Proof of Regularity of 1-, 2-, and 3-Sums of Matroids

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1 Preliminaries

1.1 Total Unimodularity

Definition 1. We say that a matrix $A \in \mathbb{Q}^{X \times Y}$ is totally unimodular, or TU for short, if for every $k \in \mathbb{Z}_{\geq 1}$, every $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 2. Let A be a TU matrix. Suppose some rows and columns of A are multiplied by $\{0, \pm 1\}$ factors. Then the resulting matrix A' is also TU.

Proof. We prove that A' is TU by Definition 1. To this end, let T' be a square submatrix of A'. Our goal is to show that $\det T' \in \{0, \pm 1\}$. Let T be the submatrix of A that represents T' before pivoting. If some of the rows or columns of T were multiplied by zeros, then T' contains zero rows or columns, and hence $\det T' = 0$. Otherwise, T' was obtained from T by multiplying certain rows and columns by -1. Since T' has finitely many rows and columns, the number of such multiplications is also finite. Since multiplying either a row or a column by -1 results in the determinant getting multiplied by -1, we get $\det T' = \pm \det T \in \{0, \pm 1\}$, as desired. $\square$

Definition 3. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-partially unimodular, or k-PU for short, if every $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 4. A matrix A is TU if and only if A is k-PU for every $k \in \mathbb{Z}_{\geq 1}$.

Proof. This follows from Definitions 1 and 3. $\square$

1.2 Pivoting

Definition 5. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. A long tableau pivot in A on (x, y) is the operation that maps A to the matrix A' where

$$\forall i \in X,\ \forall j \in Y,\ A'(i, j) = \begin{cases} \frac{A(i,j)}{A(x,y)}, & \text{if } i = x, \\ A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x. \end{cases}$$

Lemma 6. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Let A' be the result of performing a long tableau pivot in A on (x, y). Then A' can be equivalently obtained from A as follows:

1. For every row $i \in X \setminus \{x\}$, add row x multiplied by $A(i, y)/A(x, y)$ to row i.
2. Multiply row x by $1/A(x, y)$.

Proof. See implementation in Lean. $\square$

Lemma 7. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the long tableau pivot in A on (x, y) yields a TU matrix A'.

Proof. See implementation in Lean. $\square$

Definition 8. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Perform the following sequence of operations.

1. Adjoin the identity matrix $1 \in R^{X \times X}$ to A, resulting in the matrix $B = \begin{array}{|c|c|}\hline 1 & A \\ \hline\end{array} \in R^{X \times (X \oplus Y)}$.
2. Perform a long tableau pivot in B on (x, y), and let C denote the result.
3. Swap columns x and y in C, and let D be the resulting matrix.
4. Finally, remove columns indexed by X from D, and let A' be the resulting matrix.

A short tableau pivot in A on (x, y) is the operation that maps A to the matrix A' defined above.

Lemma 9. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then the short tableau pivot in A on (x, y) maps A to A' with

$$\forall i \in X,\ \forall j \in Y,\ A'(i,j) = \begin{cases} \frac{1}{A(x,y)}, & \text{if } i = x \text{ and } j = y, \\ \frac{A(x,j)}{A(x,y)}, & \text{if } i = x \text{ and } j \neq y, \\ -\frac{A(i,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j = y, \\ A(i,j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j \neq y. \end{cases}$$

Proof. Follows by direct calculation. □

Lemma 10. Let $B = \begin{array}{|c|c|}\hline B_{11} & 0 \\ \hline B_{21} & B_{22} \\ \hline\end{array} \in \mathbb{Q}^{\{X_1 \cup X_2\} \times \{Y_1 \times Y_2\}}$. Let $B' = \begin{array}{|c|c|}\hline B'_{11} & B'_{12} \\ \hline B'_{21} & B'_{22} \\ \hline\end{array}$ be the result of performing a short tableau pivot on $(x, y) \in X_1 \times Y_1$ in B. Then $B'_{12} = 0$, $B'_{22} = B_{22}$, and $\begin{array}{|c|}\hline B'_{11} \\ \hline B'_{21} \\ \hline\end{array}$ is the matrix resulting from performing a short tableau pivot on (x, y) in $\begin{array}{|c|}\hline B_{11} \\ \hline B_{21} \\ \hline\end{array}$.

Proof. This follows by a direct calculation. Indeed, because of the 0 block in B, B_{12} and B_{22} remain unchanged, and since $\begin{array}{|c|}\hline B_{11} \\ \hline B_{21} \\ \hline\end{array}$ is a submatrix of B containing the pivot element, performing a short tableau pivot in it is equivalent to performing a short tableau pivot in B and then taking the corresponding submatrix. □

Lemma 11. Let $k \in \mathbb{Z}_{\geq 1}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of performing a short tableau pivot in A on (x, y) with $x, y \in \{1, \ldots, k\}$ such that $A(x, y) \neq 0$. Then A' contains a submatrix A'' of size $(k-1) \times (k-1)$ with $|\det A''| = |\det A| / |A(x, y)|$.

Proof. Let $X = \{1, \ldots, k\} \setminus \{x\}$ and $Y = \{1, \ldots, k\} \setminus \{y\}$, and let $A'' = A'(X, Y)$. Since A'' does not contain the pivot row or the pivot column, $\forall (i, j) \in X \times Y$ we have $A''(i, j) = A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}$. For $\forall j \in Y$, let B_j be the matrix obtained from A by removing row x and column j, and let B''_j be the matrix obtained from A'' by replacing column j with $A(X, y)$ (i.e., the pivot column without the pivot element). The cofactor expansion along row x in A yields

$$\det A = \sum_{j=1}^{k} (-1)^{y+j} \cdot A(x, j) \cdot \det B_j.$$

By reordering columns of every B_j to match their order in B''_j, we get

$$\det A = (-1)^{x+y} \cdot \left(A(x, y) \cdot \det A' - \sum_{j \in Y} A(x, j) \cdot \det B''_j \right).$$

By linearity of the determinant applied to $\det A''$, we have

$$\det A'' = \det A' - \sum_{j \in Y} \frac{A(x, j)}{A(x, y)} \cdot \det B''_j$$

Therefore, $|\det A''| = |\det A| / |A(x, y)|$. □

Lemma 12. Let $k \in \mathbb{Z}_{\geq 1}$, let $A \in \mathbb{Q}^{k\times k}$, and let A' be the result of performing a short tableau pivot in A on (x, y) with $x, y \in \{1, \ldots, k\}$ such that $A(x, y) \in \{\pm 1\}$. Then A' contains a submatrix A'' of size $(k-1)\times(k-1)$ with $|\det A''| = |\det A|$.

