https://Ivan-Sergeyev.github.io/Matroid-Decomposition-Theorem-Verification
https://github.com/Ivan-Sergeyev/Matroid-Decomposition-Theorem-Verification
https://Ivan-Sergeyev.github.io/Matroid-Decomposition-Theorem-Verification/docs

Matroid Decomposition Theorem Verification

Ivan Sergeev Martin Dvorak

October 7, 2024

0.1 Basic Definitions

0.1.1 Matroid Structure

Definition 1 (matroid)**.** todo: add definition

Definition 2 (isomorphism)**.** todo: add definition

0.1.2 Matroid Classes

Definition 3 (binary matroid)**.** todo: add definition

Definition 4 (regular matroid)**.** todo: add definition

Definition 5 (graphic matroid)**.** todo: add definition

Definition 6 (cographic matroid)**.** todo: add definition

Definition 7 (planar matroid)**.** todo: add definition

Definition 8 (dual matroid)**.** todo: add definition

Definition 9 (self-dual matroid)**.** todo: add definition

0.1.3 Specific Matroids (Constructions)

Wheels

Definition 10 (wheel)**.** todo: add definition

Definition 11 (W_3)**.** todo: add definition

Definition 12 (W_4)**.** todo: add definition

R_{10}

Definition 13 (R_{10})**.** todo: add definition

R_{12}

Definition 14 (R_{12})**.** todo: add definition

Fano matroid

Definition 15 (F_7)**.** todo: add definition

$K_{3,3}$

Definition 16 ($M(K_{3,3})$)**.** todo: add definition

Definition 17 ($M(K_{3,3})^*$)**.** todo: add definition

K_5

Definition 18 ($M(K_5)$)**.** todo: add definition

Definition 19 ($M(K_5)^*$)**.** todo: add definition

0.1.4 Connectivity and Separation

Definition 20 (k-connectivity)**.** todo: add definition

Definition 21 (k-separation)**.** todo: add definition

0.1.5 Reductions

Definition 22 (deletion)**.** todo: add definition

Definition 23 (contraction)**.** todo: add definition

Definition 24 (minor)**.** todo: add definition

0.1.6 Extensions

Definition 25 (1-element extension)**.** todo: add definition

Definition 26 (2-element extension)**.** todo: add definition

0.1.7 Sums

Definition 27 (1-sum)**.** todo: add definition

Definition 28 (2-sum)**.** todo: add definition

Definition 29 (3-sum)**.** todo: add definition

Definition 30 (Δ-sum)**.** todo: add definition

Definition 31 (Y-sum)**.** todo: add definition

0.1.8 Total Unimodularity

Definition 32 (TU matrix)**.** todo: add definition

0.1.9 Auxiliary Results

Theorem 33 (Menger's theorem)**.** *A connected graph G is vertex k-connected if and only if every two nodes are connected by k internally node-disjoint paths. Equivalent is the following statement. G is vertex k-connected if and only if any $m \leq k$ nodes are joined to any $n \leq k$ nodes by k internally node-disjoint paths. One may demand that the m nodes are disjoint from the n nodes, but need not do so. Also, the k paths can be so chosen that each of the specified nodes is an endpoint of at least one of the paths.*

Definition 34 (ΔY exchange)**.** add

Theorem 35 (census from Secion 3.3)**.** *add*

0.2 Chapter 6 from Truemper

Proposition 36 (6.3.12)**.** *add*

Proposition 37 (6.3.21)**.** *add*

Proposition 38 (6.3.22)**.** *add*

Proposition 39 (6.3.23)**.** *add*

Corollary 40 (6.3.24)**.** *def:binary*$_m atroid, def : isomorphism, def : minor, def :$
$1_e lem_e xt, def : 2_e lem_e xt, prop : 6.3.12, prop : 6.3.21, prop : 6.3.22, prop : 6.3.23, def :$
$k_s ep Let M be a class of binary matroids closed under isomorphism and under taking minors. Suppose N given by B^N$ *of (6.3.12) is in* $\mathcal{M}$*, but the* 1- *and* 2-*element extensions of* N *given by (6.3.21), (6.3.22), (6.3.23), and by the accompanying conditions are not in* $\mathcal{M}$*. Assume matroid* $M \in \mathcal{M}$ *has an* N *minor. Then any* k*-separation of any such minor that corresponds to* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* N *under one of the isomorphisms induces a* k*-separation of* M*.*

