# Matroid Decomposition Theorem Verification

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# Chapter 1

# Code

## 1.1 TU Matrices

**Definition 1** (TU matrix)**.** A real matrix is *totally unimodular (TU)* if its every subdeterminant (i.e., determinant of every square submatrix) is 0 or $\pm 1$.

**Lemma 2** (entries of a TU matrix)**.** *If $A$ is TU, then every entry of $A$ is 0 or $\pm 1$.*

*Proof sketch.* Every entry is a square submatrix of size 1. □

**Lemma 3** (TUness with adjoint identity matrix)**.** *$A$ is TU iff every basis matrix of $[I \mid A]$ has determinant $\pm 1$.*

*Proof sketch.* Gaussian elimination. Basis submatrix: its columns form a basis of all columns, its rows form a basis of all rows. □

**Lemma 4** (any submatrix of a TU matrix is TU)**.** *Let $A$ be a real matrix that is TU and let $B$ be a submatrix of $A$. Then $B$ is TU.*

*Proof sketch.* Any square submatrix of $B$ is a submatrix of $A$, so its determinant is 0 or $\pm 1$. Thus, $B$ is TU. □

**Lemma 5** (block-diagonal matrix with TU blocks is TU)**.** *Let $A$ be a matrix of the form* $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & A_2 \\ \hline\end{array}$ *where $A_1$ and $A_2$ are both TU. Then $A$ is also TU.*

*Proof sketch.* Any square submatrix $T$ of $A$ has the form $\begin{array}{|c|c|}\hline T_1 & 0 \\ \hline 0 & T_2 \\ \hline\end{array}$ where $T_1$ and $T_2$ are submatrices of $A_1$ and $A_2$, respectively.

- If $T_1$ is square, then $T_2$ is also square, and $\det T = \det T_1 \cdot \det T_2 \in \{0, \pm 1\}$.
- If $T_1$ has more rows than columns, then the rows of $T$ containing $T_1$ are linearly dependent, so $\det T = 0$.
- Similar if $T_1$ has more columns than rows.

□

**Lemma 6** (transpose of TU is TU)**.** *Let $A$ be a TU matrix. Then $A^T$ is TU.*

*Proof sketch.* A submatrix $T$ of $A^T$ is a transpose of a submatrix of $A$, so $\det T \in \{0, \pm 1\}$. $\square$

**Lemma 7** (adjoining to TU matrices)**.** *Let $A$ be a TU matrix.*

- *Let $a$ be a zero row. Then $C = [A/a]$ is TU.*
- *Let $a$ be a unit row. Then $C = [A/a]$ is TU.*
- *Let $a$ be some row of $A$. Then $C = [A/a]$ is TU.*
- *Let $B$ be a matrix whose every row is a row of $A$, a zero row, or a unit row. Then $C = [A/B]$ is TU.*

*Proof sketch.*

- Let $T$ be a square submatrix of $C$. If $T$ contains a zero row, then $\det T = 0$. Otherwise $T$ is a submatrix of $A$, so $\det T \in \{0, \pm 1\}$.
- Let $T$ be a square submatrix of $C$. If $T$ contains the same row twice, then the rows are GF(2)-dependent, so $\det T = 0$. Otherwise $T$ is a submatrix of $A$, so $\det T \in \{0, \pm 1\}$.
- Let $T$ be a square submatrix of $C$. If $T$ contains the $\pm 1$ entry of the unit row, then $\det T$ equals the determinant of some submatrix of $A$ times $\pm 1$, so $\det T \in \{0, \pm 1\}$. If $T$ contains some entries of the unit row except the $\pm 1$, then $\det T = 0$. Otherwise $T$ is a submatrix of $A$, so $\det T \in \{0, \pm 1\}$.
- Either repeatedly apply the previous three items or direcly perform a similar case analysis.

$\square$

**Corollary 8** (column properties of TU matrices)**.** *Properties listed in Lemma* **??** *also hold with respect to columns.*

*Proof sketch.* Combine results of Lemma **??** and Lemma **??**. $\square$

**Definition 9** ($\mathcal{F}$-pivot)**.** Let $A$ be a matrix over a field $\mathcal{F}$ with row index set $X$ and column index set $Y$. Let $A_{xy}$ be a nonzero element. The result of a *$\mathcal{F}$-pivot* of $A$ on the *pivot element* $A_{xy}$ is the matrix $A'$ over $\mathcal{F}$ with row index set $X'$ and column index set $Y'$ defined as follows.

- For every $u \in X - x$ and $w \in Y - y$, let $A'_{uw} = A_{uw} + (A_{uy} \cdot A_{xw})/(-A_{xy})$.
- Let $A'_{xy} = -A_{xy}$, $X' = X - x + y$, and $Y' = Y - y + x$.

**Lemma 10** (pivoting preserves TUness)**.** *Let $A$ be a TU matrix and let $A_{xy}$ be a nonzero element. Let $A'$ be the matrix obtained by performing a real pivot in $A$ on $A_{xy}$. Then $A'$ is TU.*

*Proof sketch.*

- By Lemma **??** $A$ is TU iff every basis matrix of $[I \mid A]$ has determinant $\pm 1$. The same holds for $A'$ and $[I \mid A']$.
- Determinants of the basis matrices are preserved under elementary row operations in $[I \mid A]$ corresponding to the pivot in $A$, under scaling by $\pm 1$ factors, and under column exchange, all of which together convert $[I \mid A]$ to $[I \mid A']$.

$\square$

**Lemma 11** (pivoting preserves TUness)**.** *Let $A$ be a matrix and let $A_{xy}$ be a nonzero element. Let $A'$ be the matrix obtained by performing a real pivot in $A$ on $A_{xy}$. If $A'$ is TU, then $A$ is TU.*

*Proof sketch.* Reverse the row operations, scaling, and column exchange in the proof of Lemma **??**. □

### 1.1.1 Minimal Violation Matrices

**Definition 12** (minimal violation matrix)**.** Let $A$ be a real $\{0, \pm 1\}$ matrix that is not TU but all of whose proper submatrices are TU. Then $A$ is called a *minimal violation matrix of total unimodularity (minimal violation matrix).*

**Lemma 13** (simple properties of MVMs)**.** *Let $A$ be a minimal violation matrix.*

- *$A$ is square.*
- $\det A \notin \{0, \pm 1\}$.
- *If $A$ is $2 \times 2$, then $A$ does not contain a $0$.*

*Proof sketch.*

- If $A$ is not square, then since all its proper submatrices are TU, $A$ is TU, contradiction.
- If $\det A \in \{0, \pm 1\}$, then all subdeterminants of $A$ are $0$ or $\pm 1$, so $A$ is TU, contradiction.
- If $A$ is $2 \times 2$ and it contains a $0$, then $\det A \in \{\pm 1\}$, which contradicts the previous item.

□

**Lemma 14** (pivoting in MVMs)**.** *Let $A$ be a minimal violation matrix. Suppose $A$ has $\geq 3$ rows. Suppose we perform a real pivot in $A$, then delete the pivot row and column. Then the resulting matrix $A'$ is also a minimal violation matrix.*

*Proof sketch.*

- Let $A''$ denote matrix $A$ after the pivot, but before the pivot row and column are deleted.
- Since $A$ is not TU, Lemma **??** implies that $A''$ is not TU. Thus $A'$ is not TU by Lemma **??**.
- Let $T'$ be a proper square submatrix of $A'$. Let $T''$ be the submatrix of $A''$ consisting of $T'$ plus the pivot row and the pivot column, and let $T$ be the corresponding submatrix of $A$ (defined by the same row and column indices as $T''$).
- $T$ is TU as a proper submatrix of $A$. Then Lemma **??** implies that $T''$ is TU. Thus $T'$ is TU by Lemma **??**.