Proof. Apply Lemma 11 to A and use that $A(x, y) \in \{\pm 1\}$. $\square$

Lemma 13. Let $A \in \mathbb{Q}^{X\times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the short tableau pivot in A on (x, y) yields a TU matrix A'.

Proof. See implementation in Lean. $\square$

1.3 Vector Matroids

Definition 14. Let R be a semiring, let X and Y be sets, and let $A \in R^{X\times Y}$ be a matrix. The vector matroid of A is the matroid $M = (Y, \mathcal{I})$ where a set $I \subset Y$ is independent in M if and only if the columns of A indexed by I are linearly independent.

Definition 15. Let R be a semiring, let X and Y be disjoint sets, and let $S \in R^{X\times Y}$ be a matrix. Let $A = \boxed{1 \mid S} \in R^{X\times(X\cup Y)}$ be the matrix obtained from S by adjoining the identity matrix as columns, and let M be the vector matroid of A. Then S is called the standard representation of M.

Lemma 16. Let $S \in R^{X\times Y}$ be a standard representation of a vector matroid M. Then X is a base in M.

Proof. See implementation in Lean. $\square$

Lemma 17. Let $A \in \mathbb{Q}^{X\times Y}$ be a matrix, let M be the vector matroid of A, and let B be a base of M. Then there exists a standard representation matrix $S \in \mathbb{Q}^{B\times(Y\setminus B)}$ of M.

Proof. See implementation in Lean. $\square$

Lemma 18. Let $A \in \mathbb{Q}^{X\times Y}$ be a TU matrix, let M be the vector matroid of A, and let B be a base of M. Then there exists a matrix $S \in \mathbb{Q}^{B\times(Y\setminus B)}$ such that S is TU and S is a standard representation of M.

Proof. See implementation in Lean. $\square$

Definition 19. Let F be a field. The support of matrix $A \in F^{X\times Y}$ is $A^{\#} \in \{0, 1\}^{X\times Y}$ given by

$$\forall i \in X,\ \forall j \in Y,\ A^{\#}(i, j) = \begin{cases} 0, & \text{if } A(i, j) = 0, \\ 1, & \text{if } A(i, j) \neq 0. \end{cases}$$

Definition 20. Let M be a matroid, let B be a base of M, and let $e \in E\setminus B$ be an element. The fundamental circuit $C(e, B)$ of e with respect to B is the unique circuit contained in $B \cup \{e\}$.

Lemma 21. Let M be a matroid and let $S \in F^{X\times Y}$ be a standard representation matrix of M over a field F. Then $\forall y \in Y$, the fundamental circuit of y w.r.t. X is $C(y, X) = \{y\} \cup \{x \in X \mid S(x, y) \neq 0\}$.

Proof. Let $y \in Y$. Our goal is to show that $C'(y, X) = \{y\} \cup \{x \in X \mid D(x, y) \neq 0\}$ is a fundamental circuit of y with respect to X.

- $C'(y, X) \subseteq X \cup \{y\}$ by construction.
- $C'(y, X)$ is dependent, since columns of $[I \mid S]$ indexed by elements of $C(y, X)$ are linearly dependent.
- If $C \subsetneq C'(y, X)$, then C is independent. To show this, let V be the set of columns of $[I \mid S]$ indexed by elements of C and consider two cases.
  1. Suppose that $y \notin C$. Then vectors in V are linearly independent (as columns of I). Thus, C is independent.
  2. Suppose $\exists x \in X \setminus C$ such that $S(x, y) \neq 0$. Then any nontrivial linear combination of vectors in V has a non-zero entry in row x. Thus, these vectors are linearly independent, so C is independent.

□

Lemma 22. Let M be a matroid and let $S \in F^{X \times Y}$ be a standard representation matrix of M over a field F. Then $\forall y \in Y$, column $S^{\#}(\bullet, y)$ is the characteristic vector of $C(y, X) \setminus \{y\}$.

Proof. Directly follows from Lemma 21. □

Lemma 23. Let A be a TU matrix.

1. If a matroid is represented by A, then it is also represented by $A^{\#}$.
2. If a matroid is represented by $A^{\#}$, then it is also represented by A.

Proof. See implementation in Lean. □

1.4 Regular Matroids

Definition 24. A matroid M is regular if there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A.

Definition 25. We say that $A' \in \mathbb{Q}^{X \times Y}$ is a TU signing of $A \in \mathbb{Z}_2^{X \times Y}$ if A' is TU and

$$\forall i \in X,\ \forall j \in Y,\ |A'(i,j)| = A(i,j).$$

Lemma 26. Let $B \in \mathbb{Z}_2^{X \times Y}$ be a standard representation matrix of a matroid M. Then M is regular if and only if B has a TU signing.

Proof. Suppose that M is regular. By Definition 24, there exists $A \in \mathbb{Q}^{X \times Y}$ such that $M = M[A]$ and A is TU. By Lemma 16, X (the row set of B) is a base of M. By Lemma 18, A can be converted into a standard representation matrix $B' \in \mathbb{Q}^{X \times Y}$ of M such that B' is also TU. Since B' and B are both standard representations of M, by Lemma 22 the support matrices $(B')^{\#}$ and $B^{\#}$ are the same. Moreover, $B^{\#} = B$, since B has entries in $\mathbb{Z}_2$. Thus, B' is TU and $(B')^{\#} = B$, so B' is a TU signing of B.

