

Proof of Regularity of 2- and 3-Sum of Matroids

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1 2-Sum of Regular Matroids Is Regular

Lemma 1. *Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.*

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc} , the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic}a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column a''_k with the pivot column a_{rc} without the pivot element a_{rc} .

By the cofactor expansion in A along row r , we have

$$\det A = \sum_{k=1}^n (-1)^{r+k} a_{rk} \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k . By swapping the order of columns in $B_{r,k}$ to match the form of B_k , we get

$$\det A = (-1)^{r+c} (a_{rc} \det A' - \sum_{k \in C} a_{rk} \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 1. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 1. Let B_1, B_2 be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = [A_1/x]$ and $B_2 = [y \mid A_2]$, where x is a row vector, y is a column vector, and A_1, A_2 are matrices of appropriate dimensions. Let D be the outer product of y and x . The 2-sum of B_1 and B_2 is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}.$$

Definition 2. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k -TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 1. Note that a matrix is TU if and only if it is k -TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 2. *Let B_1 and B_2 be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then B is 1-TU and 2-TU.*

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B . If V is a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_1 and A_2 . Let i be the row shared by V and A_1 and j be the column shared by V and A_2 . Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij} . Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. \square

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ -TU for every $\ell < k$. Then for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also k -TU.*

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_1 and B_2 such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not k -TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row of V whose index appears in A_1 . Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_1 . Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc} . For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, \tilde{B} denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $[\tilde{A}_1/\tilde{D}]$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., A_2 remains unchanged. This holds because of the 0 block in B .
- \tilde{D} consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $[\tilde{D} \mid \tilde{A}_2]$ is TU, since this matrix consists of A_2 and copies of y scaled by factors $\{0, \pm 1\}$.
- \tilde{D} can be represented as an outer product of a column vector \tilde{y} and a row vector \tilde{x} , and we can define $\tilde{B}_1 = [\tilde{A}_1/\tilde{x}]$ and $\tilde{B}_2 = [\tilde{y} \mid \tilde{A}_2]$ similar to B_1 and B_2 , respectively. Note that \tilde{B}_1 and \tilde{B}_2 have the same size as B_1 and B_2 , respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- \tilde{B} contains a square submatrix \tilde{V} of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k - 1) \times (k - 1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B , the submatrix V'' corresponds to a submatrix \tilde{V} of \tilde{B} with the same property.

To sum up, after pivoting we obtain a matrix \tilde{B} that represents a 2-sum of TU matrices \tilde{B}_1 and \tilde{B}_2 and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k - 1)$ -TUness of \tilde{B} , which proves the lemma. \square

Lemma 4. *Let B_1 and B_2 be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_1 and B_2 , their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ -TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 2.

Step: If the Proposition holds for some k , then it also holds for $k + 1$ by Lemma 3.

Conclusion: For any TU matrices B_1 and B_2 , their 2-sum $B_1 \oplus_{2,x,y} B_2$ is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. \square

2 3-Sum of Regular Matroids Is Regular

2.1 Definition of 3-Sum

Definition 3. Let $B_1^{(0)} \in \mathbb{Z}_2^{(X_1 \cup \{x_0, x_1\}) \times (Y_1 \cup \{y_2\})}$, $B_2^{(0)} \in \mathbb{Z}_2^{(X_2 \cup \{x_2\}) \times (Y_2 \cup \{y_0, y_1\})}$ be matrices of the form

$$B_1^{(0)} = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_1^{(0)} & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_1^{(0)} & D_0^{(0)} & 1 \\ \hline & & 1 \\ \hline \end{array}, \quad B_2^{(0)} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline & D_0^{(0)} & 1 & \\ \hline & 1 & & A_2^{(0)} \\ \hline D_2^{(0)} & & & \\ \hline \end{array},$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{12}^{(0)} = D_2^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_1^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_1^{(0)}$ and $B_2^{(0)}$ is

$$B^{(0)} = B_1^{(0)} \oplus_3 B_2^{(0)} = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_1^{(0)} & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_1^{(0)} & D_0^{(0)} & 1 \\ \hline & & 1 & A_2^{(0)} \\ \hline D_{12}^{(0)} & D_2^{(0)} & \\ \hline \end{array} \in \mathbb{Z}_2^{(X_1 \cup X_2) \times (Y_1 \cup Y_2)}.$$