□

## 1.2 Matroid Definitions

**Definition 15** ((finite) matroid)**.** Let $E$ be a finite ground set. Let $\mathcal{I} \subseteq 2^E$ be a family of subsets satisfying:

- $\emptyset \in \mathcal{I}$ (non-empty)
- if $A \subseteq B \in \mathcal{I}$, then $A \in \mathcal{I}$ (down-closed)
- if $A, B \in \mathcal{I}$ and $|A| < |B|$, then $A + b \in \mathcal{I}$ for some $b \in B \setminus A$ (exchange property)

Then the pair $M = (E, \mathcal{I})$ is called a *(finite) matroid.*

**Definition 16** (binary matroid)**.** Let $B$ be a binary matrix, let $A = [I \mid B]$, and let $E$ denote the column index set of $A$. Let $\mathcal{I}$ be all index subsets $Z \subseteq E$ such that the columns of $A$ indexed by $Z$ are independent over GF(2). Then $M = (E, \mathcal{I})$ is called a *binary matroid* and $B$ is called its *(standard) representation matrix.*

**Definition 17** (regular matroid)**.** Let $M$ be a binary matroid generated from a standard representation matrix $B$. Suppose $B$ has a TU signing, i.e., there exists a real matrix $A$ such that:

- $A$ is a signed version of $B$, i.e., $|A| = B$,
- $A$ is totally unimodular.

Then $M$ is called a *regular matroid.*

## 1.3 $k$-Separation and $k$-Connectivity

**Definition 18** ($k$-separation)**.** Let $M$ be a binary matroid generated by a standard representation matrix $B$. Suppose that $B$ is partitioned as $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B_1 & D_2 \\ \hline X_2 & D_1 & B_2 \\ \hline \end{array}$ where $X_1 \sqcup X_2$ is a partition of the rows of $B$ and $Y_1 \sqcup Y_2$ is a partition of its columns. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that

- $|X_1 \cup Y_1| \geq k$ and $|X_2 \cup Y_2| \geq k$,
- GF(2)-rank $D_1$ + GF(2)-rank $D_2 \leq k - 1$.

Then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a *(Tutte) $k$-sepeartion* of $B$ and $M$.

**Definition 19** (exact $k$-separation)**.** A $k$-separation is called *exact* if the rank condition holds with equality.

**Definition 20** ($k$-separability)**.** We say that $B$ and $M$ are *(exactly) (Tutte) $k$-separable* if they have an (exact) $k$-separation.

**Definition 21** ($k$-connectivity)**.** For $k \geq 2$, $M$ and $B$ are *(Tutte) $k$-connected* if they have no $\ell$-separation for $1 \leq \ell < k$. When $M$ and $B$ are 2-connected, they are also called *connected.*

## 1.4 Sums

### 1.4.1 1-Sums

**Definition 22** (1-sum of matrices)**.** Let $B$ be a matrix that can be represented as $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B_1 & 0 \\ \hline X_2 & 0 & B_2 \\ \hline \end{array}$
Then we say that $B_1$ and $B_2$ are the two *components* of a 1*-sum decomposition* of $B$.

Conversely, a 1*-sum composition* with *components* $B_1$ and $B_2$ is the matrix $B$ above.

The expression $B = B_1 \oplus_1 B_2$ means either process.

**Definition 23** (matroid 1-sum)**.** Let $M$ be a binary matroid with a representation matrix $B$. Suppose that $B$ can be partitioned as in Definition **??** with non-zero blocks $B_1$ and $B_2$. Then the binary matroids $M_1$ and $M_2$ represented by $B_1$ and $B_2$, respectively, are the two *components* of a 1*-sum decomposition* of $M$.

Conversely, a 1*-sum composition* with *components* $M_1$ and $M_2$ is the matroid $M$ defined by the corresponding representation matrix $B$.

The expression $M = M_1 \oplus_1 M_2$ means either process.

**Lemma 24** (1-separations and 1-sums)**.** *Let $M$ be a binary matroid that is* 1*-separable. Then $M$ can be decomposed as a* 1*-sum with components given by the* 1*-separation.*

*Proof sketch.* Check by definition. □

**Lemma 25** (1-sum of regular matroids is regular)**.** *Let $M_1$ and $M_2$ be regular matroids. Then $M = M_1 \oplus_1 M_2$ is a regular matroid.*

*Conversely, if a regular matroid $M$ can be decomposed as a* 1*-sum $M = M_1 \oplus_1 M_2$, then $M_1$ and $M_2$ are both regular.*

*Proof sketch.*

extract into lemmas about TU matrices

Let $B$, $B_1$, and $B_2$ be the representation matrices of $M$, $M_1$, and $M_2$, respectively.

- Converse direction. Let $B'$ be a TU signing of $B$. Let $B_1'$ and $B_2'$ be signings of $B_1$ and $B_2$, respectively, obtained from $B$. By Lemma **??**, $B_1'$ and $B_2'$ are both TU, so $M_1$ and $M_2$ are both regular.
- Forward direction. Let $B_1'$ and $B_2'$ be TU signings of $B_1$ and $B_2$, respectively. Let $B'$ be the corresponding signing of $B$. By Lemma **??**, $B'$ is TU, so $M$ is regular.

□

### 1.4.2 2-Sums

**Definition 26** (2-sum of matrices)**.** Let $B$ be a matrix of the form $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A_1 & 0 \\ \hline X_2 & D & A_2 \\ \hline \end{array}$ Let $B_1$ be a matrix of the form $\begin{array}{c|c|} & Y_1 \\ X_1 & A_1 \\ \text{Unit} & x \end{array}$ Let $B_2$ be a matrix of the form $\begin{array}{c|c|c|} & \text{Unit} & Y_2 \\ \hline X_2 & y & A_2 \\ \hline \end{array}$ Suppose that GF(2)-rank $D = 1$, $x \neq 0$, $y \neq 0$, $D = y \cdot x$ (outer product).

Then we say that $B_1$ and $B_2$ are the two *components* of a 2*-sum decomposition* of $B$.

Conversely, a 2*-sum composition* with *components* $B_1$ and $B_2$ is the matrix $B$ above.

The expression $B = B_1 \oplus_2 B_2$ means either process.

**Definition 27** (matroid 2-sum)**.** Let $M$ be a binary matroid with a representation matrix $B$. Suppose $B$, $B_1$, and $B_2$ satisfy the assumptions of Definition **??**. Then the binary matroids $M_1$ and $M_2$ represented by $B_1$ and $B_2$, respectively, are the two *components* of a 2*-sum decomposition* of $M$.

Conversely, a 2*-sum composition* with *components* $M_1$ and $M_2$ is the matroid $M$ defined by the corresponding representation matrix $B$.

The expression $M = M_1 \oplus_2 M_2$ means either process.

**Lemma 28** (2-separations and 2-sums of connected binary matroids)**.** *Let $M$ be a binary matroid that is 2-separable. Then $M$ can be decomposed as a 2-sum with connected components given by the 2-separation.*

*Conversely, any 2-sum of two connected binary matroids is a connected binary matroid.*

*Proof sketch.* Check by definition. Connectedness of representation matrices is equivalent to connectedness of corresponding matroids. □

**Lemma 29** (2-sum of regular matroids is regular)**.** *Let $M_1$ and $M_2$ be regular matroids. Then $M = M_1 \oplus_2 M_2$ is a regular matroid.*

*Proof sketch.*

Let $B$, $B_1$, and $B_2$ be the representation matrices of $M$, $M_1$, and $M_2$, respectively. Let $B_1'$ and $B_2'$ be TU signings of $B_1$ and $B_2$, respectively. In particular, let $A_1'$, $x'$, $A_2'$, and $y'$ be the signed versions of $A_1$, $x$, $A_2$, and $y$, respectively. Let $B'$ be the signing of $B$ where the blocks of $A_1$ and $A_2$ are signed as $A_1'$ and $A_2'$, respectively, and the block of $D$ is signed as $D' = y' \cdot x'$ (outer product).

Note that $[A_1'/D']$ is TU by Lemma **??**, as every row of $D'$ is either zero or a copy of $x'$. Similarly, $[D' \mid A_2']$ is TU by Corollary **??**, as every column of $D'$ is either zero or a copy of $y'$. Additionally, $[A_1' \mid 0]$ is TU by Lemma **??**, and $[0/A_2']$ is TU by Lemma **??**.

prove lemma below, separate into statement about TU matrices

*Lemma:* Let $T$ be a square submatrix of $B'$. Then $\det T \in \{0, \pm 1\}$.

