Proof of Regularity of 2- and 3-Sum of Matroids

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1 Proof of Regularity of 2-Sum

Lemma 1. *Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.*

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc}, the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic} a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column $a''_{\cdot k}$ with the pivot column $a_{\cdot c}$ without the pivot element a_{rc}.

By the cofactor expansion in A along row r, we have

$$\det A = \sum_{k=1}^{n} (-1)^{r+k} a_{rk} \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k. By swapping the order of columns in $B_{r,k}$ to match the form of B_k, we get

$$\det A = (-1)^{r+c} (a_{rc} \det A' - \sum_{k \in C} a_{rk} \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 1. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 1. Let B_1, B_2 be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = [A_1/x]$ and $B_2 = [y \mid A_2]$, where x is a row vector, y is a column vector, and A_1, A_2 are matrices of appropriate dimensions. Let D be the outer product of y and x. The 2-sum of B_1 and B_2 is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}.$$

Definition 2. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 1. Note that a matrix is TU if and only if it is k-TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 2. *Let B_1 and B_2 be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then B is 1-TU and 2-TU.*

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B. If V is a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_1 and A_2. Let i be the row shared by V and A_1 and j be the column shared by V and A_2. Note that $V_{ij} = 0$. Thus, $\det V$ equals the product of the entries on the diagonal not containing V_{ij}. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. □

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell < k$. Then for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also k-TU.*

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_1 and B_2 such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not k-TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row of V whose index appears in A_1. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_1. Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc}. For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, $\tilde{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\left[\tilde{A}_1/\tilde{D}\right]$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., A_2 remains unchanged. This holds because of the 0 block in B.
- $\tilde{D}$ consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\left[\tilde{D} \mid \tilde{A}_2\right]$ is TU, since this matrix consists of A_2 and copies of y scaled by factors $\{0, \pm 1\}$.
- $\tilde{D}$ can be represented as an outer product of a column vector $\tilde{y}$ and a row vector $\tilde{x}$, and we can define $\tilde{B}_1 = \left[\tilde{A}_1/\tilde{x}\right]$ and $\tilde{B}_2 = \left[\tilde{y} \mid \tilde{A}_2\right]$ similar to B_1 and B_2, respectively. Note that $\tilde{B}_1$ and $\tilde{B}_2$ have the same size as B_1 and B_2, respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- $\tilde{B}$ contains a square submatrix $\tilde{V}$ of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k-1) \times (k-1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B, the submatrix V'' corresponds to a submatrix $\tilde{V}$ of $\tilde{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\tilde{B}$ that represents a 2-sum of TU matrices $\tilde{B}_1$ and $\tilde{B}_2$ and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k-1)$-TUness of $\tilde{B}$, which proves the lemma. □

Lemma 4. *Let B_1 and B_2 be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_1 and B_2, their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 2.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 3.

Conclusion: For any TU matrices B_1 and B_2, their 2-sum $B_1 \oplus_{2,x,y} B_2$ is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. □

2 3-Sums

2.1 Delta-Wye Exchange

Delta-Wye Exchange or ΔY-exchange is an operation of replacing a triangle with a 3-star or vice versa.

Definition 3. The triangle to 3-star exchange for matrices is defined as follows.

1. Let $B \in \mathbb{Z}_2^{X \times (Y \cup \{e,f,g\})}$ be a binary matrix of the form
$$B = \begin{bmatrix} \overline{B} & a & b & c \end{bmatrix}, \quad \text{where } a+b+c = 0 \text{ in } \mathbb{Z}_2.$$
Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{y\}) \times (Y \cup \{x,z\})}$ where
$$B' = \begin{bmatrix} \overline{B} & a & b \\ 0 & 1 & 1 \end{bmatrix}.$$

2. Let $B \in \mathbb{Z}_2^{(X \cup \{f\}) \times (Y \cup \{e,g\})}$ be a binary matrix of the form
$$B = \begin{bmatrix} \overline{B} & b & b \\ a & 1 & 0 \end{bmatrix}.$$
Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{z,y\}) \times (Y \cup \{x\})}$ where
$$B' = \begin{bmatrix} \overline{B} & b \\ a & 1 \\ a & 0 \end{bmatrix}.$$

3. Let $B \in \mathbb{Z}_2^{(X \cup \{e,f\}) \times (Y \cup \{g\})}$ be a binary matrix of the form
$$B = \begin{bmatrix} \overline{B} & 0 \\ a & 1 \\ b & 1 \end{bmatrix}$$
Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{x,y,z\}) \times Y}$ where
$$B' = \begin{bmatrix} \overline{B} \\ a \\ b \\ c \end{bmatrix}, \quad \text{where } a+b+c = 0 \text{ in } \mathbb{Z}_2.$$

The 3-star to triangle exchange is defined as the converse operation.

