, Matroid` Let R be a division ring, let X and Y be sets, and let $A \in R^{X \times Y}$ be a matrix. The vector matroid of A is the matroid $M = (Y, \mathcal{I})$ where a set $I \subset Y$ is independent in M if and only if the columns of A indexed by I are linearly independent.

Definition 18. `VectorMatroid` Let R be a division ring, let X and Y be disjoint sets, and let $S \in R^{X \times Y}$ be a matrix. Let $A = \begin{bmatrix} 1 & S \end{bmatrix} \in R^{X \times (X \cup Y)}$ be the matrix obtained from S by adjoining the identity matrix as columns, and let M be the vector matroid of A . Then S is called the standard representation of M .

Lemma 19. `StandardRepr` Let $S \in R^{X \times Y}$ be a standard representation of a vector matroid M . Then X is a base in M .

Proof. See implementation in Lean. \square

Lemma 20. `VectorMatroid` Adding extra zero rows to a full representation matrix of a vector matroid does not change the matroid.

Proof. See implementation in Lean. \square

Lemma 21. `Matrix.IsTotallyUnimodular, VectorMatroid, StandardRepr` Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix, let M be the vector matroid of A , and let B be a base of M . Then there exists a matrix $S \in \mathbb{Q}^{B \times (Y \setminus B)}$ such that S is TU and S is a standard representation of M .

Proof. `Matrix.IsTotallyUnimodular.longTableauPivot, Matrix.fromRows_zero, eindex_t o Matroid` See implementation in Lean.

Definition 22. `Matrix` Let R be a magma containing zero. The support of matrix $A \in R^{X \times Y}$ is $A^\# \in \{0, 1\}^{X \times Y}$ given by

$$\forall i \in X, \forall j \in Y, A^\#(i, j) = \begin{cases} 0, & \text{if } A(i, j) = 0, \\ 1, & \text{if } A(i, j) \neq 0. \end{cases}$$

Lemma 23. `Matrix.support` Transpose of a support matrix is equal to a support of the transposed matrix.

Proof. Definitional equality. □

Lemma 24. `Matrix.support` Submatrix of a support matrix is equal to a support matrix of the submatrix.

Proof. Definitional equality. □

Lemma 25. `Matrix.support` If A is a matrix over \mathbb{Z}_2 , then $A^\# = A$.

Proof. Check elementwise equality. □

Lemma 26. `Matrix.support, StandardRepr` If two standard representation matrices of the same matroid have the same base, then they have the same support.

Proof. See implementation in Lean. □

Lemma 27. `Matrix.det` A square matrix is invertible iff its determinant is invertible.

Proof. This result is proved in Mathlib. □

Lemma 28. `Matrix.IsTotallyUnimodular, Matrix.support` Let A be a rational TU matrix with finite number of rows and finite number of columns. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. `Matrix.support.submatrix, Matrix.isUnit iff isUnit det` See implementation in Lean. □

Lemma 29. `Matrix.IsTotallyUnimodular, Matrix.support` Let A be a rational TU matrix with finite number of rows. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. `Matrix.IsTotallyUnimodular.linearIndependent iff support linearIndependent of finite of finite` See implementation in Lean. □

Lemma 30. `Matrix.IsTotallyUnimodular, Matrix.support` Let A be a rational TU matrix. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. `Matrix.IsTotallyUnimodular.linearIndependent iff support linearIndependent of finite` See implementation in Lean. □

Lemma 31. `Matrix.IsTotallyUnimodular, Matrix.support, StandardRepr` Let A be a TU matrix.

1. If a matroid is represented by A , then it is also represented by $A^\#$.
2. If a matroid is represented by $A^\#$, then it is also represented by A .

Proof. `Matrix.support.transpose, Matrix.support.submatrix, Matrix.IsTotallyUnimodular.linearIndependent`

1.4 Regular Matroids

Definition 32. `Matroid, VectorMatroid, Matrix.IsTotallyUnimodular` A matroid M is regular if there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A .

Definition 33. `Matrix.IsTotallyUnimodular` We say that $A' \in \mathbb{Q}^{X \times Y}$ is a TU signing of $A \in \mathbb{Z}_2^{X \times Y}$ if A' is TU and

$$\forall i \in X, \forall j \in Y, |A'(i, j)| = A(i, j).$$

Lemma 34. `StandardRepr, Matroid.IsRegular, Matrix.IsTuSigningOf` Let $B \in \mathbb{Z}_2^{X \times Y}$ be a standard representation matrix of a matroid M . Then M is regular if and only if B has a TU signing.

Proof. `Matroid.IsRegular, Matrix.IsTuSigningOf, StandardRepr.toMatroid, isBaseX, VectorMatroid.exists_tan` $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A . By Lemma ??, X (the row set of B) is a base of M . By Lemma ??, A can be converted into a standard representation matrix $B' \in \mathbb{Q}^{X \times Y}$ of M such that B' is also TU. Since B' and B are both standard representations of M , by Lemma ?? the support matrices $(B')^\#$ and $B^\#$ are the same. Lemma ?? gives $B^\# = B$. Thus, B' is TU and $(B')^\# = B$, so B' is a TU signing of B .

