

# Proof of Regularity of 2-Sum and 3-Sum of Matroids

Ivan Sergeev

March–April 2025

## 1 The 2-Sum of Regular Matroids Is Regular

**Lemma 1.** Let  $A$  be a  $k \times k$  matrix. Let  $r, c \in \{1, \dots, k\}$  be a row and column index, respectively, such that  $a_{rc} \neq 0$ . Let  $A'$  denote the matrix obtained from  $A$  by performing a real pivot on  $a_{rc}$ . Then there exists a  $(k-1) \times (k-1)$  submatrix  $A''$  of  $A'$  with  $|\det A''| = \frac{|\det A|}{|a_{rc}|}$ .

*Proof.* Let  $A''$  be the submatrix of  $A'$  given by row index set  $R = \{1, \dots, k\} \setminus \{r\}$  and column index set  $C = \{1, \dots, k\} \setminus \{c\}$ . By the explicit formula for pivoting in  $A$  on  $a_{rc}$ , the entries of  $A''$  are given by  $a''_{ij} = a_{ij} - \frac{a_{ic} \cdot a_{rj}}{a_{rc}}$ . Using the linearity of the determinant, we can express  $\det A''$  as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \cdot \det B''_k$$

where  $B''_k$  is a matrix obtained from  $A''$  by replacing column  $a''_{\bullet k}$  with the pivot column  $a_{\bullet c}$  without the pivot element  $a_{rc}$ .

By the cofactor expansion in  $A$  along row  $r$ , we have

$$\det A = \sum_{k=1}^n (-1)^{r+k} \cdot a_{rk} \cdot \det B_{r,k}$$

where  $B_{r,k}$  is obtained from  $A$  by removing row  $r$  and column  $k$ . By swapping the order of columns in  $B_{r,k}$  to match the form of  $B_k$ , we get

$$\det A = (-1)^{r+c} (a_{rc} \cdot \det A' - \sum_{k \in C} a_{rk} \cdot \det B''_k).$$

By combining the above results, we get  $|\det A''| = \frac{|\det A|}{|a_{rc}|}$ . □

**Corollary 2.** Let  $A$  be a  $k \times k$  matrix with  $\det A \notin \{0, \pm 1\}$ . Let  $r, c \in \{1, \dots, k\}$  be a row and column index, respectively, and suppose that  $a_{rc} \in \{\pm 1\}$ . Let  $A'$  denote the matrix obtained from  $A$  by performing a real pivot on  $a_{rc}$ . Then there exists a  $(k-1) \times (k-1)$  submatrix  $A''$  of  $A'$  with  $\det A'' \notin \{0, \pm 1\}$ .

*Proof.* Since  $a_{rc} \in \{\pm 1\}$ , by Lemma 1 there exists a  $(k-1) \times (k-1)$  submatrix  $A''$  with  $|\det A| = |\det A''|$ . Since  $\det A \notin \{0, \pm 1\}$ , we have  $\det A'' \notin \{0, \pm 1\}$ . □

**Definition 3.** Let  $B_l, B_r$  be matrices with  $\{0, \pm 1\}$  entries expressed as  $B_l = \begin{bmatrix} A_l \\ x \end{bmatrix}$  and  $B_r = \begin{bmatrix} y & A_r \end{bmatrix}$ , where  $x$  is a row vector,  $y$  is a column vector, and  $A_l, A_r$  are matrices of appropriate dimensions. Let  $D$  be the outer product of  $y$  and  $x$ . The 2-sum of  $B_l$  and  $B_r$  is defined as

$$B_l \oplus_{2,x,y} B_r = \begin{bmatrix} A_l & 0 \\ D & A_r \end{bmatrix}.$$

**Definition 4.** Given  $k \in \mathbb{Z}_{\geq 1}$ , we say that a matrix  $A$  is  $k$ -TU if every square submatrix of  $A$  of size  $k$  has determinant in  $\{0, \pm 1\}$ .

**Remark 5.** Note that a matrix is TU if and only if it is  $k$ -TU for every  $k \in \mathbb{Z}_{\geq 1}$ .

**Lemma 6.** Let  $B_l$  and  $B_r$  be TU matrices and let  $B = B_l \oplus_{2,x,y} B_r$ . Then  $B$  is 1-TU and 2-TU.

*Proof.* To see that  $B$  is 1-TU, note that  $B$  is a  $\{0, \pm 1\}$  matrix by construction.

