https://Ivan-Sergeyev.github.io/Matroid-Decomposition-Theorem-Verification
https://github.com/Ivan-Sergeyev/Matroid-Decomposition-Theorem-Verification
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Matroid Decomposition Theorem Verification

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0.1 Basic Definitions

0.1.1 Matroid Structure

Definition 1 (matroid)**.** todo: todo: add definition

Definition 2 (isomorphism)**.** todo: todo: add definition

0.1.2 Matroid Classes

Definition 3 (binary matroid)**.** todo: todo: add definition

Definition 4 (regular matroid)**.** todo: todo: add definition

Definition 5 (graphic matroid)**.** todo: todo: add definition

Definition 6 (cographic matroid)**.** todo: todo: add definition

Definition 7 (planar matroid)**.** todo: todo: add definition

Definition 8 (dual matroid)**.** todo: todo: add definition

Definition 9 (self-dual matroid)**.** todo: todo: add definition

0.1.3 Specific Matroids (Constructions)

Wheels

Definition 10 (wheel)**.** todo: todo: add definition

Definition 11 (W_3)**.** todo: todo: add definition

Definition 12 (W_4)**.** todo: todo: add definition

R_{10}

Definition 13 (R_{10})**.** todo: todo: add definition

R_{12}

Definition 14 (R_{12})**.** todo: todo: add definition

Fano matroid

Definition 15 (F_7)**.** todo: todo: add definition

$K_{3,3}$

Definition 16 ($M(K_{3,3})$)**.** todo: todo: add definition

Definition 17 ($M(K_{3,3})^*$)**.** todo: todo: add definition

K_5

Definition 18 ($M(K_5)$)**.** todo: todo: add definition

Definition 19 ($M(K_5)^*$)**.** todo: todo: add definition

0.1.4 Connectivity and Separation

Definition 20 (k-connectivity)**.** todo: todo: add definition

Definition 21 (k-separation)**.** todo: todo: add definition

0.1.5 Reductions

Definition 22 (deletion)**.** todo: todo: add definition

Definition 23 (contraction)**.** todo: todo: add definition

Definition 24 (minor)**.** todo: todo: add definition

0.1.6 Extensions

Definition 25 (1-element addition)**.** todo: add name, label, uses, text

Definition 26 (1-element expansion)**.** todo: add name, label, uses, text

Definition 27 (1-element extension)**.** todo: todo: add definition

Definition 28 (2-element extension)**.** todo: todo: add definition

0.1.7 Sums

Definition 29 (1-sum)**.** todo: todo: add definition

Definition 30 (2-sum)**.** todo: todo: add definition

Definition 31 (3-sum)**.** todo: todo: add definition

Definition 32 (Δ-sum)**.** todo: todo: add definition

Definition 33 (Y-sum)**.** todo: todo: add definition

0.1.8 Total Unimodularity

Definition 34 (TU matrix)**.** todo: todo: add definition

0.1.9 Auxiliary Results

Theorem 35 (Menger's theorem)**.** *A connected graph G is vertex k-connected if and only if every two nodes are connected by k internally node-disjoint paths. Equivalent is the following statement. G is vertex k-connected if and only if any $m \leq k$ nodes are joined to any $n \leq k$ nodes by k internally node-disjoint paths. One may demand that the m nodes are disjoint from the n nodes, but need not do so. Also, the k paths can be so chosen that each of the specified nodes is an endpoint of at least one of the paths.*

Definition 36 (ΔY exchange)**.** todo: add

Theorem 37 (census from Secion 3.3)**.** *todo: add*

Definition 38 (gap)**.** todo: add

0.2 Chapter 2 from Truemper

Lemma 39 (2.3.14)**.** *todo: add name, label, uses, text*

0.3 Chapter 3 from Truemper

Lemma 40 (3.3.12)**.** *todo: add name, label, uses, text*

Lemma 41 (3.2.48)**.** *todo: add name, label, uses, text*

0.4 Chapter 4 from Truemper

0.5 Chapter 5 from Truemper

Proposition 42 (5.2.8)**.** *def:wheel Representation matrices for small wheels (from $M(W_1)$ to $M(W_4)$).*

Proposition 43 (5.2.9)**.** *def:wheel Representation matrix for $M(W_n)$, $n \geq 3$.*

Lemma 44 (5.2.10)**.** *def:binary$_m$atroid$, def : minor, def : wheelLetMbeabinarymatroidwithabinaryrepresen$ minor.*

Proof sketch. prop:5.2.8,prop:5.2.9

- $BG(B)$ is bipartite and has at least one cycle, so there is a cycle C without chords with at least 4 edges.
- Up to indices, the submatrix corresponding to C is either the matrix for $M(W_2)$ from (5.2.8) or the matrix for some $M(W_k)$, $k \geq 3$ from (5.2.9).
- In the latter case, use path shortening pivots on 1s to convert the submatrix to the former case.

□

Lemma 45 (5.2.11)**.** *def:binary$_m$atroid, $def : k_conn, def : k_sep, def : M_{W3}, def :$ minorLetMbeaconnectedbinarymatroidwithatleast4elements.ThenMhasa2$-$separationoranM(W_3) minor.*

Proof sketch. lem:5.2.10 Use Lemma 5.2.10 and apply path shortening technique. □

Corollary 46 (5.2.15)**.** *def:k$_c$onn, $def : binary_matroid, def : M_{W3}, def :$ minorEvery3$-$connectedbinarymatr minor.*

Proof sketch. lem:5.2.11 By Lemma 5.2.11, M has a 2-separation or an $M(W_3)$ minor. M is 3-connected, so the former case is impossible. □

0.6 Chapter 6 from Truemper

0.6.1 Chapter 6.2

Goal of the chapter: separation algorithm for deciding if there exists a separation of a matroid induced by a separation of its minor.

