Regularity of 1-, 2-, and 3-Sums of Matroids

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Chapter 1

Preliminaries

1.1 Total Unimodularity

Definition 1. Matrix is a function that takes a row index and returns a vector, which is a function that takes a column index and returns a value. The former aforementioned identity is definitional, the latter is syntactical. By abuse of notation $\left(R^Y\right)^X \equiv R^{X \times Y}$ we do not curry functions in this text. When a matrix happens to be finite (that is, both X and Y are finite) and its entries are numeric, we like to represent it by a table of numbers.

Definition 2. Let A be a square matrix over a commutative ring. Determinant of A is the sum over all permutations, sign of the permutation times the product of ... . .

complete definition

Definition 3. Let R be a commutative ring. We say that a matrix $A \in R^{X \times Y}$ is totally unimodular, or TU for short, if for every $k \in \mathbb{N}$, every (not necessarily contiguous) $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 4. Let A be a TU matrix. Suppose rows of A are multiplied by $\{0, \pm 1\}$ factors. Then the resulting matrix A' is also TU.

Proof. We prove that A' is TU by Definition 3. To this end, let T' be a square submatrix of A'. Our goal is to show that $\det T' \in \{0, \pm 1\}$. Let T be the submatrix of A that represents T' before pivoting. If some of the rows of T were multiplied by zeros, then T' contains zero rows, and hence $\det T' = 0$. Otherwise, T' was obtained from T by multiplying certain rows by -1. Since T' has finitely many rows, the number of such multiplications is also finite. Since multiplying a row by -1 results in the determinant getting multiplied by -1, we get $\det T' = \pm \det T \in \{0, \pm 1\}$ as desired. □

Lemma 5. Let A be a TU matrix. Suppose columns of A are multiplied by $\{0, \pm 1\}$ factors. Then the resulting matrix A' is also TU.

Proof. Apply Lemma 4 to $A^\top$. □

Definition 6. Given $k \in \mathbb{N}$, we say that a matrix A is k-partially unimodular, or k-PU for short, if every (not necessarily contiguous, not necessarily injective) $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 7. A matrix A is TU if and only if A is k-PU for every $k \in \mathbb{N}$.

Proof. This follows from Definitions 3 and 6. □

Definition 8. Matrix made of 4 blocks (2x2).

1.2 Pivoting

Definition 9. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. A long tableau pivot in A on (x, y) is the operation that maps A to the matrix A' where

$$\forall i \in X,\ \forall j \in Y,\ A'(i, j) = \begin{cases} \frac{A(i,j)}{A(x,y)}, & \text{if } i = x, \\ A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x. \end{cases}$$

Lemma 10. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the long tableau pivot in A on (x, y) yields a TU matrix.

Proof. See implementation in Lean. ☐

Definition 11. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Perform the following sequence of operations.

1. Adjoin the identity matrix $1 \in R^{X \times X}$ to A, resulting in the matrix $B = \begin{bmatrix} 1 & A \end{bmatrix} \in R^{X \times (X \oplus Y)}$.
2. Perform a long tableau pivot in B on (x, y), and let C denote the result.
3. Swap columns x and y in C, and let D be the resulting matrix.
4. Finally, remove columns indexed by X from D, and let A' be the resulting matrix.

A short tableau pivot in A on (x, y) is the operation that maps A to the matrix A' defined above.

Lemma 12. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then the short tableau pivot in A on (x, y) maps A to A' with

$$\forall i \in X,\ \forall j \in Y,\ A'(i, j) = \begin{cases} \frac{1}{A(x,y)}, & \text{if } i = x \text{ and } j = y, \\ \frac{A(x,j)}{A(x,y)}, & \text{if } i = x \text{ and } j \neq y, \\ -\frac{A(i,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j = y, \\ A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j \neq y. \end{cases}$$

Proof. Follows by direct calculation. ☐

Lemma 13. Let $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{Q}^{\{X_1 \cup X_2\} \times \{Y_1 \times Y_2\}}$. Let $B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$ be the result of performing a short tableau pivot on $(x, y) \in X_1 \times Y_1$ in B. Then $B'_{12} = 0$, $B'_{22} = B_{22}$, and $\begin{bmatrix} B'_{11} \\ B'_{21} \end{bmatrix}$ is the matrix resulting from performing a short tableau pivot on (x, y) in $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$.

Proof. This follows by a direct calculation. Indeed, because of the 0 block in B, B_{12} and B_{22} remain unchanged, and since $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is a submatrix of B containing the pivot element, performing a short tableau pivot in it is equivalent to performing a short tableau pivot in B and then taking the corresponding submatrix. ☐

Lemma 14. Let $k \in \mathbb{N}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of performing a short tableau pivot in A on (x, y) with $x, y \in \{1, \ldots, k\}$ such that $A(x, y) \neq 0$. Then A' contains a submatrix A'' of size $(k - 1) \times (k - 1)$ with $|\det A''| = |\det A|/|A(x, y)|$.

