# Proof of Regularity of 2-Sum and 3-Sum of Matroids

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## 1 The 2-Sum of Regular Matroids Is Regular

**Lemma 1.** Let $A$ be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let $A'$ denote the matrix obtained from $A$ by performing a real pivot on $a_{rc}$. Then there exists a $(k-1) \times (k-1)$ submatrix $A''$ of $A'$ with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.

*Proof.* Let $A''$ be the submatrix of $A'$ given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in $A$ on $a_{rc}$, the entries of $A''$ are given by $a''_{ij} = a_{ij} - \frac{a_{ic} \cdot a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \cdot \det B''_k$$

where $B''_k$ is a matrix obtained from $A''$ by replacing column $a''_{\bullet k}$ with the pivot column $a_{\bullet c}$ without the pivot element $a_{rc}$.

By the cofactor expansion in $A$ along row $r$, we have

$$\det A = \sum_{k=1}^{n} (-1)^{r+k} \cdot a_{rk} \cdot \det B_{r,k}$$

where $B_{r,k}$ is obtained from $A$ by removing row $r$ and column $k$. By swapping the order of columns in $B_{r,k}$ to match the form of $B_k$, we get

$$\det A = (-1)^{r+c} (a_{rc} \cdot \det A' - \sum_{k \in C} a_{rk} \cdot \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

**Corollary 2.** Let $A$ be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let $A'$ denote the matrix obtained from $A$ by performing a real pivot on $a_{rc}$. Then there exists a $(k-1) \times (k-1)$ submatrix $A''$ of $A'$ with $\det A'' \notin \{0, \pm 1\}$.

*Proof.* Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix $A''$ with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

**Definition 3.** Let $B_1, B_2$ be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = \begin{array}{|c|}\hline A_1 \\ \hline x \\ \hline\end{array}$ and $B_2 = \begin{array}{|c|c|}\hline y & A_2 \\ \hline\end{array}$, where $x$ is a row vector, $y$ is a column vector, and $A_1, A_2$ are matrices of appropriate dimensions. Let $D$ be the outer product of $y$ and $x$. The 2-sum of $B_1$ and $B_2$ is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{array}{|c|c|}\hline A_1 & 0 \\ \hline D & A_2 \\ \hline\end{array}.$$

**Definition 4.** Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix $A$ is $k$-TU if every square submatrix of $A$ of size $k$ has determinant in $\{0, \pm 1\}$.

**Remark 5.** Note that a matrix is TU if and only if it is $k$-TU for every $k \in \mathbb{Z}_{\geq 1}$.

**Lemma 6.** Let $B_1$ and $B_2$ be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then $B$ is 1-TU and 2-TU.

*Proof.* To see that $B$ is 1-TU, note that $B$ is a $\{0, \pm 1\}$ matrix by construction.

To show that $B$ is 2-TU, let $V$ be a $2 \times 2$ submatrix $V$ of $B$. If $V$ is a submatrix of $\begin{array}{|c|}\hline A_1 \\ \hline D \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline D & A_2 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline \end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_2 \\ \hline \end{array}$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise $V$ shares exactly one row and one column index with both $A_1$ and $A_2$. Let $i$ be the row shared by $V$ and $A_1$ and $j$ be the column shared by $V$ and $A_2$. Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing $V_{ij}$. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. □

**Lemma 7.** Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices $B_1$ and $B_2$ their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is $\ell$-TU for every $\ell < k$. Then for any TU matrices $B_1$ and $B_2$ their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also $k$-TU.

*Proof.* For the sake of deriving a contradiction, suppose there exist TU matrices $B_1$ and $B_2$ such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not $k$-TU. Then $B$ contains a $k \times k$ submatrix $V$ with $\det V \notin \{0, \pm 1\}$.

Note that $V$ cannot be a submatrix of $\begin{array}{|c|}\hline A_1 \\ \hline D \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline D & A_2 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline \end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_2 \\ \hline \end{array}$, as all of those four matrices are TU. Thus, $V$ shares at least one row and one column index with $A_1$ and $A_2$ each.

Consider the row of $V$ whose index appears in $A_1$. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a $\pm 1$ entry shared by $V$ and $A_1$. Let $r$ and $c$ denote the row and column index of this entry, respectively.