Here $x_2 \in X_1$, $x_0, x_1 \in X_2$, $y_0, y_1 \in Y_1$, $y_2 \in Y_2$, $A_1^{(0)} \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2^{(0)} \in \mathbb{Z}_2^{X_2 \times Y_2}$, $D_1^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_1 \setminus \{y_0, y_1\})}$, $D_2^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{12}^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times (Y_1 \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_1^{(0)}$, $B_2^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{1,12}^{(0)} = \begin{array}{|c|} \hline D_1^{(0)} \\ \hline D_{12}^{(0)} \\ \hline \end{array}, \quad D_{0,2}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_2^{(0)} \\ \hline \end{array}, \quad D_{1,0}^{(0)} = \begin{array}{|c|c|} \hline D_1^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{12,2}^{(0)} = \begin{array}{|c|c|} \hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_1^{(0)} & D_0^{(0)} \\ \hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$.

can omit

Lemma 5. Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

Proof. Verify by complete enumeration.

□

need details?

2.2 Construction of Canonical Signing

Definition 4. We call B_1 and B_2 canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively, if they have the form

$$B_1 = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_1 & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_1 & D_0 & 1 \\ \hline & & 1 \\ \hline \end{array}, \quad B_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline & D_0 & 1 & \\ \hline & 1 & & A_2 \\ \hline D_2 & & & \\ \hline \end{array}$$

where every block in B_1 and B_2 is a signing of the corresponding block in $B_1^{(0)}$ and $B_2^{(0)}$, and D_0 is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings B_1 and B_2 , the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \begin{array}{c|c|c} & A_1 & 0 \\ \hline & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & 0 \\ \hline D_1 & D_0 & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \\ & & A_2 \\ \hline D_{12} & D_2 & \end{array}$$

where $D_{12} = D_2 \cdot (D_0)^{-1} \cdot D_1$. (Note that $(D_0)^{-1}$ is over \mathbb{Q} .)

The following lemma helps construct canonical signings from arbitrary initial TU signings.

Lemma 6. *Let Q' be a TU signing of the matrix*

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0^{(0)} & 1 & \\ \hline & 1 & \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0, x_1, x_2) \times (y_0, y_1, y_2)}$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Define $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q as follows:

$$\begin{aligned} u(x_0) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0), & u(x_1) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & u(x_2) &= 1, \\ v(y_0) &= Q'(x_2, y_0), & v(y_1) &= Q'(x_2, y_1), & v(y_2) &= Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), \\ & \forall i \in \{x_0, x_1, x_2\}, \forall j \in \{y_0, y_1, y_2\}, & Q(i, j) &= Q'(i, j) \cdot u(i) \cdot v(j). \end{aligned}$$

Then Q is a TU signing of T and $Q = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline D_0 & 1 & \\ \hline & 1 & \\ \hline \end{array}$ where D_0 is the respective canonical signing of $D_0^{(0)}$.

Proof. Since Q' is a TU signing of T and Q is obtained from Q' by multiplying rows and columns by ± 1 factors, Q is also a TU signing of T . By construction, we have

$$\begin{aligned} Q(x_2, y_0) &= Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1, \\ Q(x_2, y_1) &= Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1, \\ Q(x_2, y_2) &= 0, \\ Q(x_0, y_0) &= Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1, \\ Q(x_0, y_1) &= Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1), \\ Q(x_0, y_2) &= Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1, \\ Q(x_1, y_0) &= 0, \\ Q(x_1, y_1) &= Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)), \\ Q(x_1, y_2) &= Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1. \end{aligned}$$

Thus, it remains to check that $Q(x_0, y_1)$ and $Q(x_1, y_1)$ are correct.

First, consider the entry $Q(x_0, y_1)$. If $D_0^{(0)}(x_0, y_1) = 0$, then $Q(x_0, y_1) = 0$, as needed. Otherwise, if $D_0^{(0)}(x_0, y_1) = 1$, then $Q(x_0, y_1) \in \{\pm 1\}$, as Q is a signing of T . Our goal is to show that $Q(x_0, y_1) = 1$. For the sake of deriving a contradiction suppose that $Q(x_0, y_1) = -1$. Then the determinant of the submatrix of Q indexed by $\{x_0, x_2\} \times \{y_0, y_1\}$ is

$$\det \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q . Thus, $Q(x_0, y_1) = 1$, as needed.

Consider the entry $Q(x_1, y_1)$. Since Q is a signing of T , we have $Q(x_1, y_1) \in \{\pm 1\}$. Note that we know all the other entries of Q , so we can determine the sign of $Q(x_1, y_1)$ using TUness of Q . Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $Q(x_1, y_1) = 1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q . Thus, $Q(x_1, y_1) = -1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $Q(x_1, y_1) = -1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q . Thus, $Q(x_1, y_1) = 1$, as needed.