Proof. Let $X = \{1, \ldots, k\} \setminus \{x\}$ and $Y = \{1, \ldots, k\} \setminus \{y\}$, and let $A'' = A'(X, Y)$. Since A'' does not contain the pivot row or the pivot column, $\forall (i, j) \in X \times Y$ we have $A''(i, j) = A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}$. For $\forall j \in Y$, let B_j be the matrix obtained from A by removing row x and column j, and let B''_j be the matrix obtained from A'' by replacing column j with $A(X, y)$ (i.e., the pivot column without the pivot element). The cofactor expansion along row x in A yields

$$\det A = \sum_{j=1}^{k} (-1)^{y+j} \cdot A(x, j) \cdot \det B_j.$$

By reordering columns of every B_j to match their order in B''_j, we get

$$\det A = (-1)^{x+y} \cdot \left(A(x, y) \cdot \det A' - \sum_{j \in Y} A(x, j) \cdot \det B''_j \right).$$

By linearity of the determinant applied to $\det A''$, we have

$$\det A'' = \det A' - \sum_{j \in Y} \frac{A(x, j)}{A(x, y)} \cdot \det B''_j$$

Therefore, $|\det A''| = |\det A|/|A(x, y)|$. ☐

Lemma 15. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the short tableau pivot in A on (x, y) yields a TU matrix.

Proof. See implementation in Lean. □

1.3 Vector Matroids

Definition 16.
Add definition of matroids

Definition 17. Let R be a division ring, let X and Y be sets, and let $A \in R^{X \times Y}$ be a matrix. The vector matroid of A is the matroid $M = (Y, \mathcal{I})$ where a set $I \subset Y$ is independent in M if and only if the columns of A indexed by I are linearly independent.

Definition 18. Let R be a division ring, let X and Y be disjoint sets, and let $S \in R^{X \times Y}$ be a matrix. Let $A = \begin{bmatrix} 1 & S \end{bmatrix} \in R^{X \times (X \cup Y)}$ be the matrix obtained from S by adjoining the identity matrix as columns, and let M be the vector matroid of A. Then S is called the standard representation of M.

Lemma 19. Let $S \in R^{X \times Y}$ be a standard representation of a vector matroid M. Then X is a base in M.

Proof. See implementation in Lean. □

Lemma 20. Adding extra zero rows to a full representation matrix of a vector matroid does not change the matroid.

Proof. See implementation in Lean. □

Lemma 21. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix, let M be the vector matroid of A, and let B be a base of M. Then there exists a matrix $S \in \mathbb{Q}^{B \times (Y \setminus B)}$ such that S is TU and S is a standard representation of M.

Proof. See implementation in Lean. □

Definition 22. Let R be a magma containing zero. The support of matrix $A \in R^{X \times Y}$ is $A^{\#} \in \{0, 1\}^{X \times Y}$ given by

$$\forall i \in X,\ \forall j \in Y,\ A^{\#}(i, j) = \begin{cases} 0, & \text{if } A(i, j) = 0, \\ 1, & \text{if } A(i, j) \neq 0. \end{cases}$$

Lemma 23. Transpose of a support matrix is equal to a support of the transposed matrix.

Proof. Definitional equality. □

Lemma 24. Submatrix of a support matrix is equal to a support matrix of the submatrix.

Proof. Definitional equality. □

Lemma 25. If A is a matrix over $\mathbb{Z}_2$, then $A^{\#} = A$.

Proof. Check elementwise equality. □

Lemma 26. If two standard representation matrices of the same matroid have the same base, then they have the same support.

Proof. See implementation in Lean. □

Lemma 27. A square matrix is invertible iff its determinant is invertible.

Proof. This result is proved in Mathlib. □

Lemma 28. Let A be a rational TU matrix with finite number of rows and finite number of columns. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See implementation in Lean. □

Lemma 29. Let A be a rational TU matrix with finite number of rows. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See implementation in Lean. $\square$

Lemma 30. Let A be a rational TU matrix. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See implementation in Lean. $\square$

Lemma 31. Let A be a TU matrix.

1. If a matroid is represented by A, then it is also represented by $A^{\#}$.

2. If a matroid is represented by $A^{\#}$, then it is also represented by A.

Proof. See implementation in Lean. $\square$

1.4 Regular Matroids

Definition 32. A matroid M is regular if there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A.

Definition 33. We say that $A' \in \mathbb{Q}^{X \times Y}$ is a TU signing of $A \in \mathbb{Z}_2^{X \times Y}$ if A' is TU and

$$\forall i \in X,\ \forall j \in Y,\ |A'(i,j)| = A(i,j).$$

Lemma 34. Let $B \in \mathbb{Z}_2^{X \times Y}$ be a standard representation matrix of a matroid M. Then M is regular if and only if B has a TU signing.

