Proof of Regularity of 1-, 2-, and 3-Sum of Matroids

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1 Equivalence of Definitions of Regularity

1.1 Support Matrices and Their Properties

Definition 1. Let F be a field. The support of matrix $A \in F^{X \times Y}$ is $A^{\#} \in \{0,1\}^{X \times Y}$ given by

$$\forall i \in X,\ \forall j \in Y,\ A^{\#}_{i,j} = \begin{cases} 0, & \text{if } A_{i,j} = 0, \\ 1, & \text{if } A_{i,j} \neq 0. \end{cases}$$

Definition 2. Let M be a matroid, let B be a base of M, and let $e \in E \setminus B$ be an element. The fundamental circuit $C(e, B)$ of e with respect to B is the unique circuit contained in $B \cup \{e\}$.

Lemma 3. Let M be a matroid and let $S \in F^{X \times Y}$ be a standard representation matrix of M over a field F. Then $\forall y \in Y$, the fundamental circuit of y w.r.t. X is $C(y, X) = \{y\} \cup \{x \in X \mid S(x, y) \neq 0\}$.

Proof. Let $y \in Y$. Our goal is to show that $C'(y, X) = \{y\} \cup \{x \in X \mid D(x, y) \neq 0\}$ is a fundamental circuit of y with respect to X.

- $C'(y, X) \subseteq X \cup \{y\}$ by construction.
- $C'(y, X)$ is dependent, since columns of $[I \mid S]$ indexed by elements of $C(y, X)$ are linearly dependent.
- If $C \subsetneq C'(y, X)$, then C is independent. To show this, let V be the set of columns of $[I \mid S]$ indexed by elements of C and consider two cases.
  1. Suppose that $y \notin C$. Then vectors in V are linearly independent (as columns of I). Thus, C is independent.
  2. Suppose $\exists x \in X \setminus C$ such that $S(x, y) \neq 0$. Then any nontrivial linear combination of vectors in V has a non-zero entry in row x. Thus, these vectors are linearly independent, so C is independent.

□

Lemma 4. Let M be a matroid and let $S \in F^{X \times Y}$ be a standard representation matrix of M over a field F. Then $\forall y \in Y$, column $S^{\#}(\bullet, y)$ is the characteristic vector of $C(y, X) \setminus \{y\}$.

Proof. This directly follows from Lemma 3. □

Lemma 5. Let A be a TU matrix.

1. If a matroid is represented by A, then it is also represented by $A^{\#}$.
2. If a matroid is represented by $A^{\#}$, then it is also represented by A.

Proof. See Lean implementation. □ need details?

1.2 Conversion from General to Standard Representation

Lemma 6. Let M be a matroid represented by a matrix $A \in \mathbb{Q}^{X \times Y}$ and let B be a base of M. Then there exists a matrix $S \in \mathbb{Q}^{B \times (Y \setminus B)}$ that is a standard representation matrix of M.

Proof. Let $C = \{A(\bullet, b) \mid b \in B\}$. Since B is a base of M, we can show that C is a basis in the column space $\text{span}\{A(\bullet, y) \mid y \in Y\}$. For every $y \in Y \setminus B$, let $S(\bullet, y)$ be the coordinates of $A(\bullet, y)$ in basis C. We can show that $[I \mid S]$ represents the same matroid as A, so S is a standard representation matrix of M. □ see details in implementation

Lemma 7. Let M be a matroid represented by a TU matrix $A \in \mathbb{Q}^{X \times Y}$ and let B be a base of M. Then there exists a matrix $S \in \mathbb{Q}^{B \times (Y \setminus B)}$ such that S is TU and S is a standard representation matrix of M.

Proof sketch. Apply the procedure described in the proof of Lemma 6 to A. This procedure can be represented as a sequence of elementary row operations, all of which preserve TUness. Dropping the identity matrix at the end also preserves TUness.

formalize

□

1.3 Proof of Equivalence

Definition 8. A matroid M is regular if $\exists A \in \mathbb{Q}^{X \times Y}$ such that A is TU and A represents M.