Perform a rational pivot in $B$ on the element $B_{rc}$. For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, $\tilde{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\begin{array}{|c|}\hline \tilde{A}_1 \\ \hline \tilde{D} \\ \hline \end{array}$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., $A_2$ remains unchanged. This holds because of the 0 block in $B$.
- $\tilde{D}$ consists of copies of $y$ scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\begin{array}{|c|c|}\hline \tilde{D} & \tilde{A}_2 \\ \hline \end{array}$ is TU, since this matrix consists of $A_2$ and copies of $y$ scaled by factors $\{0, \pm 1\}$.
- $\tilde{D}$ can be represented as an outer product of a column vector $\tilde{y}$ and a row vector $\tilde{x}$, and we can define $\tilde{B}_1 = \begin{array}{|c|}\hline \tilde{A}_1 \\ \hline \tilde{x} \\ \hline \end{array}$ and $\tilde{B}_2 = \begin{array}{|c|c|}\hline \tilde{y} & \tilde{A}_2 \\ \hline \end{array}$ similar to $B_1$ and $B_2$, respectively. Note that $\tilde{B}_1$ and $\tilde{B}_2$ have the same size as $B_1$ and $B_2$, respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- $\tilde{B}$ contains a square submatrix $\tilde{V}$ of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 2 from Lemma 1, pivoting in $V$ on the element $B_{rc}$ results in a matrix containing a $(k-1) \times (k-1)$ submatrix $V''$ with $\det V'' \in \{0, \pm 1\}$. Since $V$ is a submatrix of $B$, the submatrix $V''$ corresponds to a submatrix $\tilde{V}$ of $\tilde{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\tilde{B}$ that represents a 2-sum of TU matrices $\tilde{B}_1$ and $\tilde{B}_2$ and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k-1)$-TUness of $\tilde{B}$, which proves the lemma. □

**Lemma 8.** Let $B_1$ and $B_2$ be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.

*Proof.* Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices $B_1$ and $B_2$, their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is $\ell$-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 6.

Step: If the Proposition holds for some $k$, then it also holds for $k + 1$ by Lemma 7.

Conclusion: For any TU matrices $B_1$ and $B_2$, their 2-sum $B_1 \oplus_{2,x,y} B_2$ is $k$-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. $\square$

# 2 The 3-Sum of Regular Matroids Is Regular

## 2.1 Definition of 3-Sum

**Definition 9.** Let $B_1^{(0)} \in \mathbb{Z}_2^{(X_1 \cup \{x_0, x_1\}) \times (Y_1 \cup \{y_2\})}$, $B_2^{(0)} \in \mathbb{Z}_2^{(X_2 \cup \{x_2\}) \times (Y_2 \cup \{y_0, y_1\})}$ be matrices of the form

$$B_1^{(0)} = \begin{array}{|c|c|c|c|}\hline \multicolumn{3}{|c|}{A_1^{(0)}} & 0 \\ \cline{2-4} & 1 \;\; 1 & & 0 \\ \hline D_1^{(0)} & \multicolumn{2}{c|}{D_0^{(0)}} & \begin{array}{c} 1 \\ 1 \end{array} \\ \hline \end{array}, \quad B_2^{(0)} = \begin{array}{|c|c|c|}\hline 1 \;\; 1 & 0 & 0 \\ \hline D_0^{(0)} & \begin{array}{c} 1 \\ 1 \end{array} & \multirow{2}{*}{$A_2^{(0)}$} \\ \cline{1-2} D_2^{(0)} & & \\ \hline \end{array},$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{12}^{(0)} = D_2^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_1^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_1^{(0)}$ and $B_2^{(0)}$ is

$$B^{(0)} = B_1^{(0)} \oplus_3 B_2^{(0)} = \begin{array}{|c|c|c|c|}\hline \multicolumn{2}{|c|}{A_1^{(0)}} & \multicolumn{2}{c|}{0} \\ & 1 \;\; 1 & 0 & \\ \hline D_1^{(0)} & D_0^{(0)} & \begin{array}{c} 1 \\ 1 \end{array} & \multirow{2}{*}{$A_2^{(0)}$} \\ \cline{1-3} D_{12}^{(0)} & D_2^{(0)} & & \\ \hline \end{array} \in \mathbb{Z}_2^{(X_1 \cup X_2) \times (Y_1 \cup Y_2)}.$$