Proof. Suppose that M is regular. By Definition 32, there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A. By Lemma 19, X (the row set of B) is a base of M. By Lemma 21, A can be converted into a standard representation matrix $B' \in \mathbb{Q}^{X \times Y}$ of M such that B' is also TU. Since B' and B are both standard representations of M, by Lemma 26 the support matrices $(B')^{\#}$ and $B^{\#}$ are the same. Lemma 25 gives $B^{\#} = B$. Thus, B' is TU and $(B')^{\#} = B$, so B' is a TU signing of B.

Suppose that B has a TU signing $B' \in \mathbb{Q}^{X \times Y}$. Then $A = [1 \mid B']$ is TU, as it is obtained from B' by adjoining the identity matrix. Moreover, by Lemma 31, A represents the same matroid as $A^{\#} = [1 \mid B]$, which is M. Thus, A is a TU matrix representing M, so M is regular. $\square$

Chapter 2

Regularity of 1-Sum

Definition 35. Let R be a magma containing zero (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ be matrices where X_ℓ, Y_ℓ, X_r, Y_r are pairwise disjoint sets. The 1-sum $B = B_\ell \oplus_1 B_r$ of B_ℓ and B_r is

$$B = \begin{bmatrix} B_\ell & 0 \\ 0 & B_r \end{bmatrix} \in R^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}.$$

Definition 36. A matroid M is a 1-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 35.

Lemma 37. Let A be a square matrix of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Then $\det A = \det A_{11} \cdot \det A_{22}$.

Proof. This result is proved in Mathlib. □

Lemma 38. Let B_ℓ and B_r from Definition 35 be TU matrices (over $\mathbb{Q}$). Then $B = B_\ell \oplus_1 B_r$ is TU.

Proof. We prove that B is TU by Definition 3. To this end, let T be a square submatrix of B. Our goal is to show that $\det T \in \{0, \pm 1\}$.

Let T_ℓ and T_r denote the submatrices in the intersection of T with B_ℓ and B_r, respectively. Then T has the form

$$T = \begin{bmatrix} T_\ell & 0 \\ 0 & T_r \end{bmatrix}.$$

First, suppose that T_ℓ and T_r are square. Then $\det T = \det T_\ell \cdot \det T_r$ by Lemma 37. Moreover, $\det T_\ell, \det T_r \in \{0, \pm 1\}$, since T_ℓ and T_r are square submatrices of TU matrices B_ℓ and B_r, respectively. Thus, $\det T \in \{0, \pm 1\}$, as desired.

Without loss of generality we may assume that T_ℓ has fewer rows than columns. Otherwise we can transpose all matrices and use the same proof, since TUness and determinants are preserved under transposition. Thus, T can be represented in the form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where T_{11} contains T_ℓ and some zero rows, T_{22} is a submatrix of T_r, and T_{12} contains the rest of the rows of T_r (not contained in T_{22}) and some zero rows. By Lemma 37, we have $\det T = \det T_{11} \cdot \det T_{22}$. Since T_{11} contains at least one zero row, $\det T_{11} = 0$. Thus, $\det T = 0 \in \{0, \pm 1\}$, as desired. □

Theorem 39. *Let M be a 1-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 36. Since M_ℓ and M_r are regular, by Lemma 34, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_1 B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 38, and a direct calculation shows that B' is a signing of B. Thus, M is regular by Lemma 34. □

Chapter 3

Regularity of 2-Sum

Definition 40. Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ where $X_\ell \cap X_r = \{x\}$, $Y_\ell \cap Y_r = \{y\}$, X_ℓ is disjoint with Y_ℓ and Y_r, and X_r is disjoint with Y_ℓ and Y_r. Additionally, let $A_\ell = B_\ell(X_\ell \setminus \{x\}, Y_\ell)$ and $A_r = B_r(X_r, Y_r \setminus \{y\})$, and suppose $r = B_\ell(x, Y_\ell) \neq 0$ and $c = B_r(X_r, y) \neq 0$. Then the 2-sum $B = B_\ell \oplus_{2,x,y} B_r$ of B_ℓ and B_r is defined as

$$B = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix} \quad \text{where} \quad D = c \otimes r.$$

Here $D \in R^{X_r \times Y_\ell}$, and the indexing is consistent everywhere.

Definition 41. A matroid M is a 2-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 40.

Lemma 42. Let B_ℓ and B_r from Definition 40 be TU matrices (over $\mathbb{Q}$). Then $C = \begin{bmatrix} D & A_r \end{bmatrix}$ is TU.

Proof. Since B_ℓ is TU, all its entries are in $\{0, \pm 1\}$. In particular, r is a $\{0, \pm 1\}$ vector. Therefore, every column of D is a copy of y, $-y$, or the zero column. Thus, C can be obtained from B_r by adjoining zero columns, duplicating the y column, and multiplying some columns by -1. Since all these operations preserve TUess and since B_r is TU, C is also TU. ◻

Lemma 43. Let B_ℓ and B_r be matrices from Definition 40. Let B'_ℓ and B' be the matrices obtained by performing a short tableau pivot on $(x_\ell, y_\ell) \in X_\ell \times Y_\ell$ in B_ℓ and B, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B_r$.

