Matroid Decomposition Theorem Verification

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October 21, 2024

Chapter 1

Code

1.1 TU Matrices

Definition 1 (TU matrix)**.** A real matrix is *totally unimodular (TU)* if its every subdeterminant (i.e., determinant of every square submatrix) is 0 or ± 1.

Lemma 2 (entries of a TU matrix)**.** *If A is TU, then every entry of A is 0 or ± 1.*

Proof sketch. Every entry is a square submatrix of size 1. $\square$

Lemma 3 (TUness with adjoint identity matrix)**.** *A is TU iff every basis matrix of $[I \mid A]$ has determinant ± 1.*

Proof sketch. Gaussian elimination. Basis submatrix: its columns form a basis of all columns, its rows form a basis of all rows. $\square$

Lemma 4 (any submatrix of a TU matrix is TU)**.** *Let A be a real matrix that is TU and let B be a submatrix of A. Then B is TU.*

Proof sketch. Any square submatrix of B is a submatrix of A, so its determinant is 0 or ± 1. Thus, B is TU. $\square$

Lemma 5 (block-diagonal matrix with TU blocks is TU)**.** *Let A be a matrix of the form* $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & A_2 \\ \hline \end{array}$ *where A_1 and A_2 are both TU. Then A is also TU.*

Proof sketch. Any square submatrix T of A has the form $\begin{array}{|c|c|}\hline T_1 & 0 \\ \hline 0 & T_2 \\ \hline \end{array}$ where T_1 and T_2 are submatrices of A_1 and A_2, respectively.

- If T_1 is square, then T_2 is also square, and $\det T = \det T_1 \cdot \det T_2 \in \{0, \pm 1\}$.
- If T_1 has more rows than columns, then the rows of T containing T_1 are linearly dependent, so $\det T = 0$.
- Similar if T_1 has more columns than rows.

$\square$

Lemma 6 (transpose of TU is TU)**.** *Let A be a TU matrix. Then A^T is TU.*

Proof sketch. A submatrix T of A^T is a transpose of a submatrix of A, so $\det T \in \{0, \pm 1\}$. $\square$

Lemma 7 (adjoining to TU matrices)**.** *Let A be a TU matrix.*

- *Let a be a zero row. Then $C = [A/a]$ is TU.*
- *Let a be a unit row. Then $C = [A/a]$ is TU.*
- *Let a be some row of A. Then $C = [A/a]$ is TU.*
- *Let B be a matrix whose every row is a row of A, a zero row, or a unit row. Then $C = [A/B]$ is TU.*

Proof sketch.

- Let T be a square submatrix of C. If T contains a zero row, then $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$.
- Let T be a square submatrix of C. If T contains the same row twice, then the rows are GF(2)-dependent, so $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$.
- Let T be a square submatrix of C. If T contains the ± 1 entry of the unit row, then $\det T$ equals the determinant of some submatrix of A times ± 1, so $\det T \in \{0, \pm 1\}$. If T contains some entries of the unit row except the ± 1, then $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$.
- Either repeatedly apply the previous three items or direcly perform a similar case analysis.

$\square$

Corollary 8 (column properties of TU matrices)**.** *Properties listed in Lemma* **??** *also hold with respect to columns.*

Proof sketch. Combine results of Lemma **??** and Lemma **??**. $\square$

Definition 9 ($\mathcal{F}$-pivot)**.** Let A be a matrix over a field $\mathcal{F}$ with row index set X and column index set Y. Let A_{xy} be a nonzero element. The result of a *$\mathcal{F}$-pivot* of A on the *pivot element* A_{xy} is the matrix A' over $\mathcal{F}$ with row index set X' and column index set Y' defined as follows.

- For every $u \in X - x$ and $w \in Y - y$, let $A'_{uw} = A_{uw} + (A_{uy} \cdot A_{xw})/(-A_{xy})$.
- Let $A'_{xy} = -A_{xy}$, $X' = X - x + y$, and $Y' = Y - y + x$.

Lemma 10 (pivoting preserves TUness)**.** *Let A be a TU matrix and let A_{xy} be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. Then A' is TU.*

Proof sketch.

- By Lemma **??** A is TU iff every basis matrix of $[I \mid A]$ has determinant ± 1. The same holds for A' and $[I \mid A']$.
- Determinants of the basis matrices are preserved under elementary row operations in $[I \mid A]$ corresponding to the pivot in A, under scaling by ± 1 factors, and under column exchange, all of which together convert $[I \mid A]$ to $[I \mid A']$.

$\square$

Lemma 11 (pivoting preserves TUness)**.** *Let A be a matrix and let A_{xy} be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. If A' is TU, then A is TU.*

Proof sketch. Reverse the row operations, scaling, and column exchange in the proof of Lemma **??**. □

1.1.1 Minimal Violation Matrices

Definition 12 (minimal violation matrix)**.** Let A be a real $\{0, \pm 1\}$ matrix that is not TU but all of whose proper submatrices are TU. Then A is called a *minimal violation matrix of total unimodularity (minimal violation matrix).*

Lemma 13 (simple properties of MVMs)**.** *Let A be a minimal violation matrix.*

- *A is square.*
- $\det A \notin \{0, \pm 1\}$.
- *If A is 2×2, then A does not contain a* 0.

Proof sketch.

- If A is not square, then since all its proper submatrices are TU, A is TU, contradiction.
- If $\det A \in \{0, \pm 1\}$, then all subdeterminants of A are 0 or ± 1, so A is TU, contradiction.
- If A is 2×2 and it contains a 0, then $\det A \in \{\pm 1\}$, which contradicts the previous item.

□

Lemma 14 (pivoting in MVMs)**.** *Let A be a minimal violation matrix. Suppose A has ≥ 3 rows. Suppose we perform a real pivot in A, then delete the pivot row and column. Then the resulting matrix A' is also a minimal violation matrix.*

Proof sketch.

- Let A'' denote matrix A after the pivot, but before the pivot row and column are deleted.
- Since A is not TU, Lemma **??** implies that A'' is not TU. Thus A' is not TU by Lemma **??**.
- Let T' be a proper square submatrix of A'. Let T'' be the submatrix of A'' consisting of T' plus the pivot row and the pivot column, and let T be the corresponding submatrix of A (defined by the same row and column indices as T'').
- T is TU as a proper submatrix of A. Then Lemma **??** implies that T'' is TU. Thus T' is TU by Lemma **??**.

□

1.2 Matroid Definitions

Definition 15 ((finite) matroid)**.** Let E be a finite ground set. Let $\mathcal{I} \subseteq 2^E$ be a family of subsets satisfying:

- $\emptyset \in \mathcal{I}$ (non-empty)
- if $A \subseteq B \in \mathcal{I}$, then $A \in \mathcal{I}$ (down-closed)
- if $A, B \in \mathcal{I}$ and $|A| < |B|$, then $A + b \in \mathcal{I}$ for some $b \in B \setminus A$ (exchange property)

Then the pair $M = (E, \mathcal{I})$ is called a *(finite) matroid*.