Here $x_2 \in X_1$, $x_0, x_1 \in X_2$, $y_0, y_1 \in Y_1$, $y_2 \in Y_2$, $A_1^{(0)} \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2^{(0)} \in \mathbb{Z}_2^{X_2 \times Y_2}$, $D_1^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_1 \setminus \{y_0, y_1\})}$, $D_2^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{12}^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times (Y_1 \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_1^{(0)}$, $B_2^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{1,12}^{(0)} = \begin{array}{|c|}\hline D_1^{(0)} \\ \hline D_{12}^{(0)} \\ \hline \end{array}, \quad D_{0,2}^{(0)} = \begin{array}{|c|}\hline D_0^{(0)} \\ \hline D_2^{(0)} \\ \hline \end{array}, \quad D_{1,0}^{(0)} = \begin{array}{|c|c|}\hline D_1^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{12,2}^{(0)} = \begin{array}{|c|c|}\hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|}\hline D_1^{(0)} & D_0^{(0)} \\ \hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$. can omit

**Lemma 10.** Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

*Proof.* Verify by complete enumeration. $\square$ need details?

## 2.2 Construction of Canonical Signing

**Definition 11.** We call $B_1$ and $B_2$ canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively, if they have the form

$$B_1 = \begin{array}{|c|c|c|c|c|}\hline \multicolumn{4}{|c|}{A_1} & 0 \\ \cline{2-5} & & 1 & 1 & 0 \\ \hline D_1 & \multicolumn{3}{c|}{D_0} & 1 \\ \cline{5-5} & \multicolumn{3}{c|}{} & 1 \\ \hline \end{array}\;, \quad B_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 & \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 & A_2 \\ \cline{1-3} \multicolumn{2}{|c|}{D_2} & \multicolumn{2}{c|}{} \\ \hline \end{array}$$

where every block in $B_1$ and $B_2$ is a signing of the corresponding block in $B_1^{(0)}$ and $B_2^{(0)}$, and $D_0$ is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings $B_1$ and $B_2$, the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \begin{array}{|c|c|c|c|c|c|}\hline \multicolumn{4}{|c|}{A_1} & \multicolumn{2}{c|}{0} \\ \cline{2-5} & & 1 & 1 & 0 & \\ \hline D_1 & \multicolumn{3}{c|}{D_0} & 1 & \\ \cline{5-5} & \multicolumn{3}{c|}{} & 1 & A_2 \\ \cline{1-5} D_{12} & \multicolumn{3}{c|}{D_2} & \multicolumn{2}{c|}{} \\ \hline \end{array}$$

where $D_{12} = D_2 \cdot (D_0)^{-1} \cdot D_1$ (calculated over $\mathbb{Q}$).

The following lemma helps construct canonical signings from arbitrary initial TU signings.

**Lemma 12.** Let $Q'$ be a TU signing of the matrix

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0^{(0)}} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0,x_1,x_2)\times(y_0,y_1,y_2)}$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Define $u \in \{0, \pm 1\}^{\{x_0,x_1,x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0,y_1,y_2\}}$, and $Q$ as follows:

$$\begin{gathered} u(x_0) = Q'(x_2, y_0) \cdot Q'(x_0, y_0), \\ u(x_1) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), \\ u(x_2) = 1, \\ v(y_0) = Q'(x_2, y_0), \\ v(y_1) = Q'(x_2, y_1), \\ v(y_2) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), \\ \forall i \in \{x_0, x_1, x_2\},\ \forall j \in \{y_0, y_1, y_2\}, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j). \end{gathered}$$

Then $Q$ is a TU signing of $T$ and $Q = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array}$ where $D_0$ is the respective canonical signing of $D_0^{(0)}$.