Remark 2. Note that in the case distinction $\overline{B}, a, b, c$ refer to different matrices and vectors.

Definition 4. Let M be a binary matroid with the ground set E. Let $\{e,f,g\} \subseteq E$ be a triangle in M not containing a cocycle and let B be a standard binary representation matrix for M. The triangle to 3-star exchange on M results in a binary matroid M' with the ground set $E' = E \setminus \{e,f,g\} \cup \{x,y,z\}$ represented by the standard binary representation matrix B' obtained by the triangle to star exchange on B.

Conversely, let M' be a binary matroid with the ground set E'. Let $\{x,y,z\} \subseteq E'$ be a triad in M not containing a cycle and let B' be a standard binary representation matrix for M'. The 3-star to triangle exchange on M' results in a binary matroid M with the ground set $E = E' \setminus \{x,y,z\} \cup \{e,f,g\}$ represented by the standard binary representation matrix B obtained by the triangle to star exchange on B'.

Remark 3. Note that we may always choose B of the form from case 3. In this case, the condition that the triangle $\{e,f,g\}$ does not contain a cocycle is equivalent to the requirement that the row vectors a and b of B are non-zero and distinct. Hence, the row vectors a, b, and $c = a+b$ (in $\mathbb{Z}_2$) in B' are distinct, and $\{x,y,z\}$ is indeed a triad in M'.

Lemma 5. *The triangle to triad exchange in M is a triad to triangle exchange in M^*.*

Proof. By construction, if M has standard representation S, then $-S^T$ (and also S^T) is a standard representation of M^*. Plugging this into Definition 3 and reversing the operation shows the desired result. □

2.2 3-Sum and Delta-Sum Constructions

Definition 5. Let $B_1 \in \mathbb{Z}_2^{(X_1 \cup \{x_2,x_3\}) \times (Y_1 \cup \{y_3\})}$, $B_2 \in \mathbb{Z}_2^{(\{x_1\} \cup X_2) \times (\{y_1,y_2\} \cup Y_2)}$ be matrices of the form

$$B_1 = \begin{array}{|c|cc|c|} \hline & & & \\ & A_1 & & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_1 & \multicolumn{2}{c|}{\overline{D}} & 1 \\ & & & 1 \\ \hline \end{array}\,, \quad B_2 = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{\overline{D}} & 1 & \\ & & 1 & A_2 \\ \cline{1-3} \multicolumn{2}{|c|}{D_2} & & \\ \hline \end{array}\,,$$

where $\overline{D}$ is a 2×2 matrix with $\mathbb{Z}_2$ rank 2 (i.e., $\overline{D}$ is non-singular over $\mathbb{Z}_2$). Note that $x_1 \in X_1$, $x_2, x_3 \in X_2$, $y_1, y_2 \in Y_1$, $y_3 \in Y_2$, $A_1 \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2 \in \mathbb{Z}_2^{X_2 \times Y_2}$, $\overline{D} \in \mathbb{Z}_2^{(x_2,x_3) \times (y_1,y_2)}$, $D_1 \in \mathbb{Z}_2^{\{x_2,x_3\} \times (Y_1 \setminus \{y_1,y_2\})}$, $D_2 \in \mathbb{Z}_2^{(X_2 \setminus \{x_2,x_3\}) \times \{y_1,y_2\}}$. Then the 3-sum of B_1 and B_2 is defined as

$$B_1 \oplus_3 B_2 = \begin{array}{|c|cc|c|c|} \hline & & & & \\ & A_1 & & & 0 \\ \cline{2-4} & 1 & 1 & 0 & \\ \hline D_1 & \multicolumn{2}{c|}{\overline{D}} & 1 & \\ & & & 1 & A_2 \\ \cline{1-4} D_{12} & \multicolumn{2}{c|}{D_2} & & \\ \hline \end{array}\,,$$

where $D_{12} = D_2 \cdot (\overline{D})^{-1} \cdot D_1$ and the indexing is preserved.

Definition 6. To simplify notation, let $D_{1,12} = [D_1 / D_{12}]$, $D_{0,2} = [\overline{D} / D_2]$, $D_{1,0} = [D_1 \mid \overline{D}]$, $D_{12,2} = [D_{12} \mid D_2]$.

Definition 7. Let B_1, B_2 satisfy the conditions of Definition 5. Let $B_{2\Delta} \in \mathbb{Z}_2^{X_2 \times (\{z,y_1,y_2\} \cup Y_2)}$ be the matrix obtained from B_2 via a triangle-star exchange from Definition 3:

$$B_{2\Delta} = \begin{array}{|c|cc|c|c|} \hline & \multicolumn{2}{c|}{\overline{D}} & 1 & \\ d & & & 1 & A_2 \\ \cline{2-4} & \multicolumn{2}{c|}{D_2} & & \\ \hline \end{array}$$

where $d \in \mathbb{Z}_2^{Y_2}$ is such that $(D_{0,2})_{\cdot y_1} + (D_{0,2})_{\cdot y_2} + d = 0$.