*Proof:* Induction on the size of $T$. *Base:* If $T$ consists of only 1 element, then this element is 0 or $\pm 1$, so $\det T \in \{0, \pm 1\}$. *Step:* Let $T$ have size $t$ and suppose all square submatrices of $B'$ of size $\leq t - 1$ are TU.

- Suppose $T$ contains no rows of $X_1$. Then $T$ is a submatrix of $[D' \mid A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose $T$ contains no rows of $X_2$. Then $T$ is a submatrix of $[A_1' \mid 0]$, so $\det T \in \{0, \pm 1\}$.
- Suppose $T$ contains no columns of $Y_1$. Then $T$ is a submatrix of $[0/A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose $T$ contains no columns of $Y_2$. Then $T$ is a submatrix of $[A_1'/D']$, so $\det T \in \{0, \pm 1\}$.
- Remaining case: $T$ contains rows of $X_1$ and $X_2$ and columns of $Y_1$ and $Y_2$.
- If $T$ is $2 \times 2$, then $T$ is TU. Indeed, all proper submatrices of $T$ are of size $\leq 1$, which are $\{0, \pm 1\}$ entries of $A'$, and $T$ contains a zero entry (in the row of $X_2$ and column of $Y_2$), so it cannot be a minimal violation matrix by Lemma **??**. Thus, assume $T$ has size $\geq 3$.
- .

  complete proof, see last paragraph of Lemma 11.2.1 in Truemper

□

### 1.4.3 3-Sums

**Definition 30** (3-sum of matrices)**.**

add

**Definition 31** (matroid 3-sum)**.**

add

**Lemma 32** (3-separations and 3-sums)**.**

*add*

**Lemma 33** (3-sum of regular matroids is regular)**.**

*add*

# Chapter 2

# Truemper

## 2.1 Basic Definitions

### 2.1.1 Matroid Structure

**Definition 34** (matroid).

todo: add definition

**Definition 35** (isomorphism). Two matroids are isomorphic if they become equal upon a suitable relabeling of the elements.

**Definition 36** (loop).

todo: add definition

**Definition 37** (coloop).

todo: add definition

**Definition 38** (parallel elements).

todo: add definition

**Definition 39** (series elements).

todo: add definition

### 2.1.2 Matroid Classes

**Definition 40** (binary matroid).

todo: add definition

**Definition 41** (regular matroid). A binary matroid $M$ is regular if some binary representation matrix $B$ of $M$ has a totally unimodular signing (i.e., assignment of signs to the 1s in $B$ that results in a TU matrix).

**Definition 42** (graphic matroid).

todo: add definition

**Definition 43** (cographic matroid).

todo: add definition

**Definition 44** (planar matroid).

todo: add definition

**Definition 45** (dual matroid).

todo: add definition

**Definition 46** (self-dual matroid).

todo: add definition

### 2.1.3 Specific Matroids (Constructions)

**Definition 47** (wheel).

todo: add definition

**Definition 48** ($W_3$).

todo: add definition

**Definition 49** ($W_4$).

todo: add definition

**Definition 50** ($R_{10}$).

todo: add definition

**Definition 51** ($R_{12}$).

todo: add definition

**Definition 52** ($F_7$).

todo: add definition

**Definition 53** ($F_7^*$).

todo: add definition

**Definition 54** ($M(K_{3,3})$).

todo: add definition

**Definition 55** ($M(K_{3,3})^*$).

todo: add definition

**Definition 56** ($M(K_5)$).

todo: add definition

**Definition 57** ($M(K_5)^*$).

todo: add definition

### 2.1.4 Connectivity and Separation

**Definition 58** ($k$-separation)**.** See text after Proposition 3.3.18.

**Definition 59** ($k$-connectivity)**.** See text after Proposition 3.3.18.

### 2.1.5 Reductions

**Definition 60** (deletion)**.**

todo: add definition

**Definition 61** (contraction)**.**

todo: add definition

**Definition 62** (minor)**.**

todo: add definition

### 2.1.6 Extensions

**Definition 63** (1-element addition)**.**

add name, label, uses, text

**Definition 64** (1-element expansion)**.**

add name, label, uses, text

**Definition 65** (1-element extension)**.**

todo: add definition

**Definition 66** (2-element extension)**.**

todo: add definition

**Definition 67** (3-element extension)**.**

todo: add definition

### 2.1.7 Sums

**Definition 68** (1-sum)**.**

todo: add definition

**Definition 69** (2-sum)**.**

todo: add definition

**Definition 70** (3-sum)**.**

todo: add definition

**Definition 71** ($\Delta$-sum)**.**

todo: add definition

**Definition 72** ($Y$-sum)**.**

todo: add definition

### 2.1.8 Total Unimodularity

**Definition 73** (TU matrix)**.** A real matrix $A$ is totally unimodular if every square submatrix $D$ of $A$ has $\det_{\mathbb{R}} D = 0$ or $\pm 1$.

### 2.1.9 Auxiliary Results

**Theorem 74** (Menger's theorem)**.** *A connected graph $G$ is vertex $k$-connected if and only if every two nodes are connected by $k$ internally node-disjoint paths. Equivalent is the following statement. $G$ is vertex $k$-connected if and only if any $m \leq k$ nodes are joined to any $n \leq k$ nodes by $k$ internally node-disjoint paths. One may demand that the $m$ nodes are disjoint from the $n$ nodes, but need not do so. Also, the $k$ paths can be so chosen that each of the specified nodes is an endpoint of at least one of the paths.*

**Definition 75** ($\Delta Y$ exchange)**.**

add

**Definition 76** (gap)**.**

add

## 2.2 Chapter 2

**Lemma 77** (2.3.14)**.** *Let $A$ be a matrix over a field $\mathcal{F}$, with $\mathcal{F}$-rank $A = k$. If both a row submatrix and a column submatrix of $A$ have $\mathcal{F}$-rank equal to $k$, then they intersect in a submatrix of $A$ with the same $\mathcal{F}$-rank. In particular, any $k$ $\mathcal{F}$-independent rows of $A$ and any $k$ $\mathcal{F}$-independent columns of $A$ intersect in a $k \times k$ $\mathcal{F}$-nonsingular submatrix of $A$.*

*Proof sketch.* Result of linear algebra. Uses the submodularity of the $\mathcal{F}$-rank function. □

## 2.3 Chapter 3

### 2.3.1 Chapter 3.2

**Theorem 78** (3.2.25.a)**.** *Let $M$ be the graphic matroid of a connected graph $G$. Assume $(E_1, E_2)$ is a $k$-separation of $M$ with minimal $k \geq 1$. Define $G_1$ (resp. $G_2$) from $G$ by removing the edges of $E_2$ (resp. $E_1$) from $G$. Let $R_1, \ldots, R_g$ be the connected components of $G_1$, and $S_1, \ldots, S_h$ be those of $G_2$.*

*If $k = 1$, then the $R_i$ and $S_j$ are connected in tree fashion.*

*Proof sketch.* Count edges and nodes. □

**Theorem 79** (3.2.25.b)**.** *Same setting as Theorem 3.2.25.a. If $k = 2$, then the $R_i$ and $S_j$ are connected in cycle fashion.*

*Proof sketch.* Count edges and nodes. □

**Definition 80** (switching operation from section 3)**.** A swap of identification of nodes between two subgraphs induced by a 2-separation of a graph. See description and illustration on page 45.