To show that  $B$  is 2-TU, let  $V$  be a  $2 \times 2$  submatrix  $V$  of  $B$ . If  $V$  is a submatrix of  $\begin{bmatrix} A_l \\ D \end{bmatrix}$ ,  $\begin{bmatrix} D & A_r \end{bmatrix}$ ,  $\begin{bmatrix} A_l & 0 \end{bmatrix}$ , or  $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$ , then  $\det V \in \{0, \pm 1\}$ , as all of those four matrices are TU. Otherwise  $V$  shares exactly one row and one column index with both  $A_l$  and  $A_r$ . Let  $i$  be the row shared by  $V$  and  $A_l$  and  $j$  be the column shared by  $V$  and  $A_r$ . Note that  $V_{ij} = 0$ . Thus, up to sign,  $\det V$  equals the product of the entries on the diagonal not containing  $V_{ij}$ . Since both of those entries are in  $\{0, \pm 1\}$ , we have  $\det V \in \{0, \pm 1\}$ .  $\square$

**Lemma 7.** Let  $k \in \mathbb{Z}_{\geq 1}$ . Suppose that for any TU matrices  $B_l$  and  $B_r$  their 2-sum  $B = B_l \oplus_{2,x,y} B_r$  is  $\ell$ -TU for every  $\ell < k$ . Then for any TU matrices  $B_l$  and  $B_r$  their 2-sum  $B = B_l \oplus_{2,x,y} B_r$  is also  $k$ -TU.

*Proof.* For the sake of deriving a contradiction, suppose there exist TU matrices  $B_l$  and  $B_r$  such that their 2-sum  $B = B_l \oplus_{2,x,y} B_r$  is not  $k$ -TU. Then  $B$  contains a  $k \times k$  submatrix  $V$  with  $\det V \notin \{0, \pm 1\}$ .

Note that  $V$  cannot be a submatrix of  $\begin{bmatrix} A_l \\ D \end{bmatrix}$ ,  $\begin{bmatrix} D & A_r \end{bmatrix}$ ,  $\begin{bmatrix} A_l & 0 \end{bmatrix}$ , or  $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$ , as all of those four matrices are TU. Thus,  $V$  shares at least one row and one column index with  $A_l$  and  $A_r$  each.

Consider the row of  $V$  whose index appears in  $A_l$ . Note that it cannot consist of only 0 entries, as otherwise  $\det V = 0$ . Thus there exists a  $\pm 1$  entry shared by  $V$  and  $A_l$ . Let  $r$  and  $c$  denote the row and column index of this entry, respectively.

Perform a rational pivot in  $B$  on the element  $B_{rc}$ . For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example,  $\tilde{B}$  denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\begin{bmatrix} \tilde{A}_l \\ \tilde{D} \end{bmatrix}$  is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_r$ , i.e.,  $A_r$  remains unchanged. This holds because of the 0 block in  $B$ .
- $\tilde{D}$  consists of copies of  $y$  scaled by factors in  $\{0, \pm 1\}$ . This can be verified via a case distinction and a simple calculation.
- $\begin{bmatrix} \tilde{D} & \tilde{A}_r \end{bmatrix}$  is TU, since this matrix consists of  $A_r$  and copies of  $y$  scaled by factors  $\{0, \pm 1\}$ .
- $\tilde{D}$  can be represented as an outer product of a column vector  $\tilde{y}$  and a row vector  $\tilde{x}$ , and we can define  $\tilde{B}_1 = \begin{bmatrix} \tilde{A}_l \\ \tilde{x} \end{bmatrix}$  and  $\tilde{B}_2 = \begin{bmatrix} \tilde{y} & \tilde{A}_r \end{bmatrix}$  similar to  $B_l$  and  $B_r$ , respectively. Note that  $\tilde{B}_1$  and  $\tilde{B}_2$  have the same size as  $B_l$  and  $B_r$ , respectively, are both TU, and satisfy  $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$ .
- $\tilde{B}$  contains a square submatrix  $\tilde{V}$  of size  $k - 1$  with  $\det \tilde{V} \notin \{0, \pm 1\}$ . Indeed, by Corollary 2 from Lemma 1, pivoting in  $V$  on the element  $B_{rc}$  results in a matrix containing a  $(k - 1) \times (k - 1)$  submatrix  $V''$  with  $\det V'' \in \{0, \pm 1\}$ . Since  $V$  is a submatrix of  $B$ , the submatrix  $V''$  corresponds to a submatrix  $\tilde{V}$  of  $\tilde{B}$  with the same property.