Proposition 47 (6.2.1)**.** *def:k$_s$ep, $def : minor, def :$ binary$_m$atroidPartitionedversionofmatrixB^N representing a minor N of a binary matroid M, where N has an exact k-separation for some $k \geq 1$.*

Proposition 48 (6.2.3)**.** *prop:6.2.1 Matrix B for M displaying partitioned B^N*

Proposition 49 (6.2.5)**.** *prop:6.2.1,prop:6.2.3 Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Lemma 50 (6.2.6)**.** *def:binary$_m$atroid, $def : k_conn, def : 1_elem_ext, def : 2_elem_ext, def :$ $3_elem_ext, def : loop, def : coloop, def : parallel_elems, def :$ series$_e$lemsLetNbea3$-$connectedbinarymatroidon*

Proof sketch. lem:3.3.20

- Let C be a binary representation matrix of M that displays a binary representation matrix B for N.
- By assumption, B is 3-connected.
- C is connected, as otherwise by case analysis C contains a zero vector or unit vector, so M has a loop, coloop, parallel elements, or series elements, a contradiction.
- If C is not 3-connected, then by Lemma 3.3.20 there is a 2-separation of C with at least 5 rows/columns on each side. Then B has a 2-separation with at least 2 rows/columns on each side, a contradiction.
- Thus, C is 3-connected, so M is 3-connected.

□

0.6.2 Chapter 6.3

Definition 51 (6.3.2)**.** def:k$_sep, def : binary_matroid, def : minor, def : isomorphism$M$iscalledminimalifitsa$

M has an N minor.

M has no k-separation induced by the exact k-separation (F_1, F_2) of N.

The matroid M is minimal with respect to the above conditions.

Definition 52 (6.3.3)**.** def:k$_sep, def : binary_matroid, def : minor, def : isomorphism$M$iscalledminimalunder$

M has at least one N minor.

Some k-separation of at least one such minor corresponding to the exact k-separation (F_1, F_2) of N under one of the isomorphisms fails to induce a k-separation of M.

The matroid M is minimal with respect to the above conditions.

Proposition 53 (6.3.11)**.** *def:6.3.2,prop:6.2.5 Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Proposition 54 (6.3.12)**.** *def:6.3.2,prop:6.2.1,prop:6.2.5 Partitioned version of B^N: $B^N = \mathrm{diag}(A^1, A^2)$.*

Definition 55 (separation algorithm)**.** Polynomial-time recursive procedure to search for an induced partition. Described on pages 132–133 and again on pages 137–138.

Proposition 56 (6.3.13)**.** *def:6.3.2,prop:6.3.11,def:separation$_algorithmSpecialcasewhereBofaminimalMcont$ This proposition gives properties of row subvectors of row x by step 1 of the separation algorithm.*

Proposition 57 (6.3.14)**.** *def:6.3.2,prop:6.3.11,def:separation$_algorithmSpecialcasewhereBofaminimalMcont$ This proposition gives properties of column subvectors of column y by step 1 of the separation algorithm.*

Lemma 58 (6.3.15)**.** *def:6.3.2,prop:6.3.11,def:separation$_algorithmTreatsthecasewhereBhasatleasttwoadditio$*

Proof sketch. prop:6.3.13,prop:6.3.14 Argue about the structure of the matrix, applying steps 1 and 2 of the separation algorithm. □

Lemma 59 (6.3.16)**.** *def:6.3.2,prop:6.3.11,lem:6.3.15,def:separation$_algorithmExpandscase(i)ofLemma6.3.15$*

Proof sketch. Further arguments about the structure of the matrix. □

Lemma 60 (6.3.17)**.** *def:6.3.2,prop:6.3.11,lem:6.3.15,def:separation$_algorithmExpandscase(ii)ofLemma6.3.1$*

Proof sketch. Further arguments about the structure of the matrix. □

Theorem 61 (6.3.18)**.** *def:6.3.2,prop:6.3.11 Structural description of representation matrix (6.3.11) of a minimal M. Contains cases (a), (b), and (c) with sub-cases (c.1) and (c.2).*

Proof sketch. def:6.3.2,prop:6.3.13,prop:6.3.14,lem:6.3.15,lem:6.3.16,lem:6.3.17

- (6.3.13) and (6.3.14) establish (a) and (b).
- Lemmas 6.3.15, 6.3.16, and 6.3.17 prove (c.1) and (c.2).

□

Lemma 62 (6.3.19)**.** *def:6.3.3,thm:6.3.18 Additional structural statements for cases (c.1) and (c.2) of Theorem 6.3.18.*

Proof sketch. thm:6.3.18,lem:6.3.15 Reason about representation matrices using Theorem 6.3.18, Lemma 6.3.15, minimality, isomorphisms, pivots, and so on. □

Proposition 63 (6.3.21)**.** *def:6.3.3 Matrix B for M minimal under isomorphism, case (a).*

Proposition 64 (6.3.22)**.** *def:6.3.3 Matrix B for M minimal under isomorphism, case (b).*

Proposition 65 (6.3.23)**.** *def:6.3.3 Matrix $\overline{B}$ for minor $\overline{M}$ of M minimal under isomorphism.*

Theorem 66 (6.3.20)**.** *def:6.3.3,prop:6.3.21,prop:6.3.22,prop:6.3.23 Let M be minimal under isomorphism. Then one of 3 cases holds for matrix representation of M.*

Proof sketch. thm:6.3.18,lem:6.3.19 Follows directly from Theorem 6.3.18 and Lemma 6.3.19. □

Corollary 67 (6.3.24)**.** *def:binary$_m$atroid*$, def : isomorphism, def : minor, def : 1_e lem_e xt, def : 2_e lem_e xt, prop : 6.3.12, prop : 6.3.21, prop : 6.3.22, prop : 6.3.23, def : k_s epLetMbeaclassofbinarymatroidsclosedunderisomorphismandundertakingminors.SupposeNgivenbyB^N$ *of (6.3.12) is in $\mathcal{M}$, but the 1- and 2-element extensions of N given by (6.3.21), (6.3.22), (6.3.23), and by the accompanying conditions are not in $\mathcal{M}$. Assume matroid $M \in \mathcal{M}$ has an N minor. Then any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

Proof sketch. thm:6.3.20

- Let $M \in \mathcal{M}$ satisfying the assumptions. Since $\mathcal{M}$ is closed under isomorphism, suppose that N itself is a minor of M.
- Suppose the k-separation of N does not induce one in M. Then M or a minor of M containing N is minimal under isomorphism.
- By Theorem 6.3.20, M has a minor represented by (6.3.21), (6.3.22), or (6.3.23). This minor is in $\mathcal{M}$, as $\mathcal{M}$ is closed under taking minors, but this contradicts our assumptions.