**Lemma 81** (3.2.48)**.** *The matroids $M(K_5)$ and $M(K_{3,3})$ are not graphic.*

*Proof sketch.* A short proof is given on page 51. A longer, but more general proof uses the graphicness testing subroutine described on page 47. □

### 2.3.2 Chapter 3.3

**Lemma 82** (3.3.12)**.** *Let $M$ be a binary matroid with a minor $\overline{M}$, and let $\overline{B}$ be a representation matrix of $\overline{M}$. Then $M$ has a representation matrix $B$ that displays $\overline{M}$ via $\overline{B}$ and thus makes the minor $\overline{M}$ visible.*

*Proof sketch.* Follows by the definition of minor via pivots and row/column deletions. □

**Proposition 83** (3.3.17)**.** *Partitioned version of matrix $B$ representing binary matroid $M$. (same as 3.3.3)*

**Proposition 84** (3.3.18)**.** *If for some $k \geq 1$, $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k$, $\mathrm{GF}(2)$-rank $D^1 + \mathrm{GF}(2)$-rank $D^2 \leq k-1$, then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a (Tutte) $k$-separation of $B$ and $M$. This separation is exact if the rank condition holds with equality. Both $B$ and $M$ are called (Tutte) $k$-separable if they have a $k$-separation. For $k \geq 2$, $B$ and $M$ are (Tutte) $k$-connected if they have no $\ell$-separation for $1 \leq \ell < k$. When $M$ is 2-connected, we also say that $M$ is connected.*

**Lemma 85** (3.3.19)**.** *Let $M$ be a binary matroid with a representation matrix $B$. Then $M$ is connected iff $B$ is connected.*

*Proof sketch.* Check using (3.3.17) and (3.3.18) that $B$ is connecte diff it is 2-connected. Thus $M$ is 2-connected, and hence connected, iff $B$ is connected. □

**Lemma 86** (3.3.20)**.** *The following statements are equivalent for a binary matroid $M$ with set $E$ and a representation matrix $B$ of $M$.*

- *$M$ is 3-connected.*
- *$B$ is connected, has no parallel or unit vector rows and columns, and has no partition as in (3.3.17) with $\mathrm{GF}(2)$-rank $D^1 = 1$, $D^2 = 0$, and $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.*
- *Same as (ii), but $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 5$.*

*Proof sketch.*

- (i) is equivalent to (ii) by the definition of 3-connectivity.
- (iii) trivially implies (ii). (Typo in the book?)
- Assuming (ii), if the length of $B^1$ is 3 or 4, then $B$ has a zero column or row, or parallel or unit vector rows or columns, which is excluded by the first part of (ii). Thus it suffices to require $|X_1 \cup Y_1| \geq 5$ and by duality $|X_2 \cup Y_2| \geq 5$.

□

**Theorem 87** (census from Secion 3.3)**.** *A complete census of 3-connected binary matroids on $\leq 8$ elements.*

*Proof sketch.* Verified by case enumeration. □

## 2.4 Chapter 4

**Proposition 88** (4.4.5)**.** *$\Delta Y$ exchange, case 1.*

**Proposition 89** (4.4.6)**.** *$\Delta Y$ exchange, case 2.*

## 2.5 Chapter 5

**Lemma 90** (5.2.4)**.** *Let $N$ be a connected minor of a connected binary matroid $M$. Let $z \in M \backslash N$. Then $M$ has a connected minor $N'$ that is a 1-element extension of $N$ by $z$.*

*Proof sketch.*

- By Lemma 3.3.12, $M$ has a representation matrix that displays $N$ via a submatrix.
- Case distinction between $z$ being represented by a nonzero or a zero vector.
- Nonzero case: immediately get submatrix representing $N'$.
- Zero case: take a shortest path in the matrix, perform pivots, in one subcase use duality.

□

**Proposition 91** (5.2.8)**.** *Representation matrices for small wheels (from $M(W_1)$ to $M(W_4)$).*

**Proposition 92** (5.2.9)**.** *Representation matrix for $M(W_n)$, $n \geq 3$.*

**Lemma 93** (5.2.10)**.** *Let $M$ be a binary matroid with a binary representation matrix $B$. Suppose the graph $BG(B)$ contains at least one cycle. Then $M$ has an $M(W_2)$ minor.*

*Proof sketch.*

- $BG(B)$ is bipartite and has at least one cycle, so there is a cycle $C$ without chords with at least 4 edges.
- Up to indices, the submatrix corresponding to $C$ is either the matrix for $M(W_2)$ from (5.2.8) or the matrix for some $M(W_k)$, $k \geq 3$ from (5.2.9).
- In the latter case, use path shortening pivots on 1s to convert the submatrix to the former case.

□

**Lemma 94** (5.2.11)**.** *Let $M$ be a connected binary matroid with at least 4 elements. Then $M$ has a 2-separation or an $M(W_3)$ minor.*

*Proof sketch.* Use Lemma 5.2.10 and apply path shortening technique. □

**Corollary 95** (5.2.15)**.** *Every 3-connected binary matroid $M$ with at least 6 elements has an $M(W_3)$ minor.*

*Proof sketch.* By Lemma 5.2.11, $M$ has a 2-separation or an $M(W_3)$ minor. $M$ is 3-connected, so the former case is impossible. □

## 2.6 Chapter 6

### 2.6.1 Chapter 6.2

Goal of the chapter: separation algorithm for deciding if there exists a separation of a matroid induced by a separation of its minor.

**Proposition 96** (6.2.1)**.** *Partitioned version of matrix $B^N$ representing a minor $N$ of a binary matroid $M$, where $N$ has an exact $k$-separation for some $k \geq 1$.*

**Proposition 97** (6.2.3)**.** *Matrix $B$ for $M$ displaying partitioned $B^N$*

**Proposition 98** (6.2.5)**.** *Matrix $B$ for $M$ with partitioned $B^N$, row $x \in X_3$, and column $y \in Y_3$.*

**Lemma 99** (6.2.6)**.** *Let $N$ be a 3-connected binary matroid on at least 6 elements. Suppose a 1-, 2-, or 3-element binary extension of $N$, say $M$, has no loops, coloops, parallel elements, or series elements. Then $M$ is 3-connected.*

*Proof sketch.*

- Let $C$ be a binary representation matrix of $M$ that displays a binary representation matrix $B$ for $N$.
- By assumption, $B$ is 3-connected.
- $C$ is connected, as otherwise by case analysis $C$ contains a zero vector or unit vector, so $M$ has a loop, coloop, parallel elements, or series elements, a contradiction.
- If $C$ is not 3-connected, then by Lemma 3.3.20 there is a 2-separation of $C$ with at least 5 rows/columns on each side. Then $B$ has a 2-separation with at least 2 rows/columns on each side, a contradiction.
- Thus, $C$ is 3-connected, so $M$ is 3-connected.

□

### 2.6.2 Chapter 6.3

**Definition 100** (6.3.2)**.** $M$ is called minimal if it satisfies the following conditions.

- $M$ has an $N$ minor.
- $M$ has no $k$-separation induced by the exact $k$-separation $(F_1, F_2)$ of $N$.
- The matroid $M$ is minimal with respect to the above conditions.

**Definition 101** (6.3.3)**.** $M$ is called minimal under isomorphism if it satisfies the following conditions.

- $M$ has at least one $N$ minor.
- Some $k$-separation of at least one such minor corresponding to the exact $k$-separation $(F_1, F_2)$ of $N$ under one of the isomorphisms fails to induce a $k$-separation of $M$.
- The matroid $M$ is minimal with respect to the above conditions.

**Proposition 102** (6.3.11)**.** *Matrix $B$ for $M$ with partitioned $B^N$, row $x \in X_3$, and column $y \in Y_3$.*

**Proposition 103** (6.3.12)**.** *Partitioned version of $B^N$: $B^N = \mathrm{diag}(A^1, A^2)$.*

**Definition 104** (separation algorithm)**.** Polynomial-time recursive procedure to search for an induced partition. Described on pages 132–133 and again on pages 137–138.

**Proposition 105** (6.3.13)**.** *Special case where $B$ of a minimal $M$ contains just one row $x$ beyond $B^N$. This proposition gives properties of row subvectors of row $x$ by step 1 of the separation algorithm.*

**Proposition 106** (6.3.14)**.** *Special case where $B$ of a minimal $M$ contains just one column $y$ beyond $B^N$. This proposition gives properties of column subvectors of column $y$ by step 1 of the separation algorithm.*

**Lemma 107** (6.3.15)**.** *Treats the case where $B$ has at least two additional rows or columns beyond those of $B^N$.*

*Proof sketch.* Argue about the structure of the matrix, applying steps 1 and 2 of the separation algorithm. □

**Lemma 108** (6.3.16)**.** *Expands case (i) of Lemma 6.3.15.*

*Proof sketch.* Further arguments about the structure of the matrix. □

**Lemma 109** (6.3.17)**.** *Expands case (ii) of Lemma 6.3.15.*

*Proof sketch.* Further arguments about the structure of the matrix. □

**Theorem 110** (6.3.18)**.** *Structural description of representation matrix (6.3.11) of a minimal $M$. Contains cases (a), (b), and (c) with sub-cases (c.1) and (c.2).*

*Proof sketch.*

- (6.3.13) and (6.3.14) establish (a) and (b).
- Lemmas 6.3.15, 6.3.16, and 6.3.17 prove (c.1) and (c.2).