Theorem 41 (6.4.1)**.** *add*

0.3 Chapter 7 from Truemper

Definition 42 (splitter)**.** Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If every $M \in \mathcal{M}$ with a proper N minor has a 2-separation, then N is called a splitter of $\mathcal{M}$.

Theorem 43 (7.2.1.a splitter for nonwheels)**.** *def:splitter,def:wheel Let* $\mathcal{M}$ *be a class of binary matroids closed under isomorphism and under taking minors. Let* N *be a* 3*-connected minor of* $\mathcal{M}$ *on at least* 6 *elements. If* N *is not a wheel, then* N *is a splitter of* $\mathcal{M}$ *iff* $\mathcal{M}$ *does not contain a* 3*-connected* 1*-element extension of* N*.*

Proof sketch. thm:6.4.1,def:splitter,def:k$_c onn, def : 1_e lem_e xt, def : minor, def :$
$k_s ep$

If N is a splitter of $\mathcal{M}$, then clearly $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.

Prove the converse by contradiction. To this end, suppose that $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and that N is not a splitter of $\mathcal{M}$.

Thus, $\mathcal{M}$ contains a 3-connected matroid M with a proper N minor and no 2-separation.

Since $\mathcal{M}$ is closed under isomorphism, we may assume N itself to be that N minor.

By Theorem 6.4.1 (applied to M and N), M has a 3-connected minor N' that is a 3-connected 1- or 2-element extension of an N minor.

The 1-extension case has been ruled out.

In the 2-element extension case, N' is derived from the N minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take N itself to be that N minor. Thus, N' is derived from N by one addition and one expansion.

Let C be a binary matrix representing N' and displaying N. By investigating the structure of C, one can show that N' contains a 3-connected 1-element extension of an N minor, which has been ruled out.

□

Theorem 44 (7.2.1.b splitter for wheels)**.** *def:splitter,def:wheel Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If N is a wheel, then N is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and does not contain the next larger wheel.*

Proof sketch. thm:6.4.1,def:splitter,def:k$_c onn, def: 1_e lem_e xt, def: minor, def:$
$k_s ep Similar to proof of Theorem 7.2.1.a. The analysis of the matrix$ C $can be done in one go for both cases.$ □

Corollary 45 (7.2.10.a)**.** *Theorem 7.2.1.a specialized to graphs.*

Proof sketch. thm:7.2.1.a Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Corollary 46 (7.2.10.b)**.** *Theorem 7.2.1.b specialized to graphs.*

Proof sketch. thm:7.2.1.b Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Theorem 47 (7.2.11.a)**.** *def:$M_{K5}, def: M_{K33}, def: splitter, def: minor, def:$ $graphic_m atroid K_5$ is a splitter of the graphs without $K_{3,3}$ minors.*

Proof sketch. cor:7.2.10.a,def:k$_c onn, def: 1_e lem_e xt Up to isomorphism, there is just one 3-connected 1-edge exte$
To obtain it, one partitions one vertex of K_5 into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph has a $K_{3,3}$ minor. Thus, the theorem follows fromCorollary 7.2.10.a. □

Theorem 48 (7.2.11.b)**.** *def:$M_{W3}, def: M_{W4}, def: splitter, def: minor, def:$ $graphic_m atroid W_3$ is a splitter of the graphs without W_4 minors.*

Proof sketch. cor:7.2.10.b,def:k$_c onn, def: 1_e lem_e xt There is no 3-connected 1-edge extension of$ W_3, so the theorem follows from Corollary 7.2.10.b. □