□

0.6.3 Chapter 6.4

Theorem 68 (6.4.1)**.** *def:k$_c$onn, def : binary$_m$atroid, def : minor, def : 1$_e$lem$_e$xt, def :* *2$_e$lem$_e$xtLetMbea3−connectedbinarymatroidwitha3−connectedproperminorN.SupposeNhasatleast6element*

Proof sketch. lem:5.2.4,lem:6.2.6,thm:6.3.20

- Let $z \in M \setminus N$. By Lemma 5.2.4, there is a connected minor N' that is a 1-element extension of N by z. Our theorem holds iff it holds for duals, so by duality, assume that the extension is an addition.

- Reason about a matrix representation of N and N' to get a 2-separation of N'. Since M is 3-connected, this 2-separation does not induce one in M. Let M' be a minor of M that proves this fact and is minimal under isomorphism. Additionally, M' has an N' minor, so we change the element labels in M' so that N' is a minor of M'.

- Apply Theorem 6.3.20 and perform case analysis, reaching either a contradiction or a desired extension.

□

0.7 Chapter 7 from Truemper

0.7.1 Chapter 7.2

Definition 69 (splitter)**.** def:binary$_m$*atroid, def : minor, def : isomorphism, def :* *k$_c$onnLet*M*beaclassofbinarymatroidsclosedunderisomorphismandundertakingminors.Let*N*bea*3*−connected*
$\in \mathcal{M}$ with a proper N minor has a 2-separation, then N is called a splitter of $\mathcal{M}$.

Theorem 70 (7.2.1.a splitter for nonwheels)**.** *def:binary$_m$atroid, def : minor, def :* *isomorphism, def : k$_c$onn, def : wheel, def : splitterLetMbeaclassofbinarymatroidsclosedunderisomorphism*

Proof sketch. thm:6.4.1,def:splitter,def:k$_c$*onn, def : 1$_e$lem$_e$xt, def : minor, def :* *k$_s$ep*

If N is a splitter of $\mathcal{M}$, then clearly $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.

Prove the converse by contradiction. To this end, suppose that $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and that N is not a splitter of $\mathcal{M}$.

Thus, $\mathcal{M}$ contains a 3-connected matroid M with a proper N minor and no 2-separation.

Since $\mathcal{M}$ is closed under isomorphism, we may assume N itself to be that N minor.

By Theorem 6.4.1 (applied to M and N), M has a 3-connected minor N' that is a 3-connected 1- or 2-element extension of an N minor.

The 1-extension case has been ruled out.

In the 2-element extension case, N' is derived from the N minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take N itself to be that N minor. Thus, N' is derived from N by one addition and one expansion.

Let C be a binary matrix representing N' and displaying N. By investigating the structure of C, one can show that N' contains a 3-connected 1-element extension of an N minor, which has been ruled out.

□

Theorem 71 (7.2.1.b splitter for wheels)**.** *def:binary$_m$atroid*$, def : minor, def : isomorphism, def : k_conn, def : wheel, def : splitterLetMbeaclassofbinarymatroidsclosedunderisomorphism$

Proof sketch. thm:6.4.1,def:splitter,def:k$_conn, def : 1_elem_ext, def : minor, def : k_sepSimilartoproofofTheorem7.2.1.a.TheanalysisofthematrixCcanbedoneinonegoforbothcases.$ □

Corollary 72 (7.2.10.a)**.** *thm:7.2.1.a Theorem 7.2.1.a specialized to graphs.*

Proof sketch. thm:7.2.1.a Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Corollary 73 (7.2.10.b)**.** *thm:7.2.1.b Theorem 7.2.1.b specialized to graphs.*

Proof sketch. thm:7.2.1.b Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Theorem 74 (7.2.11.a)**.** *def:*$M_{K5}, def : M_{K33}, def : splitter, def : minor, def : graphic_matroidK_5$ *is a splitter of the graphs without* $K_{3,3}$ *minors.*

Proof sketch. cor:7.2.10.a,def:k$_conn, def : 1_elem_extUptoisomorphism, thereisjustone3-connected1-edgeexte$
To obtain it, one partitions one vertex of K_5 into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph has a $K_{3,3}$ minor. Thus, the theorem follows from Corollary 7.2.10.a. □

Theorem 75 (7.2.11.b)**.** *def:*$M_{W3}, def : M_{W4}, def : splitter, def : minor, def : graphic_matroidW_3$ *is a splitter of the graphs without* W_4 *minors.*

Proof sketch. cor:7.2.10.b,def:k$_conn, def : 1_elem_extThereisno3-connected1-edgeextensionof$W$_3$, so the theorem follows from Corollary 7.2.10.b. □

0.7.2 Chapter 7.3

Theorem 76 (7.3.1.a)**.** *def:*$k_conn, def : binary_matroid, def : minor, def : wheel, def : isomorphism, def : gapLetMbea3{-}connectedbinarymatroidwitha3{-}connectedproperminorNona$ ≥ 1*, there is a sequence* $M_0, \ldots, M_t = M$ *of nested* 3*-connected minors where* M_0 *is isomorphic to* N *and where the gap is* 1*.*