Definition 16 (binary matroid)**.** Let B be a binary matrix, let $A = [I \mid B]$, and let E denote the column index set of A. Let $\mathcal{I}$ be all index subsets $Z \subseteq E$ such that the columns of A indexed by Z are independent over GF(2). Then $M = (E, \mathcal{I})$ is called a *binary matroid* and B is called its *(standard) representation matrix*.

Definition 17 (regular matroid)**.** Let M be a binary matroid generated from a standard representation matrix B. Suppose B has a TU signing, i.e., there exists a real matrix A such that:

- A is a signed version of B, i.e., $|A| = B$,
- A is totally unimodular.

Then M is called a *regular matroid*.

1.3 k-Separation and k-Connectivity

Definition 18 (k-separation)**.** Let M be a binary matroid generated by a standard representation matrix B. Suppose that B is partitioned as $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B_1 & D_2 \\ \hline X_2 & D_1 & B_2 \\ \hline \end{array}$ where $X_1 \sqcup X_2$ is a partition of the rows of B and $Y_1 \sqcup Y_2$ is a partition of its columns. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that

- $|X_1 \cup Y_1| \geq k$ and $|X_2 \cup Y_2| \geq k$,
- GF(2)-rank D_1 + GF(2)-rank $D_2 \leq k - 1$.

Then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a *(Tutte) k-sepeartion* of B and M.

Definition 19 (exact k-separation)**.** A k-separation is called *exact* if the rank condition holds with equality.

Definition 20 (k-separability)**.** We say that B and M are *(exactly) (Tutte) k-separable* if they have an (exact) k-separation.

Definition 21 (k-connectivity)**.** For $k \geq 2$, M and B are *(Tutte) k-connected* if they have no ℓ-separation for $1 \leq \ell < k$. When M and B are 2-connected, they are also called *connected*.

1.4 Sums

1.4.1 1-Sums

Definition 22 (1-sum of matrices)**.** Let B be a matrix that can be represented as $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B_1 & 0 \\ \hline X_2 & 0 & B_2 \\ \hline \end{array}$
Then we say that B_1 and B_2 are the two *components* of a 1-*sum decomposition* of B.

Conversely, a 1-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_1 B_2$ means either process.

Definition 23 (matroid 1-sum)**.** Let M be a binary matroid with a representation matrix B. Suppose that B can be partitioned as in Definition **??** with non-zero blocks B_1 and B_2. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 1-*sum decomposition* of M.

Conversely, a 1-*sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_1 M_2$ means either process.

Lemma 24 (1-separations and 1-sums)**.** *Let M be a binary matroid that is* 1*-separable. Then M can be decomposed as a* 1*-sum with components given by the* 1*-separation.*

Proof sketch. Check by definition. □

Lemma 25 (1-sum of regular matroids is regular)**.** *Let M_1 and M_2 be regular matroids. Then $M = M_1 \oplus_1 M_2$ is a regular matroid.*

Conversely, if a regular matroid M can be decomposed as a 1*-sum $M = M_1 \oplus_1 M_2$, then M_1 and M_2 are both regular.*

Proof sketch.

extract into lemmas about TU matrices

Let B, B_1, and B_2 be the representation matrices of M, M_1, and M_2, respectively.

- Converse direction. Let B' be a TU signing of B. Let B_1' and B_2' be signings of B_1 and B_2, respectively, obtained from B. By Lemma **??**, B_1' and B_2' are both TU, so M_1 and M_2 are both regular.
- Forward direction. Let B_1' and B_2' be TU signings of B_1 and B_2, respectively. Let B' be the corresponding signing of B. By Lemma **??**, B' is TU, so M is regular.

□

1.4.2 2-Sums

Definition 26 (2-sum of matrices)**.** Let B be a matrix of the form $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A_1 & 0 \\ \hline X_2 & D & A_2 \\ \hline \end{array}$ Let B_1 be a matrix of the form $\begin{array}{c|c|} & Y_1 \\ X_1 & A_1 \\ \text{Unit} & x \end{array}$ Let B_2 be a matrix of the form $\begin{array}{c|c|c|} & \text{Unit} & Y_2 \\ \hline X_2 & y & A_2 \\ \hline \end{array}$ Suppose that GF(2)-rank $D = 1$, $x \neq 0$, $y \neq 0$, $D = y \cdot x$ (outer product).

Then we say that B_1 and B_2 are the two *components* of a 2-*sum decomposition* of B.

Conversely, a 2-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_2 B_2$ means either process.

Definition 27 (matroid 2-sum)**.** Let M be a binary matroid with a representation matrix B. Suppose B, B_1, and B_2 satisfy the assumptions of Definition **??**. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 2*-sum decomposition* of M.

Conversely, a 2*-sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_2 M_2$ means either process.

Lemma 28 (2-separations and 2-sums of connected binary matroids)**.** *Let M be a binary matroid that is 2-separable. Then M can be decomposed as a 2-sum with connected components given by the 2-separation.*

Conversely, any 2-sum of two connected binary matroids is a connected binary matroid.

Proof sketch. Check by definition. Connectedness of representation matrices is equivalent to connectedness of corresponding matroids. □

Lemma 29 (2-sum of regular matroids is regular)**.** *Let M_1 and M_2 be regular matroids. Then $M = M_1 \oplus_2 M_2$ is a regular matroid.*

Proof sketch.

Let B, B_1, and B_2 be the representation matrices of M, M_1, and M_2, respectively. Let B_1' and B_2' be TU signings of B_1 and B_2, respectively. In particular, let A_1', x', A_2', and y' be the signed versions of A_1, x, A_2, and y, respectively. Let B' be the signing of B where the blocks of A_1 and A_2 are signed as A_1' and A_2', respectively, and the block of D is signed as $D' = y' \cdot x'$ (outer product).

Note that $[A_1'/D']$ is TU by Lemma **??**, as every row of D' is either zero or a copy of x'. Similarly, $[D' \mid A_2']$ is TU by Corollary **??**, as every column of D' is either zero or a copy of y'. Additionally, $[A_1' \mid 0]$ is TU by Lemma **??**, and $[0/A_2']$ is TU by Lemma **??**.

prove lemma below, separate into statement about TU matrices

Lemma: Let T be a square submatrix of B'. Then $\det T \in \{0, \pm 1\}$.

Proof: Induction on the size of T. *Base:* If T consists of only 1 element, then this element is 0 or ± 1, so $\det T \in \{0, \pm 1\}$. *Step:* Let T have size t and suppose all square submatrices of B' of size $\leq t - 1$ are TU.