Definition 9. We say that $A' \in \mathbb{Q}^{X \times Y}$ is a TU signing of $A \in \mathbb{Z}_2^{X \times Y}$ if A' is TU and

$$\forall i \in X,\ \forall j \in Y,\ |A'_{i,j}| = A_{i,j}.$$

Lemma 10. Let M be a matroid given by a standard representation matrix $B \in \mathbb{Z}_2^{X \times Y}$. Then the following are equivalent.

1. M is regular.
2. B has a TU signing.

Proof.

$1 \Rightarrow 2$ Recall that X (the row set of B) is a base of M. By Lemma 7, A can be converted into a standard representation matrix $B' \in \mathbb{Q}^{X \times Y}$ of M such that B' is also TU. Since B' and B are both standard representations of M, by Lemma 4 the support matrices $(B')^{\#}$ and $B^{\#}$ are the same. Moreover, $B^{\#} = B$, since B has entries in $\mathbb{Z}_2$. Thus, B' is TU and $(B')^{\#} = B$, so B' is a TU signing of B.

$2 \Rightarrow 1$ Let $B' \in \mathbb{Q}^{X \times Y}$ be a TU signing of B. Then $A = [I \mid B']$ is TU, as it is obtained from B' by adjoining the identity matrix. Moreover, by Lemma 5, A represents the same matroid as $A^{\#} = [I \mid B]$, which is M. Thus, A is a TU matrix representing M, so M is regular.

□

2 Regularity of 1-Sum

Write up based on Lean implementation

3 Regularity of 2-Sum

3.1 Preliminaries: Properties of Matrices

Lemma 11. Let $k \in \mathbb{Z}_{\geq 1}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of pivoting in A on $A(x, y) \neq 0$ where $x, y \in \{1, \dots, k\}$. Then A' contains a submatrix A'' of size $(k-1) \times (k-1)$ with $|\det A''| = |\det A| / |A(x, y)|$.

Proof. Let $X = \{1, \ldots, k\} \setminus \{x\}$ and $Y = \{1, \ldots, k\} \setminus \{y\}$, and let $A'' = A'(X, Y)$. Since A'' does not contain the pivot row or the pivot column, $\forall (i, j) \in X \times Y$ we have $A''(i, j) = A(i, j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}$. For $\forall j \in Y$, let B_j be the matrix obtained from A by removing row x and column j, and let B''_j be the matrix obtained from A'' by replacing column j with $A(X, y)$ (i.e., the pivot column without the pivot element). The cofactor expansion along row x in A yields

$$\det A = \sum_{j=1}^{k} (-1)^{y+j} \cdot A(x, j) \cdot \det B_j.$$

By reordering columns of every B_j to match their order in B''_j, we get

$$\det A = (-1)^{x+y} \cdot \left(A(x, y) \cdot \det A' - \sum_{j \in Y} A(x, j) \cdot \det B''_j \right).$$

By linearity of the determinant applied to $\det A''$, we have

$$\det A'' = \det A' - \sum_{j \in Y} \frac{A(x, j)}{A(x, y)} \cdot \det B''_j$$

Therefore, $|\det A''| = |\det A| / |A(x, y)|$. ☐

Lemma 12. Let $k \in \mathbb{Z}_{\geq 1}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of pivoting in A on $A(x, y) \in \{\pm 1\}$ where $x, y \in \{1, \ldots, k\}$. Then A' contains a submatrix A'' of size $(k-1) \times (k-1)$ with $|\det A''| = |\det A|$.

Proof. Apply Lemma 11 to A and use that $A(x, y) \in \{\pm 1\}$. ☐

3.2 Partial Unimodularity

Definition 13. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-partially unimodular, k-PU for short, if every $k \times k$ submatrix T of A has $\det T \in \{0, \pm 1\}$.

Lemma 14. A matrix A is TU if and only if A is k-PU for every $k \in \mathbb{Z}_{\geq 1}$.