*Proof.* Since $Q'$ is a TU signing of $T$ and $Q$ is obtained from $Q'$ by multiplying rows and columns by $\pm 1$ factors, $Q$ is also a TU signing of $T$. By construction, we have

$$
\begin{aligned}
&Q(x_2, y_0) = Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1,\\
&Q(x_2, y_1) = Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1,\\
&Q(x_2, y_2) = 0,\\
&Q(x_0, y_0) = Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1,\\
&Q(x_0, y_1) = Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1),\\
&Q(x_0, y_2) = Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1,\\
&Q(x_1, y_0) = 0,\\
&Q(x_1, y_1) = Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)),\\
&Q(x_1, y_2) = Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1.
\end{aligned}
$$

Thus, it remains to check that $Q(x_0, y_1)$ and $Q(x_1, y_1)$ are correct.

First, consider the entry $Q(x_0, y_1)$. If $D_0^{(0)}(x_0, y_1) = 0$, then $Q(x_0, y_1) = 0$, as needed. Otherwise, if $D_0^{(0)}(x_0, y_1) = 1$, then $Q(x_0, y_1) \in \{\pm 1\}$, as $Q$ is a signing of $T$. Our goal is to show that $Q(x_0, y_1) = 1$. For the sake of deriving a contradiction suppose that $Q(x_0, y_1) = -1$. Then the determinant of the submatrix of $Q$ indexed by $\{x_0, x_2\} \times \{y_0, y_1\}$ is

$$\det \begin{array}{|c|c|}\hline 1 & -1 \\ \hline 1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of $Q$. Thus, $Q(x_0, y_1) = 1$, as needed.

Consider the entry $Q(x_1, y_1)$. Since $Q$ is a signing of $T$, we have $Q(x_1, y_1) \in \{\pm 1\}$. Note that we know all the other entries of $Q$, so we can determine the sign of $Q(x_1, y_1)$ using TUness of $Q$. Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q(x_1, y_1) = 1$, then $\det Q = \det \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of $Q$. Thus, $Q(x_1, y_1) = -1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q(x_1, y_1) = -1$, then $\det Q = \det \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & -1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of $Q$. Thus, $Q(x_1, y_1) = 1$, as needed.

$\square$

**Definition 13.** Let $X$ and $Y$ be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q' \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q^{(0)} \in \mathbb{Z}_2^{X \times Y}$. Let $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and $Q$ be constructed as follows:

$$
u(i) = \begin{cases} Q'(x_2, y_0) \cdot Q'(x_0, y_0), & i = x_0, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}
$$

$$
v(j) = \begin{cases} Q'(x_2, y_0), & j = y_0, \\ Q'(x_2, y_1), & j = y_1, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}
$$

$$\forall i \in X,\ \forall j \in Y, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j).$$

We call $Q$ a canonical resigning of $Q'$.

**Lemma 14.** Let $B_1'$ be a TU signing of $B_1^{(0)}$. Let $B_1$ be the canonical resigning (constructed following Definition 13) of $B_1'$. Then $B_1$ is a canonical signing of $B_1^{(0)}$ (in the sense of Definition 11) and $B_1$ is TU. Going forward, we refer to $B_1$ as a TU canonical signing for short of $B_1^{(0)}$. A TU canonical signing $B_2$ of $B_2^{(0)}$ is defined similarly (up to replacing subscripts 1 by 2).

*Proof.* This follows directly from Lemma 12. $\square$

**Lemma 15.** Let $B_2$ be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = (D_{0,2})_{\bullet,y_0}$ and $c_1 = (D_{0,2})_{\bullet,y_1}$. Then the following matrices are TU:

$$B_2^{(a)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_0 & A_2 \\ \hline\end{array}, \quad B_2^{(b)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_1 & A_2 \\ \hline\end{array}.$$

*Proof.* Pivoting in $B_2$ on $(x_2, y_0)$ and $(x_2, y_1)$ yields:

$$B_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_2 \\ \hline\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -c_0 & c_1 - c_0 & A_2 \\ \hline\end{array}$$

$$B_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_2 \\ \hline\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 - c_1 & -c_1 & A_2 \\ \hline\end{array}$$

By removing row $x_2$ from the resulting matrices and then multiplying columns $y_0$ and $y_1$ by $\{\pm 1\}$ factors, we obtain $B_2^{(a)}$ and $B_2^{(b)}$. By Lemma 14, $B_2$ is TU. Since TUness is preserved under pivoting, taking submatrices, and multiplying columns by $\pm 1$ factors, we conclude that $B_2^{(a)}$ and $B_2^{(b)}$ are TU. $\square$

**Lemma 16.** Let $B_2$ be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = D_{0,2}(\bullet, y_0)$, $c_1 = D_{0,2}(\bullet, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.