Suppose that B has a TU signing $B' \in \mathbb{Q}^{X \times Y}$. Then $A = [I \mid B']$ is TU, as it is obtained from B' by adjoining the identity matrix. Moreover, by Lemma 23, A represents the same matroid as $A^{\#} = [I \mid B]$, which is M. Thus, A is a TU matrix representing M, so M is regular. □

2 Regularity of 1-Sum

Definition 27. Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ be matrices where X_ℓ, Y_ℓ, X_r, Y_r are pairwise disjoint sets. The 1-sum $B = B_\ell \oplus_1 B_r$ of B_ℓ and B_r is

$$B = \begin{array}{|c|c|}\hline B_\ell & 0 \\ \hline 0 & B_r \\ \hline \end{array} \in R^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}.$$

Definition 28. A matroid M is a 1-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B, B_ℓ, and B_r (for M, M_ℓ, and M_r, respectively) of the form given in Definition 27.

Lemma 29. Let A be a square matrix of the form $A = \begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline 0 & A_{22} \\ \hline \end{array}$. Then $\det A = \det A_{11} \cdot \det A_{22}$.

Proof. This lemma is proved in MathLib. ☐

Lemma 30. Let B_ℓ and B_r from Definition 27 be TU matrices (over $\mathbb{Q}$). Then $B = B_\ell \oplus_1 B_r$ is TU.

Proof. We prove that B is TU by Definition 1. To this end, let T be a square submatrix of B. Our goal is to show that $\det T \in \{0, \pm 1\}$.

Let T_ℓ and T_r denote the submatrices in the intersection of T with B_ℓ and B_r, respectively. Then T has the form

$$T = \begin{array}{|c|c|}\hline T_\ell & 0 \\ \hline 0 & T_r \\ \hline \end{array}.$$

First, suppose that T_ℓ and T_r are square. Then $\det T = \det T_\ell \cdot \det T_r$ by Lemma 29. Moreover, $\det T_\ell, \det T_r \in \{0, \pm 1\}$, since T_ℓ and T_r are square submatrices of TU matrices B_ℓ and B_r, respectively. Thus, $\det T \in \{0, \pm 1\}$, as desired.

Without loss of generality we may assume that T_ℓ has fewer rows than columns. Otherwise we can transpose all matrices and use the same proof, since TUness and determinants are preserved under transposition. Thus, T can be represented in the form

$$T = \begin{array}{|c|c|}\hline T_{11} & T_{12} \\ \hline 0 & T_{22} \\ \hline \end{array},$$

where T_{11} contains T_ℓ and some zero rows, T_{22} is a submatrix of T_r, and T_{12} contains the rest of the rows of T_r (not contained in T_{22}) and some zero rows. By Lemma 29, we have $\det T = \det T_{11} \cdot \det T_{22}$. Since T_{11} contains at least one zero row, $\det T_{11} = 0$. Thus, $\det T = 0 \in \{0, \pm 1\}$, as desired. ☐

Lemma 31. Let M be a 1-sum of regular matroids M_ℓ and M_r. Then M is also regular.

Proof. Let B, B_ℓ, and B_r be standard $\mathbb{Z}_2$ representation matrices from Definition 28. Since M_ℓ and M_r are regular, by Lemma 26, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_1 B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 30, and a direct calculation shows that B' is a signing of B. Thus, M is regular by Lemma 26. ☐

3 Regularity of 2-Sum

Definition 32. Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{(X_\ell \cup \{x\}) \times Y_\ell}$ and $B_r \in R^{X_r \times (Y_r \cup \{y\})}$ be matrices of the form

$$B_\ell = \begin{array}{|c|}\hline A_\ell \\ \hline r \\ \hline \end{array}, \quad B_r = \begin{array}{|c|c|}\hline c & A_r \\ \hline \end{array}.$$

The 2-sum $B = B_\ell \oplus_{2,x,y} B_r$ of B_ℓ and B_r is defined as

$$B = \begin{array}{|c|c|}\hline A_\ell & 0 \\ \hline D & A_r \\ \hline \end{array} \quad \text{where} \quad D = c \otimes r.$$

Here $A_\ell \in R^{X_\ell \times Y_\ell}$, $A_r \in R^{X_r \times Y_r}$, $r \in R^{Y_\ell}$, $c \in R^{X_r}$, $D \in R^{X_r \times Y_\ell}$, and the indexing is consistent everywhere.

Definition 33. A matroid M is a 2-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B, B_ℓ, and B_r (for M, M_ℓ, and M_r, respectively) of the form given in Definition 32.

Lemma 34. Let B_ℓ and B_r from Definition 32 be TU matrices (over $\mathbb{Q}$). Then $C = \begin{array}{|c|c|}\hline D & A_r \\ \hline \end{array}$ is TU.

Proof. Since B_ℓ is TU, all its entries are in $\{0, \pm 1\}$. In particular, r is a $\{0, \pm 1\}$ vector. Therefore, every column of D is a copy of y, $-y$, or the zero column. Thus, C can be obtained from B_r by adjoining zero columns, duplicating the y column, and multiplying some columns by -1. Since all these operations preserve TUess and since B_r is TU, C is also TU. ☐

Lemma 35. Let B_ℓ and B_r be matrices from Definition 32. Let B'_ℓ and B' be the matrices obtained by performing a short tableau pivot on $(x_\ell, y_\ell) \in X_\ell \times Y_\ell$ in B_ℓ and B, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B_r$.

Proof. Let

$$B'_\ell = \begin{array}{|c|}\hline A'_\ell \\ \hline r' \\ \hline \end{array}, \quad B' = \begin{array}{|c|c|}\hline B'_{11} & B'_{12} \\ \hline B'_{21} & B'_{22} \\ \hline \end{array}$$

where the blocks have the same dimensions as in B_ℓ and B, respectively. By Lemma 10, $B'_{11} = A'_\ell$, $B'_{12} = 0$, and $B'_{22} = A_r$. Equality $B'_{21} = c \otimes r'$ can be verified via a direct calculation. Thus, $B' = B'_\ell \oplus_{2,x,y} B_r$. ☐

Lemma 36. Let B_ℓ and B_r from Definition 32 be TU matrices (over $\mathbb{Q}$). Then $B_\ell \oplus_{2,x,y} B_r$ is TU.