Theorem 49 (7.3.3)**.** *add*

Theorem 50 (7.3.4)**.** *add*

Theorem 51 (7.4.1)**.** *add*

0.4 Chapter 8 from Truemper

Proposition 52 (8.2.1)**.** *add*

Proposition 53 (8.2.3)**.** *add*

Proposition 54 (8.2.4)**.** *add*

Proposition 55 (8.3.10)**.** *add*

Proposition 56 (8.3.11)**.** *add*

Proposition 57 (8.5.3)**.** *add*

Lemma 58 (8.2.2)**.** *add*

Lemma 59 (8.2.6)**.** *add*

Lemma 60 (8.2.7)**.** *add*

Lemma 61 (8.3.12)**.** *add*

0.5 Chapter 10 from Truemper

Proposition 62 (10.2.4)**.** *add*

Proposition 63 (10.2.6)**.** *add*

Proposition 64 (10.2.8)**.** *add*

Proposition 65 (10.2.9)**.** *add*

Theorem 66 (10.2.11 only if)**.** *def:regular*$_m atroid, def : planar_m atroid, def : M_{K5}, def : M_{K5d} ual, def : M_{K33}, def : M_{K33d} ual, def : minorIfaregularmatroidisplanar, thenithasnoM(K$ *$M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors.*

Proof sketch.

- Planarity is preserved under taking minors.
- The listed matroids are not planar.

□

Theorem 67 (10.2.11 if)**.** *def:regular*$_m atroid, def : planar_m atroid, def : M_{K5}, def : M_{K5d} ual, def : M_{K33}, def : M_{K33d} ual, def : minorIfaregularmatroidhasnoM(K_5),$ *$M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors, then it is planar.*

Proof sketch. thm:7.4.1,lem:8.2.2,lem:8.2.6,lem:8.2.7,census sec 3.3,thm:7.3.3,prop:10.2.4,prop:10.2.6,thm:Meng

- Let M be minimally nonplanar with respect to taking minors, i.e., regular nonplanar, but with all proper minors planar.
- Goal: show that M is isomorphic to one of the listed matroids.
- By Theorem 7.4.1, M is not graphic or cographic.

- By Lemmas 8.2.2, 8.2.6, and 8.2.7, if M has a 1- or 2-separation, then M is a 1- or 2-sum. But then the components of the sum are planar, so M is also planar. Therefore, M is 3-connected.
- By the census of Section 3.3, every 3-connected $\leq$ 8-element matroid is planar, so $|M| \geq 9$.
- By the binary matroid version of the wheel Theorem 7.3.3, there exists an element z such that M
  z or M/z is 3-connected. Dualizing does not afect the assumptions, so we may assume that M
  z is 3-connected.
- Let G be a planar graph representing M
  z. Extend G to a representation of M as follows:
  - If G is a wheel, invoke (10.2.6) or (10.2.4). The latter contracdicts regularity of M, the former shows what we need.
  - If G is not a wheel, use Theorem 7.3.3 and Menger's theorem. Use a path argument and edge contraction to reduce to (10.2.6) and conclude the proof.

□

Lemma 68 (10.3.1)**.** *def:$M_{K5}, def: splitter, def: regular_matroid, def: M_{K33}, def: minor$ $M(K_5)$ is a splitter of the regular matroids with no $M(K_{3,3})$ minors.*

Proof. thm:7.2.1.a,def:k$_c onn, def: 1_e lem_e xt$

By Theorem 7.2.1.a, we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor.

Then case analysis. (The book sketches one way of checking.)