Proof sketch. thm:7.2.1.a

- Inductively for $i \geq 0$ assume the existence of a sequence $M_0, \ldots, M_i$ of 3-connected minors where M_0 is isomorphic to N, M_i is not a wheel, and the gap is 1.
- If $M_i = M$, we are done, so assume that M_i is a proper minor of M.
- Use the contrapositive of the splitter Theorem 7.2.1.a to find a larger sequence.
  - Let $\mathcal{M}$ be the collection of all matroids isomorphic to a (not necessarily proper) minor of M.
  - Since M_i is a 3-connected proper minor of the 3-conected $M \in \mathcal{M}$, it cannot be a splitter of $\mathcal{M}$. By Theorem 7.2.1.a, $\mathcal{M}$ contains a matroid M_{i+1} that is a 3-connected 1-element extension of a matroid isomorphic to M_i.
  - Since every 1-element reduction of a wheel with at least 6 elements is 2-separable, M_{i+1} is not a wheel, as otherwise M_i is 2-separable, which is a contradiction.
- If necessary, relabel $M_0, \ldots, M_i$ so that they consistute a sequence of nested minors of M_{i+1}. This sequence satisfies the induction hypothesis.
- By induction, the claimed sequence exists for M.

□

Theorem 77 (7.3.1.b)**.** *def:*$k_conn, def : binary_matroid, def : minor, def : wheel, def : isomorphism, def : gapLetMbea3{-}connectedbinarymatroidwitha3{-}connectedproperminorNona$ ≥ 1*, there is a sequence* $M_0, \ldots, M_t = M$ *of nested* 3*-connected minors where:*

- M_0 *is isomorphic to* N*,*
- *for some* $0 \leq s \leq t$ *the subsequence* $M_0, \ldots, M_s$ *consists of wheels and has gap 2,*
- *the subsequence* $M_s, \ldots, M_t$ *has gap* 1*.*

Proof sketch. thm:7.2.1.b Same as the proof of Theorem 7.3.1.a, but uses Theorem 7.2.1.b instead of 7.2.1.a to extend the sequence of minors. □

Proposition 78 (7.2.1 from 7.3.1)**.** *thm:7.3.1.a,thm:7.3.1.b,thm:7.2.1.a,thm:7.2.1.b Theorem 7.3.1 implies Theorem 7.2.1.*

Proof sketch. thm:7.3.1.a,thm:7.3.1.b,thm:7.2.1.a,thm:7.2.1.b

- Let $\mathcal{M}$ and N be as specified in Theorem 7.2.1. Suppose N is not a wheel.
- Prove the nontrivial "if" part by contradiction: let M be a 3-connected matroid of $\mathcal{M}$ with N as a proper minor.
- By Theorem 7.3.1, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where M_0 is isomorphic to N and where the gap is 1.
- Since $\mathcal{M}$ is closed under isomorphism, we may assume that M is chosen such that $M_0 = N$.
- Then $M_1 \in \mathcal{M}$ is a 3-connected 1-element extension of N, which contradicts the assumed absence of such extensions.
- If N is a wheel, the proof is analogous.

□

Corollary 79 (7.3.2.a)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H, and where each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. thm:7.3.1.a Translate Theorem 7.3.1.a directly into graph language. □

Corollary 80 (7.3.2.b)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where:*

- *G_0 is isomorphic to H,*
- *for some $0 \leq s \leq t$ the subsequence $G_0, \ldots, G_t$ consists of wheels where each G_{i+1} has exactly one additional spoke beyond those of G_i,*
- *in the subsequence $G_s, \ldots, G_t$ each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. thm:7.3.1.b Translate Theorem 7.3.1.b directly into graph language. □

Theorem 81 (7.3.3, wheel theorem)**.** *Let G be a 3-connected graph on at least 6 edges. If G is not a wheel, then G has some edge z such that at least one of the minors G/z and $G \setminus z$ is 3-connected.*

Proof sketch. cor:5.2.15,cor:7.3.2.b

- By Corollary 5.2.15, G has a W_3 minor.
- Let H be a largest wheel minor of G. Since G is not a wheel, H is a proper minor of G.
- Apply Corollary 7.3.2.b to G and H to get a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H.
- Since H is the largest wheel minor and G is not a wheel, Corollary 7.3.2.b shows that $s = 0$ and $t \geq 1$.
- Additionally, from corollary we know that $G = G_t$ has exactly one extra edge compared to G_{t-1}. In other words, $G_{t-1} = G/z$ or $G \setminus z$ for some edge z.

$\square$

Theorem 82 (7.3.3 for binary matroids)**.** *thm:7.3.1.a,thm:7.3.1.b Theorem 7.3.3 can be rewritten for binary matroids instead of graphs.*

Proof sketch. thm:7.3.1.a,thm:7.3.1.b Similar to the proof of Theorem 7.3.3, but use Theorem 7.3.1 instead of Corollary 7.3.2. $\square$

Proposition 83 (7.3.4.observation)**.** *thm:7.3.1.a,thm:7.3.1.b,lem:3.3.12,lem:6.2.6 Oservation in text on pages 160–161.*

Theorem 84 (7.3.4)**.** *def:k*$_c$*onn*$, def : binary_m atroid, def : minor, def : 1_e lem_e xpansion, def : 1_e lem_a ddition, def : 1_e lem_e xpansion Let M be a 3-connected binary matroid with a 3-connected proper minor Non$ *M where M_0 is an N minor of M and where each M_{i+1} is obtained from M_i by expansions (resp. additions) involving some series (resp. parallel) elements, possibly none, followed by a* 1*-element addition (resp. expansion).*

Proof sketch. thm:7.3.1.a,thm:7.3.1.b,prop:7.3.4.obs

- The case in parenthesis is dual to the normally stated one. Thus, only consider expansions below.
- Apply construction from observation before Theorem 7.3.4 to the sequence of minors from Theorem 7.3.1 to get the desired sequence.