To sum up, after pivoting we obtain a matrix  $\tilde{B}$  that represents a 2-sum of TU matrices  $\tilde{B}_1$  and  $\tilde{B}_2$  and contains a square submatrix of size  $k - 1$  with determinant not in  $\{0, \pm 1\}$ . This is a contradiction with  $(k - 1)$ -TUness of  $\tilde{B}$ , which proves the lemma.  $\square$

**Lemma 8.** Let  $B_l$  and  $B_r$  be TU matrices. Then  $B_l \oplus_{2,x,y} B_r$  is also TU.

*Proof.* Proof by induction.

Proposition for any  $k \in \mathbb{Z}_{\geq 1}$ : For any TU matrices  $B_l$  and  $B_r$ , their 2-sum  $B = B_l \oplus_{2,x,y} B_r$  is  $\ell$ -TU for every  $\ell \leq k$ .

Base: The Proposition holds for  $k = 1$  and  $k = 2$  by Lemma 6.

Step: If the Proposition holds for some  $k$ , then it also holds for  $k + 1$  by Lemma 7.

Conclusion: For any TU matrices  $B_l$  and  $B_r$ , their 2-sum  $B_l \oplus_{2,x,y} B_r$  is  $k$ -TU for every  $k \in \mathbb{Z}_{\geq 1}$ . Thus,  $B_l \oplus_{2,x,y} B_r$  is TU.  $\square$

## 2 The 3-Sum of Regular Matroids Is Regular

### 2.1 Definition of 3-Sum

**Definition 9.** Let  $B_l^{(0)} \in \mathbb{Z}_2^{(X_l \cup \{x_0, x_1\}) \times (Y_l \cup \{y_2\})}$ ,  $B_r^{(0)} \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0, y_1\})}$  be matrices of the form

$$B_l^{(0)} = \begin{array}{|c|cc|c|} \hline & & & 0 \\ \hline & A_l^{(0)} & & \\ \hline & 1 & 1 & 0 \\ \hline D_l^{(0)} & D_0^{(0)} & & 1 \\ & & & 1 \\ \hline \end{array}, \quad B_r^{(0)} = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline D_0^{(0)} & 1 & & \\ & 1 & & A_r^{(0)} \\ \hline D_r^{(0)} & & & \\ \hline \end{array},$$

where  $D_0^{(0)}(x_0, y_0) = 1$ ,  $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$ ,  $D_0^{(0)}(x_1, y_0) = 0$ , and  $D_0^{(0)}(x_1, y_1) = 1$ . Let  $D_{lr}^{(0)} = D_r^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_l^{(0)}$  (note that  $D_0^{(0)}$  is invertible by construction). Then the 3-sum of  $B_l^{(0)}$  and  $B_r^{(0)}$  is

$$B^{(0)} = B_l^{(0)} \oplus_3 B_r^{(0)} = \begin{array}{|c|cc|c|} \hline & & & 0 \\ \hline & A_l^{(0)} & & \\ \hline & 1 & 1 & 0 \\ \hline D_l^{(0)} & D_0^{(0)} & 1 & \\ & & 1 & A_r^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} & & \\ \hline \end{array} \in \mathbb{Z}_2^{(X_l \cup X_r) \times (Y_l \cup Y_r)}.$$

Here  $x_2 \in X_l$ ,  $x_0, x_1 \in X_r$ ,  $y_0, y_1 \in Y_l$ ,  $y_2 \in Y_r$ ,  $A_l^{(0)} \in \mathbb{Z}_2^{X_l \times Y_l}$ ,  $A_r^{(0)} \in \mathbb{Z}_2^{X_r \times Y_r}$ ,  $D_l^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_l \setminus \{y_0, y_1\})}$ ,  $D_r^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$ ,  $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$ ,  $D_{lr}^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times (Y_l \setminus \{y_0, y_1\})}$ . The indexing is kept consistent between  $B_l^{(0)}$ ,  $B_r^{(0)}$ , and  $B^{(0)}$ . To simplify notation, we use the following shorthands:

$$D_{l,lr}^{(0)} = \begin{array}{|c|} \hline D_l^{(0)} \\ \hline D_{lr}^{(0)} \\ \hline \end{array}, \quad D_{0,r}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_r^{(0)} \\ \hline \end{array}, \quad D_{l,0}^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{lr,r}^{(0)} = \begin{array}{|c|c|} \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of  $D_0^{(0)}$ .

can omit

**Lemma 10.** Let  $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$  be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

*Proof.* Verify by complete enumeration.