Proof. Let

$$B'_\ell = \begin{bmatrix} A'_\ell \\ r' \end{bmatrix}, \quad B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$$

where the blocks have the same dimensions as in B_ℓ and B, respectively. By Lemma 13, $B'_{11} = A'_\ell$, $B'_{12} = 0$, and $B'_{22} = A_r$. Equality $B'_{21} = c \otimes r'$ can be verified via a direct calculation. Thus, $B' = B'_\ell \oplus_{2,x,y} B_r$. ◻

Lemma 44. Let B_ℓ and B_r from Definition 40 be TU matrices (over $\mathbb{Q}$). Then $B_\ell \oplus_{2,x,y} B_r$ is TU.

Proof. By Lemma 7, it suffices to show that $B_\ell \oplus_{2,x,y} B_r$ is k-PU for every $k \in \mathbb{N}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since all entries of $B_\ell \oplus_{2,x,y} B_r$ are in $\{0, \pm 1\}$ by construction.

Suppose that for some $k \in \mathbb{N}$ we know that for any TU matrices B'_ℓ and B'_r (from Definition 40) their 2-sum $B'_\ell \oplus_{2,x,y} B'_r$ is k-PU. Now, given TU matrices B_ℓ and B_r (from Definition 40), our goal is to show that $B = B_\ell \oplus_{2,x,y} B_r$ is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix T of B has $\det T \in \{0, \pm 1\}$.

First, suppose that T has no rows in X_ℓ. Then T is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by Lemma 42, so $\det T \in \{0, \pm 1\}$. Thus, we may assume that T contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $T(x_\ell, y_\ell) \neq 0$. Indeed, if $T(x_\ell, y) = 0$ for all y, then $\det T = 0$ and we are done, and $T(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since B is 1-PU, all entries of T are in $\{0, \pm 1\}$, and hence $T(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 14, performing a short tableau pivot in T on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix T''

such that $|\det T| = |\det T''|$. Since T is a submatrix of B, matrix T'' is a submatrix of the matrix B' resulting from performing a short tableau pivot in B on the same entry (x_ℓ, y_ℓ). By Lemma 43, we have $B' = B'_\ell \oplus_{2,x,y} B_r$ where B'_ℓ is the result of performing a short tableau pivot in B_ℓ on (x_ℓ, y_ℓ). Since B_ℓ is TU, by Lemma 15, B'_ℓ is also TU. Thus, by the inductive hypothesis applied to T'' and $B'_\ell \oplus_{2,x,y} B_r$, we have $\det T'' \in \{0, \pm 1\}$. Since $|\det T| = |\det T''|$, we conclude that $\det T \in \{0, \pm 1\}$. ☐

Theorem 45. *Let M be a 2-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 41. Since M_ℓ and M_r are regular, by Lemma 34, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 44, and a direct calculation verifies that B' is a signing of B. Thus, M is regular by Lemma 34. ☐

Chapter 4

Regularity of 3-Sum

4.1 Definition

Definition 46. Let $B_\ell \in \mathbb{Z}_2^{(X_\ell \cup \{x_0,x_1\}) \times (Y_\ell \cup \{y_2\})}$, $B_r \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0,y_1\})}$ be matrices of the form

$$B_\ell = \left[\begin{array}{cc|c} A_\ell & & 0 \\ & 1 \;\; 1 & 0 \\ \hline D_\ell & D_0 & \begin{matrix}1\\1\end{matrix} \end{array}\right] \quad \text{and} \quad B_r = \left[\begin{array}{cc|c} 1 \;\; 1 & 0 & 0 \\ \hline D_0 & \begin{matrix}1\\1\end{matrix} & A_r \\ D_r & & \end{array}\right] \quad \text{where } D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The 3-sum $B = B_\ell \oplus_3 B_r \in \mathbb{Z}_2^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}$ of B_ℓ and B_r is defined as

$$B = \left[\begin{array}{ccc|c} A_\ell & & & 0 \\ & 1 \;\; 1 & 0 & \\ \hline D_\ell & D_0 & \begin{matrix}1\\1\end{matrix} & A_r \\ D_{\ell r} & D_r & & \end{array}\right] \quad \text{where } D_{\ell r} = D_r \cdot (D_0)^{-1} \cdot D_\ell.$$

Here $x_2 \in X_\ell$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_\ell$, $y_2 \in Y_r$, $A_\ell \in \mathbb{Z}_2^{X_\ell \times Y_\ell}$, $A_r \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_\ell \in \mathbb{Z}_2^{\{x_0,x_1\} \times (Y_\ell \setminus \{y_0,y_1\})}$, $D_r \in \mathbb{Z}_2^{(X_r \setminus \{x_0,x_1\}) \times \{y_0,y_1\}}$, $D_{\ell r} \in \mathbb{Z}_2^{(X_r \setminus \{x_0,x_1\}) \times (Y_\ell \setminus \{y_0,y_1\})}$, $D_0 \in \mathbb{Z}_2^{\{x_0,x_1\} \times \{y_0,y_1\}}$. The indexing is consistent everywhere.

