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Seymour

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Chapter 1

Code

1.1 TU Matrices

Definition 1 (TU matrix)**.** Matrix.TU A real matrix is *totally unimodular (TU)* if its every subdeterminant (i.e., determinant of every square submatrix) is 0 or ± 1.

Lemma 2 (entries of a TU matrix)**.** *def:code*$_t u_m$*atrixMatrix.TU.applyIfAisTU,theneveryentryofAis0or*± 1*.*

Proof sketch. def:code$_t u_m atrix Everyentryisasquaresubmatrixofsize 1, andthereforehasdeterminant(andva$ □

Lemma 3 (any submatrix of a TU matrix is TU)**.** *def:code*$_t u_m$*atrixMatrix.TU.submatrixLetAbearealmatrixt*

Proof sketch. def:code$_t u_m atrix Anysquaresubmatrixof$ B$isasubmatrixof$ A$, soitsdeterminantis$ 0$or \pm 1$. Thus, B is TU. □

Lemma 4 (transpose of TU is TU)**.** *def:code*$_t u_m$*atrixMatrix.TU.transposeLetAbeaTUmatrix.Then*A^T *is TU.*

Proof sketch. def:code$_t u_m atrix Asubmatrix$ Tof A^T is a transpose of a submatrix of A, so $\det T \in \{0, \pm 1\}$. □

Lemma 5 (appending zero vector to TU)**.** *def:code*$_t u_m$*atrixMatrix.TU*$_a$*djoin*$_r$*ow*0*s*$_i$*ffLetAbeamatrix.Letabe*
= [A/a] is TU exactly when A is.

Proof sketch. def:code$_t u_m atrix, lem : code_s ubmatrix_o f_t u Let$ T$beasquaresubmatrixof$ C$, andsuppose$ A$isTU.If$ 0. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. For the other direction, because A is a submatrix of C, we can apply lemma **??**. □

Lemma 6 (appending unit vector to TU)**.** *def:code*$_t u_m$*atrixLetAbeamatrix.Letabeaunitrow.ThenC*
= [A/a] is TU exactly when A is.

Proof sketch. def:code$_tu_matrix, lem: code_submatrix_of_tuLet$ T $beasquaresubmatrixof$ C$, andsuppose$ A $isTU.If$
entry of the unit row, then $\det T$ equals the determinant of some submatrix of A times ± 1, so $\det T \in \{0, \pm 1\}$. If T contains some entries of the unit row except the ± 1, then $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. For the other direction, simply note that A is a submatrix of C, and use lemma **??**. ☐

Lemma 7 (TUness with adjoint identity matrix). *def:code$_tu_matrixMatrix.TU_adjoin_id_below_iff, Matrix.TU_a$*
has determinant ± 1.

Proof sketch. def:code$_tu_matrixGaussianelimination.Basissubmatrix : itscolumnsformabasisofallcolumns,$

Lemma 8 (block-diagonal matrix with TU blocks is TU). *def:code$_tu_matrixMatrix.fromBlocks_TULetAbeama$*
and A_2 *are both TU. Then* A *is also TU.*

Proof sketch. def:code$_tu_matrixAnysquaresubmatrix$ Tof A$hastheform$ $\begin{array}{|c|c|}\hline T_1 & 0 \\ \hline 0 & T_2 \\ \hline\end{array}$ $where$ T$_1$
and T_2 are submatrices of A_1 and A_2, respectively.

- If T_1 is square, then T_2 is also square, and $\det T = \det T_1 \cdot \det T_2 \in \{0, \pm 1\}$.
- If T_1 has more rows than columns, then the rows of T containing T_1 are linearly dependent, so $\det T = 0$.
- Similar if T_1 has more columns than rows.

☐

Lemma 9 (appending parallel element to TU). *def:code$_tu_matrixLetAbeaTUmatrix.Letabesomerowof A.Then$*
= [A/a] is TU.

Proof sketch. def:code$_tu_matrixLet$ T$beasquaresubmatrixof$ C$.If$ T$containsthesamerowtwice, thentherowsare$
dependent, so$\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. ☐

Lemma 10 (appending rows to TU). *def:code$_tu_matrixLetAbeaTUmatrix.LetBbeamatrixwhoseeveryrowisar$*
= [A/B] is TU.