Proof. This follows from the definitions of TUness and k-PUness. ☐

3.3 Matrix TUness is Closed Under 2-sum

Definition 15. Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{(X_\ell \cup \{x\}) \times Y_\ell}$ and $B_r \in R^{X_r \times (Y_r \cup \{y\})}$ be matrices of the form

$$B_\ell = \begin{array}{|c|} \hline A_\ell \\ \hline r \\ \hline \end{array}, \quad B_r = \begin{array}{|c|c|} \hline c & A_r \\ \hline \end{array}.$$

Let $D = c \cdot r$ be the outer product of c and r. Then the 2-sum of B_ℓ and B_r is

$$B = B_\ell \oplus_{2,x,y} B_r = \begin{array}{|c|c|} \hline A_\ell & 0 \\ \hline D & A_r \\ \hline \end{array}.$$

Here $A_\ell \in R^{X_\ell \times Y_\ell}$, $A_r \in R^{X_r \times Y_r}$, $r \in R^{Y_\ell}$, $c \in R^{X_r}$, $D \in R^{X_r \times Y_\ell}$, and the indexing is consistent everywhere.

Lemma 16. Let B_ℓ and B_r from Definition 15 be TU matrices (over $\mathbb{Q}$). Then $C = \begin{array}{|c|c|} \hline D & A_r \\ \hline \end{array}$ is TU.

Proof. Since B_ℓ is TU, all its entries are in $\{0, \pm 1\}$. In particular, r is a $\{0, \pm 1\}$ vector. Therefore, every column of D is a copy of y, $-y$, or the zero column. Thus, C can be obtained from B_r by adjoining zero columns, duplicating the y column, and multiplying some columns by -1. Since all these operations preserve TUess and since B_r is TU, C is also TU. ☐

Lemma 17. Let B_ℓ and B_r be matrices from Definition 15. Let B'_ℓ and B' be the matrices obtained by pivoting on entry $(x_\ell, y_\ell) \in X_\ell \times Y_\ell$ in B_ℓ and B, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B_r$.

Proof. Let

$$B'_\ell = \begin{array}{|c|}\hline A'_\ell \\ \hline r' \\ \hline\end{array}, \quad B' = \begin{array}{|c|c|}\hline B'_{11} & B'_{12} \\ \hline B'_{21} & B'_{22} \\ \hline\end{array}$$

where the blocks have the same dimensions as in B_ℓ and B, respectively. Since A_ℓ is a submatrix of B, we have $B'_{11} = A'_\ell$. A direct calculation shows that $B'_{12} = 0$ and $B'_{21} = A_r$ (they remain unchanged because of the 0 block in B). Finally, $B'_{21} = c \cdot r'$ is also verified via a direct calculation. Thus, $B' = B'_\ell \oplus_{2,x,y} B_r$. $\square$

Lemma 18. Let B_ℓ and B_r from Definition 15 be TU matrices (over $\mathbb{Q}$). Then $B_\ell \oplus_{2,x,y} B_r$ is TU.

Proof. By Lemma 14, it suffices to show that $B_\ell \oplus_{2,x,y} B_r$ is k-PU for every $k \in \mathbb{Z}_{\geq 1}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since all entries of $B_\ell \oplus_{2,x,y} B_r$ are in $\{0, \pm 1\}$ by construction.

Suppose that for some $k \in \mathbb{Z}_{\geq 1}$ we know that for any TU matrices B'_ℓ and B'_r (from Definition 15) their 2-sum $B'_\ell \oplus_{2,x,y} B'_r$ is k-PU. Now, given TU matrices B_ℓ and B_r (from Definition 15), our goal is to show that $B = B_\ell \oplus_{2,x,y} B_r$ is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix T of B has $\det T \in \{0, \pm 1\}$.

First, suppose that T has no rows in X_ℓ. Then T is a submatrix of $\begin{array}{|c|c|}\hline D & A_r \\ \hline\end{array}$, which is TU by Lemma 16, so $\det T \in \{0, \pm 1\}$. Thus, we may assume that T contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $T(x_\ell, y_\ell) \neq 0$. Indeed, if $T(x_\ell, y) = 0$ for all y, then $\det T = 0$ and we are done, and $T(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since B is 1-PU, all entries of T are in $\{0, \pm 1\}$, and hence $T(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 12, pivoting in T on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix T'' such that $|\det T| = |\det T''|$. Since T is a submatrix of B, matrix T'' is a submatrix of the matrix B' resulting from pivoting in B on the same entry (x_ℓ, y_ℓ). By Lemma 17, we have $B' = B'_\ell \oplus_{2,x,y} B_r$ where B'_ℓ is the result of pivoting in B_ℓ on (x_ℓ, y_ℓ). Since TUness is preserved by pivoting and B_ℓ is TU, B'_ℓ is also TU. Thus, by the inductive hypothesis applied to T'' and $B'_\ell \oplus_{2,x,y} B_r$, we have $\det T'' \in \{0, \pm 1\}$. Since $|\det T| = |\det T''|$, we conclude that $\det T \in \{0, \pm 1\}$. $\square$

3.4 Matroid Regularity is Closed Under 2-sum

Definition 19. A matroid M is a 2-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B, B_ℓ, and B_r (for M, M_ℓ, and M_r, respectively) of the form give in Definition 15.