Proof. By Lemma 4, it suffices to show that $B_\ell \oplus_{2,x,y} B_r$ is k-PU for every $k \in \mathbb{Z}_{\geq 1}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since all entries of $B_\ell \oplus_{2,x,y} B_r$ are in $\{0, \pm 1\}$ by construction.

Suppose that for some $k \in \mathbb{Z}_{\geq 1}$ we know that for any TU matrices B'_ℓ and B'_r (from Definition 32) their 2-sum $B'_\ell \oplus_{2,x,y} B'_r$ is k-PU. Now, given TU matrices B_ℓ and B_r (from Definition 32), our goal is to show that $B = B_\ell \oplus_{2,x,y} B_r$ is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix T of B has $\det T \in \{0, \pm 1\}$.

First, suppose that T has no rows in X_ℓ. Then T is a submatrix of $\begin{array}{|c|c|}\hline D & A_r \\ \hline \end{array}$, which is TU by Lemma 34, so $\det T \in \{0, \pm 1\}$. Thus, we may assume that T contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $T(x_\ell, y_\ell) \neq 0$. Indeed, if $T(x_\ell, y) = 0$ for all y, then $\det T = 0$ and we are done, and $T(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since B is 1-PU, all entries of T are in $\{0, \pm 1\}$, and hence $T(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 12, performing a short tableau pivot in T on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix T'' such that $|\det T| = |\det T''|$. Since T is a submatrix of B, matrix T'' is a submatrix of the matrix B' resulting from performing a short tableau pivot in B on the same entry (x_ℓ, y_ℓ). By Lemma 35, we have $B' = B'_\ell \oplus_{2,x,y} B_r$ where B'_ℓ is the result of performing a short tableau pivot in B_ℓ on (x_ℓ, y_ℓ). Since TUness is preserved by pivoting and B_ℓ is TU, B'_ℓ is also TU. Thus, by the inductive hypothesis applied to T'' and $B'_\ell \oplus_{2,x,y} B_r$, we have $\det T'' \in \{0, \pm 1\}$. Since $|\det T| = |\det T''|$, we conclude that $\det T \in \{0, \pm 1\}$. ☐

Lemma 37. Let M be a 2-sum of regular matroids M_ℓ and M_r. Then M is also regular.

Proof. Let B, B_ℓ, and B_r be standard $\mathbb{Z}_2$ representation matrices from Definition 33. Since M_ℓ and M_r are regular, by Lemma 26, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 36, and a direct calculation verifies that B' is a signing of B. Thus, M is regular by Lemma 26. ☐

4 Regularity of 3-Sum

4.1 Definition

Definition 38. Let $B_l \in \mathbb{Z}_2^{(X_l \cup \{x_0,x_1\}) \times (Y_l \cup \{y_2\})}$, $B_r \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0,y_1\})}$ be matrices of the form

$$B_l = \begin{array}{|c|cc|c|}\hline \multicolumn{3}{|c|}{A_l} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_l & \multicolumn{2}{c|}{D_0} & \begin{matrix}1\\1\end{matrix} \\ \hline\end{array} \quad \text{and} \quad B_r = \begin{array}{|cc|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & \begin{matrix}1\\1\end{matrix} & \\ \cline{1-3} \multicolumn{2}{|c|}{D_r} & \multicolumn{2}{c|}{A_r} \\ \hline\end{array} \quad \text{where} \quad D_0 = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array} \quad \text{or} \quad D_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}.$$

The 3-sum $B = B_l \oplus_3 B_r \in \mathbb{Z}_2^{(X_l \cup X_r) \times (Y_l \cup Y_r)}$ of B_l and B_r is defined as

$$B = \begin{array}{|c|cc|c|c|}\hline \multicolumn{3}{|c}{A_l} & \multicolumn{2}{|c|}{0} \\ \cline{2-4} & 1 & 1 & 0 & \\ \hline D_l & \multicolumn{2}{c|}{D_0} & \begin{matrix}1\\1\end{matrix} & \\ \cline{1-4} D_{lr} & \multicolumn{2}{c|}{D_r} & \multicolumn{2}{c|}{A_r} \\ \hline\end{array} \quad \text{where} \quad D_{lr} = D_r \cdot (D_0)^{-1} \cdot D_l.$$

Here $x_2 \in X_l$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_l$, $y_2 \in Y_r$, $A_l \in \mathbb{Z}_2^{X_l \times Y_l}$, $A_r \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_l \in \mathbb{Z}_2^{\{x_0,x_1\} \times (Y_l \setminus \{y_0,y_1\})}$, $D_r \in \mathbb{Z}_2^{(X_r \setminus \{x_0,x_1\}) \times \{y_0,y_1\}}$, $D_{lr} \in \mathbb{Z}_2^{(X_r \setminus \{x_0,x_1\}) \times (Y_l \setminus \{y_0,y_1\})}$, $D_0 \in \mathbb{Z}_2^{\{x_0,x_1\} \times \{y_0,y_1\}}$. The indexing is consistent everywhere.

Remark 39. In Definition 38, D_0 is non-singular by construction, so D_{lr} and B are well-defined. Moreover, a non-singular $\mathbb{Z}_2^{2\times 2}$ matrix is either $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}$ up to re-indexing. Thus, Definition 38 can be equivalently restated with D_0 required to be non-singular and B_l, B_r, and B re-indexed appropriately.

Definition 40. A matroid M is a 3-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B, B_ℓ, and B_r (for M, M_ℓ, and M_r, respectively) of the form given in Definition 38.

