Proof of Regularity of 2- and 3-Sum of Matroids

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1 2-Sum of Regular Matroids Is Regular

Lemma 1. *Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.*

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc}, the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic} a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column $a''_{\cdot k}$ with the pivot column $a_{\cdot c}$ without the pivot element a_{rc}.

By the cofactor expansion in A along row r, we have

$$\det A = \sum_{k=1}^{n} (-1)^{r+k} a_{rk} \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k. By swapping the order of columns in $B_{r,k}$ to match the form of B_k, we get

$$\det A = (-1)^{r+c} (a_{rc} \det A' - \sum_{k \in C} a_{rk} \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. $\square$

Corollary 1. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. $\square$

Definition 1. Let B_1, B_2 be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = [A_1 / x]$ and $B_2 = [y \mid A_2]$, where x is a row vector, y is a column vector, and A_1, A_2 are matrices of appropriate dimensions. Let D be the outer product of y and x. The 2-sum of B_1 and B_2 is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}.$$

Definition 2. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 1. Note that a matrix is TU if and only if it is k-TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 2. *Let B_1 and B_2 be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then B is 1-TU and 2-TU.*

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B. If V is a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_1 and A_2. Let i be the row shared by V and A_1 and j be the column shared by V and A_2. Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij}. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. □

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell < k$. Then for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also k-TU.*

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_1 and B_2 such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not k-TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row of V whose index appears in A_1. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_1. Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc}. For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, $\tilde{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\left[\tilde{A}_1/\tilde{D}\right]$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., A_2 remains unchanged. This holds because of the 0 block in B.
- $\tilde{D}$ consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\left[\tilde{D} \mid \tilde{A}_2\right]$ is TU, since this matrix consists of A_2 and copies of y scaled by factors $\{0, \pm 1\}$.
- $\tilde{D}$ can be represented as an outer product of a column vector $\tilde{y}$ and a row vector $\tilde{x}$, and we can define $\tilde{B}_1 = \left[\tilde{A}_1/\tilde{x}\right]$ and $\tilde{B}_2 = \left[\tilde{y} \mid \tilde{A}_2\right]$ similar to B_1 and B_2, respectively. Note that $\tilde{B}_1$ and $\tilde{B}_2$ have the same size as B_1 and B_2, respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- $\tilde{B}$ contains a square submatrix $\tilde{V}$ of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k-1) \times (k-1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B, the submatrix V'' corresponds to a submatrix $\tilde{V}$ of $\tilde{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\tilde{B}$ that represents a 2-sum of TU matrices $\tilde{B}_1$ and $\tilde{B}_2$ and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k - 1)$-TUness of $\tilde{B}$, which proves the lemma. □

Lemma 4. *Let B_1 and B_2 be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_1 and B_2, their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 2.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 3.

Conclusion: For any TU matrices B_1 and B_2, their 2-sum $B_1 \oplus_{2,x,y} B_2$ is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. □

2 3-Sum of Regular Matroids Is Regular

Definition 3. Let $B_1^{(0)} \in \mathbb{Z}_2^{(X_1 \cup \{x_0, x_1\}) \times (Y_1 \cup \{y_2\})}$, $B_2^{(0)} \in \mathbb{Z}_2^{(X_2 \cup \{x_2\}) \times (Y_2 \cup \{y_0, y_1\})}$ be matrices of the form

$$B_1^{(0)} = \begin{array}{|c|cc|c|} \hline \multicolumn{3}{|c|}{A_1^{(0)}} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_1^{(0)} & \multicolumn{2}{c|}{D_0^{(0)}} & 1 \\ \cline{4-4} & & & 1 \\ \hline \end{array}\ , \quad B_2^{(0)} = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0^{(0)}} & 1 & \\ \cline{3-3} & & 1 & A_2^{(0)} \\ \hline \multicolumn{2}{|c|}{D_2^{(0)}} & & \\ \hline \end{array}\ ,$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{12}^{(0)} = D_2^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_1^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_1^{(0)}$ and $B_2^{(0)}$ is

$$B^{(0)} = B_1^{(0)} \oplus_3 B_2^{(0)} = \begin{array}{|c|cc|c|c|} \hline \multicolumn{3}{|c|}{A_1^{(0)}} & \multicolumn{2}{c|}{0} \\ \cline{2-4} & 1 & 1 & 0 & \\ \hline D_1^{(0)} & \multicolumn{2}{c|}{D_0^{(0)}} & 1 & \\ \cline{4-4} & & & 1 & A_2^{(0)} \\ \hline D_{12}^{(0)} & \multicolumn{2}{c|}{D_2^{(0)}} & & \\ \hline \end{array} \in \mathbb{Z}_2^{(X_1 \cup X_2) \times (Y_1 \times Y_2)}.$$