Lemma 20. Suppose a matroid M is a 2-sum of regular matroids M_ℓ and M_r. Then M is also regular.

Proof. Let B, B_ℓ, and B_r be standard $\mathbb{Z}_2$ representation matrices from Definition 19. Since M_ℓ and M_r are regular, by Lemma 10, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 18, and a direct calculation verifies that B' is a signing of B. Thus, M is regular by Lemma 10. $\square$

4 Regularity of 3-Sum

4.1 Definition of 3-Sum

Definition 21. Let $B_l^{(0)} \in \mathbb{Z}_2^{(X_l \cup \{x_0, x_1\}) \times (Y_l \cup \{y_2\})}$, $B_r^{(0)} \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0, y_1\})}$ be matrices of the form

$$B_l^{(0)} = \begin{array}{|c|c|c|c|}\hline \multicolumn{3}{|c|}{A_l^{(0)}} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_l^{(0)} & \multicolumn{2}{c|}{D_0^{(0)}} & 1 \\ \cline{4-4} & \multicolumn{2}{c|}{} & 1 \\ \hline\end{array}, \quad B_r^{(0)} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0^{(0)}} & 1 & \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 & A_r^{(0)} \\ \cline{1-3} \multicolumn{2}{|c|}{D_r^{(0)}} & \multicolumn{2}{c|}{} \\ \hline\end{array},$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{lr}^{(0)} = D_r^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_l^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_l^{(0)}$ and $B_r^{(0)}$ is

$$B^{(0)} = B_l^{(0)} \oplus_3 B_r^{(0)} = \left[\begin{array}{cc|c|c} \multicolumn{2}{c}{A_l^{(0)}} & & 0 \\ & 1 \;\; 1 & 0 & \\ \hline D_l^{(0)} & D_0^{(0)} & \begin{array}{c} 1 \\ 1 \end{array} & A_r^{(0)} \\ D_{lr}^{(0)} & D_r^{(0)} & & \end{array}\right] \in \mathbb{Z}_2^{(X_l \cup X_r) \times (Y_l \cup Y_r)}.$$

Here $x_2 \in X_l$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_l$, $y_2 \in Y_r$, $A_l^{(0)} \in \mathbb{Z}_2^{X_l \times Y_l}$, $A_r^{(0)} \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_l^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_l \setminus \{y_0, y_1\})}$, $D_r^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{lr}^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times (Y_l \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_l^{(0)}$, $B_r^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{l,lr}^{(0)} = \begin{array}{|c|} \hline D_l^{(0)} \\ \hline D_{lr}^{(0)} \\ \hline \end{array}, \quad D_{0,r}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_r^{(0)} \\ \hline \end{array}, \quad D_{l,0}^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{lr,r}^{(0)} = \begin{array}{|c|c|} \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$. can omit

Lemma 22. Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

Proof. Verify by complete enumeration. □ need details?

4.2 Construction of Canonical Signing

Definition 23. We call B_l and B_r canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively, if they have the form

$$B_l = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_l} & 0 \\ & 1 \;\; 1 & 0 \\ \hline D_l & D_0 & \begin{array}{c} 1 \\ 1 \end{array} \end{array}\right], \quad B_r = \left[\begin{array}{c|c|c} 1 \;\; 1 & 0 & 0 \\ \hline D_0 & \begin{array}{c} 1 \\ 1 \end{array} & A_r \\ \hline D_r & & \end{array}\right]$$

where every block in B_l and B_r is a signing of the corresponding block in $B_l^{(0)}$ and $B_r^{(0)}$, and D_0 is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings B_l and B_r, the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \left[\begin{array}{cc|c|c} \multicolumn{2}{c}{A_l} & & 0 \\ & 1 \;\; 1 & 0 & \\ \hline D_l & D_0 & \begin{array}{c} 1 \\ 1 \end{array} & A_r \\ D_{lr} & D_r & & \end{array}\right]$$

where $D_{lr} = D_r \cdot (D_0)^{-1} \cdot D_l$ (calculated over $\mathbb{Q}$).