4.2 Canonical Signing

Definition 41. We call $D_0' \in \mathbb{Q}^{\{x_0,x_1\} \times \{y_0,y_1\}}$ the canonical signing of $D_0 \in \mathbb{Z}_2^{\{x_0,x_1\} \times \{y_0,y_1\}}$ if

$$D_0 = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array} \quad \text{and} \quad D_0' = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline\end{array}, \quad \text{or} \quad D_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array} \quad \text{and} \quad D_0' = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}.$$

Similarly, we call $S' \in \mathbb{Q}^{\{x_0,x_1,x_2\} \times \{y_0,y_1,y_2\}}$ the canonical signing of $S \in \mathbb{Z}_2^{\{x_0,x_1,x_2\} \times \{y_0,y_1,y_2\}}$ if

$$S = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & \begin{matrix}1\\1\end{matrix} \\ \hline\end{array} \quad \text{and} \quad S' = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0'} & \begin{matrix}1\\1\end{matrix} \\ \hline\end{array}.$$

To simplify notation, going forward we use D_0, D_0', S, and S' to refer to the matrices of the form above.

Lemma 42. The canonical signing S' of S (from Definition 41) is TU.

Proof. Verified via a direct calculation. $\square$

Lemma 43. Let Q be a TU signing of S (from Definition 41). Let $u \in \{0, \pm 1\}^{\{x_0, x_1, x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q' be defined as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in \{x_0, x_1, x_2\},\ \forall j \in \{y_0, y_1, y_2\}.$$

Then $Q' = S'$ (from Definition 41).

Proof. Since Q is a TU signing of S and Q' is obtained from Q by multiplying rows and columns by ± 1 factors, Q' is also a TU signing of S. By construction, we have

$$\begin{aligned}
&Q'(x_2, y_0) = Q(x_2, y_0) \cdot 1 \cdot Q(x_2, y_0) = 1, \\
&Q'(x_2, y_1) = Q(x_2, y_1) \cdot 1 \cdot Q(x_2, y_1) = 1, \\
&Q'(x_2, y_2) = 0, \\
&Q'(x_0, y_0) = Q(x_0, y_0) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_0) = 1, \\
&Q'(x_0, y_1) = Q(x_0, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_1), \\
&Q'(x_0, y_2) = Q(x_0, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1, \\
&Q'(x_1, y_0) = 0, \\
&Q'(x_1, y_1) = Q(x_1, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_1)), \\
&Q'(x_1, y_2) = Q(x_1, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1.
\end{aligned}$$

Thus, it remains to show that $Q'(x_0, y_1) = S'(x_0, y_1)$ and $Q'(x_1, y_1) = S'(x_1, y_1)$.

Consider the entry $Q'(x_0, y_1)$. If $D_0(x_0, y_1) = 0$, then $Q'(x_0, y_1) = 0 = S'(x_0, y_1)$. Otherwise, we have $D_0(x_0, y_1) = 1$, and so $Q'(x_0, y_1) \in \{\pm 1\}$, as Q' is a signing of S. If $Q'(x_0, y_1) = -1$, then

$$\det Q'(\{x_0, x_2\}, \{y_0, y_1\}) = \det \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q'. Thus, $Q'(x_0, y_1) = 1 = S'(x_0, y_1)$.

Consider the entry $Q'(x_1, y_1)$. Since Q' is a signing of S, we have $Q'(x_1, y_1) \in \{\pm 1\}$. Consider two cases.

1. Suppose that $D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q'(x_1, y_1) = 1$, then $\det Q = \det \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = -1 = S'(x_1, y_1)$.

2. Suppose that $D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q'(x_1, y_1) = -1$, then $\det Q(\{x_0, x_1\}, \{y_1, y_2\}) = \det \begin{array}{|c|c|} \hline 1 & 1 \\ \hline -1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = 1 = S'(x_1, y_1)$.

□

Definition 44. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU matrix. Define $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q' as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in X, \ \forall j \in Y.$$

We call Q' the canonical re-signing of Q.

Lemma 45. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q_0 \in \mathbb{Z}_2^{X \times Y}$ such that $Q_0(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S$ (from Definition 41). Then the canonical re-signing Q' of Q is a TU signing of Q_0 and $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ (from Definition 41).

Proof. Since Q is a TU signing of Q_0 and Q' is obtained from Q by multiplying some rows and columns by ± 1 factors, Q' is also a TU signing of Q_0. Equality $Q'(\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}) = S'$ follows from Lemma 43. $\square$

Definition 46. Suppose that B_l and B_r from Definition 38 have TU signings B_l' and B_r', respectively. Let B_l'' and B_r'' be the canonical re-signings (from Definition 44) of B_l' and B_r', respectively. Let A_l'', A_r'', D_l'', D_r'', and D_0'' be blocks of B_l'' and B_r'' analogous to blocks A_l, A_r, D_l, D_r, and D_0 of B_l and B_r. The canonical signing B'' of B is defined as

$$B'' = \begin{array}{|c|c|c|c|}\hline \multicolumn{3}{|c|}{A_l''} & 0 \\ & 1 \quad 1 & 0 & \\ \hline D_l'' & D_0'' & \begin{array}{c} 1 \\ 1 \end{array} & A_r'' \\ \hline D_{lr}'' & D_r'' & & \\ \hline \end{array} \qquad \text{where} \quad D_{lr}'' = D_r'' \cdot (D_0'')^{-1} \cdot D_l''.$$

Remark 47. In Definition 46, D_0'' is non-singular by construction, so D_{lr}'' and hence B'' are well-defined.

4.3 Properties of Canonical Signing

Lemma 48. B'' from Definition 46 is a signing of B.

Proof. By Lemma 45, B_l'' and B_r'' are TU signings of B_l and B_r, respectively. As a result, blocks A_l'', A_r'', D_l'', D_r'', and D_0'' in B'' are signings of the corresponding blocks in B. Thus, it remains to show that D_{lr}'' is a signing of D_{lr}. This can be verified via a direct calculation. $\square$ need details?

Lemma 49. Suppose that B_r from Definition 38 has a TU signing B_r'. Let B_r'' be the canonical re-signing (from Definition 44) of B_r'. Let $c_0'' = B_r''(X_r, y_0)$, $c_1'' = B_r''(X_r, y_1)$, and $c_2'' = c_0'' - c_1''$. Then the following statements hold.