1. For every $i \in X_2$, we have $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}\}$.

2. $\begin{array}{|c|c|c|c|}\hline A_2 & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU.

*Proof.*

1. Let $i \in X_2$. If $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}$, then the $2 \times 2$ submatrix of $B_2$ indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has det $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & -1 \\ \hline\end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of $B_2$ (which holds by Lemma 14). Similarly, if $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}$, then the $2 \times 2$ submatrix of $B_2$ indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has det $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline -1 & 1 \\ \hline\end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of $B_2$.

2. Let $V$ be a square submatrix of $\begin{array}{|c|c|c|c|}\hline A_2 & c_0 & c_1 & c_2 \\ \hline\end{array}$. We will show that $\det V \in \{0, \pm 1\}$.

   Let $z$ denote the index of the appended column $c_2$. Suppose that column $z$ is not in $V$. Then $V$ is a submatrix of $B_2$, which is TU by Lemma 14. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column $z$ is in $V$.

   Suppose that columns $y_0$ and $y_1$ are both in $V$. Then $V$ contains columns $z$, $y_0$, and $y_1$, which are linearly dependent by construction of $c_2$. Thus, $\det V = 0$. Going forward we assume that at most one of the columns $y_0$ and $y_2$ is in $V$.

   Suppose that column $y_0$ is in $V$. Then $V$ is a submatrix of $B_2^{(b)}$ from Lemma 15, and thus $\det V \in \{0, \pm 1\}$. Otherwise, $V$ is a submatrix of $B_2^{(a)}$ from Lemma 15, and so $\det V \in \{0, \pm 1\}$.

   Thus, every square submatrix $V$ of $\tilde{T}$ has $\det V \in \{0, \pm 1\}$, and hence $\tilde{T}$ is TU.

$\square$

**Remark 17.** Vectors $c_0$, $c_1$, and $c_2$ can be defined directly in terms of entries of $B_2$, e.g., $c_2$ consists of entries of $B_2$ indexed by $(X_2 \setminus \{x_2\}) \times \{y_0\}$.

**Lemma 18.** Let $B_1$ be a TU canonical signing of $B_1^{(0)}$. Let $d_0 = D_{1,0}(x_0, \bullet)$, $d_1 = D_{1,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$. Then the following properties hold.

1. For every $j \in Y_2$, we have $\begin{array}{|c|}\hline d_0(i)\\ \hline d_1(i)\\ \hline\end{array} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \{\begin{array}{|c|}\hline 1\\ \hline -1\\ \hline\end{array}, \begin{array}{|c|}\hline -1\\ \hline 1\\ \hline\end{array}\}$.

2. $\begin{array}{|c|}\hline A_1\\ \hline d_0\\ \hline d_1\\ \hline d_2\\ \hline\end{array}$ is TU.

*Proof.* Apply Lemma 16 to $B_1^\top$, or repeat the same argument up to interchanging rows and columns. $\square$

**Lemma 19.** Let $B_1$ and $B_2$ be TU canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively.

- Let $c_0 = D_{0,2}(\bullet, y_0)$, $c_1 = D_{0,2}(\bullet, y_1)$, and $c_2 = c_0 - c_1$.
- Let $d_0 = D_{1,0}(x_0, \bullet)$, $d_1 = D_{1,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$.
- If $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0\\ \hline 0 & 1\\ \hline\end{array}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1\\ \hline 0 & 1\\ \hline\end{array}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.
- Let $D$ be the bottom-left block in the canonical signing $B$ of $B^{(0)}$ corresponding to $B_1$ and $B_2$

Then the following properties hold.

1. $D = c_0 \cdot r_0 + c_1 \cdot r_1$.

2. Rows of $D$ are in $\begin{array}{|c|}\hline \pm r_0\\ \hline \pm r_1\\ \hline \pm r_2\\ \hline 0\\ \hline\end{array}$.