The following lemma helps construct canonical signings from arbitrary initial TU signings.

Lemma 24. Let Q' be a TU signing of the matrix

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{\multirow{2}{*}{$D_0^{(0)}$}} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0,x_1,x_2)\times(y_0,y_1,y_2)}$$

where $D_0^{(0)}(x_0,y_0)=1$, $D_0^{(0)}(x_0,y_1)\in\{0,1\}$, $D_0^{(0)}(x_1,y_0)=0$, and $D_0^{(0)}(x_1,y_1)=1$. Define $u\in\{0,\pm1\}^{\{x_0,x_1,x_2\}}$, $v\in\{0,\pm1\}^{\{y_0,y_1,y_2\}}$, and Q as follows:

$$\begin{gathered}
u(x_0) = Q'(x_2,y_0)\cdot Q'(x_0,y_0),\\
u(x_1) = Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2)\cdot Q'(x_1,y_2),\\
u(x_2) = 1,\\
v(y_0) = Q'(x_2,y_0),\\
v(y_1) = Q'(x_2,y_1),\\
v(y_2) = Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2),\\
\forall i\in\{x_0,x_1,x_2\},\ \forall j\in\{y_0,y_1,y_2\},\quad Q(i,j)=Q'(i,j)\cdot u(i)\cdot v(j).
\end{gathered}$$

Then Q is a TU signing of T and $Q = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{\multirow{2}{*}{D_0}} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array}$ where D_0 is the respective canonical signing of $D_0^{(0)}$.

Proof. Since Q' is a TU signing of T and Q is obtained from Q' by multiplying rows and columns by ±1 factors, Q is also a TU signing of T. By construction, we have

$$\begin{aligned}
&Q(x_2,y_0) = Q'(x_2,y_0)\cdot 1\cdot Q'(x_2,y_0) = 1,\\
&Q(x_2,y_1) = Q'(x_2,y_1)\cdot 1\cdot Q'(x_2,y_1) = 1,\\
&Q(x_2,y_2) = 0,\\
&Q(x_0,y_0) = Q'(x_0,y_0)\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0))\cdot Q'(x_2,y_0) = 1,\\
&Q(x_0,y_1) = Q'(x_0,y_1)\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0))\cdot Q'(x_2,y_1),\\
&Q(x_0,y_2) = Q'(x_0,y_2)\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0))\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2)) = 1,\\
&Q(x_1,y_0) = 0,\\
&Q(x_1,y_1) = Q'(x_1,y_1)\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2)\cdot Q'(x_1,y_2))\cdot (Q'(x_2,y_1)),\\
&Q(x_1,y_2) = Q'(x_1,y_2)\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2)\cdot Q'(x_1,y_2))\cdot (Q'(x_2,y_0)\cdot Q'(x_0,y_0)\cdot Q'(x_0,y_2)) = 1.
\end{aligned}$$

Thus, it remains to check that $Q(x_0,y_1)$ and $Q(x_1,y_1)$ are correct.

First, consider the entry $Q(x_0,y_1)$. If $D_0^{(0)}(x_0,y_1)=0$, then $Q(x_0,y_1)=0$, as needed. Otherwise, if $D_0^{(0)}(x_0,y_1)=1$, then $Q(x_0,y_1)\in\{\pm1\}$, as Q is a signing of T. Our goal is to show that $Q(x_0,y_1)=1$. For the sake of deriving a contradiction suppose that $Q(x_0,y_1)=-1$. Then the determinant of the submatrix of Q indexed by $\{x_0,x_2\}\times\{y_0,y_1\}$ is

$$\det \begin{array}{|c|c|}\hline 1 & -1 \\ \hline 1 & 1 \\ \hline\end{array} = 2 \notin \{0,\pm1\},$$

which contradicts TUness of Q. Thus, $Q(x_0,y_1)=1$, as needed.