Note that D_0 is non-singular by construction, so $D_{\ell r}$ and B are well-defined. Moreover, a non-singular $\mathbb{Z}_2^{2 \times 2}$ matrix is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ up to re-indexing. Thus, Definition 46 can be equivalently restated with D_0 required to be non-singular and B_ℓ, B_r, and B re-indexed appropriately.

Definition 47. A matroid M is a 3-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 46.

4.2 Canonical Signing

Definition 48. We call $D_0' \in \mathbb{Q}^{\{x_0,x_1\} \times \{y_0,y_1\}}$ the canonical signing of $D_0 \in \mathbb{Z}_2^{\{x_0,x_1\} \times \{y_0,y_1\}}$ if

$$D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_0' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{or} \quad D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_0' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we call $S' \in \mathbb{Q}^{\{x_0,x_1,x_2\} \times \{y_0,y_1,y_2\}}$ the canonical signing of $S \in \mathbb{Z}_2^{\{x_0,x_1,x_2\} \times \{y_0,y_1,y_2\}}$ if

$$S = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{\multirow{2}{*}{D_0}} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array} \quad \text{and} \quad S' = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{\multirow{2}{*}{D_0'}} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array}$$

To simplify notation, going forward we use D_0, D_0', S, and S' to refer to the matrices of the form above. BTW, the canonical signing S' of S (from Definition 48) is TU.

Lemma 49. Let Q be a TU signing of S (from Definition 48). Let $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q' be defined as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in \{x_0, x_1, x_2\},\ \forall j \in \{y_0, y_1, y_2\}.$$

Then $Q' = S'$ (from Definition 48).

Proof. Since Q is a TU signing of S and Q' is obtained from Q by multiplying rows and columns by ± 1 factors, Q' is also a TU signing of S. By construction, we have

$Q'(x_2, y_0) = Q(x_2, y_0) \cdot 1 \cdot Q(x_2, y_0) = 1,$
$Q'(x_2, y_1) = Q(x_2, y_1) \cdot 1 \cdot Q(x_2, y_1) = 1,$
$Q'(x_2, y_2) = 0,$
$Q'(x_0, y_0) = Q(x_0, y_0) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_0) = 1,$
$Q'(x_0, y_1) = Q(x_0, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_1),$
$Q'(x_0, y_2) = Q(x_0, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1,$
$Q'(x_1, y_0) = 0,$
$Q'(x_1, y_1) = Q(x_1, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_1)),$
$Q'(x_1, y_2) = Q(x_1, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1.$

Thus, it remains to show that $Q'(x_0, y_1) = S'(x_0, y_1)$ and $Q'(x_1, y_1) = S'(x_1, y_1)$.

Consider the entry $Q'(x_0, y_1)$. If $D_0(x_0, y_1) = 0$, then $Q'(x_0, y_1) = 0 = S'(x_0, y_1)$. Otherwise, we have $D_0(x_0, y_1) = 1$, and so $Q'(x_0, y_1) \in \{\pm 1\}$, as Q' is a signing of S. If $Q'(x_0, y_1) = -1$, then

$$\det Q'(\{x_0, x_2\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q'. Thus, $Q'(x_0, y_1) = 1 = S'(x_0, y_1)$.

Consider the entry $Q'(x_1, y_1)$. Since Q' is a signing of S, we have $Q'(x_1, y_1) \in \{\pm 1\}$. Consider two cases.

1. Suppose that $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = 1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = -1 = S'(x_1, y_1)$.
2. Suppose that $D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = -1$, then $\det Q(\{x_0, x_1\}, \{y_1, y_2\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = 1 = S'(x_1, y_1)$.

$\square$

Definition 50. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X\times Y}$ be a TU matrix. Define $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q' as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in X, \ \forall j \in Y.$$

We call Q' the canonical re-signing of Q.

Lemma 51. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X\times Y}$ be a TU signing of $Q_0 \in \mathbb{Z}_2^{X\times Y}$ such that $Q_0(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S$ (from Definition 48). Then the canonical re-signing Q' of Q (from Definition 50) is a TU signing of Q_0 and $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ (from Definition 48).