Suppose that B has a TU signing $B' \in \mathbb{Q}^{X \times Y}$. Then $A = [1 \mid B']$ is TU, as it is obtained from B' by adjoining the identity matrix. Moreover, by Lemma ??, A represents the same matroid as $A^\# = [1 \mid B]$, which is M . Thus, A is a TU matrix representing M , so M is regular. \square

Chapter 2

Regularity of 1-Sum

Definition 35. `StandardRepr,Matrix.fromBlocks` Let R be a magma containing zero (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ be matrices where X_ℓ, Y_ℓ, X_r, Y_r are pairwise disjoint sets. The 1-sum $B = B_\ell \oplus_1 B_r$ of B_ℓ and B_r is

$$B = \begin{bmatrix} B_\ell & 0 \\ 0 & B_r \end{bmatrix} \in R^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}.$$

Definition 36. `Matroid,StandardRepr,standardRepr1sumComposition` A matroid M is a 1-sum of matroids M_ℓ and M_r if there exist standard \mathbb{Z}_2 representation matrices B_ℓ , B_r , and B (for M_ℓ , M_r , and M , respectively) of the form given in Definition ??.

Lemma 37. `Matrix.det` Let A be a square matrix of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Then $\det A = \det A_{11} \cdot \det A_{22}$.

Proof. This result is proved in Mathlib. □

Lemma 38. `standardRepr1sumComposition,Matrix.IsTotallyUnimodular` Let B_ℓ and B_r from Definition ?? be TU matrices (over \mathbb{Q}). Then $B = B_\ell \oplus_1 B_r$ is TU.

Proof. `standardRepr1sumComposition,Matrix.IsTotallyUnimodular,Matrix.det_fromBlocks` *zeroWe provethat* $\{0, \pm 1\}$.

Let T_ℓ and T_r denote the submatrices in the intersection of T with B_ℓ and B_r , respectively. Then T has the form

$$T = \begin{bmatrix} T_\ell & 0 \\ 0 & T_r \end{bmatrix}.$$

First, suppose that T_ℓ and T_r are square. Then $\det T = \det T_\ell \cdot \det T_r$ by Lemma ??. Moreover, $\det T_\ell, \det T_r \in \{0, \pm 1\}$, since T_ℓ and T_r are square submatrices of TU matrices B_ℓ and B_r , respectively. Thus, $\det T \in \{0, \pm 1\}$, as desired.

Without loss of generality we may assume that T_ℓ has fewer rows than columns. Otherwise we can transpose all matrices and use the same proof, since TUness and determinants are preserved under transposition. Thus, T can be represented in the form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where T_{11} contains T_ℓ and some zero rows, T_{22} is a submatrix of T_r , and T_{12} contains the rest of the rows of T_r (not contained in T_{22}) and some zero rows. By Lemma ??, we have $\det T = \det T_{11} \cdot \det T_{22}$. Since T_{11} contains at least one zero row, $\det T_{11} = 0$. Thus, $\det T = 0 \in \{0, \pm 1\}$, as desired. \square

Theorem 39. *Matroid.Is1sumOf,Matroid.IsRegular* Let M be a 1-sum of regular matroids M_ℓ and M_r . Then M is also regular.

Proof. StandardRepr,Matroid.Is1sumOf,Matroid.IsRegular,StandardRepr.toMatroid_isRegular_iff_hasTuSigni
 B_r , and B be standard \mathbb{Z}_2 representation matrices from Definition ??. Since M_ℓ and M_r are regular, by Lemma ??, B_ℓ and B_r have TU signings B'_ℓ and B'_r , respectively. Then $B' = B'_\ell \oplus_1 B'_r$ is a TU signing of B . Indeed, B' is TU by Lemma ??, and a direct calculation shows that B' is a signing of B . Thus, M is regular by Lemma ??. \square

Chapter 3

Regularity of 2-Sum

Definition 40. `StandardRepr,Matrix.fromBlocks` Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{(X_\ell \cup \{x\}) \times Y_\ell}$ and $B_r \in R^{X_r \times (Y_r \cup \{y\})}$ be matrices of the form

$$B_\ell = \begin{bmatrix} A_\ell \\ r \end{bmatrix}, \quad B_r = \begin{bmatrix} c & A_r \end{bmatrix}.$$

The 2-sum $B = B_\ell \oplus_{2,x,y} B_r$ of B_ℓ and B_r is defined as

$$B = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix} \quad \text{where} \quad D = c \otimes r.$$

Here $A_\ell \in R^{X_\ell \times Y_\ell}$, $A_r \in R^{X_r \times Y_r}$, $r \in R^{Y_\ell}$, $c \in R^{X_r}$, $D \in R^{X_r \times Y_\ell}$, and the indexing is consistent everywhere.

Definition 41. `Matroid,StandardRepr,matrix2sumComposition` A matroid M is a 2-sum of matroids M_ℓ and M_r if there exist standard \mathbb{Z}_2 representation matrices B_ℓ , B_r , and B (for M_ℓ , M_r , and M , respectively) of the form given in Definition ??.