□

**Lemma 111** (6.3.19)**.** *Additional structural statements for cases (c.1) and (c.2) of Theorem 6.3.18.*

*Proof sketch.* Reason about representation matrices using Theorem 6.3.18, Lemma 6.3.15, minimality, isomorphisms, pivots, and so on. □

**Proposition 112** (6.3.21)**.** *Matrix $B$ for $M$ minimal under isomorphism, case (a).*

**Proposition 113** (6.3.22)**.** *Matrix $B$ for $M$ minimal under isomorphism, case (b).*

**Proposition 114** (6.3.23)**.** *Matrix $\overline{B}$ for minor $\overline{M}$ of $M$ minimal under isomorphism.*

**Theorem 115** (6.3.20)**.** *Let $M$ be minimal under isomorphism. Then one of 3 cases holds for matrix representation of $M$.*

*Proof sketch.* Follows directly from Theorem 6.3.18 and Lemma 6.3.19. □

**Corollary 116** (6.3.24)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Suppose $N$ given by $B^N$ of (6.3.12) is in $\mathcal{M}$, but the 1- and 2-element extensions of $N$ given by (6.3.21), (6.3.22), (6.3.23), and by the accompanying conditions are not in $\mathcal{M}$. Assume matroid $M \in \mathcal{M}$ has an $N$ minor. Then any $k$-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of $N$ under one of the isomorphisms induces a $k$-separation of $M$.*

*Proof sketch.*

- Let $M \in \mathcal{M}$ satisfying the assumptions. Since $\mathcal{M}$ is closed under isomorphism, suppose that $N$ itself is a minor of $M$.
- Suppose the $k$-separation of $N$ does not induce one in $M$. Then $M$ or a minor of $M$ containing $N$ is minimal under isomorphism.
- By Theorem 6.3.20, $M$ has a minor represented by (6.3.21), (6.3.22), or (6.3.23). This minor is in $\mathcal{M}$, as $\mathcal{M}$ is closed under taking minors, but this contradicts our assumptions.

□

### 2.6.3 Chapter 6.4

**Theorem 117** (6.4.1)**.** *Let $M$ be a 3-connected binary matroid with a 3-connected proper minor $N$. Suppose $N$ has at least 6 elements. Then $M$ has a 3-connected minor $N'$ that is a 1- or 2-element extension of some $N$ minor of $M$. In the 2-element case, $N'$ is derived from the $N$ minor by one addition and one expansion.*

*Proof sketch.*

- Let $z \in M \setminus N$. By Lemma 5.2.4, there is a connected minor $N'$ that is a 1-element extension of $N$ by $z$. Our theorem holds iff it holds for duals, so by duality, assume that the extension is an addition.
- Reason about a matrix representation of $N$ and $N'$ to get a 2-separation of $N'$. Since $M$ is 3-connected, this 2-separation does not induce one in $M$. Let $M'$ be a minor of $M$ that proves this fact and is minimal under isomorphism. Additionally, $M'$ has an $N'$ minor, so we change the element labels in $M'$ so that $N'$ is a minor of $M'$.
- Apply Theorem 6.3.20 and perform case analysis, reaching either a contradiction or a desired extension.

□

## 2.7 Chapter 7

### 2.7.1 Chapter 7.2

**Definition 118** (splitter)**.** Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let $N$ be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If every $M \in \mathcal{M}$ with a proper $N$ minor has a 2-separation, then $N$ is called a splitter of $\mathcal{M}$.

**Theorem 119** (7.2.1.a splitter for nonwheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let $N$ be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If $N$ is not a wheel, then $N$ is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of $N$.*

*Proof sketch.*

- If $N$ is a splitter of $\mathcal{M}$, then clearly $\mathcal{M}$ does not contain a 3-connected 1-element extension of $N$.

- Prove the converse by contradiction. To this end, suppose that $\mathcal{M}$ does not contain a 3-connected 1-element extension of $N$ and that $N$ is not a splitter of $\mathcal{M}$.
- Thus, $\mathcal{M}$ contains a 3-connected matroid $M$ with a proper $N$ minor and no 2-separation.
- Since $\mathcal{M}$ is closed under isomorphism, we may assume $N$ itself to be that $N$ minor.
- By Theorem 6.4.1 (applied to $M$ and $N$), $M$ has a 3-connected minor $N'$ that is a 3-connected 1- or 2-element extension of an $N$ minor.
- The 1-extension case has been ruled out.
- In the 2-element extension case, $N'$ is derived from the $N$ minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take $N$ itself to be that $N$ minor. Thus, $N'$ is derived from $N$ by one addition and one expansion.
- Let $C$ be a binary matrix representing $N'$ and displaying $N$. By investigating the structure of $C$, one can show that $N'$ contains a 3-connected 1-element extension of an $N$ minor, which has been ruled out.

$\square$

**Theorem 120** (7.2.1.b splitter for wheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let $N$ be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If $N$ is a wheel, then $N$ is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of $N$ and does not contain the next larger wheel.*

*Proof sketch.* Similar to proof of Theorem 7.2.1.a. The analysis of the matrix $C$ can be done in one go for both cases. $\square$

**Corollary 121** (7.2.10.a)**.** *Theorem 7.2.1.a specialized to graphs.*

*Proof sketch.* Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. $\square$

**Corollary 122** (7.2.10.b)**.** *Theorem 7.2.1.b specialized to graphs.*

*Proof sketch.* Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. $\square$

**Theorem 123** (7.2.11.a)**.** *$K_5$ is a splitter of the graphs without $K_{3,3}$ minors.*

*Proof sketch.* Up to isomorphism, there is just one 3-connected 1-edge extension of $K_5$. To obtain it, one partitions one vertex of $K_5$ into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph has a $K_{3,3}$ minor. Thus, the theorem follows from Corollary 7.2.10.a. $\square$

**Theorem 124** (7.2.11.b)**.** *$W_3$ is a splitter of the graphs without $W_4$ minors.*

*Proof sketch.* There is no 3-connected 1-edge extension of $W_3$, so the theorem follows from Corollary 7.2.10.b. $\square$

### 2.7.2 Chapter 7.3

**Theorem 125** (7.3.1.a)**.** *Let $M$ be a 3-connected binary matroid with a 3-connected proper minor $N$ on at least 6 elements. Assume $N$ is not a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \dots, M_t = M$ of nested 3-connected minors where $M_0$ is isomorphic to $N$ and where the gap is 1.*

*Proof sketch.*

- Inductively for $i \geq 0$ assume the existence of a sequence $M_0, \dots, M_i$ of 3-connected minors where $M_0$ is isomorphic to $N$, $M_i$ is not a wheel, and the gap is 1.
- If $M_i = M$, we are done, so assume that $M_i$ is a proper minor of $M$.
- Use the contrapositive of the splitter Theorem 7.2.1.a to find a larger sequence.
  - Let $\mathcal{M}$ be the collection of all matroids isomorphic to a (not necessarily proper) minor of $M$.
  - Since $M_i$ is a 3-connected proper minor of the 3-conected $M \in \mathcal{M}$, it cannot be a splitter of $\mathcal{M}$. By Theorem 7.2.1.a, $\mathcal{M}$ contains a matroid $M_{i+1}$ that is a 3-connected 1-element extension of a matroid isomorphic to $M_i$.
  - Since every 1-element reduction of a wheel with at least 6 elements is 2-separable, $M_{i+1}$ is not a wheel, as otherwise $M_i$ is 2-separable, which is a contradiction.
- If necessary, relabel $M_0, \dots, M_i$ so that they consistute a sequence of nested minors of $M_{i+1}$. This sequence satisfies the induction hypothesis.
- By induction, the claimed sequence exists for $M$.