- Suppose T contains no rows of X_1. Then T is a submatrix of $[D' \mid A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no rows of X_2. Then T is a submatrix of $[A_1' \mid 0]$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_1. Then T is a submatrix of $[0/A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_2. Then T is a submatrix of $[A_1'/D']$, so $\det T \in \{0, \pm 1\}$.
- Remaining case: T contains rows of X_1 and X_2 and columns of Y_1 and Y_2.
- If T is 2×2, then T is TU. Indeed, all proper submatrices of T are of size ≤ 1, which are $\{0, \pm 1\}$ entries of A', and T contains a zero entry (in the row of X_2 and column of Y_2), so it cannot be a minimal violation matrix by Lemma **??**. Thus, assume T has size ≥ 3.
- .

  complete proof, see last paragraph of Lemma 11.2.1 in Truemper

□

1.4.3 3-Sums

Definition 30 (3-sum of matrices).

add

Definition 31 (matroid 3-sum).

add

Lemma 32 (3-separations and 3-sums).

add

Lemma 33 (3-sum of regular matroids is regular).

add

Chapter 2

Truemper

2.1 Basic Definitions

2.1.1 Matroid Structure

Definition 34 (matroid).

todo: add definition

Definition 35 (isomorphism). Two matroids are isomorphic if they become equal upon a suitable relabeling of the elements.

Definition 36 (loop).

todo: add definition

Definition 37 (coloop).

todo: add definition

Definition 38 (parallel elements).

todo: add definition

Definition 39 (series elements).

todo: add definition

2.1.2 Matroid Classes

Definition 40 (binary matroid).

todo: add definition

Definition 41 (regular matroid). A binary matroid M is regular if some binary representation matrix B of M has a totally unimodular signing (i.e., assignment of signs to the 1s in B that results in a TU matrix).

Definition 42 (graphic matroid).

todo: add definition

Definition 43 (cographic matroid).

todo: add definition

Definition 44 (planar matroid).

todo: add definition

Definition 45 (dual matroid).

todo: add definition

Definition 46 (self-dual matroid).

todo: add definition

2.1.3 Specific Matroids (Constructions)

Definition 47 (wheel).

todo: add definition

Definition 48 (W_3).

todo: add definition

Definition 49 (W_4).

todo: add definition

Definition 50 (R_{10}).

todo: add definition

Definition 51 (R_{12}).

todo: add definition

Definition 52 (F_7).

todo: add definition

Definition 53 (F_7^*).

todo: add definition

Definition 54 ($M(K_{3,3})$).

todo: add definition

Definition 55 ($M(K_{3,3})^*$).

todo: add definition

Definition 56 ($M(K_5)$).

todo: add definition

Definition 57 ($M(K_5)^*$).

todo: add definition

2.1.4 Connectivity and Separation

Definition 58 (k-separation)**.** See text after Proposition 3.3.18.

Definition 59 (k-connectivity)**.** See text after Proposition 3.3.18.

2.1.5 Reductions

Definition 60 (deletion)**.**

todo: add definition

Definition 61 (contraction)**.**

todo: add definition

Definition 62 (minor)**.**

todo: add definition

2.1.6 Extensions

Definition 63 (1-element addition)**.**

add name, label, uses, text

Definition 64 (1-element expansion)**.**

add name, label, uses, text

Definition 65 (1-element extension)**.**

todo: add definition

Definition 66 (2-element extension)**.**

todo: add definition

Definition 67 (3-element extension)**.**

todo: add definition

2.1.7 Sums

Definition 68 (1-sum)**.**

todo: add definition

Definition 69 (2-sum)**.**

todo: add definition

Definition 70 (3-sum)**.**

todo: add definition

Definition 71 (Δ-sum)**.**

todo: add definition

Definition 72 (Y-sum)**.**

todo: add definition

2.1.8 Total Unimodularity

Definition 73 (TU matrix)**.** A real matrix A is totally unimodular if every square submatrix D of A has $\det_{\mathbb{R}} D = 0$ or ± 1.

2.1.9 Auxiliary Results

Theorem 74 (Menger's theorem)**.** *A connected graph G is vertex k-connected if and only if every two nodes are connected by k internally node-disjoint paths. Equivalent is the following statement. G is vertex k-connected if and only if any $m \leq k$ nodes are joined to any $n \leq k$ nodes by k internally node-disjoint paths. One may demand that the m nodes are disjoint from the n nodes, but need not do so. Also, the k paths can be so chosen that each of the specified nodes is an endpoint of at least one of the paths.*

Definition 75 (ΔY exchange)**.**

add

Definition 76 (gap)**.**

add

2.2 Chapter 2

Lemma 77 (2.3.14)**.** *Let A be a matrix over a field $\mathcal{F}$, with $\mathcal{F}$-rank $A = k$. If both a row submatrix and a column submatrix of A have $\mathcal{F}$-rank equal to k, then they intersect in a submatrix of A with the same $\mathcal{F}$-rank. In particular, any k $\mathcal{F}$-independent rows of A and any k $\mathcal{F}$-independent columns of A intersect in a $k \times k$ $\mathcal{F}$-nonsingular submatrix of A.*

Proof sketch. Result of linear algebra. Uses the submodularity of the $\mathcal{F}$-rank function. □

2.3 Chapter 3

2.3.1 Chapter 3.2

Theorem 78 (3.2.25.a)**.** *Let M be the graphic matroid of a connected graph G. Assume (E_1, E_2) is a k-separation of M with minimal $k \geq 1$. Define G_1 (resp. G_2) from G by removing the edges of E_2 (resp. E_1) from G. Let $R_1, \ldots, R_g$ be the connected components of G_1, and $S_1, \ldots, S_h$ be those of G_2.*

If $k = 1$, then the R_i and S_j are connected in tree fashion.

Proof sketch. Count edges and nodes. □

Theorem 79 (3.2.25.b)**.** *Same setting as Theorem 3.2.25.a. If $k = 2$, then the R_i and S_j are connected in cycle fashion.*

Proof sketch. Count edges and nodes. □

Definition 80 (switching operation from section 3)**.** A swap of identification of nodes between two subgraphs induced by a 2-separation of a graph. See description and illustration on page 45.

Lemma 81 (3.2.48)**.** *The matroids $M(K_5)$ and $M(K_{3,3})$ are not graphic.*

Proof sketch. A short proof is given on page 51. A longer, but more general proof uses the graphicness testing subroutine described on page 47. □

2.3.2 Chapter 3.3

Lemma 82 (3.3.12)**.** *Let M be a binary matroid with a minor $\overline{M}$, and let $\overline{B}$ be a representation matrix of $\overline{M}$. Then M has a representation matrix B that displays $\overline{M}$ via $\overline{B}$ and thus makes the minor $\overline{M}$ visible.*

Proof sketch. Follows by the definition of minor via pivots and row/column deletions. □

Proposition 83 (3.3.17)**.** *Partitioned version of matrix B representing binary matroid M. (same as 3.3.3)*

Proposition 84 (3.3.18)**.** *If for some $k \geq 1$, $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k$, $\mathrm{GF}(2)$-rank $D^1 + \mathrm{GF}(2)$-rank $D^2 \leq k - 1$, then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a (Tutte) k-separation of B and M. This separation is exact if the rank condition holds with equality. Both B and M are called (Tutte) k-separable if they have a k-separation. For $k \geq 2$, B and M are (Tutte) k-connected if they have no ℓ-separation for $1 \leq \ell < k$. When M is 2-connected, we also say that M is connected.*

Lemma 85 (3.3.19)**.** *Let M be a binary matroid with a representation matrix B. Then M is connected iff B is connected.*

Proof sketch. Check using (3.3.17) and (3.3.18) that B is connecte diff it is 2-connected. Thus M is 2-connected, and hence connected, iff B is connected. □

Lemma 86 (3.3.20)**.** *The following statements are equivalent for a binary matroid M with set E and a representation matrix B of M.*

- *M is 3-connected.*
- *B is connected, has no parallel or unit vector rows and columns, and has no partition as in (3.3.17) with $\mathrm{GF}(2)$-rank $D^1 = 1$, $D^2 = 0$, and $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.*
- *Same as (ii), but $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 5$.*

Proof sketch.