Definition 8. Let B_1, B_2, and $B_{2\Delta}$ be matrices from Definitions 5 and 7. Then the Δ-sum of B_1 and $B_{2\Delta}$ is $B_1 \oplus_\Delta B_{2\Delta} = B_1 \oplus_3 B_2$.

2.3 Regularity of 3-Sum

Lemma 6. *Suppose A and A' are TU signings of the same matrix $B \in \mathbb{Z}_2^{m \times n}$. Then there exist vectors $u \in \{\pm 1\}^m$ and $v \in \{\pm 1\}^n$ such that $a'_{ij} = u_i v_j a_{ij}$ for every $i \in [m]$, $j \in [n]$.*

Proof.

adapt from Lemma 9.2.6 in Truemper

□

Lemma 7. *Let B_2 be a matrix from Definition 5. If B_2 is regular, then it has a TU signing $\tilde{B}_2$ where all entries in columns y_1 and y_2 are in $\{0, 1\}$.*

Proof. Since B_2 is regular, it has a TU signing B'_2. Recall that multiplying rows and columns of a TU matrix by factors in $\{0, \pm 1\}$ preserves TUness.

If $B'_2(x_1, y_1) = -1$, multiply column y_1 by -1. Similarly, if $B'_2(x_1, y_2) = -1$, multiply column y_2 by -1. Thus, we may assume that B'_2 has $B'_2(x_1, y_1) = B'_2(x_1, y_2) = 1$.

Next, consider each row of B'_2. It can have one of the following forms.

- $[0 \mid 0]$, $[0 \mid 1]$, $[1 \mid 0]$, $[1 \mid 1]$. In this case, we do not need to modify the signing.
- $[0 \mid -1]$, $[-1 \mid 0]$, $[-1 \mid -1]$. In this case, we can multiply this row by -1 to make all its non-negative.
- $[1 \mid -1]$, $[-1 \mid 1]$. This case leads to a contradiction, as the matrix composed of this row and row x_1 has
$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \quad \text{or} \quad \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2,$$
which is impossible as B_2' is a TU signing.

Thus, we can multiply columns and rows of B_2' to obtain a TU signing $\tilde{B}_2$ where all entries in columns y_1 and y_2 are in $\{0, 1\}$, as desired. $\square$

Lemma 8. *Let B_2 be a matrix from Definition 5 and let $\tilde{B}_2$ be a TU signing of B_2 from Lemma 7. To simplify notation, let $\tilde{a} = (\tilde{D}_{0,2})_{\cdot y_1}$ and $\tilde{b} = (\tilde{D}_{0,2})_{\cdot y_2}$. Then pivoting in $\tilde{B}_2$ on (x_1, y_1) and (x_1, y_2) yields:*

$$\tilde{B}_2 = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline -\tilde{a} & \tilde{b} - \tilde{a} & \tilde{A}_2 \\ \hline \end{array}$$

$$\tilde{B}_2 = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \tilde{a} - \tilde{b} & -\tilde{b} & \tilde{A}_2 \\ \hline \end{array}$$

Proof. Recall that a real pivot in matrix A on entry $a_{rc} \neq 0$ transforms the matrix as follows:

$$\begin{array}{|c|c|} \hline a_{rc} & a_{rj} \\ \hline a_{ic} & a_{ij} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|} \hline \frac{1}{a_{rc}} & \frac{a_{rj}}{a_{rc}} \\ \hline -\frac{a_{ic}}{a_{rc}} & a_{ij} - \frac{a_{rj} a_{ic}}{a_{rc}} \\ \hline \end{array}$$

A direct calculation proves the claim. $\square$

Corollary 2. Let B_2 be a matrix from Definition 5 and let $\tilde{B}_2$ be a TU signing of B_2 from Lemma 7. Then the following matrices are TU:

$$\tilde{B}_2^{(a)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{a} & \tilde{A}_2 \\ \hline \end{array}, \quad \tilde{B}_2^{(b)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array}.$$

Proof. Recall that pivoting, taking submatrices, and multiplying columns by ± 1 factors preserves TUness. Combining these facts with Lemma 8 gives the corollary. $\square$

Lemma 9. *Let B_2 and $B_{2\Delta}$ be matrices from Definitions 5 and 7. If B_2 is regular, then $B_{2\Delta}$ is regular.*

Proof. Let $\tilde{B}_2$ be a TU signing of B_2 from Lemma 7. Let $\tilde{\overline{D}}$, $\tilde{D}_2$, and $\tilde{A}_2$ be the signings of $\overline{D}$, D_2, and A_2, respectively, etc. Let $\tilde{d} = (\tilde{D}_{0,2})_{\cdot y_1} - (\tilde{D}_{0,2})_{\cdot y_2}$ and $\tilde{B}_{2\Delta} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2]$. Since $\tilde{D_{0,2}} \in \{0, 1\}^{X_2 \times \{y_1, y_2\}}$ by Lemma 7, we have $\tilde{d} \in \{0, \pm 1\}^{X_2}$, so $\tilde{B}_{2\Delta}$ is a signing of $B_{2\Delta}$. Our goal is to prove that $\tilde{B}_{2\Delta}$ is TU. To this end, let V be a square submatrix of $\tilde{B}_{2\Delta}$. We will show that $\det V \in \{0, \pm 1\}$.