□

**Theorem 126** (7.3.1.b)**.** *Let $M$ be a 3-connected binary matroid with a 3-connected proper minor $N$ on at least 6 elements. Assume $N$ is a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \dots, M_t = M$ of nested 3-connected minors where:*

- *$M_0$ is isomorphic to $N$,*
- *for some $0 \leq s \leq t$ the subsequence $M_0, \dots, M_s$ consists of wheels and has gap 2,*
- *the subsequence $M_s, \dots, M_t$ has gap 1.*

*Proof sketch.* Same as the proof of Theorem 7.3.1.a, but uses Theorem 7.2.1.b instead of 7.2.1.a to extend the sequence of minors. □

**Proposition 127** (7.2.1 from 7.3.1)**.** *Theorem 7.3.1 implies Theorem 7.2.1.*

*Proof sketch.*

- Let $\mathcal{M}$ and $N$ be as specified in Theorem 7.2.1. Suppose $N$ is not a wheel.
- Prove the nontrivial "if" part by contradiction: let $M$ be a 3-connected matroid of $\mathcal{M}$ with $N$ as a proper minor.
- By Theorem 7.3.1, there is a sequence $M_0, \dots, M_t = M$ of nested 3-connected minors where $M_0$ is isomorphic to $N$ and where the gap is 1.

- Since $\mathcal{M}$ is closed under isomorphism, we may assume that $M$ is chosen such that $M_0 = N$.
- Then $M_1 \in \mathcal{M}$ is a 3-connected 1-element extension of $N$, which contradicts the assumed absence of such extensions.
- If $N$ is a wheel, the proof is analogous.

□

**Corollary 128** (7.3.2.a)**.** *Let $G$ be a 3-connected graph with a 3-connected proper minor $H$ with at least 6 edges. Assume $H$ is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \dots, G_t = G$ where $G_0$ is isomorphic to $H$, and where each $G_{i+1}$ has exactly one edge beyond those of $G_i$.*

*Proof sketch.* Translate Theorem 7.3.1.a directly into graph language. □

**Corollary 129** (7.3.2.b)**.** *Let $G$ be a 3-connected graph with a 3-connected proper minor $H$ with at least 6 edges. Assume $H$ is a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \dots, G_t = G$ where:*

- *$G_0$ is isomorphic to $H$,*
- *for some $0 \leq s \leq t$ the subsequence $G_0, \dots, G_t$ consists of wheels where each $G_{i+1}$ has exactly one additional spoke beyond those of $G_i$,*
- *in the subsequence $G_s, \dots, G_t$ each $G_{i+1}$ has exactly one edge beyond those of $G_i$.*

*Proof sketch.* Translate Theorem 7.3.1.b directly into graph language. □

**Theorem 130** (7.3.3, wheel theorem)**.** *Let $G$ be a 3-connected graph on at least 6 edges. If $G$ is not a wheel, then $G$ has some edge $z$ such that at least one of the minors $G/z$ and $G \setminus z$ is 3-connected.*

*Proof sketch.*

- By Corollary 5.2.15, $G$ has a $W_3$ minor.
- Let $H$ be a largest wheel minor of $G$. Since $G$ is not a wheel, $H$ is a proper minor of $G$.
- Apply Corollary 7.3.2.b to $G$ and $H$ to get a sequence of nested 3-connected minors $G_0, \dots, G_t = G$ where $G_0$ is isomorphic to $H$.
- Since $H$ is the largest wheel minor and $G$ is not a wheel, Corollary 7.3.2.b shows that $s = 0$ and $t \geq 1$.
- Additionally, from corollary we know that $G = G_t$ has exactly one extra edge compared to $G_{t-1}$. In other words, $G_{t-1} = G/z$ or $G \setminus z$ for some edge $z$.

□

**Theorem 131** (7.3.3 for binary matroids)**.** *Theorem 7.3.3 can be rewritten for binary matroids instead of graphs.*

*Proof sketch.* Similar to the proof of Theorem 7.3.3, but use Theorem 7.3.1 instead of Corollary 7.3.2. □

**Proposition 132** (7.3.4.observation)**.** *Oservation in text on pages 160–161.*

**Theorem 133** (7.3.4)**.** *Let $M$ be a 3-connected binary matroid with a 3-connected proper minor $N$ on at least 6 elements. If $M$ does not contain a 3-connected 1-element expansion (resp. addition) of any $N$ minor, then $M$ has a sequence of nested 3-connected minors $M_0, \dots, M_t = M$ where $M_0$ is an $N$ minor of $M$ and where each $M_{i+1}$ is obtained from $M_i$ by expansions (resp. additions) involving some series (resp. parallel) elements, possibly none, followed by a 1-element addition (resp. expansion).*

*Proof sketch.*

- The case in parenthesis is dual to the normally stated one. Thus, only consider expansions below.
- Apply construction from observation before Theorem 7.3.4 to the sequence of minors from Theorem 7.3.1 to get the desired sequence.

□

**Corollary 134** (7.3.5)**.** *Specializes Theorem 7.3.4 to graphs.*

### 2.7.3 Chapter 7.4

**Theorem 135** (7.4.1 planarity characterization)**.** *A graph is planar if and only if it has no $K_{3,3}$ or $K_5$ minors.*

*Proof sketch.*

- "Only if": planarity is preserved by taking minors, and by Lemma 3.2.48 both $K_{3,3}$ and $K_5$ are not planar.
- Let $G$ be a connected nonplanar graph with all proper minors planar. Goal: show that $G$ is isomorphic to $K_{3,3}$ or $K_5$.
- Prove that $G$ cannot be 1- or 2-separable. Thus $G$ is 3-connected.
- By Corollary 5.2.15, $G$ has a $W_3$ minor, say $H$. Note: no $H$ minor of $G$ can be extended to a minor of $G$ by addition of an edge that connects two nonadjacent nodes.
- Then by Corollary 7.3.5.b, there exists a sequence $G_0, \dots, G_t = G$ of 3-connected minors where $G_0$ is an $H$ minor and $G_{i+1}$ is constructed from $G_i$ following very specific steps.
- By minimality, $G_{t-1}$ is planar and $G$ is not. Argue about a planar drawing of $G_{t-1}$ and how $G$ can be derived from it. Show that this must result in a subdivision of $K_{3,3}$ or $K_5$.

□

**Theorem 136** (Kuratowski)**.** *A graph is planar if and only if it has no subdivision of $K_{3,3}$ or $K_5$.*

*Proof.* Note: Theorem 7.4.1 is equivalent to Kuratowski's theorem: a $K_{3,3}$ minor induces a subdivision of $K_{3,3}$ and a $K_5$ minor also leads to a subdivision of $K_5$ or $K_{3,3}$ (the latter in the case when an expansion step splits a vertex of degree 4 into two vertices of degree 3 after the new edge is inserted). □

## 2.8 Chapter 8

### 2.8.1 Chapter 8.2

This chapter is about deducing and manipulating 1- and 2-sum decompositions and compositions.

**Proposition 137** (8.2.1)**.** *Matrix of* 1*-separation.*

**Lemma 138** (8.2.2)**.** *Let $M$ be a binary matroid. Assume $M$ to be a* 1*-sum of two matroids $M_1$ and $M_2$.*

- *If $M$ is graphic, then there eixst graphs $G$, $G_1$, $G_2$ for $M$, $M_1$, $M_2$, respectively, such that identification of a node of $G_1$ with one of $G_2$ creates $G$.*
- *If $M_1$ and $M_2$ are graphic (resp. planar), then $M$ is graphic (resp. planar).*

*Proof sketch.* Elementary application of Theorem 3.2.25.a. □

**Proposition 139** (8.2.3)**.** *Matrix of exact* 2*-separation.*

**Proposition 140** (8.2.4)**.** *Matrices $B^1$ and $B^2$ of* 2*-sum.*

**Lemma 141** (8.2.6)**.** *Any* 2*-separation of a connected binary matroid $M$ produces a* 2*-sum with connected components $M_1$ and $M_2$. Conversely, any* 2*-sum of two connected binary matroids $M_1$ and $M_2$ is a connected binary matroid $M$.*

*Proof sketch.*

- Definitions imply everything except connectedness.
- It is easy to check that connectedness of (8.2.3) implies connectedness of (8.2.4) and vice versa.
- By Lemma 3.3.19, connectedness of representation matrices is equivalent to connectedness of the corresponding matroids.