$\square$

Corollary 85 (7.3.5)**.** *thm:7.3.4 Specializes Theorem 7.3.4 to graphs.*

0.7.3 Chapter 7.4

Theorem 86 (7.4.1 planarity characterization)**.** *def:planar*$_m$*atroid*$, def : M_{K33}, def : M_{K5} A graph is planar if and only if it has no K_{3,3}$ *or* K_5 *minors.*

Proof sketch. lem:3.2.48,cor:5.2.15,cor:7.3.5

- "Only if": planarity is preserved by taking minors, and by Lemma 3.2.48 both $K_{3,3}$ and K_5 are not planar.
- Let G be a connected nonplanar graph with all proper minors planar. Goal: show that G is isomorphic to $K_{3,3}$ or K_5.
- Prove that G cannot be 1- or 2-separable. Thus G is 3-connected.
- By Corollary 5.2.15, G has a W_3 minor, say H. Note: no H minor of G can be extended to a minor of G by addition of an edge that connects two nonadjacent nodes.
- Then by Corollary 7.3.5.b, there exists a sequence $G_0, \ldots, G_t = G$ of 3-connected minors where G_0 is an H minor and G_{i+1} is constructed from G_i following very specific steps.
- By minimality, G_{t-1} is planar and G is not. Argue about a planar drawing of G_{t-1} and how G can be derived from it. Show that this must result in a subdivision of $K_{3,3}$ or K_5.

□

Theorem 87 (Kuratowski)**.** *def:planar$_m$atroid*$, def: M_{K33}, def: M_{K5} Agraphisplanarifandonlyifithasnosu$
or K_5*.*

Proof. thm:7.4.1 Note: Theorem 7.4.1 is equivalent to Kuratowski's theorem: a $K_{3,3}$ minor induces a subdivision of $K_{3,3}$ and a K_5 minor also leads to a subdivision of K_5 or $K_{3,3}$ (the latter in the case when an expansion step splits a vertex of degree 4 into two vertices of degree 3 after the new edge is inserted). □

0.8 Chapter 8 from Truemper

0.8.1 Chapter 8.2

This chapter is about deducing and manipulating 1- and 2-sum decompositions and compositions.

Proposition 88 (8.2.1)**.** *def:k$_s$epMatrixof1−separation.*

Lemma 89 (8.2.2)**.** *def:binary$_m$atroid*$, def: 1_sum, def: graphic_matroid, def:$
$planar_m atroid Let M be a binary matroid. Assume M to be a 1-sum of two matroids M_1$
and M_2*.*

- *If* M *is graphic, then there eixst graphs* G*,* G_1*,* G_2 *for* M*,* M_1*,* M_2*, respectively, such that identification of a node of* G_1 *with one of* G_2 *creates* G*.*
- *If* M_1 *and* M_2 *are graphic (resp. planar), then* M *is graphic (resp. planar).*

Proof sketch. thm:3.2.25.a Elementary application of Theorem 3.2.25.a. □

Proposition 90 (8.2.3)**.** *def:*$k_sepMatrixofexact2{-}separation.$

Proposition 91 (8.2.4)**.** *prop:8.2.3 Matrices* B^1 *and* B^2 *of* 2*-sum.*

Lemma 92 (8.2.6)**.** *def:*$binary_matroid, def : k_sep, def : k_conn, def : 2_sumAny2{-}separationofaconnectedbina$ *and* M_2*. Conversely, any* 2*-sum of two connected binary matroids* M_1 *and* M_2 *is a connected binary matroid* M*.*

Proof sketch. prop:8.2.3,prop:8.2.4,lem:3.3.19

- Definitions imply everything except connectedness.
- It is easy to check that connectedness of (8.2.3) implies connectedness of (8.2.4) and vice versa.
- By Lemma 3.3.19, connectedness of representation matrices is equivalent to connectedness of the corresponding matroids.

□

Lemma 93 (8.2.7)**.** *def:*$binary_matroid, def : k_conn, def : 2_sum, prop : 8.2.3, prop : 8.2.4, def : graphic_matroid, def : planar_matroidLetMbeaconnectedbinarymatroidthatisa2{-}sumofM_1$ *and* M_2*, as given via* B*,* B_1*, and* B_2 *of (8.2.3) and (8.2.4).*

- *If* M *is graphic, then there exist* 2*-connected graphs* G*,* G_1*, and* G_2 *for* M*,* M_1*, and* M_2*, respectively, with the following feature. The graph* G *is produced when one identifies the edge* x *of* G_1 *with the edge* y *of* G_2*, and when subsequently the edge so created is deleted.*
- *If* M_1 *and* M_2 *are graphic (resp. planar), then* M *is graphic (resp. planar).*

Proof sketch. def:$k_sep, thm : 3.2.25.b, lem : 8.2.6, prop : 8.2.3, prop : 8.2.4, switchingopsec3$

Ingredients: look at a 2-separation and the corresponding subgraphs, use Theorem 3.2.25.b, use the switching operation of Section 3.2, use Lemma 8.2.6 and representations (8.2.3) and (8.2.4).

Use the construction from the drawing, check that fundamental circuits match, conclude that M is graphic. For planar graphs, the edge identification can be done in a planar way.

□

0.8.2 Chapter 8.3

Proposition 94 (8.3.1)**.** *def:*$k_sepMatrixBwithexactk{-}separation.$

Proposition 95 (8.3.2)**.** *prop:8.3.1,def:*$3_sumPartitionofBdisplayingk{-}sum.$

Proposition 96 (8.3.9)**.** *prop:8.3.2,def:*$3_sumThe(well-chosen)matrix\overline{B}$ *representing the connecting minor* $\overline{M}$ *of a* 3*-sum.*

Proposition 97 (8.3.10). *prop:8.3.2,prop:8.3.9,def:3*$_sumThematrixBrepresentinga3-sum(afterreasoning)$.