Suppose that column $\tilde{d}$ (with index z) is not in V. Then V is a submatrix of $[\tilde{D}_{0,2} \mid \tilde{A}_2]$ and hence a submatrix of $\tilde{B}_2$. Since $\tilde{B}_2$ is TU, we have $\det V \in \{0, \pm 1\}$. Going forward we assume that column $\tilde{d}$ (with index z) is in V.

Suppose that columns $(\tilde{D}_{0,2})_{\cdot y_1}$ and $(\tilde{D}_{0,2})_{\cdot y_2}$ (with indices y_1 and y_2, respectively) are both in V. Then V contains three linearly dependent columns: $\tilde{d}$, $(\tilde{D}_{0,2})_{\cdot y_1}$, and $(\tilde{D}_{0,2})_{\cdot y_2}$ (with indices z, y_1, and y_2, respectively). Thus, $\det V = 0$. Going forward we assume that at most one of the columns $(\tilde{D}_{0,2})_{\cdot y_1}$ and $(\tilde{D}_{0,2})_{\cdot y_2}$ is in V.

Suppose that column $(\tilde{D}_{0,2})_{\cdot y_1}$ (with index y_1) is in V. Then V is a submatrix of $\tilde{B}_2^{(b)}$ from Corollary 2, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $\tilde{B}_2^{(a)}$ from Corollary 2, and so $\det V \in \{0, \pm 1\}$.

Since our case distinction is exhaustive, we showed that every square submatrix V of $\tilde{B}_{2\Delta}$ has $\det V \in \{0, \pm 1\}$. Thus, $\tilde{B}_{2\Delta}$ is TU, and so $B_{2\Delta}$ is regular. ☐

Lemma 10. *Let B_2 and $B_{2\Delta}$ be matrices from Definitions 5 and 7. If $B_{2\Delta}$ is regular, then B_2 is regular.*

Proof. Since $B_{2\Delta}$ is regular, $B_{2\Delta}^*$ is also regular. Since $B_{2\Delta}$ is obtained from B_2 via a ΔY-exchange, B_2^* can be obtained from $B_{2\Delta}^*$ via the same operation. Therefore, B_2^* is regular by Lemma 9. Thus, B_2 is regular. ☐

Corollary 3. B_2 from Definition 5 is regular if and only if $B_{2\Delta}$ from Definition 7 is regular.

Proof. Combine the results of Lemmas 9 and 10. ☐

Lemma 11. *Assume the notation of Definitions 5 and 7. Then the columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$.*

Proof. Columns of $[d \mid D_{0,2}]$ trivially satisfy the claim, so it only remains to show that columns of $D_{1,12}$ are in $[d \mid D_{0,2} \mid 0]$. Note that $D_{1,12} = D_{0,2} \cdot ((\overline{D})^{-1} \cdot D_1)$, i.e., every column of $D_{1,12}$ can be expressed as a linear combination of the columns of $D_{0,2}$ (over $\mathbb{Z}_2$). In particular, every column of $D_{1,12}$ is either zero, one of the columns of $D_{0,2}$, or their sum. By construction, $(D_{0,2})_{\cdot y_1} + (D_{0,2})_{\cdot y_2} = d$. Thus, the desired result holds. ☐

Corollary 4. As a direct corollary of Lemma 11, columns of $[d \mid D \mid A_2]$ are in $[d \mid D_{0,2} \mid A_2 \mid 0]$.

Lemma 12. *Let B_1 and B_2 be matrices from Definition 5. If B_1 and B_2 are regular, then $B_1 \oplus_3 B_2$ is regular.*

Proof. Let $B_{2\Delta}$ be the matrix from Definition 7. By Lemma 9, $B_{2\Delta}$ is regular. Since $B_1 \oplus_\Delta B_{2\Delta} = B_1 \oplus_3 B_2$, to prove the desired result it suffices to show that $B_1 \oplus_\Delta B_{2\Delta}$ is regular.

similar argument as for 2-sums; tight point: after signing $B_{2\Delta}$, need to propagate signing from $\tilde{D}$ to A_1/D, which involves an argument relying on traversing bipartite graphs; keep track of the form of D, the form is preserved under pivoting

☐