□

Lemma 69 (10.3.6)**.** *def:k$_c onn, def: 1_e lem_e xt, def: M_{K33}, def: regular_m atroid, def: binary_m atroid Every 3-connected binary 1-element expansion of$ $M(K_{3,3})$ is non-regular.*

Proof sketch. By case analysis via graphs plus T sets. □

Theorem 70 (10.3.11)**.** *def:k$_c onn, def: regular_m atroid, def: M_{K33}, def: minor, def: graphic_m atroid, def: cographic_m atroid, def: isomorphism, def: R10, def: R12 Let M be a 3-connected regular matroid with an$ $M(K_{3,3})$ minor. Assume that M is not graphic and not cographic, but that each proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof. lem:10.3.6,thm:7.3.4,thm:Menger This proof is extremely long and technical. It involves case distinctions and graph constructions. □

Theorem 71 (10.4.1 only if)**.** *def:k$_c$onn*$, def : regular_matroid, def : graphic_matroid, def :$ $cographic_matroid, def : R10, def : R12, def : minorIf 3{-}connectedregularmatroidisgraphicorcographic, ther$ *or R_{12} minors.*

Proof sketch. prop:10.2.8,prop:10.2.9 Representations (10.2.8) and (10.2.9) for R_{10} and R_{12} show that these are nongraphic and isomorphic to their duals, hence also noncographic, so we are done. □

Theorem 72 (10.4.1 if)**.** *def:k$_c$onn*$, def : regular_matroid, def : graphic_matroid, def :$ $cographic_matroid, def : R10, def : R12, def : minorIf a3{-}connectedregularmatroidhasnoR_{10}$ *or R_{12} minors, then it is graphic or cographic.*

Proof sketch. thm:10.2.11.if,lem:10.3.1,thm:10.3.11

- Let M be 3-connected, regular, nongraphic, and noncographic matroid.
- Thus M is not planar, so by Theorem 10.2.11 it has a minor isomorphic to $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$.
- By Lemma 10.3.1, $M(K_5)$ is a splitter for the regular matroids with no $M(K_{3,3})$ minors.
- These results imply that M has a minor isomorphic to $M(K_{3,3})$, or $M(K_{3,3})^*$, or M is isomorphic to $M(K_5)$ or $M(K_5)^*$.
- The latter is a contradiction, so M or M^* has an $M(K_{3,3})$ minor.
- Theorem 10.3.11 implies that M or M^* has R_{10} or R_{12} as a minor.
- Since R_{10} and R_{12} are self-dual, M has R_{10} or R_{12} as a minor.

□

Note: Truemper's proof of **??** and **??** relies on representing matroids via graphs plus T sets. An alternative proof, which utilizes the notion of graph signings, can be found in J. Geelen, B. Gerards - Regular matroid decomposition via signed graphs. Although the proof appears shorter than Truemper's, it heavily relies certain relatively advanced graph-theoretic results.

Bonus: Whitney's characterization of planar graphs (Corollary 10.2.13).

0.6 Chapter 11 from Truemper

0.6.1 Chapter 11.2

The goal of this chapter is to prove the "simple" direction of the regular matroid decomposition theorem.

todo: move ingredients to respective sections, add them as "uses" clauses

Ingredients from Section 9.2:

- A matrix is TU if all its subdeterminants are 0, ± 1.

- A binary matroid is regular if it has a signing that is TU.
- By Lemma 9.2.6 and Corollary 9.2.7, this signing is unique up to scaling by ± 1 factors.
- The signing can be accomplished by signing one arbitrarily selected row or column at a time.

Ingredients from minimal violation matrices:

- Definition: a minimal violation matrix of total unimodularity (minimal violation matrix, MVM) is a $\{0, \pm 1\}$ matrix that is not TU, but all its submatrices are TU.
- MVMs are square and have determinant not equal to $0, \pm 1$.
- In particular, a 2×2 violation matrix has four ± 1's.
- Cosider a MVM of order ≥ 3. Perform a pivot in it, then delete the pivot row and column. Then the resulting matrix is also MVM ("by a simple cofactor argument").

Lemma 73 (11.2.1)**.** *def:regular$_m$atroid, def : $1_s um, def$: $2_s umAny1{-}or2{-}sumoftworegularmatroidsisalso$*

Proof sketch. prop:8.2.1,prop:8.2.3,prop:8.2.4

- 1-sum case: $M_1 \oplus_1 M_2$ is represented by a matrix $B = \text{diag}(A_1, A_2)$ where A_1 and A_2 represent M_1 and M_2. Use the same signings for A_1 and A_2 in B to prove that B is TU and hence the 1-sum is regular.
- 2-sum case: Slightly more complicated signing process. Similarly, reuse signings from M_1 and M_2, define signing on remaining nonzero elements via a concrete formula, then prove that the resulting matrix is TU.