1. For every $i \in X_r$, $\boxed{c_0''(i) \mid c_1''(i)} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\boxed{1 \mid -1}, \boxed{-1 \mid 1}\}$.
2. For every $i \in X_r$, $c_2''(i) \in \{0, \pm 1\}$.
3. $\boxed{c_0'' \mid c_2'' \mid A_r''}$ is TU.
4. $\boxed{c_1'' \mid c_2'' \mid A_r''}$ is TU.

5. $\begin{array}{|c|c|c|c|}\hline c_0'' & c_1'' & c_2'' & A_r'' \\ \hline\end{array}$ is TU.

Proof. Throughout the proof we use that B_r'' is TU, which holds by Lemma 45.

1. Since B_r'' is TU, all its entries are in $\{0, \pm 1\}$, and in particular $\begin{array}{|c|c|}\hline c_0''(i) & c_1''(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}}$. If $\begin{array}{|c|c|}\hline c_0'(i) & c_1''(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}$, then

   $$\det B_r''(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & -1 \\ \hline\end{array} = -2 \notin \{0, \pm 1\},$$

   which contradicts TUness of B_r''. Similarly, if $\begin{array}{|c|c|}\hline c_0''(i) & c_1''(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}$, then

   $$\det B_r''(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{array}{|c|c|}\hline 1 & 1 \\ \hline -1 & 1 \\ \hline\end{array} = 2 \notin \{0, \pm 1\},$$

   which contradicts TUness of B_r''. Thus, the desired statement holds.

2. Follows from item 1 and a direct calculation.

3. Performing a short tableau pivot in B_r'' on (x_2, y_0) yields:

   $$B_r'' = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0'' & c_1'' & A_r'' \\ \hline\end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -c_0'' & c_1'' - c_0'' & A_r'' \\ \hline\end{array}$$

   The resulting matrix can be transformed into $\begin{array}{|c|c|c|}\hline c_0'' & c_2'' & A_r'' \\ \hline\end{array}$ by removing row x_2 and multiplying columns y_0 and y_1 by -1. Since B_r'' is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, we conclude that $\begin{array}{|c|c|c|}\hline c_0'' & c_2'' & A_r'' \\ \hline\end{array}$ is TU.

4. Similar to item 4, performing a short tableau pivot in B_r'' on (x_2, y_1) yields:

   $$B_r'' = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0'' & c_1'' & A_r'' \\ \hline\end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0'' - c_1'' & -c_1'' & A_r'' \\ \hline\end{array}$$

   The resulting matrix can be transformed into $\begin{array}{|c|c|c|}\hline c_1'' & c_2'' & A_r'' \\ \hline\end{array}$ by removing row x_2, multiplying column y_1 by -1, and swapping the order of columns y_0 and y_1. Since B_r'' is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, and re-ordering columns, we conclude that $\begin{array}{|c|c|c|}\hline c_1'' & c_2'' & A_r'' \\ \hline\end{array}$ is TU.

5. Let V be a square submatrix of $\begin{array}{|c|c|c|c|}\hline c_0'' & c_1'' & c_2'' & A_r'' \\ \hline\end{array}$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

   Suppose that column c_2'' is not in V. Then V is a submatrix of B_r'', which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

   Suppose that columns c_0'' and c_1'' are both in V. Then V contains columns c_0'', c_1'', and $c_2'' = c_0'' - c_1''$, which are linearly. Thus, $\det V = 0$. Going forward we assume that at least one of the columns c_0'' and c_1'' is not in V.

   Suppose that column c_1'' is not in V. Then V is a submatrix of $\begin{array}{|c|c|c|}\hline c_0'' & c_2'' & A_r'' \\ \hline\end{array}$, which is TU by item 3. Thus, $\det V \in \{0, \pm 1\}$. Similarly, if column c_0'' is not in V, then V is a submatrix of $\begin{array}{|c|c|c|}\hline c_1'' & c_2'' & A_r'' \\ \hline\end{array}$, which is TU by item 4. Thus, $\det V \in \{0, \pm 1\}$.

$\square$

Lemma 50. Suppose that B_l from Definition 38 has a TU signing B_l'. Let B_l'' be the canonical re-signing (from Definition 44) of B_l'. Let $d_0'' = B_l''(x_0, Y_l)$, $d_1'' = B_l''(x_1, Y_l)$, and $d_2'' = d_0'' - d_1''$. Then the following statements hold.

1. For every $j \in Y_l$, $\begin{array}{|c|}\hline d_0''(i) \\ \hline d_1''(j) \\ \hline \end{array} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline \end{array}, \begin{array}{|c|}\hline -1 \\ \hline 1 \\ \hline \end{array} \right\}$.
2. For every $j \in Y_l$, $d_2''(j) \in \{0, \pm 1\}$.
3. $\begin{array}{|c|}\hline A_l'' \\ \hline d_0'' \\ \hline d_2'' \\ \hline \end{array}$ is TU.
4. $\begin{array}{|c|}\hline A_l'' \\ \hline d_1'' \\ \hline d_2'' \\ \hline \end{array}$ is TU.
5. $\begin{array}{|c|}\hline A_l'' \\ \hline d_0'' \\ \hline d_1'' \\ \hline d_2'' \\ \hline \end{array}$ is TU.

Proof. Apply Lemma 49 to $B_l^\top$, or repeat the same arguments up to transposition. □

Lemma 51. Let B'' be from Definition 46. Let $c_0'' = B''(X_r, y_0)$, $c_1'' = B''(X_r, y_1)$, and $c_2'' = c_0'' - c_1''$. Similarly, let $d_0'' = B''(x_0, Y_l)$, $d_1'' = B''(x_1, Y_l)$, and $d_2'' = d_0'' - d_1''$. Then the following statements hold.

1. For every $i \in X_r$, $c_2''(i) \in \{0, \pm 1\}$.
2. If $D_0'' = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}$, then $D'' = c_0'' \otimes d_0'' - c_1'' \otimes d_1''$. If $D_0'' = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$, then $D'' = c_0'' \otimes d_0'' - c_0'' \otimes d_1'' + c_1'' \otimes d_1''$.
3. For every $j \in Y_l$, $D''(X_r, j) \in \{0, \pm c_0'', \pm c_1'', \pm c_2''\}$.
4. For every $i \in X_r$, $D''(i, Y_l) \in \{0, \pm d_0'', \pm d_1'', \pm d_2''\}$.
5. $\begin{array}{|c|c|}\hline D'' & A_r'' \\ \hline \end{array}$ is TU.
6. $\begin{array}{|c|}\hline A_l'' \\ \hline D'' \\ \hline \end{array}$ is TU.