$\square$

need details?

## 2.2 Construction of Canonical Signing

**Definition 11.** We call  $B_l$  and  $B_r$  canonical signings of  $B_l^{(0)}$  and  $B_r^{(0)}$ , respectively, if they have the form

$$B_l = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_l & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_l & D_0 & 1 \\ \hline & & 1 \\ \hline \end{array}, \quad B_r = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline & D_0 & 1 & \\ \hline & & 1 & A_r \\ \hline D_r & & & \\ \hline \end{array}$$

where every block in  $B_l$  and  $B_r$  is a signing of the corresponding block in  $B_l^{(0)}$  and  $B_r^{(0)}$ , and  $D_0$  is the canonical signing of  $D_0^{(0)}$ , which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings  $B_l$  and  $B_r$ , the corresponding canonical signing of  $B^{(0)}$  is defined as

$$B = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_l & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_l & D_0 & 1 \\ \hline & & 1 & A_r \\ \hline D_{lr} & D_r & & \\ \hline \end{array}$$

where  $D_{lr} = D_r \cdot (D_0)^{-1} \cdot D_l$  (calculated over  $\mathbb{Q}$ ).

The following lemma helps construct canonical signings from arbitrary initial TU signings.

**Lemma 12.** Let  $Q'$  be a TU signing of the matrix

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0^{(0)} & 1 & \\ \hline & & 1 \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0, x_1, x_2) \times (y_0, y_1, y_2)}$$

where  $D_0^{(0)}(x_0, y_0) = 1$ ,  $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$ ,  $D_0^{(0)}(x_1, y_0) = 0$ , and  $D_0^{(0)}(x_1, y_1) = 1$ . Define  $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$ ,  $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$ , and  $Q$  as follows:

$$\begin{aligned} u(x_0) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0), \\ u(x_1) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), \\ u(x_2) &= 1, \\ v(y_0) &= Q'(x_2, y_0), \\ v(y_1) &= Q'(x_2, y_1), \\ v(y_2) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), \\ \forall i \in \{x_0, x_1, x_2\}, \forall j \in \{y_0, y_1, y_2\}, \quad Q(i, j) &= Q'(i, j) \cdot u(i) \cdot v(j). \end{aligned}$$

Then  $Q$  is a TU signing of  $T$  and  $Q = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0 & 1 & \\ \hline & & 1 \\ \hline \end{array}$  where  $D_0$  is the respective canonical signing of  $D_0^{(0)}$ .

*Proof.* Since  $Q'$  is a TU signing of  $T$  and  $Q$  is obtained from  $Q'$  by multiplying rows and columns by  $\pm 1$  factors,  $Q$  is also a TU signing of  $T$ . By construction, we have

$$\begin{aligned}
Q(x_2, y_0) &= Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1, \\
Q(x_2, y_1) &= Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1, \\
Q(x_2, y_2) &= 0, \\
Q(x_0, y_0) &= Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1, \\
Q(x_0, y_1) &= Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1), \\
Q(x_0, y_2) &= Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1, \\
Q(x_1, y_0) &= 0, \\
Q(x_1, y_1) &= Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)), \\
Q(x_1, y_2) &= Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1.
\end{aligned}$$

Thus, it remains to check that  $Q(x_0, y_1)$  and  $Q(x_1, y_1)$  are correct.

First, consider the entry  $Q(x_0, y_1)$ . If  $D_0^{(0)}(x_0, y_1) = 0$ , then  $Q(x_0, y_1) = 0$ , as needed. Otherwise, if  $D_0^{(0)}(x_0, y_1) = 1$ , then  $Q(x_0, y_1) \in \{\pm 1\}$ , as  $Q$  is a signing of  $T$ . Our goal is to show that  $Q(x_0, y_1) = 1$ . For the sake of deriving a contradiction suppose that  $Q(x_0, y_1) = -1$ . Then the determinant of the submatrix of  $Q$  indexed by  $\{x_0, x_2\} \times \{y_0, y_1\}$  is

$$\det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of  $Q$ . Thus,  $Q(x_0, y_1) = 1$ , as needed.