- (i) is equivalent to (ii) by the definition of 3-connectivity.
- (iii) trivially implies (ii). (Typo in the book?)
- Assuming (ii), if the length of B^1 is 3 or 4, then B has a zero column or row, or parallel or unit vector rows or columns, which is excluded by the first part of (ii). Thus it suffices to require $|X_1 \cup Y_1| \geq 5$ and by duality $|X_2 \cup Y_2| \geq 5$.

□

Theorem 87 (census from Secion 3.3)**.** *A complete census of 3-connected binary matroids on ≤ 8 elements.*

Proof sketch. Verified by case enumeration. □

2.4 Chapter 4

Proposition 88 (4.4.5)**.** *ΔY exchange, case 1.*

Proposition 89 (4.4.6)**.** *ΔY exchange, case 2.*

2.5 Chapter 5

Lemma 90 (5.2.4)**.** *Let N be a connected minor of a connected binary matroid M. Let $z \in M \backslash N$. Then M has a connected minor N' that is a 1-element extension of N by z.*

Proof sketch.

- By Lemma 3.3.12, M has a representation matrix that displays N via a submatrix.
- Case distinction between z being represented by a nonzero or a zero vector.
- Nonzero case: immediately get submatrix representing N'.
- Zero case: take a shortest path in the matrix, perform pivots, in one subcase use duality.

□

Proposition 91 (5.2.8)**.** *Representation matrices for small wheels (from $M(W_1)$ to $M(W_4)$).*

Proposition 92 (5.2.9)**.** *Representation matrix for $M(W_n)$, $n \geq 3$.*

Lemma 93 (5.2.10)**.** *Let M be a binary matroid with a binary representation matrix B. Suppose the graph $BG(B)$ contains at least one cycle. Then M has an $M(W_2)$ minor.*

Proof sketch.

- $BG(B)$ is bipartite and has at least one cycle, so there is a cycle C without chords with at least 4 edges.
- Up to indices, the submatrix corresponding to C is either the matrix for $M(W_2)$ from (5.2.8) or the matrix for some $M(W_k)$, $k \geq 3$ from (5.2.9).
- In the latter case, use path shortening pivots on 1s to convert the submatrix to the former case.

□

Lemma 94 (5.2.11)**.** *Let M be a connected binary matroid with at least 4 elements. Then M has a 2-separation or an $M(W_3)$ minor.*

Proof sketch. Use Lemma 5.2.10 and apply path shortening technique. □

Corollary 95 (5.2.15)**.** *Every 3-connected binary matroid M with at least 6 elements has an $M(W_3)$ minor.*

Proof sketch. By Lemma 5.2.11, M has a 2-separation or an $M(W_3)$ minor. M is 3-connected, so the former case is impossible. □

2.6 Chapter 6

2.6.1 Chapter 6.2

Goal of the chapter: separation algorithm for deciding if there exists a separation of a matroid induced by a separation of its minor.

Proposition 96 (6.2.1)**.** *Partitioned version of matrix B^N representing a minor N of a binary matroid M, where N has an exact k-separation for some $k \geq 1$.*

Proposition 97 (6.2.3)**.** *Matrix B for M displaying partitioned B^N*

Proposition 98 (6.2.5)**.** *Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Lemma 99 (6.2.6)**.** *Let N be a 3-connected binary matroid on at least 6 elements. Suppose a 1-, 2-, or 3-element binary extension of N, say M, has no loops, coloops, parallel elements, or series elements. Then M is 3-connected.*

Proof sketch.

- Let C be a binary representation matrix of M that displays a binary representation matrix B for N.
- By assumption, B is 3-connected.
- C is connected, as otherwise by case analysis C contains a zero vector or unit vector, so M has a loop, coloop, parallel elements, or series elements, a contradiction.
- If C is not 3-connected, then by Lemma 3.3.20 there is a 2-separation of C with at least 5 rows/columns on each side. Then B has a 2-separation with at least 2 rows/columns on each side, a contradiction.
- Thus, C is 3-connected, so M is 3-connected.

□

2.6.2 Chapter 6.3

Definition 100 (6.3.2)**.** M is called minimal if it satisfies the following conditions.

- M has an N minor.
- M has no k-separation induced by the exact k-separation (F_1, F_2) of N.
- The matroid M is minimal with respect to the above conditions.

Definition 101 (6.3.3)**.** M is called minimal under isomorphism if it satisfies the following conditions.

- M has at least one N minor.
- Some k-separation of at least one such minor corresponding to the exact k-separation (F_1, F_2) of N under one of the isomorphisms fails to induce a k-separation of M.
- The matroid M is minimal with respect to the above conditions.

Proposition 102 (6.3.11)**.** *Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Proposition 103 (6.3.12)**.** *Partitioned version of B^N: $B^N = \mathrm{diag}(A^1, A^2)$.*

Definition 104 (separation algorithm)**.** Polynomial-time recursive procedure to search for an induced partition. Described on pages 132–133 and again on pages 137–138.

Proposition 105 (6.3.13)**.** *Special case where B of a minimal M contains just one row x beyond B^N. This proposition gives properties of row subvectors of row x by step 1 of the separation algorithm.*

Proposition 106 (6.3.14)**.** *Special case where B of a minimal M contains just one column y beyond B^N. This proposition gives properties of column subvectors of column y by step 1 of the separation algorithm.*

Lemma 107 (6.3.15)**.** *Treats the case where B has at least two additional rows or columns beyond those of B^N.*

Proof sketch. Argue about the structure of the matrix, applying steps 1 and 2 of the separation algorithm. □

Lemma 108 (6.3.16)**.** *Expands case (i) of Lemma 6.3.15.*

Proof sketch. Further arguments about the structure of the matrix. □

Lemma 109 (6.3.17)**.** *Expands case (ii) of Lemma 6.3.15.*

Proof sketch. Further arguments about the structure of the matrix. □

Theorem 110 (6.3.18)**.** *Structural description of representation matrix (6.3.11) of a minimal M. Contains cases (a), (b), and (c) with sub-cases (c.1) and (c.2).*

Proof sketch.