Proof. Since Q is a TU signing of Q_0 and Q' is obtained from Q by multiplying some rows and columns by ± 1 factors, Q' is also a TU signing of Q_0. Equality $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ follows from Lemma 49. □

Definition 52. Suppose that B_ℓ and B_r from Definition 46 have TU signings B'_ℓ and B'_r, respectively. Let B''_ℓ and B''_r be the canonical re-signings (from Definition 50) of B'_ℓ and B'_r, respectively. Let A''_ℓ, A''_r, D''_ℓ, D''_r, and D''_0 be blocks of B''_ℓ and B''_r analogous to blocks A_ℓ, A_r, D_ℓ, D_r, and D_0 of B_ℓ and B_r. The canonical signing B'' of B is defined as

$$B'' = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{A''_\ell} & \multicolumn{2}{c}{0} \\ & 1 \;\; 1 & 0 & \\ \hline D''_\ell & D''_0 & \begin{array}{c} 1 \\ 1 \end{array} & A''_r \\ D''_{\ell r} & D''_r & & \end{array}\right] \quad \text{where } D''_{\ell r} = D''_r \cdot (D''_0)^{-1} \cdot D''_\ell.$$

Note that D''_0 is non-singular by construction, so $D''_{\ell r}$ and hence B'' are well-defined.

4.3 Properties of Canonical Signing

Lemma 53. B'' from Definition 52 is a signing of B.

Proof. By Lemma 51, B''_ℓ and B''_r are TU signings of B_ℓ and B_r, respectively. As a result, blocks A''_ℓ, A''_r, D''_ℓ, D''_r, and D''_0 in B'' are signings of the corresponding blocks in B. Thus, it remains to show that $D''_{\ell r}$ is a signing of $D_{\ell r}$. This can be verified via a direct calculation. □ Need details?

Lemma 54. Suppose that B_r from Definition 46 has a TU signing B'_r. Let B''_r be the canonical re-signing (from Definition 50) of B'_r. Let $c''_0 = B''_r(X_r, y_0)$, $c''_1 = B''_r(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Then the following statements hold.

1. For every $i \in X_r$, $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$.
2. For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.
3. $\begin{bmatrix} c''_0 & c''_2 & A''_r \end{bmatrix}$ is TU.

4. $\begin{bmatrix} c_1'' & c_2'' & A_r'' \end{bmatrix}$ is TU.
5. $\begin{bmatrix} c_0'' & c_1'' & c_2'' & A_r'' \end{bmatrix}$ is TU.

Proof. Throughout the proof we use that B_r'' is TU, which holds by Lemma 51.

1. Since B_r'' is TU, all its entries are in $\{0, \pm 1\}$, and in particular $\begin{bmatrix} c_0''(i) & c_1''(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}}$. If $\begin{bmatrix} c_0'(i) & c_1''(i) \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, then

$$\det B_r''(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\},$$

which contradicts TUness of B_r''. Similarly, if $\begin{bmatrix} c_0''(i) & c_1''(i) \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$, then

$$\det B_r''(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of B_r''. Thus, the desired statement holds.

2. Follows from item 1 and a direct calculation.

3. Performing a short tableau pivot in B_r'' on (x_2, y_0) yields:

$$B_r'' = \begin{bmatrix} \boxed{1} & 1 & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c_1'' - c_0 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $\begin{bmatrix} c_0'' & c_2'' & A_r'' \end{bmatrix}$ by removing row x_2 and multiplying columns y_0 and y_1 by -1. Since B_r'' is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, we conclude that $\begin{bmatrix} c_0'' & c_2'' & A_r'' \end{bmatrix}$ is TU.

4. Similar to item 4, performing a short tableau pivot in B_r'' on (x_2, y_1) yields:

$$B_r'' = \begin{bmatrix} 1 & \boxed{1} & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ c_0'' - c_1 & -c_1 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $\begin{bmatrix} c_1'' & c_2'' & A_r'' \end{bmatrix}$ by removing row x_2, multiplying column y_1 by -1, and swapping the order of columns y_0 and y_1. Since B_r'' is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, and re-ordering columns, we conclude that $\begin{bmatrix} c_1'' & c_2'' & A_r'' \end{bmatrix}$ is TU.

5. Let V be a square submatrix of $\begin{bmatrix} c_0'' & c_1'' & c_2'' & A_r'' \end{bmatrix}$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

   Suppose that column c_2'' is not in V. Then V is a submatrix of B_r'', which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

   Suppose that columns c_0'' and c_1'' are both in V. Then V contains columns c_0'', c_1'', and $c_2'' = c_0'' - c_1''$, which are linearly. Thus, $\det V = 0$. Going forward we assume that at least one of the columns c_0'' and c_1'' is not in V.

   Suppose that column c_1'' is not in V. Then V is a submatrix of $\begin{bmatrix} c_0'' & c_2'' & A_r'' \end{bmatrix}$, which is TU by item 3. Thus, $\det V \in \{0, \pm 1\}$. Similarly, if column c_0'' is not in V, then V is a submatrix of $\begin{bmatrix} c_1'' & c_2'' & A_r'' \end{bmatrix}$, which is TU by item 4. Thus, $\det V \in \{0, \pm 1\}$.