Proof. 1. Holds by Lemma 49.2.

2. Note that

$$\begin{array}{|c|}\hline D_l'' \\ \hline D_{lr}'' \\ \hline \end{array} = \begin{array}{|c|}\hline D_0'' \\ \hline D_r'' \\ \hline \end{array} \cdot (D_0'')^{-1} \cdot D_l'', \quad \begin{array}{|c|}\hline D_0'' \\ \hline D_r'' \\ \hline \end{array} = \begin{array}{|c|}\hline D_0'' \\ \hline D_r'' \\ \hline \end{array} \cdot (D_0'')^{-1} \cdot D_0'', \quad \begin{array}{|c|}\hline D_0'' \\ \hline D_r'' \\ \hline \end{array} = \begin{array}{|c|c|}\hline c_0'' & c_1'' \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline D_l'' & D_0'' \\ \hline \end{array} = \begin{array}{|c|}\hline d_0'' \\ \hline d_1'' \\ \hline \end{array}.$$

Thus,

$$D'' = \begin{array}{|c|c|}\hline D_l'' & D_0'' \\ \hline D_{lr}'' & D_r'' \\ \hline \end{array} = \begin{array}{|c|}\hline D_0'' \\ \hline D_r'' \\ \hline \end{array} \cdot (D_0'')^{-1} \cdot \begin{array}{|c|c|}\hline D_l'' & D_0'' \\ \hline \end{array} = \begin{array}{|c|c|}\hline c_0'' & c_1'' \\ \hline \end{array} \cdot (D_0'')^{-1} \cdot \begin{array}{|c|}\hline d_0'' \\ \hline d_1'' \\ \hline \end{array}.$$

Considering the two cases for D_0'' and performing the calculations yields the desired results.

3. Let $j \in Y_l$. By Lemma 50.1, $\begin{array}{|c|}\hline d_0''(i) \\ \hline d_1''(j) \\ \hline \end{array} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \left\{ \begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline \end{array}, \begin{array}{|c|}\hline -1 \\ \hline 1 \\ \hline \end{array} \right\}$. Consider two cases.

   (a) If $D_0'' = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}$, then by item 2 we have $D''(X_r, j) = d_0''(j) \cdot c_0'' + (-d_1''(j)) \cdot c_1''$. By considering all possible cases for $d_0''(j)$ and $d_1''(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c_0'', \pm c_1'', \pm (c_0'' - c_1'')\}$.

(b) If $D''_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}$, then by item 2 we have $D''(X_r, j) = (d''_0(j) - d''_1(j)) \cdot c''_0 + d''_1(j) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm(c''_0 - c''_1)\}$.

4. Let $i \in X_r$. By Lemma 49.1, $\begin{array}{|c|c|}\hline c''_0(i) & c''_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}\}$. Consider two cases.

   (a) If $D''_0 = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline\end{array}$, then by item 2 we have $D''(i, Y_l) = c''_0(i) \cdot d''_0 + (-c''_1(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_l) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

   (b) If $D''_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}$, then by item 2 we have $D''(i, Y_l) = c''_0(i) \cdot d''_0 + (c''_1(i) - c''_0(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_l) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

5. By Lemma 49.5, $\begin{array}{|c|c|c|c|}\hline c''_0 & c''_1 & c''_2 & A''_r \\ \hline\end{array}$ is TU. Since TUness is preserved under adjoining zero columns, copies of existing columns, and multiplying columns by ± 1 factors, $\begin{array}{|c|c|c|c|c|}\hline 0 & \pm c''_0 & \pm c''_1 & \pm c''_2 & A''_r \\ \hline\end{array}$ is also TU. By item 3, $\begin{array}{|c|c|}\hline D'' & A''_r \\ \hline\end{array}$ is a submatrix of the latter matrix, hence it is also TU.

6. By Lemma 50.5, $\begin{array}{|c|}\hline A''_l \\ \hline d''_0 \\ \hline d''_1 \\ \hline d''_2 \\ \hline\end{array}$ is TU. Since TUness is preserved under adjoining zero rows, copies of existing rows, and multiplying rows by ± 1 factors, $\begin{array}{|c|}\hline A''_l \\ \hline 0 \\ \hline \pm d''_0 \\ \hline \pm d''_1 \\ \hline \pm d''_2 \\ \hline\end{array}$ is also TU. By item 4, $\begin{array}{|c|}\hline A''_l \\ \hline D'' \\ \hline\end{array}$ is a submatrix of the latter matrix, hence it is also TU.

□

4.4 Proof of Regularity

Definition 52. Let X_l, Y_l, X_r, Y_r be sets and let $c_0, c_1 \in \mathbb{Q}^{X_r}$ be column vectors such that for every $i \in X_r$ we have $c_0(i),\ c_1(i),\ c_0(i) - c_1(i) \in \{0, \pm 1\}$. Define $\mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$ to be the family of matrices of the form $\begin{array}{|c|c|}\hline A_l & 0 \\ \hline D & A_r \\ \hline\end{array}$ where $A_l \in \mathbb{Q}^{X_l \times Y_l}$, $A_r \in \mathbb{Q}^{X_r \times Y_r}$, and $D \in \mathbb{Q}^{X_r \times Y_l}$ are such that: (a) for every $j \in Y_r$, $D(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$, (b) $\begin{array}{|c|c|c|c|}\hline c_0 & c_1 & c_0 - c_1 & A_r \\ \hline\end{array}$ is TU, (c) $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$ is TU.

Lemma 53. Let B'' be from Definition 46. Then $B'' \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c''_0, c''_1)$ where $c''_0 = B''(X_r, y_0)$ and $c''_1 = B''(X_r, y_1)$.