- (6.3.13) and (6.3.14) establish (a) and (b).
- Lemmas 6.3.15, 6.3.16, and 6.3.17 prove (c.1) and (c.2).

□

Lemma 111 (6.3.19)**.** *Additional structural statements for cases (c.1) and (c.2) of Theorem 6.3.18.*

Proof sketch. Reason about representation matrices using Theorem 6.3.18, Lemma 6.3.15, minimality, isomorphisms, pivots, and so on. □

Proposition 112 (6.3.21)**.** *Matrix B for M minimal under isomorphism, case (a).*

Proposition 113 (6.3.22)**.** *Matrix B for M minimal under isomorphism, case (b).*

Proposition 114 (6.3.23)**.** *Matrix $\overline{B}$ for minor $\overline{M}$ of M minimal under isomorphism.*

Theorem 115 (6.3.20)**.** *Let M be minimal under isomorphism. Then one of* 3 *cases holds for matrix representation of M.*

Proof sketch. Follows directly from Theorem 6.3.18 and Lemma 6.3.19. □

Corollary 116 (6.3.24)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Suppose N given by B^N of (6.3.12) is in $\mathcal{M}$, but the* 1- *and* 2-*element extensions of N given by (6.3.21), (6.3.22), (6.3.23), and by the accompanying conditions are not in $\mathcal{M}$. Assume matroid $M \in \mathcal{M}$ has an N minor. Then any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

Proof sketch.

- Let $M \in \mathcal{M}$ satisfying the assumptions. Since $\mathcal{M}$ is closed under isomorphism, suppose that N itself is a minor of M.
- Suppose the k-separation of N does not induce one in M. Then M or a minor of M containing N is minimal under isomorphism.
- By Theorem 6.3.20, M has a minor represented by (6.3.21), (6.3.22), or (6.3.23). This minor is in $\mathcal{M}$, as $\mathcal{M}$ is closed under taking minors, but this contradicts our assumptions.

□

2.6.3 Chapter 6.4

Theorem 117 (6.4.1)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N. Suppose N has at least 6 elements. Then M has a 3-connected minor N' that is a 1- or 2-element extension of some N minor of M. In the 2-element case, N' is derived from the N minor by one addition and one expansion.*

Proof sketch.

- Let $z \in M \setminus N$. By Lemma 5.2.4, there is a connected minor N' that is a 1-element extension of N by z. Our theorem holds iff it holds for duals, so by duality, assume that the extension is an addition.
- Reason about a matrix representation of N and N' to get a 2-separation of N'. Since M is 3-connected, this 2-separation does not induce one in M. Let M' be a minor of M that proves this fact and is minimal under isomorphism. Additionally, M' has an N' minor, so we change the element labels in M' so that N' is a minor of M'.
- Apply Theorem 6.3.20 and perform case analysis, reaching either a contradiction or a desired extension.

□

2.7 Chapter 7

2.7.1 Chapter 7.2

Definition 118 (splitter)**.** Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If every $M \in \mathcal{M}$ with a proper N minor has a 2-separation, then N is called a splitter of $\mathcal{M}$.

Theorem 119 (7.2.1.a splitter for nonwheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If N is not a wheel, then N is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.*

Proof sketch.

- If N is a splitter of $\mathcal{M}$, then clearly $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.

- Prove the converse by contradiction. To this end, suppose that $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and that N is not a splitter of $\mathcal{M}$.
- Thus, $\mathcal{M}$ contains a 3-connected matroid M with a proper N minor and no 2-separation.
- Since $\mathcal{M}$ is closed under isomorphism, we may assume N itself to be that N minor.
- By Theorem 6.4.1 (applied to M and N), M has a 3-connected minor N' that is a 3-connected 1- or 2-element extension of an N minor.
- The 1-extension case has been ruled out.
- In the 2-element extension case, N' is derived from the N minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take N itself to be that N minor. Thus, N' is derived from N by one addition and one expansion.
- Let C be a binary matrix representing N' and displaying N. By investigating the structure of C, one can show that N' contains a 3-connected 1-element extension of an N minor, which has been ruled out.

□

Theorem 120 (7.2.1.b splitter for wheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If N is a wheel, then N is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and does not contain the next larger wheel.*

Proof sketch. Similar to proof of Theorem 7.2.1.a. The analysis of the matrix C can be done in one go for both cases. □

Corollary 121 (7.2.10.a)**.** *Theorem 7.2.1.a specialized to graphs.*

Proof sketch. Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Corollary 122 (7.2.10.b)**.** *Theorem 7.2.1.b specialized to graphs.*

Proof sketch. Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Theorem 123 (7.2.11.a)**.** *K_5 is a splitter of the graphs without $K_{3,3}$ minors.*

Proof sketch. Up to isomorphism, there is just one 3-connected 1-edge extension of K_5. To obtain it, one partitions one vertex of K_5 into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph has a $K_{3,3}$ minor. Thus, the theorem follows from Corollary 7.2.10.a. □

Theorem 124 (7.2.11.b)**.** *W_3 is a splitter of the graphs without W_4 minors.*

Proof sketch. There is no 3-connected 1-edge extension of W_3, so the theorem follows from Corollary 7.2.10.b. □

2.7.2 Chapter 7.3

Theorem 125 (7.3.1.a)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. Assume N is not a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where M_0 is isomorphic to N and where the gap is 1.*

Proof sketch.

- Inductively for $i \geq 0$ assume the existence of a sequence $M_0, \ldots, M_i$ of 3-connected minors where M_0 is isomorphic to N, M_i is not a wheel, and the gap is 1.
- If $M_i = M$, we are done, so assume that M_i is a proper minor of M.
- Use the contrapositive of the splitter Theorem 7.2.1.a to find a larger sequence.
  - Let $\mathcal{M}$ be the collection of all matroids isomorphic to a (not necessarily proper) minor of M.
  - Since M_i is a 3-connected proper minor of the 3-conected $M \in \mathcal{M}$, it cannot be a splitter of $\mathcal{M}$. By Theorem 7.2.1.a, $\mathcal{M}$ contains a matroid M_{i+1} that is a 3-connected 1-element extension of a matroid isomorphic to M_i.
  - Since every 1-element reduction of a wheel with at least 6 elements is 2-separable, M_{i+1} is not a wheel, as otherwise M_i is 2-separable, which is a contradiction.
- If necessary, relabel $M_0, \ldots, M_i$ so that they consistute a sequence of nested minors of M_{i+1}. This sequence satisfies the induction hypothesis.
- By induction, the claimed sequence exists for M.

□

Theorem 126 (7.3.1.b)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. Assume N is a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where:*

- *M_0 is isomorphic to N,*
- *for some $0 \leq s \leq t$ the subsequence $M_0, \ldots, M_s$ consists of wheels and has gap 2,*
- *the subsequence $M_s, \ldots, M_t$ has gap 1.*

Proof sketch. Same as the proof of Theorem 7.3.1.a, but uses Theorem 7.2.1.b instead of 7.2.1.a to extend the sequence of minors. □

Proposition 127 (7.2.1 from 7.3.1)**.** *Theorem 7.3.1 implies Theorem 7.2.1.*

Proof sketch.