Proposition 98 (8.3.11). *def:3*$_sumRepresentationmatricesB^1$ *and* B^2 *of the components* M_1 *and* M_2 *of a 3-sum (after reasoning).*

Lemma 99 (8.3.12). *def:k*$_conn, def : k_sep, def : binary_matroid, def : 3_sumLetMbea3-connectedbinarymatro$ *of* M *with* $|E_1|, |E_2| \geq 4$ *produces a 3-sum, and vice versa.*

Proof. prop:8.3.1,prop:8.3.10,lem:2.3.14,prop:8.3.9

- The converse easily follows from (8.3.10), which directly produces a desired 3-separation.
- Take a 3-separation. Since M is 3-connected, it must be exact. Consider the representation matrix (8.3.11). Reason about that matrix.
- Analyse shortest paths in a bipartite graph based on the matrix.
- Apply path shortening technique from Chapter 5 to reduce a shortest path by pivots to one with exactly two arcs.
- Reason about the corresponding entries and about the effects of the pivots on the matrix.
- Apply Lemma 2.3.14. Eventually get an instance of (8.3.10) with (8.3.9). Thus, M is a 3-sum.

□

0.8.3 Chapter 8.5

Proposition 100 (8.5.3). *prop:8.3.10,prop:8.3.11,prop:4.4.5 Matrix* $B^{2\Delta}$ *for* $M_{2\Delta}$.

0.9 Chapter 9 from Truemper

Proposition 101 (9.2.14). *def:R12 Matrix* B^{12} *of regular matroid* R_{12}

0.10 Chapter 10 from Truemper

Proposition 102 (10.2.4). *Derivation of a graph with* T *nodes for* F_7.

Proposition 103 (10.2.6). *Derivation of a graph with* T *nodes for* $M(K_{3,3})^*$.

Proposition 104 (10.2.8). *Derivation of a graph with* T *nodes for* R_{10}.

Proposition 105 (10.2.9). *Derivation of a graph with* T *nodes for* R_{12}.

Theorem 106 (10.2.11 only if). *def:regular*$_matroid, def : planar_matroid, def : M_{K5}, def : M_{K5d}ual, def : M_{K33}, def : M_{K33d}ual, def : minorIfaregularmatroidisplanar, thenithasnoM(K$ $M(K_5)^*$, $M(K_{3,3})$, *or* $M(K_{3,3})^*$ *minors.*

Proof sketch.
- Planarity is preserved under taking minors.
- The listed matroids are not planar.

□

Theorem 107 (10.2.11 if)**.** *def:regular$_m$atroid, def : planar$_m$atroid, def : M_{K5}, def : M_{K5d}ual, def : M_{K33}, def : M_{K33d}ual, def : minorIfaregularmatroidhasno$M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors, then it is planar.*

Proof sketch. thm:7.4.1,lem:8.2.2,lem:8.2.6,lem:8.2.7,census sec 3.3,thm:7.3.3,prop:10.2.4,prop:10.2.6,thm:Meng

- Let M be minimally nonplanar with respect to taking minors, i.e., regular nonplanar, but with all proper minors planar.
- Goal: show that M is isomorphic to one of the listed matroids.
- By Theorem 7.4.1, M is not graphic or cographic.
- By Lemmas 8.2.2, 8.2.6, and 8.2.7, if M has a 1- or 2-separation, then M is a 1- or 2-sum. But then the components of the sum are planar, so M is also planar. Therefore, M is 3-connected.
- By the census of Section 3.3, every 3-connected $\leq$ 8-element matroid is planar, so $|M| \geq 9$.
- By the binary matroid version of the wheel Theorem 7.3.3, there exists an element z such that $M \setminus z$ or M/z is 3-connected. Dualizing does not afect the assumptions, so we may assume that $M \setminus z$ is 3-connected.
- Let G be a planar graph representing $M \setminus z$. Extend G to a representation of M as follows:
  - If G is a wheel, invoke (10.2.6) or (10.2.4). The latter contracdicts regularity of M, the former shows what we need.
  - If G is not a wheel, use Theorem 7.3.3 and Menger's theorem. Use a path argument and edge contraction to reduce to (10.2.6) and conclude the proof.

□

Lemma 108 (10.3.1)**.** *def:M_{K5}, def : splitter, def : regular$_m$atroid, def : M_{K33}, def : minor$M(K_5)$ is a splitter of the regular matroids with no $M(K_{3,3})$ minors.*

Proof. thm:7.2.1.a,def:k$_c$onn, def : 1$_e$lem$_e$xt

By Theorem 7.2.1.a, we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor.

Then case analysis. (The book sketches one way of checking.)

□

Lemma 109 (10.3.6)**.** *def:k$_c$onn, $def: 1_e lem_e xt, def: M_{K33}, def: regular_m atroid, def:$ $binary_m atroid$ Every 3−connected binary 1−element expansion of $M(K_{3,3})$ is nonregular.*

Proof sketch. By case analysis via graphs plus T sets. □

Theorem 110 (10.3.11)**.** *def:k$_c$onn, $def: regular_m atroid, def: M_{K33}, def:$ $minor, def: graphic_m atroid, def: cographic_m atroid, def: isomorphism, def:$ $R10, def: R12$ Let M be a 3−connected regular matroid with an $M(K_{3,3})$ minor. Assume that M is not graphic and not cographic, but that each proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof. lem:10.3.6,thm:7.3.4,thm:Menger This proof is extremely long and technical. It involves case distinctions and graph constructions. □

Theorem 111 (10.4.1 only if)**.** *def:k$_c$onn, $def: regular_m atroid, def: graphic_m atroid, def:$ $cographic_m atroid, def: R10, def: R12, def: minor$ If 3−connected regular matroid is graphic or cographic, ther or R_{12} minors.*