Lemma 42. `matrix2sumComposition,Matrix.IsTotallyUnimodular` Let B_ℓ and B_r from Definition ?? be TU matrices (over \mathbb{Q}). Then $C = \begin{bmatrix} D & A_r \end{bmatrix}$ is TU.

Proof. `matrix2sumComposition,Matrix.IsTotallyUnimodular,Matrix.IsTotallyUnimodular.mul_cols` Since B_ℓ is TU, all its entries are in $\{0, \pm 1\}$. In particular, r is a $\{0, \pm 1\}$ vector. Therefore, every column of D is a copy of y , $-y$, or the zero column. Thus, C can be obtained from B_r by adjoining zero columns, duplicating the y column, and multiplying some columns by -1 . Since all these operations preserve TUess and since B_r is TU, C is also TU. \square

Lemma 43. `matrix2sumComposition,Matrix.shortTableauPivot` Let B_ℓ and B_r be matrices from Definition ??. Let B'_ℓ and B' be the matrices obtained by performing a short tableau pivot on $(x_\ell, y_\ell) \in X_\ell \times Y_\ell$ in B_ℓ and B , respectively. Then $B' = B'_\ell \oplus_{2,x,y} B_r$.

Proof. `matrix2sumComposition, Matrix.shortTableauPivot, Matrix.shortTableauPivotzero, Matrix.shortTableauPivotzero`

$\begin{bmatrix} A'_\ell \\ r' \end{bmatrix}$, $B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$ where the blocks have the same dimensions as in B_ℓ and B_r , respectively. By Lemma ??, $B'_{11} = A'_\ell$, $B'_{12} = 0$, and $B'_{22} = A_r$. Equality $B'_{21} = c \otimes r'$ can be verified via a direct calculation. Thus, $B' = B'_\ell \oplus_{2,x,y} B_r$. \square

Lemma 44. `matrix2sumComposition, Matrix.IsTotallyUnimodular` Let B_ℓ and B_r from Definition ?? be TU matrices (over \mathbb{Q}). Then $B_\ell \oplus_{2,x,y} B_r$ is TU.

Proof. `Matrix.isTotallyUnimodular, if for all is PartiallyUnimodular, matrix2sumComposition, Matrix.IsTotallyUnimodular` B_r is k -PU for every $k \in \mathbb{N}$. We prove this claim by induction on k . The base case with $k = 1$ holds, since all entries of $B_\ell \oplus_{2,x,y} B_r$ are in $\{0, \pm 1\}$ by construction.

Suppose that for some $k \in \mathbb{N}$ we know that for any TU matrices B'_ℓ and B'_r (from Definition ??) their 2-sum $B'_\ell \oplus_{2,x,y} B'_r$ is k -PU. Now, given TU matrices B_ℓ and B_r (from Definition ??), our goal is to show that $B = B_\ell \oplus_{2,x,y} B_r$ is $(k+1)$ -PU, i.e., that every $(k+1) \times (k+1)$ submatrix T of B has $\det T \in \{0, \pm 1\}$.

First, suppose that T has no rows in X_ℓ . Then T is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by Lemma ??, so $\det T \in \{0, \pm 1\}$. Thus, we may assume that T contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $T(x_\ell, y_\ell) \neq 0$. Indeed, if $T(x_\ell, y) = 0$ for all y , then $\det T = 0$ and we are done, and $T(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since B is 1-PU, all entries of T are in $\{0, \pm 1\}$, and hence $T(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma ??, performing a short tableau pivot in T on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix T'' such that $|\det T| = |\det T''|$. Since T is a submatrix of B , matrix T'' is a submatrix of the matrix B' resulting from performing a short tableau pivot in B on the same entry (x_ℓ, y_ℓ) . By Lemma ??, we have $B' = B'_\ell \oplus_{2,x,y} B_r$ where B'_ℓ is the result of performing a short tableau pivot in B_ℓ on (x_ℓ, y_ℓ) . Since B_ℓ is TU, by Lemma ??, B'_ℓ is also TU. Thus, by the inductive hypothesis applied to T'' and $B'_\ell \oplus_{2,x,y} B_r$, we have $\det T'' \in \{0, \pm 1\}$. Since $|\det T| = |\det T''|$, we conclude that $\det T \in \{0, \pm 1\}$. \square

Theorem 45. `Matroid.IsRegular, Matroid.Is2sumOf` Let M be a 2-sum of regular matroids M_ℓ and M_r . Then M is also regular.