□

**Lemma 142** (8.2.7)**.** *Let $M$ be a connected binary matroid that is a* 2*-sum of $M_1$ and $M_2$, as given via $B$, $B_1$, and $B_2$ of (8.2.3) and (8.2.4).*

- *If $M$ is graphic, then there exist* 2*-connected graphs $G$, $G_1$, and $G_2$ for $M$, $M_1$, and $M_2$, respectively, with the following feature. The graph $G$ is produced when one identifies the edge $x$ of $G_1$ with the edge $y$ of $G_2$, and when subsequently the edge so created is deleted.*
- *If $M_1$ and $M_2$ are graphic (resp. planar), then $M$ is graphic (resp. planar).*

*Proof sketch.*

- Ingredients: look at a 2-separation and the corresponding subgraphs, use Theorem 3.2.25.b, use the switching operation of Section 3.2, use Lemma 8.2.6 and representations (8.2.3) and (8.2.4).
- Use the construction from the drawing, check that fundamental circuits match, conclude that $M$ is graphic. For planar graphs, the edge identification can be done in a planar way.

□

### 2.8.2 Chapter 8.3

**Proposition 143** (8.3.1)**.** *Matrix $B$ with exact $k$-separation.*

**Proposition 144** (8.3.2)**.** *Partition of $B$ displaying $k$-sum.*

**Proposition 145** (8.3.9)**.** *The (well-chosen) matrix $\overline{B}$ representing the connecting minor $\overline{M}$ of a 3-sum.*

**Proposition 146** (8.3.10)**.** *The matrix $B$ representing a 3-sum (after reasoning).*

**Proposition 147** (8.3.11)**.** *Representation matrices $B^1$ and $B^2$ of the components $M_1$ and $M_2$ of a 3-sum (after reasoning).*

**Lemma 148** (8.3.12)**.** *Let $M$ be a 3-connected binary matroid on a set $E$. Then any 3-separation $(E_1, E_2)$ of $M$ with $|E_1|, |E_2| \geq 4$ produces a 3-sum, and vice versa.*

*Proof.*

- The converse easily follows from (8.3.10), which directly produces a desired 3-separation.
- Take a 3-separation. Since $M$ is 3-connected, it must be exact. Consider the representation matrix (8.3.11). Reason about that matrix.
- Analyse shortest paths in a bipartite graph based on the matrix.
- Apply path shortening technique from Chapter 5 to reduce a shortest path by pivots to one with exactly two arcs.
- Reason about the corresponding entries and about the effects of the pivots on the matrix.
- Apply Lemma 2.3.14. Eventually get an instance of (8.3.10) with (8.3.9). Thus, $M$ is a 3-sum.

□

### 2.8.3 Chapter 8.5

**Proposition 149** (8.5.3)**.** *Matrix $B^{2\Delta}$ for $M_{2\Delta}$.*

## 2.9 Chapter 9

**Proposition 150** (9.2.14)**.** *Matrix $B^{12}$ of regular matroid $R_{12}$.*

## 2.10 Chapter 10

**Proposition 151** (10.2.4)**.** *Derivation of a graph with $T$ nodes for $F_7$.*

**Proposition 152** (10.2.6)**.** *Derivation of a graph with $T$ nodes for $M(K_{3,3})^*$.*

**Proposition 153** (10.2.8)**.** *Derivation of a graph with $T$ nodes for $R_{10}$.*

**Proposition 154** (10.2.9)**.** *Derivation of a graph with $T$ nodes for $R_{12}$.*

**Theorem 155** (10.2.11 only if)**.** *If a regular matroid is planar, then it has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors.*

*Proof sketch.*

- Planarity is preserved under taking minors.
- The listed matroids are not planar.

□

**Theorem 156** (10.2.11 if)**.** *If a regular matroid has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors, then it is planar.*

*Proof sketch.*

- Let $M$ be minimally nonplanar with respect to taking minors, i.e., regular nonplanar, but with all proper minors planar.
- Goal: show that $M$ is isomorphic to one of the listed matroids.
- By Theorem 7.4.1, $M$ is not graphic or cographic.
- By Lemmas 8.2.2, 8.2.6, and 8.2.7, if $M$ has a 1- or 2-separation, then $M$ is a 1- or 2-sum. But then the components of the sum are planar, so $M$ is also planar. Therefore, $M$ is 3-connected.
- By the census of Section 3.3, every 3-connected $\leq$ 8-element matroid is planar, so $|M| \geq 9$.
- By the binary matroid version of the wheel Theorem 7.3.3, there exists an element $z$ such that $M \setminus z$ or $M/z$ is 3-connected. Dualizing does not afect the assumptions, so we may assume that $M \setminus z$ is 3-connected.
- Let $G$ be a planar graph representing $M \setminus z$. Extend $G$ to a representation of $M$ as follows:
  - If $G$ is a wheel, invoke (10.2.6) or (10.2.4). The latter contracdicts regularity of $M$, the former shows what we need.
  - If $G$ is not a wheel, use Theorem 7.3.3 and Menger's theorem. Use a path argument and edge contraction to reduce to (10.2.6) and conclude the proof.

□

**Lemma 157** (10.3.1)**.** *$M(K_5)$ is a splitter of the regular matroids with no $M(K_{3,3})$ minors.*

*Proof.*

- By Theorem 7.2.1.a, we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor.
- Then case analysis. (The book sketches one way of checking.)

□

**Lemma 158** (10.3.6)**.** *Every 3-connected binary 1-element expansion of $M(K_{3,3})$ is nonregular.*

*Proof sketch.* By case analysis via graphs plus $T$ sets. □

**Theorem 159** (10.3.11)**.** *Let $M$ be a 3-connected regular matroid with an $M(K_{3,3})$ minor. Assume that $M$ is not graphic and not cographic, but that each proper minor of $M$ is graphic or cographic. Then $M$ is isomorphic to $R_{10}$ or $R_{12}$.*

*Proof.* This proof is extremely long and technical. It involves case distinctions and graph constructions. □

**Theorem 160** (10.4.1 only if)**.** *If 3-connected regular matroid is graphic or cographic, then it has no $R_{10}$ or $R_{12}$ minors.*

*Proof sketch.* Representations (10.2.8) and (10.2.9) for $R_{10}$ and $R_{12}$ show that these are nongraphic and isomorphic to their duals, hence also noncographic, so we are done. □

**Theorem 161** (10.4.1 if)**.** *If a 3-connected regular matroid has no $R_{10}$ or $R_{12}$ minors, then it is graphic or cographic.*

*Proof sketch.*

- Let $M$ be 3-connected, regular, nongraphic, and noncographic matroid.
- Thus $M$ is not planar, so by Theorem 10.2.11 it has a minor isomorphic to $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$.
- By Lemma 10.3.1, $M(K_5)$ is a splitter for the regular matroids with no $M(K_{3,3})$ minors.
- These results imply that $M$ has a minor isomorphic to $M(K_{3,3})$, or $M(K_{3,3})^*$, or $M$ is isomorphic to $M(K_5)$ or $M(K_5)^*$.
- The latter is a contradiction, so $M$ or $M^*$ has an $M(K_{3,3})$ minor.
- Theorem 10.3.11 implies that $M$ or $M^*$ has $R_{10}$ or $R_{12}$ as a minor.
- Since $R_{10}$ and $R_{12}$ are self-dual, $M$ has $R_{10}$ or $R_{12}$ as a minor.

□

Note: Truemper's proof of **??** and **??** relies on representing matroids via graphs plus $T$ sets. An alternative proof, which utilizes the notion of graph signings, can be found in J. Geelen, B. Gerards - Regular matroid decomposition via signed graphs. Although the proof appears shorter than Truemper's, it heavily relies certain relatively advanced graph-theoretic results.

Bonus: Whitney's characterization of planar graphs (Corollary 10.2.13).

## 2.11 Chapter 11

### 2.11.1 Chapter 11.2

The goal of this chapter is to prove the "simple" direction of the regular matroid decomposition theorem.