Consider the entry  $Q(x_1, y_1)$ . Since  $Q$  is a signing of  $T$ , we have  $Q(x_1, y_1) \in \{\pm 1\}$ . Note that we know all the other entries of  $Q$ , so we can determine the sign of  $Q(x_1, y_1)$  using TUness of  $Q$ . Consider two cases.

1. Suppose that  $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . If  $Q(x_1, y_1) = 1$ , then  $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$ , which contradicts TUness of  $Q$ . Thus,  $Q(x_1, y_1) = -1$ , as needed.

2. Suppose that  $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . If  $Q(x_1, y_1) = -1$ , then  $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$ , which contradicts TUness of  $Q$ . Thus,  $Q(x_1, y_1) = 1$ , as needed.

□

**Definition 13.** Let  $X$  and  $Y$  be sets with  $\{x_0, x_1, x_2\} \subseteq X$  and  $\{y_0, y_1, y_2\} \subseteq Y$ . Let  $Q' \in \mathbb{Q}^{X \times Y}$  be a TU signing of  $Q^{(0)} \in \mathbb{Z}_2^{X \times Y}$ . Let  $u \in \{0, \pm 1\}^X$ ,  $v \in \{0, \pm 1\}^Y$ , and  $Q$  be constructed as follows:

$$\begin{aligned}
u(i) &= \begin{cases} Q'(x_2, y_0) \cdot Q'(x_0, y_0), & i = x_0, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases} \\
v(j) &= \begin{cases} Q'(x_2, y_0), & j = y_0, \\ Q'(x_2, y_1), & j = y_1, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases} \\
&\quad \forall i \in X, \forall j \in Y, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j).
\end{aligned}$$

We call  $Q$  a canonical resigning of  $Q'$ .

**Lemma 14.** Let  $B'_l$  be a TU signing of  $B_l^{(0)}$ . Let  $B_l$  be the canonical resigning (constructed following Definition 13) of  $B'_l$ . Then  $B_l$  is a canonical signing of  $B_l^{(0)}$  (in the sense of Definition 11) and  $B_l$  is TU. Going forward, we refer to  $B_l$  as a TU canonical signing for short of  $B_l^{(0)}$ . A TU canonical signing  $B_r$  of  $B_r^{(0)}$  is defined similarly (up to replacing subscripts 1 by 2).

*Proof.* This follows directly from Lemma 12.  $\square$

**Lemma 15.** Let  $B_r$  be a TU canonical signing of  $B_r^{(0)}$ . Let  $c_0 = (D_{0,r})_{\bullet, y_0}$  and  $c_1 = (D_{0,r})_{\bullet, y_1}$ . Then the following matrices are TU:

$$B_r^{(a)} = \begin{bmatrix} c_0 - c_1 & c_0 & A_r \end{bmatrix}, \quad B_r^{(b)} = \begin{bmatrix} c_0 - c_1 & c_1 & A_r \end{bmatrix}.$$

*Proof.* Pivoting in  $B_r$  on  $(x_2, y_0)$  and  $(x_2, y_1)$  yields:

$$\begin{array}{ccc} B_r = \begin{bmatrix} \textcircled{1} & 1 & 0 \\ c_0 & c_1 & A_r \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c_1 - c_0 & A_r \end{bmatrix} \\ \\ B_r = \begin{bmatrix} 1 & \textcircled{1} & 0 \\ c_0 & c_1 & A_r \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 1 & 0 \\ c_0 - c_1 & -c_1 & A_r \end{bmatrix} \end{array}$$

By removing row  $x_2$  from the resulting matrices and then multiplying columns  $y_0$  and  $y_1$  by  $\{\pm 1\}$  factors, we obtain  $B_r^{(a)}$  and  $B_r^{(b)}$ . By Lemma 14,  $B_r$  is TU. Since TUness is preserved under pivoting, taking submatrices, and multiplying columns by  $\pm 1$  factors, we conclude that  $B_r^{(a)}$  and  $B_r^{(b)}$  are TU.  $\square$

**Lemma 16.** Let  $B_r$  be a TU canonical signing of  $B_r^{(0)}$ . Let  $c_0 = D_{0,r}(\bullet, y_0)$ ,  $c_1 = D_{0,r}(\bullet, y_1)$ , and  $c_2 = c_0 - c_1$ . Then the following properties hold.

1. For every  $i \in X_r$ , we have  $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$ .
2.  $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$  is TU.