Proof. `StandardRepr, Matroid.Is2sumOf, StandardRepr.toMatroid, isRegular, if has TuSigning, matrix2sumComposition` B_r , and B be standard \mathbb{Z}_2 representation matrices from Definition ??. Since M_ℓ and M_r are regular, by Lemma ??, B_ℓ and B_r have TU signings B'_ℓ and B'_r , respectively. Then $B' = B'_\ell \oplus_{2,x,y} B'_r$ is a TU signing of B . Indeed, B' is TU by Lemma ??, and a direct calculation verifies that B' is a signing of B . Thus, M is regular by Lemma ??. \square

Chapter 4

Regularity of 3-Sum

4.1 Definition

Definition 46. `StandardRepr,Matrix.fromBlocks` Let $B_\ell \in \mathbb{Z}_2^{(X_\ell \cup \{x_0, x_1\}) \times (Y_\ell \cup \{y_2\})}$, $B_r \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0, y_1\})}$ be matrices of the form

$$B_\ell = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_\ell & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_\ell & D_0 & 1 \\ & & 1 \\ \hline \end{array} \quad \text{and} \quad B_r = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline D_0 & 1 & & \\ & 1 & & A_r \\ \hline D_r & & & \\ \hline \end{array} \quad \text{where } D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The 3-sum $B = B_\ell \oplus_3 B_r \in \mathbb{Z}_2^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}$ of B_ℓ and B_r is defined as

$$B = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_\ell & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_\ell & D_0 & 1 \\ & & 1 \\ \hline D_{\ell r} & D_r & A_r \\ \hline \end{array} \quad \text{where } D_{\ell r} = D_r \cdot (D_0)^{-1} \cdot D_\ell.$$

Here $x_2 \in X_\ell$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_\ell$, $y_2 \in Y_r$, $A_\ell \in \mathbb{Z}_2^{X_\ell \times Y_\ell}$, $A_r \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_\ell \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_\ell \setminus \{y_0, y_1\})}$, $D_r \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_{\ell r} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times (Y_\ell \setminus \{y_0, y_1\})}$, $D_0 \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$. The indexing is consistent everywhere.

Note that D_0 is non-singular by construction, so $D_{\ell r}$ and B are well-defined. Moreover, a non-singular $\mathbb{Z}_2^{2 \times 2}$ matrix is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ up to re-indexing. Thus, Definition ?? can be equivalently restated with D_0 required to be non-singular and B_ℓ , B_r , and B re-indexed appropriately.

Definition 47. Matroid,StandardRepr,MatrixSum3.matrix A matroid M is a 3-sum of matroids M_ℓ and M_r if there exist standard \mathbb{Z}_2 representation matrices B_ℓ , B_r , and B (for M_ℓ , M_r , and M , respectively) of the form given in Definition ??.

4.2 Canonical Signing

Definition 48. Matrix.fromBlocks We call $D'_0 \in \mathbb{Q}^{\{x_0, x_1\} \times \{y_0, y_1\}}$ the canonical signing of $D_0 \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$ if

$$D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D'_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{or} \quad D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D'_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we call $S' \in \mathbb{Q}^{\{x_0, x_1, x_2\} \times \{y_0, y_1, y_2\}}$ the canonical signing of $S \in \mathbb{Z}_2^{\{x_0, x_1, x_2\} \times \{y_0, y_1, y_2\}}$ if

$$S = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0 & 1 & \\ \hline & 1 & \\ \hline \end{array} \quad \text{and} \quad S' = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D'_0 & 1 & \\ \hline & 1 & \\ \hline \end{array}$$

To simplify notation, going forward we use D_0 , D'_0 , S , and S' to refer to the matrices of the form above. BTW, the canonical signing S' of S (from Definition ??) is TU.

Lemma 49. Matrix.IsTuSigningOf,matrix3x3signed Let Q be a TU signing of S (from Definition ??). Let $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q' be defined as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in \{x_0, x_1, x_2\}, \quad \forall j \in \{y_0, y_1, y_2\}.$$

Then $Q' = S'$ (from Definition ??).

Proof. `Matrix.IsTuSigningOf, Matrix.IsTotallyUnimodular.mul_rows, Matrix.IsTotallyUnimodular.mul_cols, Matrix.IsTotallyUnimodular.mul_rows` factors, Q' is also a TU signing of S . By construction, we have

$$\begin{aligned}
Q'(x_2, y_0) &= Q(x_2, y_0) \cdot 1 \cdot Q(x_2, y_0) = 1, \\
Q'(x_2, y_1) &= Q(x_2, y_1) \cdot 1 \cdot Q(x_2, y_1) = 1, \\
Q'(x_2, y_2) &= 0, \\
Q'(x_0, y_0) &= Q(x_0, y_0) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_0) = 1, \\
Q'(x_0, y_1) &= Q(x_0, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_1), \\
Q'(x_0, y_2) &= Q(x_0, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1, \\
Q'(x_1, y_0) &= 0, \\
Q'(x_1, y_1) &= Q(x_1, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_1)), \\
Q'(x_1, y_2) &= Q(x_1, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1.
\end{aligned}$$

Thus, it remains to show that $Q'(x_0, y_1) = S'(x_0, y_1)$ and $Q'(x_1, y_1) = S'(x_1, y_1)$.