□

Lemma 7. Let B'_1 be a TU signing of $B_1^{(0)}$. Let $u \in \{0, \pm 1\}^{X_1 \cup \{x_2\}}$, $v \in \{0, \pm 1\}^{Y_1 \cup \{y_2\}}$, and B_1 be:

$$u(i) = \begin{cases} B'_1(x_2, y_0) \cdot B'_1(x_0, y_0), & i = x_0, \\ B'_1(x_2, y_0) \cdot B'_1(x_0, y_0) \cdot B'_1(x_0, y_2) \cdot B'_1(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X_1, \end{cases}$$

$$v(j) = \begin{cases} B'_1(x_2, y_0), & j = y_0, \\ B'_1(x_2, y_1), & j = y_1, \\ B'_1(x_2, y_0) \cdot B'_1(x_0, y_0) \cdot B'_1(x_0, y_2), & j = y_2, \\ 1, & j \in Y_1, \end{cases}$$

$$\forall i \in X_1 \cup \{x_2\}, \forall j \in Y_1 \cup \{y_2\}, \quad B_1(i, j) = B'_1(i, j) \cdot u(i) \cdot v(j).$$

Then B_1 is a canonical signing of $B_1^{(0)}$ and is TU. We will refer to B_1 as TU canonical signing for short.

The TU canonical signing B_2 of $B_2^{(0)}$ is defined similarly by replacing B'_1 , $B_1^{(0)}$, and B_1 by B'_2 , $B_2^{(0)}$, and B_2 , respectively.

Proof. This follows directly from Lemma 6. □

Lemma 8. Let B_2 be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = (D_{0,2})_{\cdot, y_0}$ and $c_1 = (D_{0,2})_{\cdot, y_1}$. Then the following matrices are TU:

$$B_2^{(a)} = [c_0 - c_1 \mid c_0 \mid A_2], \quad B_2^{(b)} = [c_0 - c_1 \mid c_1 \mid A_2].$$

Proof. Pivoting in B_2 on (x_2, y_0) and (x_2, y_1) yields:

$$B_2 = \begin{bmatrix} \textcircled{1} & 1 & 0 \\ c_0 & c_1 & A_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c_1 - c_0 & A_2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & \textcircled{1} & 0 \\ c_0 & c_1 & A_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ c_0 - c_1 & -c_1 & A_2 \end{bmatrix}$$

By removing row x_2 from the resulting matrices and then multiplying columns y_0 and y_1 by $\{\pm 1\}$ factors, we obtain $B_2^{(a)}$ and $B_2^{(b)}$. Since B_2 is TU and TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $B_2^{(a)}$ and $B_2^{(b)}$ are TU. □

Lemma 9. Let B_2 be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = D_{0,2}(\cdot, y_0)$, $c_1 = D_{0,2}(\cdot, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.

1. For every $i \in X_2$, we have $[c_0(i) \mid c_1(i)] \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{[1 \mid -1], [-1 \mid 1]\}$.
2. c_2 is a signing of $D_{0,2}^{(0)}(\cdot, y_0) + D_{0,2}^{(0)}(\cdot, y_1)$.
3. $[A_2 \mid c_0 \mid c_1 \mid c_2]$ is TU.

Proof. 1. Let $i \in X_2$. If $[c_0(i) \mid c_1(i)] = [1 \mid -1]$, then the 2×2 submatrix of B_2 indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of B_2 . Similarly, if $[c_0(i) \mid c_1(i)] = [-1 \mid 1]$, then the 2×2 submatrix of B_2 indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of B_2 .

2. By item 1, we know that $[c_0(i) \mid c_1(i)] \neq [1 \mid -1]$ and $[c_0(i) \mid c_1(i)] \neq [-1 \mid 1]$. In all other cases, a direct calculation shows that $c_2(i) \in \{0, \pm 1\}$ and $|c_2(i)| = D_{0,2}^{(0)}(\cdot, y_0) + D_{0,2}^{(0)}(\cdot, y_1)$.

3. Let V be a square submatrix of $[A_2 \mid c_0 \mid c_1 \mid c_2]$. We will show that $\det V \in \{0, \pm 1\}$.

Let z denote the index of the appended column c_2 . Suppose that column z is not in V . Then V is a submatrix of B_2 , which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V .