Here $x_2 \in X_1$, $x_0, x_1 \in X_2$, $y_0, y_1 \in Y_1$, $y_2 \in Y_2$, $A_1^{(0)} \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2^{(0)} \in \mathbb{Z}_2^{X_2 \times Y_2}$, $D_1^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_1 \setminus \{y_0, y_1\})}$, $D_2^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{12}^{(0)} \in \mathbb{Z}_2^{(X_2 \setminus \{x_0, x_1\}) \times (Y_1 \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_1^{(0)}$, $B_2^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{1,12}^{(0)} = \begin{array}{|c|} \hline D_1^{(0)} \\ \hline D_{12}^{(0)} \\ \hline \end{array}, \quad D_{0,2}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_2^{(0)} \\ \hline \end{array}, \quad D_{1,0}^{(0)} = \begin{array}{|c|c|} \hline D_1^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{12,2}^{(0)} = \begin{array}{|c|c|} \hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_1^{(0)} & D_0^{(0)} \\ \hline D_{12}^{(0)} & D_2^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$.

Lemma 5. *Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)*

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

Proof. Verify by complete enumeration. □ need details?

Definition 4. We call B_1 and B_2 canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively, if they have the form

$$B_1 = \begin{array}{|c|cc|c|} \hline \multicolumn{3}{|c|}{A_1} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_1 & \multicolumn{2}{c|}{D_0} & 1 \\ \cline{4-4} & & & 1 \\ \hline \end{array}\ , \quad B_2 = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 & \\ \cline{3-3} & & 1 & A_2 \\ \hline \multicolumn{2}{|c|}{D_2} & & \\ \hline \end{array}$$

where every block in B_1 and B_2 is a signing of the corresponding block in $B_1^{(0)}$ and $B_2^{(0)}$, and D_0 is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings B_1 and B_2, the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \begin{array}{|c|c|c|c|}
\hline
\multicolumn{2}{|c}{A_1} & & 0 \\
 & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} & \begin{array}{|c|}\hline 0 \\ \hline\end{array} & \\
\hline
D_1 & D_0 & \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} & A_2 \\
\hline
D_{12} & D_2 & & \\
\hline
\end{array}$$

where $D_{12} = D_2 \cdot (D_0)^{-1} \cdot D_1$. (Note that $(D_0)^{-1}$ is over $\mathbb{Q}$.)

The following lemma helps construct canonical signings from arbitrary initial TU signings.

Lemma 6. *Let Q' be a TU signing of the matrix*

$$T = \begin{array}{|c|c|c|}
\hline
1 & 1 & 0 \\
\hline
\multicolumn{2}{|c|}{D_0^{(0)}} & 1 \\
\cline{3-3}
\multicolumn{2}{|c|}{} & 1 \\
\hline
\end{array} \in \mathbb{Z}_2^{(x_0,x_1,x_2)\times(y_0,y_1,y_2)}$$

where $D_0^{(0)}(x_0, x_0) = 1$, $D_0^{(0)}(x_0, x_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, x_0) = 0$, and $D_0^{(0)}(x_1, x_1) = 1$. Let $u \in \{0, \pm 1\}^{(x_0,x_1,x_2)}$, $v \in \{0, \pm 1\}^{(y_0,y_1,y_2)}$, and Q be the following vectors and matrices:

$$u(x_0) = Q'(x_2, y_0) \cdot Q'(x_0, y_0), \quad u(x_1) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), \quad u(x_2) = 1,$$
$$v(y_0) = Q'(x_2, y_0), \quad v(y_1) = Q'(x_2, y_1), \quad v(y_2) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2),$$
$$Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in \{x_0, x_1, x_2\},\ j \in \{y_0, y_1, y_2\}.$$

Then Q is a TU signing of T and $Q = \begin{array}{|c|c|c|}
\hline
1 & 1 & 0 \\
\hline
\multicolumn{2}{|c|}{D_0} & 1 \\
\cline{3-3}
\multicolumn{2}{|c|}{} & 1 \\
\hline
\end{array}$ *where D_0 is the respective canonical signing of $D_0^{(0)}$.*

Direct proof of Lemma 6. Since Q' is a TU signing of T and Q is obtained from Q' by multiplying rows and columns by ± 1 factors, Q is also a TU signing of T. By construction, we have

$Q(x_2, y_0) = Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1,$
$Q(x_2, y_1) = Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1,$
$Q(x_2, y_2) = 0,$
$Q(x_0, y_0) = Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1,$
$Q(x_0, y_1) = Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1),$
$Q(x_0, y_2) = Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1,$
$Q(x_1, y_0) = 0,$
$Q(x_1, y_1) = Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)),$
$Q(x_1, y_2) = Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1.$

Thus, it remains to check that $Q(x_0, y_1)$ and $Q(x_1, y_1)$ are correct.