Proof sketch. prop:10.2.8,prop:10.2.9 Representations (10.2.8) and (10.2.9) for R_{10} and R_{12} show that these are nongraphic and isomorphic to their duals, hence also noncographic, so we are done. □

Theorem 112 (10.4.1 if)**.** *def:k$_c$onn, $def: regular_m atroid, def: graphic_m atroid, def:$ $cographic_m atroid, def: R10, def: R12, def: minor$ If a 3−connected regular matroid has no R_{10} or R_{12} minors, then it is graphic or cographic.*

Proof sketch. thm:10.2.11.if,lem:10.3.1,thm:10.3.11

- Let M be 3-connected, regular, nongraphic, and noncographic matroid.
- Thus M is not planar, so by Theorem 10.2.11 it has a minor isomorphic to $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$.
- By Lemma 10.3.1, $M(K_5)$ is a splitter for the regular matroids with no $M(K_{3,3})$ minors.
- These results imply that M has a minor isomorphic to $M(K_{3,3})$, or $M(K_{3,3})^*$, or M is isomorphic to $M(K_5)$ or $M(K_5)^*$.
- The latter is a contradiction, so M or M^* has an $M(K_{3,3})$ minor.
- Theorem 10.3.11 implies that M or M^* has R_{10} or R_{12} as a minor.
- Since R_{10} and R_{12} are self-dual, M has R_{10} or R_{12} as a minor.

□

Note: Truemper's proof of **??** and **??** relies on representing matroids via graphs plus T sets. An alternative proof, which utilizes the notion of graph signings, can be found in J. Geelen, B. Gerards - Regular matroid decomposition via signed graphs. Although the proof appears shorter than Truemper's, it heavily relies certain relatively advanced graph-theoretic results.

Bonus: Whitney's characterization of planar graphs (Corollary 10.2.13).

0.11 Chapter 11 from Truemper

0.11.1 Chapter 11.2

The goal of this chapter is to prove the “simple” direction of the regular matroid decomposition theorem.

Ingredients from Section 9.2:

- A matrix is TU if all its subdeterminants are 0, ±1.
- A binary matroid is regular if it has a signing that is TU.
- By Lemma 9.2.6 and Corollary 9.2.7, this signing is unique up to scaling by ±1 factors.
- The signing can be accomplished by signing one arbitrarily selected row or column at a time.

Ingredients from minimal violation matrices:

- Definition: a minimal violation matrix of total unimodularity (minimal violation matrix, MVM) is a $\{0, \pm 1\}$ matrix that is not TU, but all its submatrices are TU.
- MVMs are square and have determinant not equal to $0, \pm 1$.
- In particular, a 2×2 violation matrix has four ±1’s.
- Cosider a MVM of order ≥ 3. Perform a pivot in it, then delete the pivot row and column. Then the resulting matrix is also MVM ("by a simple cofactor argument").

Lemma 113 (11.2.1)**.** *def:regular*$_m$*atroid*$, def : 1_sum, def : 2_sumAny1{-}or2{-}sumoftworegularmatroidsisals$

Proof sketch. prop:8.2.1,prop:8.2.3,prop:8.2.4

- 1-sum case: $M_1 \oplus_1 M_2$ is represented by a matrix $B = \mathrm{diag}(A_1, A_2)$ where A_1 and A_2 represent M_1 and M_2. Use the same signings for A_1 and A_2 in B to prove that B is TU and hence the 1-sum is regular.
- 2-sum case: Slightly more complicated signing process. Similarly, reuse signings from M_1 and M_2, define signing on remaining nonzero elements via a concrete formula, then prove that the resulting matrix is TU.

□

Lemma 114 (11.2.7)**.** *prop:8.3.10,prop:8.3.11,prop:8.5.3 M_2 of (8.3.10) and (8.3.11) is regular iff $M_{2\Delta}$ of (8.5.3) (M_2 converted by a ΔY exchange) is regular.*

Proof sketch. Utilize signings, minimal violation matrices, intersections (inside matrices), column dependence, pivot, duality. □

Corollary 115 (11.2.8)**.** *def:Delta$_{Y}$$_e$xchange,$def$: regular$_m$atroidΔY exchanges maintain regularity.*

Proof. lem:11.2.7 Follows by Lemma 11.2.7. □

Lemma 116 (11.2.9)**.** *def:regular$_m$atroid,def : 3$_s$umAny3$-$sumoftworegularmatroidsisalsoregular.*

Proof sketch. lem:11.2.7,cor:11.2.8 Yet more complicated, but similar. Uses the result that "ΔY exchanges maintain regularity" (Corollary 11.2.8 of Lemma 11.2.7). The rest of the arguments are similar to the 2-sum case: prove that submatrices are TU, then prove that the whole matrix is TU. □

Theorem 117 (11.2.10)**.** *def:regular$_m$atroid,def : 1$_s$um,def : 2$_s$um,def : 3$_s$umAny1$-$,2$-$,or3$-$sumoftworegularmatroidsisregular.*

Proof sketch. lem:11.2.1,lem:11.2.9 Combine Lemmas 11.2.1 and 11.2.9. □

Corollary 118 (11.2.12)**.** *def:regular$_m$atroid,def : Delta$_s$um,def : $Y_s$umAnyΔ-sum or Y-sum of two regular matroids is also regular.*

Proof sketch. def:Delta$_s$um,$def : Y_sum, thm : 11.2.10, cor : 11.2.8 Followsfromdefinitionsof \Delta$-sums and Y-sum, together with Theorem 11.2.10 and Corollary 11.2.8. □

0.11.2 Chapter 11.3

Proposition 119 (11.3.3)**.** *prop:10.2.8 Graph plus T set representing R_{10}*

Proposition 120 (11.3.5)**.** *prop:10.2.4 Graph plus T set representing F_7.*

Proposition 121 (11.3.11)**.** *prop:9.2.14 The binary representation matrix B^{12} for R_{12}.*

The goal of the chapter is to prove the "hard" direction of the regular matroid decomposition theorem.