- Let $\mathcal{M}$ and N be as specified in Theorem 7.2.1. Suppose N is not a wheel.
- Prove the nontrivial "if" part by contradiction: let M be a 3-connected matroid of $\mathcal{M}$ with N as a proper minor.
- By Theorem 7.3.1, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where M_0 is isomorphic to N and where the gap is 1.

- Since $\mathcal{M}$ is closed under isomorphism, we may assume that M is chosen such that $M_0 = N$.
- Then $M_1 \in \mathcal{M}$ is a 3-connected 1-element extension of N, which contradicts the assumed absence of such extensions.
- If N is a wheel, the proof is analogous.

□

Corollary 128 (7.3.2.a)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H, and where each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. Translate Theorem 7.3.1.a directly into graph language. □

Corollary 129 (7.3.2.b)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where:*

- *G_0 is isomorphic to H,*
- *for some $0 \leq s \leq t$ the subsequence $G_0, \ldots, G_t$ consists of wheels where each G_{i+1} has exactly one additional spoke beyond those of G_i,*
- *in the subsequence $G_s, \ldots, G_t$ each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. Translate Theorem 7.3.1.b directly into graph language. □

Theorem 130 (7.3.3, wheel theorem)**.** *Let G be a 3-connected graph on at least 6 edges. If G is not a wheel, then G has some edge z such that at least one of the minors G/z and $G \setminus z$ is 3-connected.*

Proof sketch.

- By Corollary 5.2.15, G has a W_3 minor.
- Let H be a largest wheel minor of G. Since G is not a wheel, H is a proper minor of G.
- Apply Corollary 7.3.2.b to G and H to get a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H.
- Since H is the largest wheel minor and G is not a wheel, Corollary 7.3.2.b shows that $s = 0$ and $t \geq 1$.
- Additionally, from corollary we know that $G = G_t$ has exactly one extra edge compared to G_{t-1}. In other words, $G_{t-1} = G/z$ or $G \setminus z$ for some edge z.

□

Theorem 131 (7.3.3 for binary matroids)**.** *Theorem 7.3.3 can be rewritten for binary matroids instead of graphs.*

Proof sketch. Similar to the proof of Theorem 7.3.3, but use Theorem 7.3.1 instead of Corollary 7.3.2. □

Proposition 132 (7.3.4.observation)**.** *Oservation in text on pages 160–161.*

Theorem 133 (7.3.4)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. If M does not contain a 3-connected 1-element expansion (resp. addition) of any N minor, then M has a sequence of nested 3-connected minors $M_0, \ldots, M_t = M$ where M_0 is an N minor of M and where each M_{i+1} is obtained from M_i by expansions (resp. additions) involving some series (resp. parallel) elements, possibly none, followed by a 1-element addition (resp. expansion).*

Proof sketch.

- The case in parenthesis is dual to the normally stated one. Thus, only consider expansions below.
- Apply construction from observation before Theorem 7.3.4 to the sequence of minors from Theorem 7.3.1 to get the desired sequence.

□

Corollary 134 (7.3.5)**.** *Specializes Theorem 7.3.4 to graphs.*

2.7.3 Chapter 7.4

Theorem 135 (7.4.1 planarity characterization)**.** *A graph is planar if and only if it has no $K_{3,3}$ or K_5 minors.*

Proof sketch.

- "Only if": planarity is preserved by taking minors, and by Lemma 3.2.48 both $K_{3,3}$ and K_5 are not planar.
- Let G be a connected nonplanar graph with all proper minors planar. Goal: show that G is isomorphic to $K_{3,3}$ or K_5.
- Prove that G cannot be 1- or 2-separable. Thus G is 3-connected.
- By Corollary 5.2.15, G has a W_3 minor, say H. Note: no H minor of G can be extended to a minor of G by addition of an edge that connects two nonadjacent nodes.
- Then by Corollary 7.3.5.b, there exists a sequence $G_0, \ldots, G_t = G$ of 3-connected minors where G_0 is an H minor and G_{i+1} is constructed from G_i following very specific steps.
- By minimality, G_{t-1} is planar and G is not. Argue about a planar drawing of G_{t-1} and how G can be derived from it. Show that this must result in a subdivision of $K_{3,3}$ or K_5.

□

Theorem 136 (Kuratowski)**.** *A graph is planar if and only if it has no subdivision of $K_{3,3}$ or K_5.*

Proof. Note: Theorem 7.4.1 is equivalent to Kuratowski's theorem: a $K_{3,3}$ minor induces a subdivision of $K_{3,3}$ and a K_5 minor also leads to a subdivision of K_5 or $K_{3,3}$ (the latter in the case when an expansion step splits a vertex of degree 4 into two vertices of degree 3 after the new edge is inserted). □

2.8 Chapter 8

2.8.1 Chapter 8.2

This chapter is about deducing and manipulating 1- and 2-sum decompositions and compositions.

Proposition 137 (8.2.1)**.** *Matrix of* 1*-separation.*

Lemma 138 (8.2.2)**.** *Let M be a binary matroid. Assume M to be a* 1*-sum of two matroids M_1 and M_2.*

- *If M is graphic, then there eixst graphs G, G_1, G_2 for M, M_1, M_2, respectively, such that identification of a node of G_1 with one of G_2 creates G.*
- *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Proof sketch. Elementary application of Theorem 3.2.25.a. □

Proposition 139 (8.2.3)**.** *Matrix of exact* 2*-separation.*

Proposition 140 (8.2.4)**.** *Matrices B^1 and B^2 of* 2*-sum.*

Lemma 141 (8.2.6)**.** *Any* 2*-separation of a connected binary matroid M produces a* 2*-sum with connected components M_1 and M_2. Conversely, any* 2*-sum of two connected binary matroids M_1 and M_2 is a connected binary matroid M.*

Proof sketch.

- Definitions imply everything except connectedness.
- It is easy to check that connectedness of (8.2.3) implies connectedness of (8.2.4) and vice versa.
- By Lemma 3.3.19, connectedness of representation matrices is equivalent to connectedness of the corresponding matroids.

□

Lemma 142 (8.2.7)**.** *Let M be a connected binary matroid that is a* 2*-sum of M_1 and M_2, as given via B, B_1, and B_2 of (8.2.3) and (8.2.4).*

- *If M is graphic, then there exist* 2*-connected graphs G, G_1, and G_2 for M, M_1, and M_2, respectively, with the following feature. The graph G is produced when one identifies the edge x of G_1 with the edge y of G_2, and when subsequently the edge so created is deleted.*
- *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Proof sketch.

- Ingredients: look at a 2-separation and the corresponding subgraphs, use Theorem 3.2.25.b, use the switching operation of Section 3.2, use Lemma 8.2.6 and representations (8.2.3) and (8.2.4).
- Use the construction from the drawing, check that fundamental circuits match, conclude that M is graphic. For planar graphs, the edge identification can be done in a planar way.