Proof sketch. def:code$_tu_matrix, lem: code_tu_add_zero_row, lem: code_tu_add_unit_row, lem:$
$code_tu_add_copy_rowEitherrepeatedlyapplyLemmas$ **??**, **??**, and**??**$orperformasimilarcaseanalysisdirectly.$ ☐

Corollary 11 (appending columns to TU). *def:code$_tu_matrix, lem: code_tu_add_zero_row, lem:$*
$code_tu_add_unit_row, lem: code_tu_add_copy_rowLetAbeaTUmatrix.LetBbeamatrixwhoseeverycolumnisacolumno$
= [A | B] is TU.

Proof sketch. def:code$_tu_matrix, lem: code_tu_add_zero_row, lem: code_tu_add_unit_row, lem:$
$code_tu_add_copy_row, lem: code_tu_transpose$ C^T is TU by Lemma **??** and construction, so C is TU by Lemma **??**. ☐

Definition 12 ($\mathcal{F}$-pivot)**.** Let A be a matrix over a field $\mathcal{F}$ with row index set X and column index set Y. Let A_{xy} be a nonzero element. The result of a *$\mathcal{F}$-pivot* of A on the *pivot element* A_{xy} is the matrix A' over $\mathcal{F}$ with row index set X' and column index set Y' defined as follows.

- For every $u \in X - x$ and $w \in Y - y$, let $A'_{uw} = A_{uw} + (A_{uy} \cdot A_{xw})/(-A_{xy})$.
- Let $A'_{xy} = -A_{xy}$, $X' = X - x + y$, and $Y' = Y - y + x$.

Lemma 13 (pivoting preserves TUness)**.** *def:code*$_t u_m atrix, def : code_p ivotLetAbeaTUmatrixandletA_{xy}$ *be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. Then A' is TU.*

Proof sketch. def:code$_t u_m atrix, def : code_p ivot, lem : code_t u_a djoin_i d$

By Lemma **??** A is TU iff every basis matrix of $[I \mid A]$ has determinant ± 1. The same holds for A' and $[I \mid A']$.

Determinants of the basis matrices are preserved under elementary row operations in $[I \mid A]$ corresponding to the pivot in A, under scaling by ± 1 factors, and under column exchange, all of which together convert $[I \mid A]$ to $[I \mid A']$.

□

Lemma 14 (pivoting preserves TUness)**.** *def:code*$_t u_m atrix, def : code_p ivotLetAbeamatrixandletA_{xy}$ *be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. If A' is TU, then A is TU.*

Proof sketch. def:code$_t u_m atrix, def : code_p ivot, lem : code_p ivot_t uReversetherowoperations, scaling, andcolum$

1.1.1 Minimal Violation Matrices

Definition 15 (minimal violation matrix)**.** def:code$_t u_m atrixLet$A$beareal\{0,$ $\pm 1\}$ matrix that is not TU but all of whose proper submatrices are TU. Then A is called a *minimal violation matrix of total unimodularity (minimal violation matrix).*

Lemma 16 (simple properties of MVMs)**.** *def:code*$_m vmLetAbeaminimalviolationmatrix.$

A is square.

$\det A \notin \{0, \pm 1\}$.

If A is 2×2, then A does not contain a 0.

Proof sketch. def:code$_m vm$

If A is not square, then since all its proper submatrices are TU, A is TU, contradiction.

If $\det A \in \{0, \pm 1\}$, then all subdeterminants of A are 0 or ± 1, so A is TU, contradiction.

If A is 2×2 and it contains a 0, then $\det A \in \{\pm 1\}$, which contradicts the previous item.

$\square$

Lemma 17 (pivoting in MVMs)**.** *def:code*$_m vm, def : code_p ivotLetAbeaminimalviolationmatrix.SupposeAhas$
3 rows. Suppose we perform a real pivot in A, then delete the pivot row and column. Then the resulting matrix A' is also a minimal violation matrix.

Proof sketch. def:code$_m vm, lem : code_d iagonal_w ith_t u_b locks, lem : code_r everse_p ivot_t u, lem :$ $code_p ivot_t u, lem : code_s ubmatrix_o f_t u$

Let A'' denote matrix A after the pivot, but before the pivot row and column are deleted.

Since A is not TU, Lemma **??** implies that A'' is not TU. Thus A' is not TU by Lemma **??**.

Let T' be a proper square submatrix of A'. Let T'' be the submatrix of A'' consisting of T' plus the pivot row and the pivot column, and let T be the corresponding submatrix of A (defined by the same row and column indices as T'').

T is TU as a proper submatrix of A. Then Lemma **??** implies that T'' is TU. Thus T' is TU by Lemma **??**.