□

Lemma 74 (11.2.7)**.** *prop:8.3.10,prop:8.3.11,prop:8.5.3 todo: add lemma*

Corollary 75 (11.2.8)**.** *def:Delta$_{Y e}$xchange, def : $regular_m atroid\Delta Y$ exchanges maintain regularity.*

Proof. lem:11.2.7 Follows by Lemma 11.2.7. □

Lemma 76 (11.2.9)**.** *def:regular$_m$atroid, def : $3_s umAny3{-}sumoftworegularmatroidsisalsoregular$.*

Proof sketch. lem:11.2.7,cor:11.2.8 Yet more complicated, but similar. Uses the result that "ΔY exchanges maintain regularity" (Corollary 11.2.8 of Lemma 11.2.7). The rest of the arguments are similar to the 2-sum case: prove that submatrices are TU, then prove that the whole matrix is TU. □

Theorem 77 (11.2.10)**.** *def:regular$_m$atroid, def : $1_s um, def$: $2_s um, def$: $3_s umAny1{-}, 2{-}, or3{-}sumoftworegularmatroidsisregular$.*

Proof sketch. lem:11.2.1,lem:11.2.9 Combine Lemmas 11.2.1 and 11.2.9. □

Corollary 78 (11.2.12)**.** *def:regular$_m$atroid*, $def : Delta_s um, def : Y_s umAny\Delta$-*sum or* Y*-sum of two regular matroids is also regular.*

Proof sketch. def:Delta$_s um, def : Y_s um, thm : 11.2.10, cor : 11.2.8 Followsfromdefinitionsof \Delta$-sums and Y-sum, together with Theorem 11.2.10 and Corollary 11.2.8. □

0.6.2 Chapter 11.3

Proposition 79 (11.3.3)**.** *add prop*

Proposition 80 (11.3.5)**.** *add prop*

Proposition 81 (11.3.11)**.** *add prop*

The goal of the chapter is to prove the "hard" direction of the regular matroid decomposition theorem.

Theorem 82 (11.3.2)**.** *def:regular$_m$atroid*, $def : R10, def : splitter R_{10}$ *is a splitter of the class of regular matroids.*

In short: up to isomorphism, the only 3*-connected regular matroid with* R_{10} *minor is* R_{10}*.*

Proof sketch. thm:7.2.1.a,prop:11.3.3,def:isomorphism,def:1$_e lem_e xt, prop : 11.3.5, def : F7, def : contraction$

Splitter theorem case (a)

R_{10} is self-dual, so it suffices to consider 1-element additions.

Represent R_{10} by (11.3.3)

Up to isomorphism, there are only 3 distinct 3-connected 1-element extensions.

Case 1 (graphic): contract a certain edge, the resulting graph contains a subdivision of (11.3.5), which represents F_7. Thus, this extension is nonregular.

Cases 2, 3 (nongraphic): reduce instances to (11.3.5), same conclusion.

□

Theorem 83 (11.3.10)**.** *cor:6.3.24,def:R12 In short: Restatement of* **??** *for* R_{12}*. Replacements:* $\mathcal{M}$ *is the class of regular matroids,* N *is* R_{12}*, (6.3.12) is (11.3.6), (6.3.21-23) are (11.3.7-9).*

Theorem 84 (11.3.12)**.** *def:regular$_m$atroid*, $def : R12, def : minor, def : k_s ep, prop : 11.3.11, def : isomorphism Let M bearegularmatroidwith R_{12}$ *minor. Then any* 3*-separation of that minor corresponding to the* 3*-separation* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* R_{12} *(see (11.3.11) – matrix* B^{12} *for* R_{12} *defining the* 3*-separation) under one of the isomorphisms induces a* 3*-separation of* M*.*

In short: every regular matroid with R_{12} *minor is a* 3*-sum of two proper minors.*

Proof sketch. def:$1_e lem_e xt, prop : 10.2.9, thm : 11.3.10$

Preparation: calculate all 3-connected regular 1-element additions of R_{12}. This involves somewhat tedious case checking. (Representation of R_{12} in (10.2.9) helps a lot.) By the symmetry of B^{12} and thus by duality, this effectively gives all 3-connected 1-element extensions as well.