Suppose that columns y_0 and y_1 are both in V . Then V contains columns z , y_0 , and y_1 , which are linearly dependent by construction of c_2 . Thus, $\det V = 0$. Going forward we assume that at most one of the columns y_0 and y_1 is in V .

Suppose that column y_0 is in V . Then V is a submatrix of $B_2^{(b)}$ from Lemma 8, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $B_2^{(a)}$ from Lemma 8, and so $\det V \in \{0, \pm 1\}$.

Thus, every square submatrix V of \tilde{T} has $\det V \in \{0, \pm 1\}$, and hence \tilde{T} is TU. \square

Remark 2. Vectors c_0 , c_1 , and c_2 can be defined directly in terms of entries of B_2 , e.g., c_2 is composed of entries of B_2 indexed by $(X_2 \setminus \{x_2\}) \times \{y_0\}$.

Lemma 10. Let B_1 be a TU canonical signing of $B_1^{(0)}$. Let $d_0 = D_{1,0}(x_0, \cdot)$, $d_1 = D_{1,0}(x_1, \cdot)$, and $d_2 = d_0 - d_1$. Then the following properties hold.

1. For every $j \in Y_2$, we have $[d_0(j)/d_1(j)] \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \{[1/ \mid -1], [-1/ \mid 1]\}$.
2. d_2 is a signing of $D_{1,0}^{(0)}(x_0, \cdot) + D_{1,0}^{(0)}(x_1, \cdot)$.
3. $[A_1/d_0/d_1/d_2]$ is TU.

Proof. Apply Lemma 9 to B_1^T , or repeat the same argument up to interchanging rows and columns. \square

Lemma 11. Let B_1 and B_2 be TU canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively.

- Let $c_0 = D_{0,2}(\cdot, y_0)$, $c_1 = D_{0,2}(\cdot, y_1)$, and $c_2 = c_0 - c_1$.
- Let $d_0 = D_{1,0}(x_0, \cdot)$, $d_1 = D_{1,0}(x_1, \cdot)$, and $d_2 = d_0 - d_1$.
- If $D_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.
- Let D be the bottom-left block in the canonical signing B of $B^{(0)}$ corresponding to B_1 and B_2 .

Then the following properties hold.

1. $D = c_0 \cdot r_0 + c_1 \cdot r_1$.
2. Rows of D are in $[\pm r_0 / \pm r_1 / \pm r_2 / 0]$.
3. Columns of D are in $[\pm c_0 \mid \pm c_1 \mid \pm c_2 \mid 0]$.

4. $[A_2 \mid c_0 \mid c_1 \mid c_2]$ is TU.
5. $[A_2 \mid D]$ is TU.
6. $[A_1/r_0/r_1/r_2]$ is TU.
7. $[A_1/D]$ is TU.
8. $[c_0 \mid c_1]$ contains D_0 (the canonical signing of $D_0^{(0)}$) as a submatrix.

Proof. 1. Follows via a direct calculation.

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2. By item 1, for every $i \in X_2$ we have $D(i, \cdot) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$. By Lemma 8.1, we know that $[c_0(i) \mid c_1(i)] \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{[1 \mid -1], [-1 \mid 1]\}$. Therefore, $D(i, \cdot)$ is equal to either 0, $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.
3. Holds by the same argument as item 2 up to interchanging rows and columns.
4. Holds by Lemma 9.3.
5. By item 3, columns of $[A_2 \mid D]$ are in $[A_2 \mid \pm c_0 \mid \pm c_1 \mid \pm c_2 \mid 0]$. Since $[A_2 \mid c_0 \mid c_1 \mid c_2]$ is TU and since adding zero columns and copies of columns multiplied by ± 1 factors preserves TUness, $[A_2 \mid D]$ is also TU.
6. By Lemma 10.3 (or by the same argument as item 4 up to interchanging rows and columns), $[A_1/d_0/d_1/d_2]$ is TU. Since TUness is preserved under multiplication of rows by ± 1 factors (and changing row order), $[A_1/r_0/r_1/r_2]$ is also TU.
7. Holds by the same argument as item 5 up to interchanging rows and columns.
8. Holds by construction.

□

Definition 5. Let $A_1 \in \mathbb{Q}^{X_1 \cup Y_1}$, $A_2 \in \mathbb{Q}^{X_2 \cup Y_2}$, $c_0, c_1 \in \mathbb{Q}^{X_2}$, $r_0, r_1 \in \mathbb{Q}^{Y_1}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–8 from the statement of Lemma 11 are satisfied for $A_1, A_2, c_0, c_1, r_0, r_1$. Given $k \in \mathbb{Z}_{>1}$,

refactor

define $\text{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, k)$ to mean “ $C = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}$ is k -TU”.