Consider the entry $Q(x_1,y_1)$. Since Q is a signing of T, we have $Q(x_1,y_1)\in\{\pm1\}$. Note that we know all the other entries of Q, so we can determine the sign of $Q(x_1,y_1)$ using TUness of Q. Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$. If $Q(x_1,y_1)=1$, then $\det Q = \det \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline\end{array} = -2 \notin \{0,\pm1\}$, which contradicts TUness of Q. Thus, $Q(x_1,y_1)=-1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q(x_1, y_1) = -1$, then $\det Q_{\{x_0,x_1\},\{y_1,y_2\}} = \det \begin{array}{|c|c|}\hline 1 & 1 \\ \hline -1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q. Thus, $Q(x_1, y_1) = 1$, as needed.

$\square$

Definition 25. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q' \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q^{(0)} \in \mathbb{Z}_2^{X \times Y}$. Let $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q be constructed as follows:

$$u(i) = \begin{cases} Q'(x_2, y_0) \cdot Q'(x_0, y_0), & i = x_0, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}$$

$$v(j) = \begin{cases} Q'(x_2, y_0), & j = y_0, \\ Q'(x_2, y_1), & j = y_1, \\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}$$

$$\forall i \in X,\ \forall j \in Y, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j).$$

We call Q a canonical resigning of Q'.

Lemma 26. Let B_l' be a TU signing of $B_l^{(0)}$. Let B_l be the canonical resigning (constructed following Definition 25) of B_l'. Then B_l is a canonical signing of $B_l^{(0)}$ (in the sense of Definition 23) and B_l is TU. Going forward, we refer to B_l as a TU canonical signing for short of $B_l^{(0)}$. A TU canonical signing B_r of $B_r^{(0)}$ is defined similarly (up to replacing subscripts 1 by 2).

Proof. This follows directly from Lemma 24. $\square$

4.3 Properties of Canonical Signing

Lemma 27. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = (D_{0,r})_{\bullet,y_0}$ and $c_1 = (D_{0,r})_{\bullet,y_1}$. Then the following matrices are TU:

$$B_r^{(a)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_0 & A_r \\ \hline \end{array}, \quad B_r^{(b)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_1 & A_r \\ \hline \end{array}.$$

Proof. Pivoting in B_r on (x_2, y_0) and (x_2, y_1) yields:

$$B_r = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_r \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -c_0 & c_1 - c_0 & A_r \\ \hline \end{array}$$

$$B_r = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_r \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 - c_1 & -c_1 & A_r \\ \hline \end{array}$$

By removing row x_2 from the resulting matrices and then multiplying columns y_0 and y_1 by $\{\pm 1\}$ factors, we obtain $B_r^{(a)}$ and $B_r^{(b)}$. By Lemma 26, B_r is TU. Since TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $B_r^{(a)}$ and $B_r^{(b)}$ are TU. $\square$

Lemma 28. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.

1. For every $i \in X_r$, we have $\boxed{c_0(i) \mid c_1(i)} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\boxed{1 \mid -1}, \boxed{-1 \mid 1}\}$.

2. $\boxed{A_r \mid c_0 \mid c_1 \mid c_2}$ is TU.

Proof. 1. Let $i \in X_r$. If $\boxed{c_0(i) \mid c_1(i)} = \boxed{1 \mid -1}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & -1 \\ \hline \end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r (which holds by Lemma 26). Similarly, if $\boxed{c_0(i) \mid c_1(i)} = \boxed{-1 \mid 1}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has $\det \begin{array}{|c|c|} \hline 1 & 1 \\ \hline -1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r.

2. Let V be a square submatrix of $\boxed{A_r \mid c_0 \mid c_1 \mid c_2}$. We will show that $\det V \in \{0, \pm 1\}$.

   Let z denote the index of the appended column c_2. Suppose that column z is not in V. Then V is a submatrix of B_r, which is TU by Lemma 26. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

   Suppose that columns y_0 and y_1 are both in V. Then V contains columns z, y_0, and y_1, which are linearly dependent by construction of c_2. Thus, $\det V = 0$. Going forward we assume that at most one of the columns y_0 and y_2 is in V.

   Suppose that column y_0 is in V. Then V is a submatrix of $B_r^{(b)}$ from Lemma 27, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $B_r^{(a)}$ from Lemma 27, and so $\det V \in \{0, \pm 1\}$.