Ingredients from Section 9.2:

- A matrix is TU if all its subdeterminants are 0, ±1.
- A binary matroid is regular if it has a signing that is TU.
- By Lemma 9.2.6 and Corollary 9.2.7, this signing is unique up to scaling by ±1 factors.
- The signing can be accomplished by signing one arbitrarily selected row or column at a time.

Ingredients from minimal violation matrices:

- Definition: a minimal violation matrix of total unimodularity (minimal violation matrix, MVM) is a $\{0, \pm 1\}$ matrix that is not TU, but all its submatrices are TU.
- MVMs are square and have determinant not equal to $0, \pm 1$.
- In particular, a $2 \times 2$ violation matrix has four $\pm 1$'s.
- Cosider a MVM of order $\geq 3$. Perform a pivot in it, then delete the pivot row and column. Then the resulting matrix is also MVM ("by a simple cofactor argument").

**Lemma 162** (11.2.1)**.** *Any* 1- *or* 2-*sum of two regular matroids is also regular.*

*Proof sketch.*

- 1-sum case: $M_1 \oplus_1 M_2$ is represented by a matrix $B = \text{diag}(A_1, A_2)$ where $A_1$ and $A_2$ represent $M_1$ and $M_2$. Use the same signings for $A_1$ and $A_2$ in $B$ to prove that $B$ is TU and hence the 1-sum is regular.
- 2-sum case: Slightly more complicated signing process. Similarly, reuse signings from $M_1$ and $M_2$, define signing on remaining nonzero elements via a concrete formula, then prove that the resulting matrix is TU.

□

**Lemma 163** (11.2.7)**.** *$M_2$ of (8.3.10) and (8.3.11) is regular iff $M_{2\Delta}$ of (8.5.3) ($M_2$ converted by a $\Delta Y$ exchange) is regular.*

*Proof sketch.* Utilize signings, minimal violation matrices, intersections (inside matrices), column dependence, pivot, duality. □

**Corollary 164** (11.2.8)**.** *$\Delta Y$ exchanges maintain regularity.*

*Proof.* Follows by Lemma 11.2.7. □

**Lemma 165** (11.2.9)**.** *Any* 3-*sum of two regular matroids is also regular.*

*Proof sketch.* Yet more complicated, but similar. Uses the result that "$\Delta Y$ exchanges maintain regularity" (Corollary 11.2.8 of Lemma 11.2.7). The rest of the arguments are similar to the 2-sum case: prove that submatrices are TU, then prove that the whole matrix is TU. □

**Theorem 166** (11.2.10)**.** *Any* 1-, 2-, *or* 3-*sum of two regular matroids is regular.*

*Proof sketch.* Combine Lemmas 11.2.1 and 11.2.9. □

**Corollary 167** (11.2.12)**.** *Any $\Delta$-sum or $Y$-sum of two regular matroids is also regular.*

*Proof sketch.* Follows from definitions of $\Delta$-sums and $Y$-sum, together with Theorem 11.2.10 and Corollary 11.2.8. □

### 2.11.2 Chapter 11.3

**Proposition 168** (11.3.3)**.** *Graph plus $T$ set representing $R_{10}$*

**Proposition 169** (11.3.5)**.** *Graph plus $T$ set representing $F_7$.*

**Proposition 170** (11.3.11)**.** *The binary representation matrix $B^{12}$ for $R_{12}$.*

The goal of the chapter is to prove the "hard" direction of the regular matroid decomposition theorem.

**Theorem 171** (11.3.2)**.** *$R_{10}$ is a splitter of the class of regular matroids.*
*In short: up to isomorphism, the only 3-connected regular matroid with $R_{10}$ minor is $R_{10}$.*

*Proof sketch.*

- Splitter theorem case (a)
- $R_{10}$ is self-dual, so it suffices to consider 1-element additions.
- Represent $R_{10}$ by (11.3.3)
- Up to isomorphism, there are only 3 distinct 3-connected 1-element extensions.
- Case 1 (graphic): contract a certain edge, the resulting graph contains a subdivision of (11.3.5), which represents $F_7$. Thus, this extension is nonregular.
- Cases 2, 3 (nongraphic): reduce instances to (11.3.5), same conclusion.

$\square$

**Theorem 172** (11.3.10)**.** *In short: Restatement of* **??** *for $R_{12}$. Replacements: $\mathcal{M}$ is the class of regular matroids, $N$ is $R_{12}$, (6.3.12) is (11.3.6), (6.3.21-23) are (11.3.7-9).*

**Theorem 173** (11.3.12)**.** *Let $M$ be a regular matroid with $R_{12}$ minor. Then any 3-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of $R_{12}$ (see (11.3.11) – matrix $B^{12}$ for $R_{12}$ defining the 3-separation) under one of the isomorphisms induces a 3-separation of $M$.*
*In short: every regular matroid with $R_{12}$ minor is a 3-sum of two proper minors.*

*Proof sketch.*

- Preparation: calculate all 3-connected regular 1-element additions of $R_{12}$. This involves somewhat tedious case checking. (Representation of $R_{12}$ in (10.2.9) helps a lot.) By the symmetry of $B^{12}$ and thus by duality, this effectively gives all 3-connected 1-element extensions as well.
- Verify conditions of theorem 11.3.10 (which implies the result).
- (11.3.7) and (11.3.9) are ruled out immediately from preparatory calculations.
- The rest is case checking ((c.1) and (c.2)), simpified by preparatory calculations.

$\square$

**Theorem 174** (11.3.14 regular matroid decomposition, easy direction)**.** *Every binary matroid produced from graphic, cographic, and matroids isomorphic to $R_{10}$ by repeated 1-, 2-, and 3-sum compositions is regular.*

*Proof sketch.* Follows from theorem 11.2.10. □

**Theorem 175** (11.3.14 regular matroid decomposition, hard direction)**.** *Every regular matroid $M$ can be decomposed into graphic and cographic matroids and matroids isomorphic to $R_{10}$ by repeated* 1-*,* 2-*, and* 3- *sum decompositions. Specifically: If $M$ is a regular* 3*-connected matroid that is not graphic and not cographic, then $M$ is isomorphic to $R_{10}$ or has an $R_{12}$ minor. In the latter case, any* 3*-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of $R_{12}$ ((11.3.11)) under one of the isomorphisms induces a* 3*-separation of $M$.*

*Proof sketch.*

- Let $M$ be a regular matroid. Assume $M$ is not graphic and not cographic.
- If $M$ is 1-separable, then it is a 1-sum. If $M$ is 2-separable, then it is a 2-sum. Thus assume $M$ is 3-connected.
- By theorem 10.4.1, $M$ has an $R_{10}$ or an $R_{12}$ minor.
- $R_{10}$ case: by theorem 11.3.2, $M$ is isomorphic to $R_{10}$.
- $R_{12}$ case: by theorem 11.3.12, $M$ has an induced by 3-separation, so by lemma 8.3.12, $M$ is a 3-sum.

□

### 2.11.3 Extensions of Regular Matroid Decomposition

- Theorem 11.3.14 remains valid when 3-sums are replaced by $\Delta$- and $Y$-sums (Theorem 11.3.16).
- Theorem 11.3.14 (and 11.3.16) can also be proved for matroids with no $F_7$ minors or with no $F_7^*$ minors. (Uses Lemma 11.3.19: $F_7$ ($F_7^*$) is a splitter of the binary matroids with no $F_7^*$ ($F_7$) minors.)

### 2.11.4 Applications of Regular Matroid Decomposition

- Efficient algorithm for testing if a binary matroid is regular (Section 11.4).
- Efficient algorithm for deciding if a real matrix is TU (Section 11.4).
- Constructing TU matrices (Theorem 11.5.9). (Translate 3-sum version of theorem 11.3.16 into matrix language.)
- Constructing $0, 1$ TU matrices (Theorem 11.5.13).
- Characterization of the cycle polytope (theorem 11.5.17). (Problem: let $M$ be a connected binary matroid with ground set $E$ and element weighs $w_e$ for all $e \in E$. Find a disjoint union $C$ of circuits of $M$ such that $\sum_{e \in C} w_e$ is maximized.)
- Number of nonzeros in TU matrices (Theorem 11.5.18).
- Triples in circuits (Theorem 11.5.18).
- Odd cycles (Theorem 11.5.20).