Consider the entry $Q'(x_0, y_1)$. If $D_0(x_0, y_1) = 0$, then $Q'(x_0, y_1) = 0 = S'(x_0, y_1)$. Otherwise, we have $D_0(x_0, y_1) = 1$, and so $Q'(x_0, y_1) \in \{\pm 1\}$, as Q' is a signing of S . If $Q'(x_0, y_1) = -1$, then

$$\det Q'(\{x_0, x_2\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q' . Thus, $Q'(x_0, y_1) = 1 = S'(x_0, y_1)$.

Consider the entry $Q'(x_1, y_1)$. Since Q' is a signing of S , we have $Q'(x_1, y_1) \in \{\pm 1\}$. Consider two cases.

1. Suppose that $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = 1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q' . Thus, $Q'(x_1, y_1) = -1 = S'(x_1, y_1)$.
2. Suppose that $D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = -1$, then $\det Q(\{x_0, x_1\}, \{y_1, y_2\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q' . Thus, $Q'(x_1, y_1) = 1 = S'(x_1, y_1)$.

□

Definition 50. `Matrix.IsTotallyUnimodular` Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU matrix. Define $u \in \{0, \pm 1\}^X$,

$v \in \{0, \pm 1\}^Y$, and Q' as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in X, \forall j \in Y.$$

We call Q' the canonical re-signing of Q .

Lemma 51. `Matrix.IsTuSigningOf,matrix3x3signed,Matrix.toCanonicalSigning`
Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q_0 \in \mathbb{Z}_2^{X \times Y}$ such that $Q_0(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S$ (from Definition ??). Then the canonical re-signing Q' of Q (from Definition ??) is a TU signing of Q_0 and $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ (from Definition ??).

Proof. `Matrix.IsTuSigningOf,Matrix.IsTotallyUnimodular.mul_rows,Matrix.IsTotallyUnimodular.mul_cols,M`
and Q' is obtained from Q by multiplying some rows and columns by ± 1 factors, Q' is also a TU signing of Q_0 . Equality $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ follows from Lemma ??. \square

Definition 52. `MatrixSum3.matrix,Matrix.IsTuSigningOf,Matrix.toCanonicalSigning`
Suppose that B_ℓ and B_r from Definition ?? have TU signings B'_ℓ and B'_r , respectively. Let B''_ℓ and B''_r be the canonical re-signings (from Definition ??) of B'_ℓ and B'_r , respectively. Let $A''_\ell, A''_r, D''_\ell, D''_r$, and D''_0 be blocks of B''_ℓ and B''_r analogous to blocks A_ℓ, A_r, D_ℓ, D_r , and D_0 of B_ℓ and B_r . The canonical signing B'' of B is defined as

$$B'' = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & A''_\ell & & 0 \\ \hline & 1 & 1 & 0 \\ \hline D''_\ell & D''_0 & 1 & \\ & & 1 & A''_r \\ \hline D''_{\ell r} & D''_r & & \\ \hline \end{array} \quad \text{where } D''_{\ell r} = D''_r \cdot (D''_0)^{-1} \cdot D''_\ell.$$

Note that D''_0 is non-singular by construction, so $D''_{\ell r}$ and hence B'' are well-defined.

4.3 Properties of Canonical Signing

Lemma 53. `MatrixSum3.toCanonicalSigning` B'' from Definition ?? is a signing of B .

Proof. `Matrix.HasTuCanonicalSigning.toCanonicalSigning,Matrix.IsTuSigningOf` By Lemma ??, B''_ℓ and B''_r are TU signings of B_ℓ and B_r , respectively. As a result, blocks A''_ℓ , A''_r , D''_ℓ , D''_r , and D''_0 in B'' are signings of the corresponding blocks in B . Thus, it remains to show that $D''_{\ell r}$ is a signing of $D_{\ell r}$. This can be verified via a direct calculation. (Todo: Need details?) \square

Lemma 54. `MatrixSum3.matrix,Matrix.IsTuSigningOf,Matrix.toCanonicalSigning` Suppose that B_r from Definition ?? has a TU signing B'_r . Let B''_r be the canonical re-signing (from Definition ??) of B'_r . Let $c''_0 = B''_r(X_r, y_0)$, $c''_1 = B''_r(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Then the following statements hold.

1. For every $i \in X_r$, $[c''_0(i) \ c''_1(i)] \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{[1 \ -1], [-1 \ 1]\}$.
2. For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.
3. $[c''_0 \ c''_2 \ A''_r]$ is TU.
4. $[c''_1 \ c''_2 \ A''_r]$ is TU.
5. $[c''_0 \ c''_1 \ c''_2 \ A''_r]$ is TU.

Proof. `Matrix.HasTuCanonicalSigning.toCanonicalSigning,Matrix.IsTotallyUnimodular,Matrix.shortTableauP` is TU, which holds by Lemma ??.

1. Since B''_r is TU, all its entries are in $\{0, \pm 1\}$, and in particular $[c''_0(i) \ c''_1(i)] \in \{0, \pm 1\}^{\{y_0, y_1\}}$. If $[c''_0(i) \ c''_1(i)] = [1 \ -1]$, then

$$\det B''_r(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\},$$

which contradicts TUness of B''_r . Similarly, if $[c''_0(i) \ c''_1(i)] = [-1 \ 1]$, then

$$\det B''_r(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of B''_r . Thus, the desired statement holds.

