Matroids in Lean: Status Update

Ivan Sergeev

ISTA

8 January 2025

Totally Unimodular Matrices

In Mathlib:

- Definition of TU matrices
- TUness is preserved under:
  - transposition
  - taking of submatrices, incl. adjoining parallel rows/columns
  - adjoining zero rows/columns
  - unit rows/columns

In repo:

- Finite block-diagonal matrix with TU blocks is TU

Next up:

- TUness is preserved under pivoting
- Generalize to infinite matrices

Matroid API

Additions:

- Notions: circuit, loop, coloop, separator
- Constructors: circuit matroid, vector matroid
- Classes: representable matroids, graphic and cographic matroids

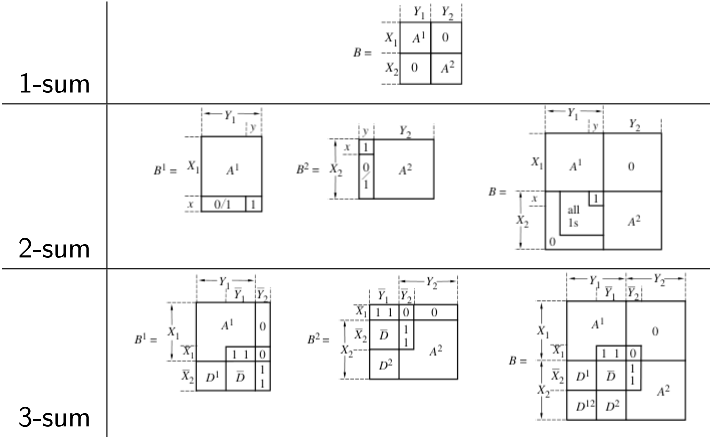
Changes:

- Binary matroid definition now uses vector matroid
- Different, more general approach to 1-, 2-, and 3-sums*

Next up:

- Fill in blanks: sorry's and missing useful lemmas

Old Approach to k-Sums



[Truemper 1998]; binary matroids M_1, M_2, $M_1 \oplus_k M_2$ have standard matrix representation B_1, B_2, B

New Approach: 1-Sum

- [Oxley 2011]: $M_1 \oplus_1 M_2$ for general matroids $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$
- Assumption: $E_1 \cap E_2 = \emptyset$
- Ground set: $E_1 \cup E_2$
- Independent sets: $\{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$
- In MathLib: Matroid.disjointSum

New Approach: 2-Sum

- [Oxley 2011]: $M_1 \oplus_2 M_2$ for general matroids $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$
- Assumptions:
  - $|E_1|, |E_2| \geq 2$
  - $E_1 \cap E_2 = \{p\}$
  - p is not a loop or a coloop in M_1 or M_2
- Ground set: $E_1 \cup E_2 \setminus \{p\}$
- Circuits:

$$\mathcal{C}(M_1 \setminus \{p\}) \cup \mathcal{C}(M_2 \setminus \{p\}) \cup \{C_2 \cup C_2 \setminus \{p\} \mid p \in C_1 \in \mathcal{C}(M_1),\ p \in C_2 \in \mathcal{C}(M_2)\}$$

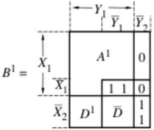
- In repo: Matroid.TwoSum

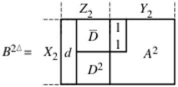
New Approach: 3-Sum

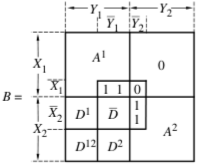
- [Oxley 2011]: $M_1 \oplus_3 M_2$ is for binary matroids $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$
- Assumptions:
  - $|E_1|, |E_2| \geq 7$
  - $E_1 \cap E_2 = T$, T is a triangle in M_1 and M_2
  - Neither M_1 nor M_2 has cocircuit contained in T
- Ground set: $E = E_1 \Delta E_2$
- Circuits: $\mathcal{C}(M_1 \setminus T) \cup \mathcal{C}(M_2 \setminus T) \cup \mathcal{C}_\Delta$ where $\mathcal{C}_\Delta =$ minimal sets of form $C_1 \Delta C_2$ where C_i is a circuit of M_i, $C_1 \cap T = C_2 \cap T$, and $C_i \cap T$ has exactly one element
- Note: 1-sum and 2-sum are special cases
- In repo: BinaryMatroid.DeltaSum

Note on New 3-Sum

- 3-sum in [Oxley] corresponds to Δ-sum in [Truemper]
- Regularity results and decomposition theorem hold for both







New Approach: Regularity of k-Sum

- Matroid is regular iff it can be represented over any field
- If M_1 and M_2 are regular, they can be represented over any field
- After wlog conversion, can connect representations of M_1 and M_2 with $M_1 \oplus_k M_2$
- Thus $M_1 \oplus_k M_2$ can be represented over any field, hence is regular

- Below: example for 2-sum (last matrix without column p represents $M_1 \oplus_2 M_2$)

$$
\begin{array}{cc}
E(M_1)-p & p \\
\multicolumn{2}{c}{\left[\begin{array}{c|c}
 & 0 \\
 & 0 \\
A_1 & \vdots \\
 & 0 \\
 & 1
\end{array}\right]}
\end{array}
\qquad
\begin{array}{cc}
p & E(M_2)-p \\
\multicolumn{2}{c}{\left[\begin{array}{c|c}
1 & \\
0 & \\
\vdots & A_2 \\
0 & \\
0 &
\end{array}\right]}
\end{array}
\qquad
\begin{array}{ccc}
E(M_1)-p & p & E(M_2)-p \\
\multicolumn{3}{c}{\left[\begin{array}{c|c|c}
 & 0 & \\
 & 0 & \\
A_1 & \vdots & 0 \\
 & 0 & \\
\cline{2-3}
 & 1 & \\
\cline{1-2}
 & 0 & \\
 & \vdots & \\
0 & \vdots & A_2 \\
 & 0 & \\
 & 0 &
\end{array}\right]}
\end{array}
$$

Next Steps

For old approach:

- TUness is preserved under pivoting
- TUness of explicit matrix representations of 2-sum and 3-sum

For new approach:

- Characterization of regular matroids
- Matrix representations for 2-sum and 3-sum

Nice to have:

- Updated statement of hard direction of Seymour's theorem
  - Prove up to Kuratowsky's theorem?
- TUness properties for infinite matrices
- Circuit matroid construction for infinite case