□

2.8.2 Chapter 8.3

Proposition 143 (8.3.1). *Matrix B with exact k-separation.*

Proposition 144 (8.3.2). *Partition of B displaying k-sum.*

Proposition 145 (8.3.9). *The (well-chosen) matrix $\overline{B}$ representing the connecting minor $\overline{M}$ of a 3-sum.*

Proposition 146 (8.3.10). *The matrix B representing a 3-sum (after reasoning).*

Proposition 147 (8.3.11). *Representation matrices B^1 and B^2 of the components M_1 and M_2 of a 3-sum (after reasoning).*

Lemma 148 (8.3.12). *Let M be a 3-connected binary matroid on a set E. Then any 3-separation (E_1, E_2) of M with $|E_1|, |E_2| \geq 4$ produces a 3-sum, and vice versa.*

Proof.

- The converse easily follows from (8.3.10), which directly produces a desired 3-separation.
- Take a 3-separation. Since M is 3-connected, it must be exact. Consider the representation matrix (8.3.11). Reason about that matrix.
- Analyse shortest paths in a bipartite graph based on the matrix.
- Apply path shortening technique from Chapter 5 to reduce a shortest path by pivots to one with exactly two arcs.
- Reason about the corresponding entries and about the effects of the pivots on the matrix.
- Apply Lemma 2.3.14. Eventually get an instance of (8.3.10) with (8.3.9). Thus, M is a 3-sum.

□

2.8.3 Chapter 8.5

Proposition 149 (8.5.3). *Matrix $B^{2\Delta}$ for $M_{2\Delta}$.*

2.9 Chapter 9

Proposition 150 (9.2.14). *Matrix B^{12} of regular matroid R_{12}.*

2.10 Chapter 10

Proposition 151 (10.2.4). *Derivation of a graph with T nodes for F_7.*

Proposition 152 (10.2.6). *Derivation of a graph with T nodes for $M(K_{3,3})^*$.*

Proposition 153 (10.2.8). *Derivation of a graph with T nodes for R_{10}.*

Proposition 154 (10.2.9). *Derivation of a graph with T nodes for R_{12}.*

Theorem 155 (10.2.11 only if)**.** *If a regular matroid is planar, then it has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors.*

Proof sketch.

- Planarity is preserved under taking minors.
- The listed matroids are not planar.

□

Theorem 156 (10.2.11 if)**.** *If a regular matroid has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors, then it is planar.*

Proof sketch.

- Let M be minimally nonplanar with respect to taking minors, i.e., regular nonplanar, but with all proper minors planar.
- Goal: show that M is isomorphic to one of the listed matroids.
- By Theorem 7.4.1, M is not graphic or cographic.
- By Lemmas 8.2.2, 8.2.6, and 8.2.7, if M has a 1- or 2-separation, then M is a 1- or 2-sum. But then the components of the sum are planar, so M is also planar. Therefore, M is 3-connected.
- By the census of Section 3.3, every 3-connected $\leq$ 8-element matroid is planar, so $|M| \geq 9$.
- By the binary matroid version of the wheel Theorem 7.3.3, there exists an element z such that $M \setminus z$ or M/z is 3-connected. Dualizing does not afect the assumptions, so we may assume that $M \setminus z$ is 3-connected.
- Let G be a planar graph representing $M \setminus z$. Extend G to a representation of M as follows:
  - If G is a wheel, invoke (10.2.6) or (10.2.4). The latter contracdicts regularity of M, the former shows what we need.
  - If G is not a wheel, use Theorem 7.3.3 and Menger's theorem. Use a path argument and edge contraction to reduce to (10.2.6) and conclude the proof.

□

Lemma 157 (10.3.1)**.** *$M(K_5)$ is a splitter of the regular matroids with no $M(K_{3,3})$ minors.*

Proof.

- By Theorem 7.2.1.a, we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor.
- Then case analysis. (The book sketches one way of checking.)

□

Lemma 158 (10.3.6)**.** *Every 3-connected binary 1-element expansion of $M(K_{3,3})$ is nonregular.*

Proof sketch. By case analysis via graphs plus T sets. □

Theorem 159 (10.3.11)**.** *Let M be a 3-connected regular matroid with an $M(K_{3,3})$ minor. Assume that M is not graphic and not cographic, but that each proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof. This proof is extremely long and technical. It involves case distinctions and graph constructions. □

Theorem 160 (10.4.1 only if)**.** *If* 3*-connected regular matroid is graphic or cographic, then it has no* R_{10} *or* R_{12} *minors.*

Proof sketch. Representations (10.2.8) and (10.2.9) for R_{10} and R_{12} show that these are nongraphic and isomorphic to their duals, hence also noncographic, so we are done. □

Theorem 161 (10.4.1 if)**.** *If a* 3*-connected regular matroid has no* R_{10} *or* R_{12} *minors, then it is graphic or cographic.*

Proof sketch.

- Let M be 3-connected, regular, nongraphic, and noncographic matroid.
- Thus M is not planar, so by Theorem 10.2.11 it has a minor isomorphic to $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$.
- By Lemma 10.3.1, $M(K_5)$ is a splitter for the regular matroids with no $M(K_{3,3})$ minors.
- These results imply that M has a minor isomorphic to $M(K_{3,3})$, or $M(K_{3,3})^*$, or M is isomorphic to $M(K_5)$ or $M(K_5)^*$.
- The latter is a contradiction, so M or M^* has an $M(K_{3,3})$ minor.
- Theorem 10.3.11 implies that M or M^* has R_{10} or R_{12} as a minor.
- Since R_{10} and R_{12} are self-dual, M has R_{10} or R_{12} as a minor.

□

Note: Truemper's proof of **??** and **??** relies on representing matroids via graphs plus T sets. An alternative proof, which utilizes the notion of graph signings, can be found in J. Geelen, B. Gerards - Regular matroid decomposition via signed graphs. Although the proof appears shorter than Truemper's, it heavily relies certain relatively advanced graph-theoretic results.

Bonus: Whitney's characterization of planar graphs (Corollary 10.2.13).

2.11 Chapter 11

2.11.1 Chapter 11.2

The goal of this chapter is to prove the "simple" direction of the regular matroid decomposition theorem.

Ingredients from Section 9.2:

- A matrix is TU if all its subdeterminants are 0, ± 1.
- A binary matroid is regular if it has a signing that is TU.
- By Lemma 9.2.6 and Corollary 9.2.7, this signing is unique up to scaling by ± 1 factors.
- The signing can be accomplished by signing one arbitrarily selected row or column at a time.

Ingredients from minimal violation matrices:

- Definition: a minimal violation matrix of total unimodularity (minimal violation matrix, MVM) is a $\{0, \pm 1\}$ matrix that is not TU, but all its submatrices are TU.
- MVMs are square and have determinant not equal to $0, \pm 1$.
- In particular, a 2×2 violation matrix has four ± 1's.
- Cosider a MVM of order ≥ 3. Perform a pivot in it, then delete the pivot row and column. Then the resulting matrix is also MVM ("by a simple cofactor argument").