Lemma 12. Assume the notation of Definition 5. Then $\text{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, 1)$ holds.

Proof. Every entry of C is in one of four blocks: 0, A_1 , D , A_2 . By the assumptions of Definition 5, all of these blocks are TU. Thus, C is 1-TU. □

Lemma 13. Assume the notation of Definition 5. Let $i \in X_1$, let $T = [A_1(i, \cdot)/D]$. Suppose we pivot on entry $T(i, j) \in \{\pm 1\}$ in T and obtain matrix $T' = [a'/D']$. Then columns of D' are in $[\pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1) \mid 0]$.

Proof. Since T is a submatrix of $[A_1/D]$, which is TU by assumptions of Definition 5, we have that T is TU. Since pivoting preserves TUness, T' is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column p in T could be and what another column q in T could be. This uniquely determines the resulting columns p' and q' in T' by the pivot formula. In every case, either $[p' \mid q']$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of T' , or the restriction of p' and q' to X_2 is in $[\pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1) \mid 0]$. □

need details?

Lemma 14. Assume the notation of Definition 5. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose $\text{Proposition}(A'_1, A_2, c_0, c_1, r'_0, r'_1, k-1)$ holds for all A'_1, r'_0 , and r'_1 satisfying the assumptions of Definition 5 (together with A_2, c_0 , and c_1). Then $\text{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, k)$ holds.

Proof. Let V be a $k \times k$ submatrix of C . For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that V is a submatrix of $[A_1/D]$, $[A_1 \mid 0]$, $[D \mid A_2]$, or $[0/A_2]$. Since all of those four matrices are TU by the assumptions of Definition 5, we have $\det V \in \{0, \pm 1\}$. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row index shared by V and A_1 . Note that this row in V cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a ± 1 entry shared by V and A_1 . Let i and j denote the row and the column index of this entry, respectively.

Perform a pivot in C on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, B' denotes the entire matrix after the pivot. Note that the following statements hold.

- C' contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix V' of C' with the same row and column index sets as V minus the pivot row i and pivot column j .
- $C' = \begin{bmatrix} A'_1 & 0 \\ D' & A_2 \end{bmatrix}$, i.e., the 0 and the A_2 blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.
- $[A'_1/D']$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.
- The columns of D' are in $[0 \mid \pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1)]$. This holds by Lemma 13.
- There exist r'_0 and r'_1 such that $D' = c_0 \cdot r'_0 + c_1 \cdot r'_1$ and the assumptions of Definition 5 are satisfied for $A'_1, A_2, c_0, c_1, r'_0, r'_1$. This follows from the previous bullet point by carefully checking assumptions. need details?
- C' is $(k-1)$ -TU. This follows from the hypothesis: $\text{Proposition}(A'_1, A_2, c_0, c_1, r'_0, r'_1, k-1)$ holds.

To sum up, after pivoting we obtain a matrix C' (which can be obtained in the manner of Definition 5) that is $(k-1)$ -TU and contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. \square

Lemma 15. *Let B_1 and B_2 be TU canonical signings. Then the corresponding canonical signing B is TU.*

Proof. Define $A_1, A_2, c_0, c_1, r_0, r_1$ as in Lemma 11. Note that canonical signing B has the form of C in the notation of Definition 5.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: $\text{Proposition}(A'_1, A_2, c_0, c_1, r'_0, r'_1, k)$ holds for all A'_1, r'_0 , and r'_1 satisfying the assumptions of Definition 5.

Base: The Proposition holds for $k = 1$ by Lemma 12.

Step: If the Proposition holds for some k , then it also holds for $k+1$ by Lemma 14.

Conclusion: $\text{Proposition}(A'_1, A_2, c_0, c_1, r'_0, r'_1, k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$.

Specializing the conclusion to $A_1, A_2, c_0, c_1, r_0, r_1$ (obtained from B_1 and B_2 as described in the statement of Lemma 11) shows that canonical signing B is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, B is TU. \square

Corollary 2. Suppose that $B_1^{(0)}$ and $B_2^{(0)}$ have TU signings. Then $B_1 \oplus_3 B_2$ has a TU signing.

Proof sketch. Start with some TU signings, obtain canonical signings, apply Lemma 15. \square