2. Follows from item ?? and a direct calculation.
3. Performing a short tableau pivot in B''_r on (x_2, y_0) yields:

$$B''_r = \begin{bmatrix} \boxed{1} & 1 & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c''_1 - c_0 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $[c''_0 \ c''_2 \ A''_r]$ by removing row x_2 and multiplying columns y_0 and y_1 by -1 . Since B''_r is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, we conclude that $[c''_0 \ c''_2 \ A''_r]$ is TU.

4. Similar to item ??, performing a short tableau pivot in B_r'' on (x_2, y_1) yields:

$$B_r'' = \begin{bmatrix} 1 & \boxed{1} & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ c_0'' - c_1 & -c_1 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $[c_1'' \ c_2'' \ A_r'']$ by removing row x_2 , multiplying column y_1 by -1 , and swapping the order of columns y_0 and y_1 . Since B_r'' is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, and re-ordering columns, we conclude that $[c_1'' \ c_2'' \ A_r'']$ is TU.

5. Let V be a square submatrix of $[c_0'' \ c_1'' \ c_2'' \ A_r'']$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

Suppose that column c_2'' is not in V . Then V is a submatrix of B_r'' , which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V .

Suppose that columns c_0'' and c_1'' are both in V . Then V contains columns c_0'' , c_1'' , and $c_2'' = c_0'' - c_1''$, which are linearly dependent. Thus, $\det V = 0$. Going forward we assume that at least one of the columns c_0'' and c_1'' is not in V .

Suppose that column c_1'' is not in V . Then V is a submatrix of $[c_0'' \ c_2'' \ A_r'']$, which is TU by item ??. Thus, $\det V \in \{0, \pm 1\}$. Similarly, if column c_0'' is not in V , then V is a submatrix of $[c_1'' \ c_2'' \ A_r'']$, which is TU by item ??. Thus, $\det V \in \{0, \pm 1\}$.

□

Lemma 55. `MatrixSum3.matrix, Matrix.IsTuSigningOf, Matrix.toCanonicalSigning`

Suppose that B_ℓ from Definition ?? has a TU signing B'_ℓ . Let B''_ℓ be the canonical re-signing (from Definition ??) of B'_ℓ . Let $d_0'' = B''_\ell(x_0, Y_\ell)$, $d_1'' = B''_\ell(x_1, Y_\ell)$, and $d_2'' = d_0'' - d_1''$. Then the following statements hold.

1. For every $j \in Y_\ell$, $\begin{bmatrix} d_0''(j) \\ d_1''(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
2. For every $j \in Y_\ell$, $d_2''(j) \in \{0, \pm 1\}$.
3. $\begin{bmatrix} A''_\ell \\ d_0'' \\ d_2'' \end{bmatrix}$ is TU.
4. $\begin{bmatrix} A''_\ell \\ d_1'' \\ d_2'' \end{bmatrix}$ is TU.
5. $\begin{bmatrix} A''_\ell \\ d_0'' \\ d_1'' \\ d_2'' \end{bmatrix}$ is TU.

Proof. MatrixSum3.HasTuBr.cccAr_isTotallyUnimodularApplyLemma ??toB_ℓ[⊤], or repeat the same arguments up to transposition. \square

Lemma 56. MatrixSum3.toCanonicalSigning,Matrix.IsTotallyUnimodular Let B'' be from Definition ?. Let $c''_0 = B''(X_r, y_0)$, $c''_1 = B''(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Similarly, let $d''_0 = B''(x_0, Y_\ell)$, $d''_1 = B''(x_1, Y_\ell)$, and $d''_2 = d''_0 - d''_1$. Then the following statements hold.

1. For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.
2. If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_1 \otimes d''_1$. If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_0 \otimes d''_1 + c''_1 \otimes d''_1$.
3. For every $j \in Y_\ell$, $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm c''_2\}$.
4. For every $i \in X_r$, $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.
5. $\begin{bmatrix} D'' & A''_r \end{bmatrix}$ is TU.
6. $\begin{bmatrix} A''_\ell \\ D'' \end{bmatrix}$ is TU.

Proof. MatrixSum3.HasTuBr.cccAr_isTotallyUnimodular, lem : three_sum_signing_{B1p}rops, Matrix.IsTotallyU

Holds by Lemma ??.??.

Note that

$$\begin{bmatrix} D''_\ell \\ D''_{\ell r} \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_\ell, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_0, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix}, \quad \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

Thus,

$$D'' = \begin{bmatrix} D''_\ell & D''_0 \\ D''_{\ell r} & D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

Considering the two cases for D''_0 and performing the calculations yields the desired results.

Let $j \in Y_\ell$. By Lemma ??.??, $\begin{bmatrix} d''_0(j) \\ d''_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Consider two cases.