Lemma 162 (11.2.1)**.** *Any* 1- *or* 2-*sum of two regular matroids is also regular.*

Proof sketch.

- 1-sum case: $M_1 \oplus_1 M_2$ is represented by a matrix $B = \text{diag}(A_1, A_2)$ where A_1 and A_2 represent M_1 and M_2. Use the same signings for A_1 and A_2 in B to prove that B is TU and hence the 1-sum is regular.
- 2-sum case: Slightly more complicated signing process. Similarly, reuse signings from M_1 and M_2, define signing on remaining nonzero elements via a concrete formula, then prove that the resulting matrix is TU.

□

Lemma 163 (11.2.7)**.** *M_2 of (8.3.10) and (8.3.11) is regular iff $M_{2\Delta}$ of (8.5.3) (M_2 converted by a ΔY exchange) is regular.*

Proof sketch. Utilize signings, minimal violation matrices, intersections (inside matrices), column dependence, pivot, duality. □

Corollary 164 (11.2.8)**.** *ΔY exchanges maintain regularity.*

Proof. Follows by Lemma 11.2.7. □

Lemma 165 (11.2.9)**.** *Any* 3-*sum of two regular matroids is also regular.*

Proof sketch. Yet more complicated, but similar. Uses the result that "ΔY exchanges maintain regularity" (Corollary 11.2.8 of Lemma 11.2.7). The rest of the arguments are similar to the 2-sum case: prove that submatrices are TU, then prove that the whole matrix is TU. □

Theorem 166 (11.2.10)**.** *Any* 1-, 2-, *or* 3-*sum of two regular matroids is regular.*

Proof sketch. Combine Lemmas 11.2.1 and 11.2.9. □

Corollary 167 (11.2.12)**.** *Any Δ-sum or Y-sum of two regular matroids is also regular.*

Proof sketch. Follows from definitions of Δ-sums and Y-sum, together with Theorem 11.2.10 and Corollary 11.2.8. □

2.11.2 Chapter 11.3

Proposition 168 (11.3.3)**.** *Graph plus T set representing R_{10}*

Proposition 169 (11.3.5)**.** *Graph plus T set representing F_7.*

Proposition 170 (11.3.11)**.** *The binary representation matrix B^{12} for R_{12}.*

The goal of the chapter is to prove the "hard" direction of the regular matroid decomposition theorem.

Theorem 171 (11.3.2)**.** *R_{10} is a splitter of the class of regular matroids.*
In short: up to isomorphism, the only 3-connected regular matroid with R_{10} minor is R_{10}.

Proof sketch.

- Splitter theorem case (a)
- R_{10} is self-dual, so it suffices to consider 1-element additions.
- Represent R_{10} by (11.3.3)
- Up to isomorphism, there are only 3 distinct 3-connected 1-element extensions.
- Case 1 (graphic): contract a certain edge, the resulting graph contains a subdivision of (11.3.5), which represents F_7. Thus, this extension is nonregular.
- Cases 2, 3 (nongraphic): reduce instances to (11.3.5), same conclusion.

$\square$

Theorem 172 (11.3.10)**.** *In short: Restatement of **??** for R_{12}. Replacements: $\mathcal{M}$ is the class of regular matroids, N is R_{12}, (6.3.12) is (11.3.6), (6.3.21-23) are (11.3.7-9).*

Theorem 173 (11.3.12)**.** *Let M be a regular matroid with R_{12} minor. Then any 3-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of R_{12} (see (11.3.11) – matrix B^{12} for R_{12} defining the 3-separation) under one of the isomorphisms induces a 3-separation of M.*
In short: every regular matroid with R_{12} minor is a 3-sum of two proper minors.

Proof sketch.

- Preparation: calculate all 3-connected regular 1-element additions of R_{12}. This involves somewhat tedious case checking. (Representation of R_{12} in (10.2.9) helps a lot.) By the symmetry of B^{12} and thus by duality, this effectively gives all 3-connected 1-element extensions as well.
- Verify conditions of theorem 11.3.10 (which implies the result).
- (11.3.7) and (11.3.9) are ruled out immediately from preparatory calculations.
- The rest is case checking ((c.1) and (c.2)), simpified by preparatory calculations.

$\square$

Theorem 174 (11.3.14 regular matroid decomposition, easy direction)**.** *Every binary matroid produced from graphic, cographic, and matroids isomorphic to R_{10} by repeated 1-, 2-, and 3-sum compositions is regular.*

Proof sketch. Follows from theorem 11.2.10. □

Theorem 175 (11.3.14 regular matroid decomposition, hard direction)**.** *Every regular matroid M can be decomposed into graphic and cographic matroids and matroids isomorphic to R_{10} by repeated* 1-*,* 2-*, and* 3- *sum decompositions. Specifically: If M is a regular* 3*-connected matroid that is not graphic and not cographic, then M is isomorphic to R_{10} or has an R_{12} minor. In the latter case, any* 3*-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of R_{12} ((11.3.11)) under one of the isomorphisms induces a* 3*-separation of M.*

Proof sketch.

- Let M be a regular matroid. Assume M is not graphic and not cographic.
- If M is 1-separable, then it is a 1-sum. If M is 2-separable, then it is a 2-sum. Thus assume M is 3-connected.
- By theorem 10.4.1, M has an R_{10} or an R_{12} minor.
- R_{10} case: by theorem 11.3.2, M is isomorphic to R_{10}.
- R_{12} case: by theorem 11.3.12, M has an induced by 3-separation, so by lemma 8.3.12, M is a 3-sum.

□

2.11.3 Extensions of Regular Matroid Decomposition

- Theorem 11.3.14 remains valid when 3-sums are replaced by Δ- and Y-sums (Theorem 11.3.16).
- Theorem 11.3.14 (and 11.3.16) can also be proved for matroids with no F_7 minors or with no F_7^* minors. (Uses Lemma 11.3.19: F_7 (F_7^*) is a splitter of the binary matroids with no F_7^* (F_7) minors.)

2.11.4 Applications of Regular Matroid Decomposition

- Efficient algorithm for testing if a binary matroid is regular (Section 11.4).
- Efficient algorithm for deciding if a real matrix is TU (Section 11.4).
- Constructing TU matrices (Theorem 11.5.9). (Translate 3-sum version of theorem 11.3.16 into matrix language.)
- Constructing $0, 1$ TU matrices (Theorem 11.5.13).
- Characterization of the cycle polytope (theorem 11.5.17). (Problem: let M be a connected binary matroid with ground set E and element weighs w_e for all $e \in E$. Find a disjoint union C of circuits of M such that $\sum_{e \in C} w_e$ is maximized.)
- Number of nonzeros in TU matrices (Theorem 11.5.18).
- Triples in circuits (Theorem 11.5.18).
- Odd cycles (Theorem 11.5.20).