3. Columns of $D$ are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm c_2 & 0\\ \hline\end{array}$.

4. $\begin{array}{|c|c|c|c|}\hline A_2 & c_0 & c_1 & c_2\\ \hline\end{array}$ is TU.

5. $\begin{array}{|c|c|}\hline A_2 & D\\ \hline\end{array}$ is TU.

6. $\begin{array}{|c|}\hline A_1\\ \hline r_0\\ \hline r_1\\ \hline r_2\\ \hline\end{array}$ is TU.

7. $\begin{array}{|c|}\hline A_1\\ \hline D\\ \hline\end{array}$ is TU.

8. $\begin{array}{|c|c|}\hline c_0 & c_1\\ \hline\end{array}$ contains $D_0$ (the canonical signing of $D_0^{(0)}$) as a submatrix.

*Proof.* 1. Follows via a direct calculation. need details?

2. By item 1, for every $i \in X_2$ we have $D(i, \bullet) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$. By Lemma 15.1, we know that $\begin{array}{|c|c|}\hline c_0(i) & c_1(i)\\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1\\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1\\ \hline\end{array}\}$. Therefore, $D(i, \bullet)$ is equal to either $0$, $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.

3. Holds by the same argument as item 2 up to interchanging rows and columns.

4. Holds by Lemma 16.2.

5. By item 3, columns of $\begin{array}{|c|c|}\hline A_2 & D \\ \hline\end{array}$ are in $\begin{array}{|c|c|c|c|c|}\hline A_2 & \pm c_0 & \pm c_1 & \pm c_2 & 0 \\ \hline\end{array}$. Since $\begin{array}{|c|c|c|c|}\hline A_2 & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU and since adding zero columns and copies of columns multiplied by $\pm 1$ factors preserves TUness, $\begin{array}{|c|c|}\hline A_2 & D \\ \hline\end{array}$ is also TU.

6. By Lemma 18.2 (or by the same argument as item 4 up to interchanging rows and columns), $\begin{array}{|c|}\hline A_1 \\ \hline d_0 \\ \hline d_1 \\ \hline d_2 \\ \hline\end{array}$ is TU. Since TUness is preserved under multiplication of rows by $\pm 1$ and exchanging rows, $\begin{array}{|c|}\hline A_1 \\ \hline r_0 \\ \hline r_1 \\ \hline r_2 \\ \hline\end{array}$ is also TU.

7. Holds by the same argument as item 5 up to interchanging rows and columns.

8. Holds by construction.

$\square$

**Definition 20.** Let $A_1 \in \mathbb{Q}^{X_1 \cup Y_1}$, $A_2 \in \mathbb{Q}^{X_2 \cup Y_2}$, $c_0, c_1 \in \mathbb{Q}^{X_2}$, $r_0, r_1 \in \mathbb{Q}^{Y_1}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–8 from the statement of Lemma 19 are satisfied for $A_1$, $A_2$, $c_0$, $c_1$, $r_0$, $r_1$. Given $k \in \mathbb{Z}_{\geq 1}$, define Proposition$(A_1, A_2, c_0, c_1, r_0, r_1, k)$ to mean "$C = \begin{array}{|c|c|}\hline A_1 & 0 \\ \hline D & A_2 \\ \hline\end{array}$ is $k$-TU".

**Lemma 21.** Assume the notation of Definition 20. Then Proposition$(A_1, A_2, c_0, c_1, r_0, r_1, 1)$ holds.

*Proof.* Every entry of $C$ is in one of four blocks: 0, $A_1$, $D$, $A_2$. By the assumptions of Definition 20, all of these blocks are TU. Thus, $C$ is 1-TU. $\square$

**Lemma 22.** Assume the notation of Definition 20. Let $i \in X_1$, let $T = \begin{array}{|c|}\hline A_1(i, \bullet) \\ \hline D \\ \hline\end{array}$. Suppose we pivot on entry $T(i, j) \in \{\pm 1\}$ in $T$ and obtain matrix $T' = \begin{array}{|c|}\hline a' \\ \hline D' \\ \hline\end{array}$. Then columns of $D'$ are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$.