Theorem 122 (11.3.2)**.** *def:regular$_m$atroid,def : R10,def : splitterR_{10} is a splitter of the class of regular matroids.*

In short: up to isomorphism, the only 3*-connected regular matroid with R_{10} minor is R_{10}.*

Proof sketch. thm:7.2.1.a,prop:11.3.3,def:isomorphism,def:1$_e$$lem_ext, prop : 11.3.5, def : F7, def : contraction$

Splitter theorem case (a)

R_{10} is self-dual, so it suffices to consider 1-element additions.

Represent R_{10} by (11.3.3)

Up to isomorphism, there are only 3 distinct 3-connected 1-element extensions.

Case 1 (graphic): contract a certain edge, the resulting graph contains a subdivision of (11.3.5), which represents F_7. Thus, this extension is nonregular.

Cases 2, 3 (nongraphic): reduce instances to (11.3.5), same conclusion.

□

Theorem 123 (11.3.10)**.** *cor:6.3.24,def:R12 In short: Restatement of* **??** *for* R_{12}*. Replacements:* $\mathcal{M}$ *is the class of regular matroids,* N *is* R_{12}*, (6.3.12) is (11.3.6), (6.3.21-23) are (11.3.7-9).*

Theorem 124 (11.3.12)**.** *def:regular*$_matroid, def : R12, def : minor, def : k_sep, prop : 11.3.11, def : isomorphismLetMbearegularmatroidwith R_{12}$ *minor. Then any* 3*-separation of that minor corresponding to the* 3*-separation* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* R_{12} *(see (11.3.11) – matrix* B^{12} *for* R_{12} *defining the* 3*-separation) under one of the isomorphisms induces a* 3*-separation of* M*.*

In short: every regular matroid with R_{12} *minor is a* 3*-sum of two proper minors.*

Proof sketch. def:$1_elem_ext, prop : 10.2.9, thm : 11.3.10$

Preparation: calculate all 3-connected regular 1-element additions of R_{12}. This involves somewhat tedious case checking. (Representation of R_{12} in (10.2.9) helps a lot.) By the symmetry of B^{12} and thus by duality, this effectively gives all 3-connected 1-element extensions as well.

Verify conditions of theorem 11.3.10 (which implies the result).

(11.3.7) and (11.3.9) are ruled out immediately from preparatory calculations.

The rest is case checking ((c.1) and (c.2)), simpified by preparatory calculations.

□

Theorem 125 (11.3.14 regular matroid decomposition, easy direction)**.** *def:regular*$_matroid, def : graphic_matroid, def : cographic_matroid, def : isomorphism, def : R10, def : 1_sum, def : 2_sum, def : 3_sumEverybinarymatroidproducedfromgraphic, cographic, andmatroidsisomorphict$ *by repeated* 1*-,* 2*-, and* 3*-sum compositions is regular.*

Proof sketch. thm:11.2.10 Follows from theorem 11.2.10. □

Theorem 126 (11.3.14 regular matroid decomposition, hard direction)**.** *def:regular*$_matroid, def : graphic_matroid, def : cographic_matroid, def : isomorphism, def : R10, def : R12, def : 1_sum, def : 2_sum, def : 3_sum, def : k_conn, def : k_sep, prop : 11.3.11 EveryregularmatroidMcanbedecomposedintographicandcographicmatroidsandmatroidsisomorphicto$ *by repeated* 1*-,* 2*-, and* 3*- sum decompositions. Specifically: If* M *is a regular* 3*-connected matroid that is not graphic and not cographic, then* M *is isomorphic to* R_{10} *or has an* R_{12} *minor. In the latter case, any* 3*-separation of that minor corresponding to the 3-separation* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* R_{12} *((11.3.11)) under one of the isomorphisms induces a* 3*-separation of* M*.*

Proof sketch. thm:10.4.1.if,thm:11.3.2,thm:11.3.12,lem:8.3.12

- Let M be a regular matroid. Assume M is not graphic and not cographic.

- If M is 1-separable, then it is a 1-sum. If M is 2-separable, then it is a 2-sum. Thus assume M is 3-connected.
- By theorem 10.4.1, M has an R_{10} or an R_{12} minor.
- R_{10} case: by theorem 11.3.2, M is isomorphic to R_{10}.
- R_{12} case: by theorem 11.3.12, M has an induced by 3-separation, so by lemma 8.3.12, M is a 3-sum.

□

0.11.3 Extensions of Regular Matroid Decomposition

- Theorem 11.3.14 remains valid when 3-sums are replaced by Δ- and Y-sums (Theorem 11.3.16).
- Theorem 11.3.14 (and 11.3.16) can also be proved for matroids with no F_7 minors or with no F_7^* minors. (Uses Lemma 11.3.19: F_7 (F_7^*) is a splitter of the binary matroids with no F_7^* (F_7) minors.)

0.11.4 Applications of Regular Matroid Decomposition

- Efficient algorithm:for.testing.if a binary matroid is regular (Section 11.4).
- Efficient algorithm:for.deciding.if a real matrix is TU (Section 11.4).
- Constructing TU matrices (Theorem 11.5.9). (Translate 3-sum version of theorem 11.3.16 into matrix language.)
- Constructing $0, 1$ TU matrices (Theorem 11.5.13).
- Characterization of the cycle polytope (theorem 11.5.17). (Problem: let M be a connected binary matroid with ground set E and element weighs w_e for all $e \in E$. Find a disjoint union C of circuits of M such that $\sum_{e \in C} w_e$ is maximized.)
- Number of nonzeros in TU matrices (Theorem 11.5.18).
- Triples in circuits (Theorem 11.5.18).
- Odd cycles (Theorem 11.5.20).