1. If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item ?? we have $D''(X_r, j) = d''_0(j) \cdot c''_0 + (-d''_1(j)) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm(c''_0 - c''_1)\}$.
2. If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item ?? we have $D''(X_r, j) = (d''_0(j) - d''_1(j)) \cdot c''_0 + d''_1(j) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm(c''_0 - c''_1)\}$.

Let $i \in X_r$. By Lemma ??, $[c_0''(i) \ c_1''(i)] \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{[1 \ -1], [-1 \ 1]\}$. Consider two cases.

1. If $D_0'' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item ?? we have $D''(i, Y_\ell) = c_0''(i) \cdot d_0'' + (-c_1''(i)) \cdot d_1''$. By considering all possible cases for $c_0''(i)$ and $c_1''(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d_0'', \pm d_1'', \pm d_2''\}$.
2. If $D_0'' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item ?? we have $D''(i, Y_\ell) = c_0''(i) \cdot d_0'' + (c_1''(i) - c_0''(i)) \cdot d_1''$. By considering all possible cases for $c_0''(i)$ and $c_1''(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d_0'', \pm d_1'', \pm d_2''\}$.

By Lemma ??, $[c_0'' \ c_1'' \ c_2'' \ A_r'']$ is TU. Since TUness is preserved under adjoining zero columns, copies of existing columns, and multiplying columns by ± 1 factors, $[0 \ \pm c_0'' \ \pm c_1'' \ \pm c_2'' \ A_r'']$ is also TU. By item ??, $[D'' \ A_r'']$ is a submatrix of the latter matrix, hence it is also TU.

By Lemma ??, $\begin{bmatrix} A_\ell'' \\ d_0'' \\ d_1'' \\ d_2'' \end{bmatrix}$ is TU. Since TUness is preserved under adjoining zero

rows, copies of existing rows, and multiplying rows by ± 1 factors, $\begin{bmatrix} A_\ell'' \\ 0 \\ \pm d_0'' \\ \pm d_1'' \\ \pm d_2'' \end{bmatrix}$ is also

TU. By item ??, $\begin{bmatrix} A_\ell'' \\ D'' \end{bmatrix}$ is a submatrix of the latter matrix, hence it is also TU.

□

4.4 Proof of Regularity

Definition 57. Matrix.IsTotallyUnimodular Let X_ℓ, Y_ℓ, X_r, Y_r be sets and let $c_0, c_1 \in \mathbb{Q}^{X_r}$ be column vectors such that for every $i \in X_r$ we have $c_0(i), c_1(i), c_0(i) - c_1(i) \in \{0, \pm 1\}$. Define $\mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ to be the family of matrices of the form $\begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ where $A_\ell \in \mathbb{Q}^{X_\ell \times Y_\ell}$, $A_r \in \mathbb{Q}^{X_r \times Y_r}$, and $D \in \mathbb{Q}^{X_r \times Y_\ell}$ are such that:

1. for every $j \in Y_\ell$, $D(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$,
2. $[c_0 \ c_1 \ c_0 - c_1 \ A_r]$ is TU,
3. $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU.

Lemma 58. MatrixSum3.toCanonicalSigning, MatrixLikeSum3 Let B'' be from Definition ?. Then $B'' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0'', c_1'')$ where $c_0'' = B''(X_r, y_0)$ and $c_1'' = B''(X_r, y_1)$.

Proof. `lem:three_sum_signing_B_props, MatrixLikeSum3Recall` that $c_0'' - c_1'' \in \{0, \pm 1\}^{X_r}$ by Lemma `??`, so $\mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0'', c_1'')$ is well-defined. To see that $B'' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0'', c_1'')$, note that all properties from Definition `??` are satisfied: property `??` holds by Lemma `??`, property `??` holds by Lemma `??`, and property `??` holds by Lemma `??`. \square

Lemma 59. `MatrixLikeSum3, Matrix.shortTableauPivot` Let $C \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ from Definition `??`. Let $x \in X_\ell$ and $y \in Y_\ell$ be such that $A_\ell(x, y) \neq 0$, and let C' be the result of performing a short tableau pivot in C on (x, y) . Then $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$.

Proof. `Matrix.shortTableauPivot_zero, Matrix.IsTotallyUnimodular, Matrix.IsTotallyUnimodular.shortTableauPivot` Let $C' = \begin{bmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{bmatrix}$, and let $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ be the result of performing a short tableau pivot on (x, y) in $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$. Observe the following.

- By Lemma `??`, $C'_{11} = A'_\ell$, $C'_{12} = 0$, $C'_{21} = D'$, and $C'_{22} = A_r$.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property `??` for C , all entries of A_ℓ are in $\{0, \pm 1\}$.
- $A_\ell(x, y) \in \{\pm 1\}$, as $A_\ell(x, y) \in \{0, \pm 1\}$ by the above observation and $A_\ell(x, y) \neq 0$ by the assumption.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property `??` for C , and since pivoting preserves TUness, $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ is also TU.