$\square$

1.2 Matroid Definitions

Definition 18 (binary matroid)**.** BinaryMatroid Let B be a binary matrix, let $A = [I \mid B]$, and let E denote the column index set of A. Let $\mathcal{I}$ be all index subsets $Z \subseteq E$ such that the columns of A indexed by Z are independent over GF(2). Then $M = (E, \mathcal{I})$ is called a *binary matroid* and B is called its *(standard) representation matrix.*

Definition 19 (regular matroid)**.** def:code$_b inary_m atroid, def : code_t u_m atrixBinaryMatroid.IsRegularLet$M

A is a signed version of B, i.e., $|A| = B$,

A is totally unimodular.

Then M is called a *regular matroid.*

1.3 k-Separation and k-Connectivity

Definition 20 (k-separation)**.** def:code$_binary_matroidLetMbeabinarymatroidgeneratedbyastandardrepresent$

X_2 is a partition of the rows of B and $Y_1 \sqcup Y_2$ is a partition of its columns. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that

- $|X_1 \cup Y_1| \geq k$ and $|X_2 \cup Y_2| \geq k$,
- GF(2)-rank D_1 + GF(2)-rank $D_2 \leq k - 1$.

Then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a *(Tutte) k-separation* of B and M.

Definition 21 (exact k-separation)**.** def:code$_{k_s}epA$k$-separationiscalledexactiftherankconditionholdswitheq$

Definition 22 (k-separability)**.** def:code$_{k_s}epWesaythat$BandM*are (exactly) (Tutte) k-separable*$iftheyhavean($

Definition 23 (k-connectivity)**.** def:code$_{k_s}epFor$k ≥ 2, M and B are *(Tutte) k-connected* if they have no ℓ-separation for $1 \leq \ell < k$. When M and B are 2-connected, they are also called *connected*.

1.4 Sums

1.4.1 1-Sums

Definition 24 (1-sum of matrices)**.** Matrix$_1sumCompositionLet$B$beamatrixthatcanberepresentedas$ $\begin{array}{cc} & Y \\ X_1 & \boxed{B} \\ X_2 & \boxed{0} \end{array}$

and B_2 are the two *components* of a 1-*sum decomposition* of B.

Conversely, a 1-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_1 B_2$ means either process.

Definition 25 (matroid 1-sum)**.** def:code$_binary_matroid, def: code_{1_s}um_of_matricesBinaryMatroid.Is1sumO$ $zeroblocks$B$_1$ and B_2. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 1-*sum decomposition* of M.

Conversely, a 1-*sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_1 M_2$ means either process.

Lemma 26 (1-sum is commutative)**.** *def:code*$_{1_s}um_of_binaryBinaryMatroid.Is1sumOf.commtodo:$ *add*

Lemma 27 (1-sum of regular matroids is regular)**.** *BinaryMatroid.Is1sumOf.isRegular def:code*$_{1_s}um_of_binary, def: code_regular_matroidLetM_1$ *and* M_2 *be regular matroids. Then* $M = M_1 \oplus_1 M_2$ *is a regular matroid.*

Conversely, if a regular matroid M *can be decomposed as a* 1*-sum* $M = M_1 \oplus_1 M_2$*, then* M_1 *and* M_2 *are both regular.*

Proof sketch. def:code$_{1s}um_of_binary, def : code_regular_matroidtodo : extractintolemmasaboutTUmatricesLet$
and B_2 be the representation matrices of M, M_1, and M_2, respectively.

- Converse direction. Let B' be a TU signing of B. Let B'_1 and B'_2 be signings of B_1 and B_2, respectively, obtained from B. By Lemma **??**, B'_1 and B'_2 are both TU, so M_1 and M_2 are both regular.

- Forward direction. Let B'_1 and B'_2 be TU signings of B_1 and B_2, respectively. Let B' be the corresponding signing of B. By Lemma **??**, B' is TU, so M is regular.

□

Lemma 28 (left summand of regular 1-sum is regular)**.** *def:code$_{1s}um_of_binaryBinaryMatroid.Is1sumOf.isRe$*
add

Lemma 29 (right summand of regular 1-sum is regular)**.** *def:code$_{1s}um_of_binaryBinaryMatroid.Is1sumOf.isR$*
add

1.4.2 2-Sums

Definition 30 (2-sum of matrices)**.** Matrix$_2sumCompositionLet$B$beamatrixoftheform$ $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A_1 & 0 \\ \hline X_2 & D & A_2 \\ \hline \end{array}$ LetB

be a matrix of the form $\begin{array}{c|c|} & Y_1 \\ X_1 & A_1 \\ \text{Unit} & x \end{array}$ Let B_2 be a matrix of the form $\begin{array}{c|c|c|} & \text{Unit} & Y_2 \\ \hline X_2 & y & A_2 \\ \hline \end{array}$

Suppose that GF(2)-rank $D = 1$, $x \neq 0$, $y \neq 0$, $D = y \cdot x$ (outer product).