*Proof.* 1. Let  $i \in X_r$ . If  $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$ , then the  $2 \times 2$  submatrix of  $B_r$  indexed by  $\{x_2, i\} \times \{y_0, y_1\}$  has  $\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$ , which contradicts TUness of  $B_r$  (which holds by Lemma 14). Similarly, if  $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ , then the  $2 \times 2$  submatrix of  $B_r$  indexed by  $\{x_2, i\} \times \{y_0, y_1\}$  has  $\det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$ , which contradicts TUness of  $B_r$ .

2. Let  $V$  be a square submatrix of  $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$ . We will show that  $\det V \in \{0, \pm 1\}$ .

Let  $z$  denote the index of the appended column  $c_2$ . Suppose that column  $z$  is not in  $V$ . Then  $V$  is a submatrix of  $B_r$ , which is TU by Lemma 14. Thus,  $\det V \in \{0, \pm 1\}$ . Going forward we assume that column  $z$  is in  $V$ .

Suppose that columns  $y_0$  and  $y_1$  are both in  $V$ . Then  $V$  contains columns  $z$ ,  $y_0$ , and  $y_1$ , which are linearly dependent by construction of  $c_2$ . Thus,  $\det V = 0$ . Going forward we assume that at most one of the columns  $y_0$  and  $y_1$  is in  $V$ .

Suppose that column  $y_0$  is in  $V$ . Then  $V$  is a submatrix of  $B_r^{(b)}$  from Lemma 15, and thus  $\det V \in \{0, \pm 1\}$ . Otherwise,  $V$  is a submatrix of  $B_r^{(a)}$  from Lemma 15, and so  $\det V \in \{0, \pm 1\}$ .

Thus, every square submatrix  $V$  of  $\tilde{T}$  has  $\det V \in \{0, \pm 1\}$ , and hence  $\tilde{T}$  is TU.  $\square$

**Remark 17.** Vectors  $c_0$ ,  $c_1$ , and  $c_2$  can be defined directly in terms of entries of  $B_r$ , e.g.,  $c_2$  consists of entries of  $B_r$  indexed by  $(X_r \setminus \{x_2\}) \times \{y_0\}$ .

**Lemma 18.** Let  $B_l$  be a TU canonical signing of  $B_l^{(0)}$ . Let  $d_0 = D_{l,0}(x_0, \bullet)$ ,  $d_1 = D_{l,0}(x_1, \bullet)$ , and  $d_2 = d_0 - d_1$ . Then the following properties hold.

1. For every  $j \in Y_r$ , we have  $\begin{bmatrix} d_0(j) \\ d_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

2.  $\begin{bmatrix} A_l \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$  is TU.

*Proof.* Apply Lemma 16 to  $B_l^\top$ , or repeat the same argument up to interchanging rows and columns.  $\square$

**Lemma 19.** Let  $B_l$  and  $B_r$  be TU canonical signings of  $B_l^{(0)}$  and  $B_r^{(0)}$ , respectively.

- Let  $c_0 = D_{0,r}(\bullet, y_0)$ ,  $c_1 = D_{0,r}(\bullet, y_1)$ , and  $c_2 = c_0 - c_1$ .
- Let  $d_0 = D_{l,0}(x_0, \bullet)$ ,  $d_1 = D_{l,0}(x_1, \bullet)$ , and  $d_2 = d_0 - d_1$ .
- If  $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , let  $r_0 = d_0$ ,  $r_1 = -d_1$ ,  $r_2 = d_2$ . If  $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , let  $r_0 = d_2$ ,  $r_1 = d_1$ ,  $r_2 = d_0$ .
- Let  $D$  be the bottom-left block in the canonical signing  $B$  of  $B^{(0)}$  corresponding to  $B_l$  and  $B_r$ .

Then the following properties hold.

1.  $D = c_0 \cdot r_0 + c_1 \cdot r_1$ .
2. Rows of  $D$  are in  $\begin{bmatrix} \pm r_0 \\ \pm r_1 \\ \pm r_2 \\ 0 \end{bmatrix}$ .
3. Columns of  $D$  are in  $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm c_2 & 0 \end{bmatrix}$ .
4.  $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$  is TU.
5.  $\begin{bmatrix} A_r & D \end{bmatrix}$  is TU.
6.  $\begin{bmatrix} A_l \\ r_0 \\ r_1 \\ r_2 \end{bmatrix}$  is TU.
7.  $\begin{bmatrix} A_l \\ D \end{bmatrix}$  is TU.
8.  $\begin{bmatrix} c_0 & c_1 \end{bmatrix}$  contains  $D_0$  (the canonical signing of  $D_0^{(0)}$ ) as a submatrix.