Verify conditions of theorem 11.3.10 (which implies the result).

(11.3.7) and (11.3.9) are ruled out immediately from preparatory calculations.

The rest is case checking ((c.1) and (c.2)), simpified by preparatory calculations.

□

Theorem 85 (11.3.14 regular matroid decomposition, easy direction)**.** *def:regular$_m$atroid, def : graphic$_m$atroid, def : cographic$_m$atroid, def : isomorphism, def : R10, def : $1_s um, def : 2_s um, def : 3_s umEverybinarymatroidproducedfromgraphic, cographic, andmatroidsisomorphict$ by repeated 1-, 2-, and 3-sum compositions is regular.*

Proof sketch. thm:11.2.10 Follows from theorem 11.2.10. □

Theorem 86 (11.3.14 regular matroid decomposition, hard direction)**.** *def:regular$_m$atroid, def : graphic$_m$atroid, def : cographic$_m$atroid, def : isomorphism, def : R10, def : R12, def : $1_s um, def : 2_s um, def : 3_s um, def : k_c onn, def : k_s ep, prop :$ $11.3.11 Everyregular matroid M can be decomposed into graphic and cographic matroids and matroids isomorphic to$ by repeated 1-, 2-, and 3- sum decompositions. Specifically: If M is a regular 3-connected matroid that is not graphic and not cographic, then M is isomorphic to R_{10} or has an R_{12} minor. In the latter case, any 3-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of R_{12} ((11.3.11)) under one of the isomorphisms induces a 3-separation of M.*

Proof sketch. thm:10.4.1.if,thm:11.3.2,thm:11.3.12,lem:8.3.12

- Let M be a regular matroid. Assume M is not graphic and not cographic.
- If M is 1-separable, then it is a 1-sum. If M is 2-separable, then it is a 2-sum. Thus assume M is 3-connected.
- By theorem 10.4.1, M has an R_{10} or an R_{12} minor.
- R_{10} case: by theorem 11.3.2, M is isomorphic to R_{10}.
- R_{12} case: by theorem 11.3.12, M has an induced by 3-separation, so by lemma 8.3.12, M is a 3-sum.

□

0.6.3 Extensions of Regular Matroid Decomposition

- Theorem 11.3.14 remains valid when 3-sums are replaced by Δ- and Y-sums (Theorem 11.3.16).
- Theorem 11.3.14 (and 11.3.16) can also be proved for matroids with no F_7 minors or with no F_7^* minors. (Uses Lemma 11.3.19: F_7 (F_7^*) is a splitter of the binary matroids with no F_7^* (F_7) minors.)

0.6.4 Applications of Regular Matroid Decomposition

- Efficient algorithm:for.testing.if a binary matroid is regular (Section 11.4).
- Efficient algorithm:for.deciding.if a real matrix is TU (Section 11.4).
- Constructing TU matrices (Theorem 11.5.9). (Translate 3-sum version of theorem 11.3.16 into matrix language.)
- Constructing $0, 1$ TU matrices (Theorem 11.5.13).
- Characterization of the cycle polytope (theorem 11.5.17). (Problem: let M be a connected binary matroid with ground set E and element weighs w_e for all $e \in E$. Find a disjoint union C of circuits of M such that $\sum_{e \in C} w_e$ is maximized.)
- Number of nonzeros in TU matrices (Theorem 11.5.18).
- Triples in circuits (Theorem 11.5.18).
- Odd cycles (Theorem 11.5.20).