Then we say that B_1 and B_2 are the two *components* of a 2-*sum decomposition* of B.

Conversely, a 2-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_2 B_2$ means either process.

Definition 31 (matroid 2-sum)**.** def:code$_binary_matroid, def : code_{2s}um_of_matricesBinaryMatroid.Is2sumO$
and B_2 satisfy the assumptions of Definition **??**. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 2-*sum decomposition* of M.

Conversely, a 2-*sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_2 M_2$ means either process.

Lemma 32 (2-sum of regular matroids is regular)**.** *def:code$_{2s}um_of_binary, def :$ $code_regular_matroidBinaryMatroid.Is2sumOf.isRegularLetM_1$ and M_2 be regular matroids. Then $M = M_1 \oplus_2 M_2$ is a regular matroid.*

Proof sketch. def:code$_{2s}um_of_binary, def : code_regular_matroid, lem : code_tu_add_ok_rows, cor :$
$code_tu_add_ok_cols$

Let B, B_1, and B_2 be the representation matrices of M, M_1, and M_2, respectively. Let B_1' and B_2' be TU signings of B_1 and B_2, respectively. In particular, let A_1', x', A_2', and y' be the signed versions of A_1, x, A_2, and y, respectively. Let B' be the signing of B where the blocks of A_1 and A_2 are signed as A_1' and A_2', respectively, and the block of D is signed as $D' = y' \cdot x'$ (outer product).

Note that $[A_1'/D']$ is TU by Lemma **??**, as every row of D' is either zero or a copy of x'. Similarly, $[D' \mid A_2']$ is TU by Corollary **??**, as every column of D' is either zero or a copy of y'. Additionally, $[A_1' \mid 0]$ is TU by Corollary **??**, and $[0/A_2']$ is TU by Lemma **??**.

todo: prove lemma below, separate into statement about TU matrices

Lemma: Let T be a square submatrix of B'. Then $\det T \in \{0, \pm 1\}$.

Proof: Induction on the size of T. *Base:* If T consists of only 1 element, then this element is 0 or ± 1, so $\det T \in \{0, \pm 1\}$. *Step:* Let T have size t and suppose all square submatrices of B' of size $\leq t - 1$ are TU.

- Suppose T contains no rows of X_1. Then T is a submatrix of $[D' \mid A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no rows of X_2. Then T is a submatrix of $[A_1' \mid 0]$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_1. Then T is a submatrix of $[0/A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_2. Then T is a submatrix of $[A_1'/D']$, so $\det T \in \{0, \pm 1\}$.
- Remaining case: T contains rows of X_1 and X_2 and columns of Y_1 and Y_2.
- If T is 2×2, then T is TU. Indeed, all proper submatrices of T are of size ≤ 1, which are $\{0, \pm 1\}$ entries of A', and T contains a zero entry (in the row of X_2 and column of Y_2), so it cannot be a minimal violation matrix by Lemma **??**. Thus, assume T has size ≥ 3.
- . todo: complete proof, see last paragraph of Lemma 11.2.1 in Truemper

□

Lemma 33 (left summand of regular 2-sum is regular)**.** *def:code$_{2s}um_of_binaryBinaryMatroid.Is2sumOf.isRe$*
add

Lemma 34 (right summand of regular 2-sum is regular)**.** *def:code$_{2s}um_of_binaryBinaryMatroid.Is2sumOf.isR$*
add

1.4.3 3-Sums

Definition 35 (3-sum of matrices)**.** Matrix$_3$*sumCompositiontodo* : *add*

Definition 36 (matroid 3-sum)**.** def:code$_{b}inary_{m}atroidBinaryMatroid.Is3sumOftodo:$
add

Lemma 37 (3-sum of regular matroids is regular)**.** *def:code*$_{3_s}um_of_binaryBinaryMatroid.Is3sumOf.isRegula$
add

Lemma 38 (left summand of regular 3-sum is regular)**.** *def:code*$_{3_s}um_of_binaryBinaryMatroid.Is3sumOf.isRe$
add

Lemma 39 (right summand of regular 3-sum is regular)**.** *def:code*$_{3_s}um_of_binaryBinaryMatroid.Is3sumOf.isR$
add