   Thus, every square submatrix V of $\tilde{T}$ has $\det V \in \{0, \pm 1\}$, and hence $\tilde{T}$ is TU.

$\square$

Remark 29. Vectors c_0, c_1, and c_2 can be defined directly in terms of entries of B_r, e.g., c_2 consists of entries of B_r indexed by $(X_r \setminus \{x_2\}) \times \{y_0\}$.

Lemma 30. Let B_l be a TU canonical signing of $B_l^{(0)}$. Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$. Then the following properties hold.

1. For every $j \in Y_r$, we have $\begin{array}{|c|} \hline d_0(i) \\ \hline d_1(i) \\ \hline \end{array} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \{\begin{array}{|c|} \hline 1 \\ \hline -1 \\ \hline \end{array}, \begin{array}{|c|} \hline -1 \\ \hline 1 \\ \hline \end{array}\}$.

2. $\begin{array}{|c|} \hline A_l \\ \hline d_0 \\ \hline d_1 \\ \hline d_2 \\ \hline \end{array}$ is TU.

Proof. Apply Lemma 28 to $B_l^\top$, or repeat the same argument up to interchanging rows and columns. $\square$

Lemma 31. Let B_l and B_r be TU canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively.

- Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$.
- Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$.
- If $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.
- Let D be the bottom-left block in the canonical signing B of $B^{(0)}$ corresponding to B_l and B_r

Then the following properties hold.

1. $D = c_0 \cdot r_0 + c_1 \cdot r_1$.

2. Rows of D are in $\begin{array}{|c|} \hline \pm r_0 \\ \hline \pm r_1 \\ \hline \pm r_2 \\ \hline 0 \\ \hline \end{array}$.

3. Columns of D are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm c_2 & 0 \\ \hline\end{array}$.

4. $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU.

5. $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ is TU.

6. $\begin{array}{|c|}\hline A_l \\ \hline r_0 \\ \hline r_1 \\ \hline r_2 \\ \hline\end{array}$ is TU.

7. $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$ is TU.

8. $\begin{array}{|c|c|}\hline c_0 & c_1 \\ \hline\end{array}$ contains D_0 (the canonical signing of $D_0^{(0)}$) as a submatrix.

Proof. 1. Follows via a direct calculation. need details?

2. By item 1, for every $i \in X_r$ we have $D(i, \bullet) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$. By Lemma 28.1, we know that $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}\}$. Therefore, $D(i, \bullet)$ is equal to either 0, $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.

3. Holds by the same argument as item 2 up to interchanging rows and columns.

4. Holds by Lemma 28.2.

5. By item 3, columns of $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ are in $\begin{array}{|c|c|c|c|c|}\hline A_r & \pm c_0 & \pm c_1 & \pm c_2 & 0 \\ \hline\end{array}$. Since $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU and since adding zero columns and copies of columns multiplied by ± 1 factors preserves TUness, $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ is also TU.

6. By Lemma 30.2 (or by the same argument as item 4 up to interchanging rows and columns), $\begin{array}{|c|}\hline A_l \\ \hline d_0 \\ \hline d_1 \\ \hline d_2 \\ \hline\end{array}$ is TU. Since TUness is preserved under multiplication of rows by ± 1 and exchanging rows, $\begin{array}{|c|}\hline A_l \\ \hline r_0 \\ \hline r_1 \\ \hline r_2 \\ \hline\end{array}$ is also TU.

7. Holds by the same argument as item 5 up to interchanging rows and columns.

8. Holds by construction.

$\square$

4.4 Proof of Regularity

Definition 32. Let $A_l \in \mathbb{Q}^{X_l \cup Y_l}$, $A_r \in \mathbb{Q}^{X_r \cup Y_r}$, $c_0, c_1 \in \mathbb{Q}^{X_r}$, $r_0, r_1 \in \mathbb{Q}^{Y_l}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–8 from the statement of Lemma 31 are satisfied for A_l, A_r, c_0, c_1, r_0, r_1. Given $k \in \mathbb{Z}_{\geq 1}$, define Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, k)$ to mean "$C = \begin{array}{|c|c|}\hline A_l & 0 \\ \hline D & A_r \\ \hline\end{array}$ is k-TU".

Lemma 33. Assume the notation of Definition 32. Then Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, 1)$ holds.