□

Lemma 55. Suppose that B_ℓ from Definition 46 has a TU signing B_ℓ'. Let B_ℓ'' be the canonical re-signing (from Definition 50) of B_ℓ'. Let $d_0'' = B_\ell''(x_0, Y_\ell)$, $d_1'' = B_\ell''(x_1, Y_\ell)$, and $d_2'' = d_0'' - d_1''$. Then the following statements hold.

1. For every $j \in Y_\ell$, $\begin{bmatrix} d_0''(i) \\ d_1''(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
2. For every $j \in Y_\ell$, $d_2''(j) \in \{0, \pm 1\}$.

3. $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_2 \end{bmatrix}$ is TU.

4. $\begin{bmatrix} A''_\ell \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU.

5. $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU.

Proof. Apply Lemma 54 to $B_\ell^\top$, or repeat the same arguments up to transposition. $\square$

Lemma 56. Let B'' be from Definition 52. Let $c''_0 = B''(X_r, y_0)$, $c''_1 = B''(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Similarly, let $d''_0 = B''(x_0, Y_\ell)$, $d''_1 = B''(x_1, Y_\ell)$, and $d''_2 = d''_0 - d''_1$. Then the following statements hold.

1. For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.

2. If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_1 \otimes d''_1$. If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_0 \otimes d''_1 + c''_1 \otimes d''_1$.

3. For every $j \in Y_\ell$, $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm c''_2\}$.

4. For every $i \in X_r$, $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

5. $\begin{bmatrix} A''_\ell \\ D'' \end{bmatrix}$ is TU.

Proof.

1. Holds by Lemma 54.2.

2. Note that

$$\begin{bmatrix} D''_\ell \\ D''_{\ell r} \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_\ell, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_0, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix}, \quad \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

   Thus,

$$D'' = \begin{bmatrix} D''_\ell & D''_0 \\ D''_{\ell r} & D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

   Considering the two cases for D''_0 and performing the calculations yields the desired results.

3. Let $j \in Y_\ell$. By Lemma 55.1, $\begin{bmatrix} d''_0(j) \\ d''_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Consider two cases.

   (a) If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item 2 we have $D''(X_r, j) = d''_0(j) \cdot c''_0 + (-d''_1(j)) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm(c''_0 - c''_1)\}$.

   (b) If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item 2 we have $D''(X_r, j) = (d''_0(j) - d''_1(j)) \cdot c''_0 + d''_1(j) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm(c''_0 - c''_1)\}$.

4. Let $i \in X_r$. By Lemma 54.1, $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix}\}$. Consider two cases.

   (a) If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item 2 we have $D''(i, Y_\ell) = c''_0(i) \cdot d''_0 + (-c''_1(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

   (b) If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item 2 we have $D''(i, Y_\ell) = c''_0(i) \cdot d''_0 + (c''_1(i) - c''_0(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

5. By Lemma 55.5, $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU. Since TUness is preserved under adjoining zero rows, copies of existing rows, and multiplying rows by ± 1 factors, $\begin{bmatrix} A''_\ell \\ 0 \\ \pm d''_0 \\ \pm d''_1 \\ \pm d''_2 \end{bmatrix}$ is also TU. By item 4, $\begin{bmatrix} A''_\ell \\ D'' \end{bmatrix}$ is a submatrix of the latter matrix, hence it is also TU.

□

4.4 Proof of Regularity

Definition 57. Let X_ℓ, Y_ℓ, X_r, Y_r be sets and let $c_0, c_1 \in \mathbb{Q}^{X_r}$ be column vectors such that for every $i \in X_r$ we have $c_0(i),\ c_1(i),\ c_0(i) - c_1(i) \in \{0, \pm 1\}$. Define $\mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ to be the family of matrices of the form $\begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ where $A_\ell \in \mathbb{Q}^{X_\ell \times Y_\ell}$, $A_r \in \mathbb{Q}^{X_r \times Y_r}$, and $D \in \mathbb{Q}^{X_r \times Y_\ell}$ are such that:

1. for every $j \in Y_\ell$, $D(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$,
2. $\begin{bmatrix} c_0 & c_1 & c_0 - c_1 & A_r \end{bmatrix}$ is TU,
3. $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU.

Lemma 58. Let B'' be from Definition 52. Then $B'' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c''_0, c''_1)$ where $c''_0 = B''(X_r, y_0)$ and $c''_1 = B''(X_r, y_1)$.

Proof. Recall that $c''_0 - c''_1 \in \{0, \pm 1\}^{X_r}$ by Lemma 56.1, so $\mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c''_0, c''_1)$ is well-defined. To see that $B'' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c''_0, c''_1)$, note that all properties from Definition 57 are satisfied: property 1 holds by Lemma 56.3, property 2 holds by Lemma 54.5, and property 3 holds by Lemma 56.5. □

Lemma 59. Let $C \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ from Definition 57. Let $x \in X_\ell$ and $y \in Y_\ell$ be such that $A_\ell(x, y) \neq 0$, and let C' be the result of performing a short tableau pivot in C on (x, y). Then $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$.