These observations immediately imply properties `??` and `??` for C' . Indeed, property `??` holds for C' , since $C'_{22} = A_r$ and $\begin{bmatrix} c_0 & c_1 & c_0 - c_1 & A_r \end{bmatrix}$ is TU by property `??` for C . On the other hand, property `??` follows from $C'_{11} = A'_\ell$, $C'_{21} = D'$, and $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ being TU. Thus, it only remains to show that C' satisfies property `??`. Let $j \in Y_r$. Our goal is to prove that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Suppose $j = y$. By the pivot formula, $D'(X_r, y) = -\frac{D(X_r, y)}{A_\ell(x, y)}$. Since $D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property `??` for C and since $A_\ell(x, y) \in \{\pm 1\}$, we get $D'(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Now suppose $j \in Y_\ell \setminus \{y\}$. By the pivot formula, $D'(X_r, j) = D(X_r, j) - \frac{A_\ell(x, j)}{A_\ell(x, y)} \cdot D(X_r, y)$. Here $D(X_r, j), D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property `??` for C , and $A_\ell(x, j) \in \{0, \pm 1\}$ and $A_\ell(x, y) \in \{\pm 1\}$ by the prior observations. Perform an exhaustive case distinction on $D(X_r, j), D(X_r, y), A_\ell(x, j)$, and $A_\ell(x, y)$. In every case, we can show that either $\begin{bmatrix} A_\ell(x, y) & A_\ell(x, j) \\ D(X_r, y) & D(X_r, j) \end{bmatrix}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$, or that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$, as desired. (Todo: need details?) \square

Lemma 60. `MatrixLikeSum3,Matrix.IsTotallyUnimodular` Let $C \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ from Definition ???. Then C is TU.

Proof. `MatrixLikeSum3,Matrix.IsTotallyUnimodular iff for all is PartiallyUnimodular, shortTableauPivotSubmatrixLikePivotByLemma ???, itsufficestoshowthatCisk-PUforeveryk \in \mathbb{N}`. We prove this claim by induction on k . The base case with $k = 1$ holds, since properties ?? and ?? in Definition ??? imply that A_ℓ , A_r , and D are TU, so all their entries of $C = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ are in $\{0, \pm 1\}$, as desired.

Suppose that for some $k \in \mathbb{N}$ we know that every $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ is k -PU. Our goal is to show that C is $(k+1)$ -PU, i.e., that every $(k+1) \times (k+1)$ submatrix S of C has $\det V \in \{0, \pm 1\}$.

First, suppose that V has no rows in X_ℓ . Then V is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by property ?? in Definition ???, so $\det V \in \{0, \pm 1\}$. Thus, we may assume that S contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $V(x_\ell, y_\ell) \neq 0$. Indeed, if $V(x_\ell, y) = 0$ for all y , then $\det V = 0$ and we are done, and $V(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since C is 1-PU, all entries of V are in $\{0, \pm 1\}$, and hence $V(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma ??, performing a short tableau pivot in V on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix S'' such that $|\det V| = |\det V''|$. Since V is a submatrix of C , matrix V'' is a submatrix of the matrix C' resulting from performing a short tableau pivot in C on the same entry (x_ℓ, y_ℓ) . By Lemma ??, we have $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$. Thus, by the inductive hypothesis applied to V'' and C' , we have $\det V'' \in \{0, \pm 1\}$. Since $|\det V| = |\det V''|$, we conclude that $\det V \in \{0, \pm 1\}$. \square

Lemma 61. `MatrixSum3.toCanonicalSigning,Matrix.IsTotallyUnimodular` B'' from Definition ??? is TU.

Proof. `lem:three_sum_like_signing_B, lem : three_sum_like_tuCombines the resultsof Lemmas ???and ???`. \square

Theorem 62. `Matroid.Is3sumOf,Matroid.IsRegular` Let M be a 3-sum of regular matroids M_ℓ and M_r . Then M is also regular.

Proof. `StandardRepr,Matroid.Is3sumOf,StandardRepr.toMatroid,IsRegular iff hasTuSigning, MatrixSum3.tothree_sum_signing_Bvalid, lem : three_sum_signing_Btu` Let B_ℓ , B_r , and B be standard \mathbb{Z}_2 representation matrices from Definition ???. Since M_ℓ and M_r are regular, by Lemma ??, B_ℓ and B_r have TU signings. Then the canonical signing B'' from Definition ?? is a TU signing of B . Indeed, B'' is a signing of B by Lemma ??, and B'' is TU by Lemma ???. Thus, M is regular by Lemma ???. \square

Chapter 5

Conclusion

Definition 63. `Matroid.IsRegular, Matroid.Is1sumOf, Matroid.Is2sumOf, Matroid.Is3sumOf`
Regular matroid is good. Any 1-sum of good matroids is a good matroid. Any 2-sum of good matroids is a good matroid. Any 3-sum of good matroids is a good matroid.

Corollary 64. `Matroid.IsGood, Matroid.IsRegular` Any good matroid is regular. This is the easy direction of the Seymour theorem.

Proof. `Matroid.Is1sumOf.isRegular, Matroid.Is2sumOf.isRegular, Matroid.Is3sumOf.isRegular`
Structural induction. □