Proof. Every entry of C is in one of four blocks: 0, A_l, D, A_r. By the assumptions of Definition 32, all of these blocks are TU. Thus, C is 1-TU. $\square$

Lemma 34. Assume the notation of Definition 32. Let $i \in X_l$, let $T = \begin{array}{|c|}\hline A_l(i, \bullet) \\ \hline D \\ \hline\end{array}$. Suppose we pivot on entry $T(i, j) \in \{\pm 1\}$ in T and obtain matrix $T' = \begin{array}{|c|}\hline a' \\ \hline D' \\ \hline\end{array}$. Then columns of D' are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$.

Proof. Since T is a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$, which is TU by assumptions of Definition 32, we have that T is TU. Since pivoting preserves TUness, T' is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column p in T could be and what another column q in T could be. This uniquely determines the resulting columns p' and q' in T' by the pivot formula. In every case, either $\begin{array}{|c|c|}\hline p' & q' \\ \hline\end{array}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of T', or the restriction of p' and q' to X_r is in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$). ◻ need details?

Lemma 35. Assume the notation of Definition 32. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose Proposition$(A_l', A_r, c_0, c_1, r_0', r_1', k-1)$ holds for all A_l', r_0', and r_1' satisfying the assumptions of Definition 32 (together with A_r, c_0, and c_1). Then Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, k)$ holds.

Proof. Let V be a $k \times k$ submatrix of C. For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that V is a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline A_l & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline D & A_r \\ \hline\end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_r \\ \hline\end{array}$. Since all of those four matrices are TU by the assumptions of Definition 32, we have $\det V \in \{0, \pm 1\}$. Thus, V shares at least one row and one column index with A_l and A_r each.

Consider the row index shared by V and A_l. Note that this row in V cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a ± 1 entry shared by V and A_l. Let i and j denote the row and the column index of this entry, respectively.

Perform a pivot in C on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, B' denotes the entire matrix after the pivot. Note that the following statements hold.

- C' contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix V' of C' with the same row and column index sets as V minus the pivot row i and pivot column j.
- $C' = \begin{array}{|c|c|}\hline A_l' & 0 \\ \hline D' & A_r \\ \hline\end{array}$, i.e., the 0 and the A_r blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.
- $\begin{array}{|c|}\hline A_l' \\ \hline D' \\ \hline\end{array}$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.
- The columns of D' are in $\begin{array}{|c|c|c|c|}\hline 0 & \pm c_0 & \pm c_1 & \pm(c_0 - c_1) \\ \hline\end{array}$. This holds by Lemma 34.
- There exist r_0' and r_1' such that $D' = c_0 \cdot r_0' + c_1 \cdot r_1'$ and the assumptions of Definition 32 are satisfied for A_l', A_r, c_0, c_1, r_0', r_1'. This follows from the previous bullet point by carefully checking assumptions. need details?
- C' is $(k-1)$-TU. This follows from the hypothesis: Proposition$(A_l', A_r, c_0, c_1, r_0', r_1', k-1)$ holds.

To sum up, after pivoting we obtain a matrix C' (which can be obtained in the manner of Definition 32) that is $(k-1)$-TU and contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. ◻

Lemma 36. Let B_l and B_r be TU canonical signings. Then the corresponding canonical signing B is TU.

Proof. Define A_l, A_r, c_0, c_1, r_0, r_1 as in Lemma 31. Note that canonical signing B has the form of C in the notation of Definition 32.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: Proposition$(A_l', A_r, c_0, c_1, r_0', r_1', k)$ holds for all A_l', r_0', and r_1' satisfying the assumptions of Definition 32.

Base: The Proposition holds for $k = 1$ by Lemma 33.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 35.

Conclusion: Proposition$(A_l', A_r, c_0, c_1, r_0', r_1', k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$.

Specializing the conclusion to A_l, A_r, c_0, c_1, r_0, r_1 (obtained from B_l and B_r as described in the statement of Lemma 31) shows that canonical signing B is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, B is TU. ◻

Corollary 37. Suppose that $B_l^{(0)}$ and $B_r^{(0)}$ have TU signings. Then $B_l \oplus_3 B_r$ has a TU signing.

Proof sketch. Start with some TU signings, obtain canonical signings, apply Lemma 36. $\square$