Proof. Our goal is to show that C' satisfies all properties from Definition 57. Let $C' = \begin{bmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{bmatrix}$, and let $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ be the result of performing a short tableau pivot on (x, y) in $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$. Observe the following.

- By Lemma 13, $C'_{11} = A'_\ell$, $C'_{12} = 0$, $C'_{21} = D'$, and $C'_{22} = A_r$.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property 3 for C, all entries of A_ℓ are in $\{0, \pm 1\}$.
- $A_\ell(x, y) \in \{\pm 1\}$, as $A_\ell(x, y) \in \{0, \pm 1\}$ by the above observation and $A_\ell(x, y) \neq 0$ by the assumption.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property 3 for C, and since pivoting preserves TUness, $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ is also TU.

These observations immediately imply properties 2 and 3 for C'. Indeed, property 2 holds for C', since $C'_{22} = A_r$ and $\begin{bmatrix} c_0 & c_1 & c_0 - c_1 & A_r \end{bmatrix}$ is TU by property 2 for C. On the other hand, property 3 follows from $C'_{11} = A'_\ell$, $C'_{21} = D'$, and $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ being TU. Thus, it only remains to show that C' satisfies property 1. Let $j \in Y_r$. Our goal is to prove that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Suppose $j = y$. By the pivot formula, $D'(X_r, y) = -\frac{D(X_r, y)}{A_\ell(x, y)}$. Since $D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property 1 for C and since $A_\ell(x, y) \in \{\pm 1\}$, we get $D'(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Now suppose $j \in Y_\ell \setminus \{y\}$. By the pivot formula, $D'(X_r, j) = D(X_r, j) - \frac{A_\ell(x, j)}{A_\ell(x, y)} \cdot D(X_r, y)$. Here $D(X_r, j),\ D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property 1 for C, and $A_\ell(x, j) \in \{0, \pm 1\}$ and $A_\ell(x, y) \in$

$\{\pm 1\}$ by the prior observations. Perform an exhaustive case distinction on $D(X_r, j)$, $D(X_r, y)$, $A_\ell(x, j)$, and $A_\ell(x, y)$. In every case, we can show that either $\begin{bmatrix} A_\ell(x,y) & A_\ell(x,j) \\ D(X_r,y) & D(X_r,j) \end{bmatrix}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$, or that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$, as desired. $\square$

need details?

Lemma 60. Let $C \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ from Definition 57. Then C is TU.

Proof. By Lemma 7, it suffices to show that C is k-PU for every $k \in \mathbb{N}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since properties 2 and 3 in Definition 57 imply that A_ℓ, A_r, and D are TU, so all their entries of $C = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ are in $\{0, \pm 1\}$, as desired.

Suppose that for some $k \in \mathbb{N}$ we know that every $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$ is k-PU. Our goal is to show that C is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix S of C has $\det V \in \{0, \pm 1\}$.

First, suppose that V has no rows in X_ℓ. Then V is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by property 2 in Definition 57, so $\det V \in \{0, \pm 1\}$. Thus, we may assume that S contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $V(x_\ell, y_\ell) \neq 0$. Indeed, if $V(x_\ell, y) = 0$ for all y, then $\det V = 0$ and we are done, and $V(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since C is 1-PU, all entries of V are in $\{0, \pm 1\}$, and hence $V(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 14, performing a short tableau pivot in V on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix S'' such that $|\det V| = |\det V''|$. Since V is a submatrix of C, matrix V'' is a submatrix of the matrix C' resulting from performing a short tableau pivot in C on the same entry (x_ℓ, y_ℓ). By Lemma 59, we have $C' \in \mathcal{C}(X_\ell, Y_\ell, X_r, Y_r; c_0, c_1)$. Thus, by the inductive hypothesis applied to V'' and C', we have $\det V'' \in \{0, \pm 1\}$. Since $|\det V| = |\det V''|$, we conclude that $\det V \in \{0, \pm 1\}$. $\square$

Lemma 61. B'' from Definition 52 is TU.

Proof. Combine the results of Lemmas 58 and 60. $\square$

Theorem 62. *Let M be a 3-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 47. Since M_ℓ and M_r are regular, by Lemma 34, B_ℓ and B_r have TU signings. Then the canonical signing B'' from Definition 52 is a TU signing of B. Indeed, B'' is a signing of B by Lemma 53, and B'' is TU by Lemma 61. Thus, M is regular by Lemma 34. $\square$

Chapter 5

Conclusion

Definition 63. Regular matroid is good. Any 1-sum of good matroids is a good matroid. Any 2-sum of good matroids is a good matroid. Any 3-sum of good matroids is a good matroid.

Corollary 64. Any good matroid is regular. This is the easy direction of the Seymour theorem.

Proof. Structural induction. $\square$