*Proof.* 1. Follows via a direct calculation.

need details?

2. By item 1, for every  $i \in X_r$  we have  $D(i, \bullet) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$ . By Lemma 15.1, we know that  $\begin{bmatrix} c_0(i) & c_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$ . Therefore,  $D(i, \bullet)$  is equal to either  $0$ ,  $\pm r_0$ ,  $\pm r_1$ , or  $\pm(r_0 + r_1) = \pm r_2$ .

3. Holds by the same argument as item 2 up to interchanging rows and columns.

4. Holds by Lemma 16.2.

5. By item 3, columns of  $\begin{bmatrix} A_r & D \end{bmatrix}$  are in  $\begin{bmatrix} A_r & \pm c_0 & \pm c_1 & \pm c_2 & 0 \end{bmatrix}$ . Since  $\begin{bmatrix} A_r & c_0 & c_1 & c_2 \end{bmatrix}$  is TU and since adding zero columns and copies of columns multiplied by  $\pm 1$  factors preserves TUness,  $\begin{bmatrix} A_r & D \end{bmatrix}$  is also TU.

6. By Lemma 18.2 (or by the same argument as item 4 up to interchanging rows and columns),

$\begin{bmatrix} A_l \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$  is TU. Since TUness is preserved under multiplication of rows by  $\pm 1$  and exchanging rows,  $\begin{bmatrix} A_l \\ r_0 \\ r_1 \\ r_2 \end{bmatrix}$  is also TU.

7. Holds by the same argument as item 5 up to interchanging rows and columns.

8. Holds by construction. □

**Definition 20.** Let  $A_l \in \mathbb{Q}^{X_l \cup Y_l}$ ,  $A_r \in \mathbb{Q}^{X_r \cup Y_r}$ ,  $c_0, c_1 \in \mathbb{Q}^{X_r}$ ,  $r_0, r_1 \in \mathbb{Q}^{Y_l}$ . Let  $D = c_0 \cdot r_0 + c_1 \cdot r_1$ . Suppose that properties 2–8 from the statement of Lemma 19 are satisfied for  $A_l, A_r, c_0, c_1, r_0, r_1$ . Given  $k \in \mathbb{Z}_{\geq 1}$ ,

define  $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, k)$  to mean “ $C = \begin{bmatrix} A_l & 0 \\ D & A_r \end{bmatrix}$  is  $k$ -TU”.

**Lemma 21.** Assume the notation of Definition 20. Then  $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, 1)$  holds.

*Proof.* ++ Every entry of  $C$  is in one of four blocks: 0,  $A_l$ ,  $D$ ,  $A_r$ . By the assumptions of Definition 20, all of these blocks are TU. Thus,  $C$  is 1-TU. □

**Lemma 22.** Assume the notation of Definition 20. Let  $i \in X_l$ , let  $T = \begin{bmatrix} A_l(i, \bullet) \\ D \end{bmatrix}$ . Suppose we pivot on entry

$T(i, j) \in \{\pm 1\}$  in  $T$  and obtain matrix  $T' = \begin{bmatrix} a' \\ D' \end{bmatrix}$ . Then columns of  $D'$  are in  $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \end{bmatrix}$ .

*Proof.* Since  $T$  is a submatrix of  $\begin{bmatrix} A_l \\ D \end{bmatrix}$ , which is TU by assumptions of Definition 20, we have that  $T$  is TU.

Since pivoting preserves TUness,  $T'$  is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column  $p$  in  $T$  could be and what another column  $q$  in  $T$  could be. This uniquely determines the resulting columns  $p'$  and  $q'$  in  $T'$  by the pivot formula. In every case, either  $\begin{bmatrix} p' & q' \end{bmatrix}$  contains a submatrix with determinant not in  $\{0, \pm 1\}$ , which contradicts TUness of  $T'$ , or the restriction of  $p'$  and  $q'$  to  $X_r$  is in  $\begin{bmatrix} \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \end{bmatrix}$ . □

need details?

**Lemma 23.** Assume the notation of Definition 20. Let  $k \in \mathbb{Z}_{\geq 2}$ . Suppose  $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k-1)$  holds for all  $A'_l, r'_0$ , and  $r'_1$  satisfying the assumptions of Definition 20 (together with  $A_r, c_0$ , and  $c_1$ ). Then  $\text{Proposition}(A_l, A_r, c_0, c_1, r_0, r_1, k)$  holds.