Proof. Recall that $c''_0 - c''_1 \in \{0, \pm 1\}^{X_r}$ by Lemma 51.1, so $\mathcal{C}(X_l, Y_l, X_r, Y_r; c''_0, c''_1)$ is well-defined. To see that $B'' \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c''_0, c''_1)$, note that all properties from Definition 52 are satisfied: property (a) holds by Lemma 51.3, property (b) holds by Lemma 49.5, and property (c) holds by Lemma 51.6. □

Lemma 54. Let $C \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$ from Definition 52. Let $x \in X_l$ and $y \in Y_l$ be such that $A_l(x, y) \neq 0$, and let C' be the result of performing a short tableau pivot in C on (x, y). Then $C' \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$.

Proof. Our goal is to show that C' satisfies all properties from Definition 52. Let $C' = \begin{array}{|c|c|}\hline C'_{11} & C'_{12} \\ \hline C'_{21} & C'_{22} \\ \hline\end{array}$, and let $\begin{array}{|c|}\hline A'_l \\ \hline D' \\ \hline\end{array}$ be the result of performing a short tableau pivot on (x, y) in $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$. Observe the following.

- By Lemma 10, $C'_{11} = A'_l$, $C'_{12} = 0$, $C'_{21} = D'$, and $C'_{22} = A_r$.

- Since $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$ is TU by property (c) for C, all entries of A_l are in $\{0, \pm 1\}$.
- $A_l(x, y) \in \{\pm 1\}$, as $A_l(x, y) \in \{0, \pm 1\}$ by the above observation and $A_l(x, y) \neq 0$ by the assumption.
- Since $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$ is TU by property (c) for C and since pivoting preserves TUness, $\begin{array}{|c|}\hline A_l' \\ \hline D' \\ \hline\end{array}$ is also TU.

These observations immediately imply properties (b) and (c) for C'. Indeed, property (b) holds for C', since $C'_{22} = A_r$ and $\begin{array}{|c|c|c|c|}\hline c_0 & c_1 & c_0 - c_1 & A_r \\ \hline\end{array}$ is TU by property (b) for C. On the other hand, property (c) follows from $C'_{11} = A'_l$, $C'_{21} = D'$, and $\begin{array}{|c|}\hline A_l' \\ \hline D' \\ \hline\end{array}$ being TU. Thus, it only remains to show that C' satisfies property (a). Let $j \in Y_r$. Our goal is to prove that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Suppose $j = y$. By the pivot formula, $D'(X_r, y) = -\frac{D(X_r, y)}{A_l(x, y)}$. Since $D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property (a) for C and since $A_l(x, y) \in \{\pm 1\}$, we get $D'(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Now suppose $j \in Y_l \setminus \{y\}$. By the pivot formula, $D'(X_r, j) = D(X_r, j) - \frac{A_l(x,j)}{A_l(x,y)} \cdot D(X_r, y)$. Here $D(X_r, j)$, $D(X_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property (a) for C, and $A_l(x, j) \in \{0, \pm 1\}$ and $A_l(x, y) \in \{\pm 1\}$ by the prior observations. Perform an exhaustive case distinction on $D(X_r, j)$, $D(X_r, y)$, $A_l(x, j)$, and $A_l(x, y)$. In every case, we can show that either $\begin{array}{|c|c|}\hline A_l(x,y) & A_l(x,j) \\ \hline D(X_r,y) & D(X_r,j) \\ \hline\end{array}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$, or that $D'(X_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$, as desired. $\square$ need details?

Lemma 55. Let $C \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$ from Definition 52. Then C is TU.

Proof. By Lemma 4, it suffices to show that C is k-PU for every $k \in \mathbb{Z}_{\geq 1}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since properties (b) and (c) in Definition 52 imply that A_l, A_r, and D are TU, so all their entries of $C = \begin{array}{|c|c|}\hline A_l & 0 \\ \hline D & A_r \\ \hline\end{array}$ are in $\{0, \pm 1\}$, as desired.

Suppose that for some $k \in \mathbb{Z}_{\geq 1}$ we know that every $C' \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$ is k-PU. Our goal is to show that C is k-PU, i.e., that every $(k+1) \times (k+1)$ submatrix S of C has $\det V \in \{0, \pm 1\}$.

First, suppose that V has no rows in X_ℓ. Then V is a submatrix of $\begin{array}{|c|c|}\hline D & A_r \\ \hline\end{array}$, which is TU by property (b) in Definition 52, so $\det V \in \{0, \pm 1\}$. Thus, we may assume that S contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $V(x_\ell, y_\ell) \neq 0$. Indeed, if $V(x_\ell, y) = 0$ for all y, then $\det V = 0$ and we are done, and $V(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since C is 1-PU, all entries of V are in $\{0, \pm 1\}$, and hence $V(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 12, performing a short tableau pivot in V on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix S'' such that $|\det V| = |\det V''|$. Since V is a submatrix of C, matrix V'' is a submatrix of the matrix C' resulting from performing a short tableau pivot in C on the same entry (x_ℓ, y_ℓ). By Lemma 54, we have $C' \in \mathcal{C}(X_l, Y_l, X_r, Y_r; c_0, c_1)$. Thus, by the inductive hypothesis applied to V'' and C', we have $\det V'' \in \{0, \pm 1\}$. Since $|\det V| = |\det V''|$, we conclude that $\det V \in \{0, \pm 1\}$. $\square$

Lemma 56. B'' from Definition 46 is TU.

Proof. Combine the results of Lemmas 53 and 55. $\square$

Lemma 57. Let M be a 3-sum of regular matroids M_ℓ and M_r. Then M is also regular.

Proof. Let B, B_ℓ, and B_r be standard $\mathbb{Z}_2$ representation matrices from Definition 40. Since M_ℓ and M_r are regular, by Lemma 26, B_ℓ and B_r have TU signings. Then the canonical signing B'' from Definition 46 is a TU signing of B. Indeed, B'' is a signing of B by Lemma 48, and B'' is TU by Lemma 56. Thus, M is regular by Lemma 26. $\square$