*Proof.* Since $T$ is a submatrix of $\begin{array}{|c|}\hline A_1 \\ \hline D \\ \hline\end{array}$, which is TU by assumptions of Definition 20, we have that $T$ is TU. Since pivoting preserves TUness, $T'$ is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column $p$ in $T$ could be and what another column $q$ in $T$ could be. This uniquely determines the resulting columns $p'$ and $q'$ in $T'$ by the pivot formula. In every case, either $\begin{array}{|c|c|}\hline p' & q' \\ \hline\end{array}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of $T'$, or the restriction of $p'$ and $q'$ to $X_2$ is in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$). $\square$ need details?

**Lemma 23.** Assume the notation of Definition 20. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k-1)$ holds for all $A_1'$, $r_0'$, and $r_1'$ satisfying the assumptions of Definition 20 (together with $A_2$, $c_0$, and $c_1$). Then Proposition$(A_1, A_2, c_0, c_1, r_0, r_1, k)$ holds.

*Proof.* Let $V$ be a $k \times k$ submatrix of $C$. For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that $V$ is a submatrix of $\begin{array}{|c|}\hline A_1 \\ \hline D \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline D & A_2 \\ \hline\end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_2 \\ \hline\end{array}$. Since all of those four matrices are TU by the assumptions of Definition 20, we have $\det V \in \{0, \pm 1\}$. Thus, $V$ shares at least one row and one column index with $A_1$ and $A_2$ each.

Consider the row index shared by $V$ and $A_1$. Note that this row in $V$ cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a $\pm 1$ entry shared by $V$ and $A_1$. Let $i$ and $j$ denote the row and the column index of this entry, respectively.

Perform a pivot in $C$ on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, $B'$ denotes the entire matrix after the pivot. Note that the following statements hold.

- $C'$ contains a $(k-1) \times (k-1)$ submatrix $V'$ with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix $V'$ of $C'$ with the same row and column index sets as $V$ minus the pivot row $i$ and pivot column $j$.

- $C' = \begin{array}{|c|c|} \hline A_1' & 0 \\ \hline D' & A_2 \\ \hline \end{array}$, i.e., the 0 and the $A_2$ blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.

- $\begin{array}{|c|} \hline A_1' \\ \hline D' \\ \hline \end{array}$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.

- The columns of $D'$ are in $\begin{array}{|c|c|c|c|} \hline 0 & \pm c_0 & \pm c_1 & \pm(c_0 - c_1) \\ \hline \end{array}$. This holds by Lemma 22.

- There exist $r_0'$ and $r_1'$ such that $D' = c_0 \cdot r_0' + c_1 \cdot r_1'$ and the assumptions of Definition 20 are satisfied for $A_1'$, $A_2$, $c_0$, $c_1$, $r_0'$, $r_1'$. This follows from the previous bullet point by carefully checking assumptions. need details?

- $C'$ is $(k-1)$-TU. This follows from the hypothesis: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k-1)$ holds.

To sum up, after pivoting we obtain a matrix $C'$ (which can be obtained in the manner of Definition 20) that is $(k-1)$-TU and contains a $(k-1) \times (k-1)$ submatrix $V'$ with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. □

**Lemma 24.** Let $B_1$ and $B_2$ be TU canonical signings. Then the corresponding canonical signing $B$ is TU.

*Proof.* Define $A_1$, $A_2$, $c_0$, $c_1$, $r_0$, $r_1$ as in Lemma 19. Note that canonical signing $B$ has the form of $C$ in the notation of Definition 20.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k)$ holds for all $A_1'$, $r_0'$, and $r_1'$ satisfying the assumptions of Definition 20.

Base: The Proposition holds for $k = 1$ by Lemma 21.

Step: If the Proposition holds for some $k$, then it also holds for $k+1$ by Lemma 23.

Conclusion: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$.

Specializing the conclusion to $A_1$, $A_2$, $c_0$, $c_1$, $r_0$, $r_1$ (obtained from $B_1$ and $B_2$ as described in the statement of Lemma 19) shows that canonical signing $B$ is $k$-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B$ is TU. □

**Corollary 25.** Suppose that $B_1^{(0)}$ and $B_2^{(0)}$ have TU signings. Then $B_1 \oplus_3 B_2$ has a TU signing.

*Proof sketch.* Start with some TU signings, obtain canonical signings, apply Lemma 24. □