*Proof.* Let  $V$  be a  $k \times k$  submatrix of  $C$ . For the sake of deriving a contradiction assume that  $\det V \notin \{0, \pm 1\}$ .

Suppose that  $V$  is a submatrix of  $\begin{bmatrix} A_l \\ D \end{bmatrix}$ ,  $\begin{bmatrix} A_l & 0 \end{bmatrix}$ ,  $\begin{bmatrix} D & A_r \end{bmatrix}$ , or  $\begin{bmatrix} 0 \\ A_r \end{bmatrix}$ . Since all of those four matrices are TU by the assumptions of Definition 20, we have  $\det V \in \{0, \pm 1\}$ . Thus,  $V$  shares at least one row and one column index with  $A_l$  and  $A_r$  each.

Consider the row index shared by  $V$  and  $A_l$ . Note that this row in  $V$  cannot consist of only 0 entries, as otherwise  $\det V = 0$ . Thus, there exists a  $\pm 1$  entry shared by  $V$  and  $A_l$ . Let  $i$  and  $j$  denote the row and the column index of this entry, respectively.

Perform a pivot in  $C$  on the element  $C(i, j)$ . For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example,  $B'$  denotes the entire matrix after the pivot. Note that the following statements hold.



- $C'$  contains a  $(k-1) \times (k-1)$  submatrix  $V'$  with  $\det V' \notin \{0, \pm 1\}$ . This holds by the same argument as for the 2-sum: look at the submatrix  $V'$  of  $C'$  with the same row and column index sets as  $V$  minus the pivot row  $i$  and pivot column  $j$ .
- $C' = \begin{bmatrix} A'_l & 0 \\ D' & A_r \end{bmatrix}$ , i.e., the 0 and the  $A_r$  blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.
- $\begin{bmatrix} A'_l \\ D' \end{bmatrix}$  is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.
- The columns of  $D'$  are in  $\begin{bmatrix} 0 & \pm c_0 & \pm c_1 & \pm(c_0 - c_1) \end{bmatrix}$ . This holds by Lemma 22.
- There exist  $r'_0$  and  $r'_1$  such that  $D' = c_0 \cdot r'_0 + c_1 \cdot r'_1$  and the assumptions of Definition 20 are satisfied for  $A'_l, A_r, c_0, c_1, r'_0, r'_1$ . This follows from the previous bullet point by carefully checking assumptions. need details?
- $C'$  is  $(k-1)$ -TU. This follows from the hypothesis:  $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k-1)$  holds.

To sum up, after pivoting we obtain a matrix  $C'$  (which can be obtained in the manner of Definition 20) that is  $(k-1)$ -TU and contains a  $(k-1) \times (k-1)$  submatrix  $V'$  with  $\det V' \notin \{0, \pm 1\}$ . This contradiction proves the lemma.  $\square$

**Lemma 24.** Let  $B_l$  and  $B_r$  be TU canonical signings. Then the corresponding canonical signing  $B$  is TU.

*Proof.* Define  $A_l, A_r, c_0, c_1, r_0, r_1$  as in Lemma 19. Note that canonical signing  $B$  has the form of  $C$  in the notation of Definition 20.

Proof by induction.

Proposition for any  $k \in \mathbb{Z}_{\geq 1}$ :  $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$  holds for all  $A'_l, r'_0$ , and  $r'_1$  satisfying the assumptions of Definition 20.

Base: The Proposition holds for  $k = 1$  by Lemma 21.

Step: If the Proposition holds for some  $k$ , then it also holds for  $k+1$  by Lemma 23.

Conclusion:  $\text{Proposition}(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$  holds for all  $k \in \mathbb{Z}_{\geq 1}$ .

Specializing the conclusion to  $A_l, A_r, c_0, c_1, r_0, r_1$  (obtained from  $B_l$  and  $B_r$  as described in the statement of Lemma 19) shows that canonical signing  $B$  is  $k$ -TU for every  $k \in \mathbb{Z}_{\geq 1}$ . Thus,  $B$  is TU.  $\square$

**Corollary 25.** Suppose that  $B_l^{(0)}$  and  $B_r^{(0)}$  have TU signings. Then  $B_l \oplus_3 B_r$  has a TU signing.

*Proof sketch.* Start with some TU signings, obtain canonical signings, apply Lemma 24.  $\square$