First, consider the entry $Q(x_0, y_1)$. If $D_0^{(0)}(x_0, y_1) = 0$, then $Q(x_0, y_1) = 0$, as needed. Otherwise, if $D_0^{(0)}(x_0, y_1) = 1$, then $Q(x_0, y_1) \in \{\pm 1\}$, as Q is a signing of T. Our goal is to show that $Q(x_0, y_1) = 1$. For the sake of deriving a contradiction suppose that $Q(x_0, y_1) = -1$. Then the determinant of the submatrix of Q indexed by $\{x_0, x_2\} \times \{y_0, y_1\}$ is

$$\det \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q. Thus, $Q(x_0, y_1) = 1$, as needed.

Consider the entry $Q(x_1, y_1)$. Since Q is a signing of T, we have $Q(x_1, y_1) \in \{\pm 1\}$. Note that we know all the other entries of Q, so we can determine the sign of $Q(x_1, y_1)$ using TUness of Q. Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$. If $Q(x_1, y_1) = 1$, then $\det Q = \det \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline\end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q. Thus, $Q(x_1, y_1) = -1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}$. If $Q(x_1, y_1) = -1$, then $\det Q = \det \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & -1 & 1 \\ \hline\end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q. Thus, $Q(x_1, y_1) = 1$, as needed.

$\square$

Lemma 7. *Let B_1' and B_2' be TU signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively. Let B_1 and B_2 be obtained from B_1' and B_2', respectively, by multiplying their rows and columns by ± 1 factors from Lemma 6. Then B_1 and B_2 are TU canonical signings.*

Proof. This follows directly from Lemma 6. $\square$

Lemma 8. *Let B_2 be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = (D_{0,2})_{\cdot, y_0}$ and $c_1 = (D_{0,2})_{\cdot, y_1}$. Then the following matrices are TU:*

$$B_2^{(a)} = [c_0 - c_1 \mid c_0 \mid A_2], \quad B_2^{(b)} = [c_0 - c_1 \mid c_1 \mid A_2].$$

Proof. Recall that pivoting in matrix A on entry $a_{rc} \neq 0$ transforms the matrix as follows:

$$\begin{array}{|c|c|}\hline a_{rc} & a_{rj} \\ \hline a_{ic} & a_{ij} \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline \frac{1}{a_{rc}} & \frac{a_{rj}}{a_{rc}} \\ \hline -\frac{a_{ic}}{a_{rc}} & a_{ij} - \frac{a_{rj} a_{ic}}{a_{rc}} \\ \hline\end{array}$$

Pivoting in B_2 on (x_2, y_0) and (x_2, y_1) yields:

$$B_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_2 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -c_0 & c_1 - c_0 & A_2 \\ \hline\end{array}$$

$$B_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_2 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 - c_1 & -c_1 & A_2 \\ \hline\end{array}$$

By removing row x_2 from the resulting matrices and then multiplying columns y_0 and y_1 by $\{\pm 1\}$ factors, we obtain $B_2^{(a)}$ and $B_2^{(b)}$. Since B_2 is TU and TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $B_2^{(a)}$ and $B_2^{(b)}$ are TU. $\square$

Lemma 9. *Let B_2 be a TU canonical signing of $B_2^{(0)}$. Let $c_0 = D_{0,2}(\cdot, y_0)$, $c_1 = D_{0,2}(\cdot, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.*

1. *c_2 is a signing of $D_{0,2}(\cdot, y_0) + D_{0,2}(\cdot, y_1)$.*
2. *$[A_2 \mid c_0 \mid c_1 \mid c_2]$ is TU.*

Proof. 1. Let $r(i)$ denote a two-element row vector consisting of entries y_0 and y_1 in row i of B_2. Note that $r(x_2) = [1 \mid 1]$.

First, suppose that $r(i) = [1 \mid -1]$ or $r(i) = [-1 \mid 1]$. Then the matrix composed of rows i and x_2 has

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \quad \text{or} \quad \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2,$$

which is impossible as B_2 is a TU signing. Thus, these cases are not realized.

Now, suppose that $r(i)$ is one of the following: $[0 \mid 0]$, $[0 \mid 1]$, $[1 \mid 0]$, $[1 \mid 1]$, $[0 \mid -1]$, $[-1 \mid 0]$, $[-1 \mid -1]$. In all of these cases, a direct calculation shows that $c_2(i) \in \{0, \pm 1\}$ and $|c_2(i)| = D_{0,2}(\cdot, y_0) + D_{0,2}(\cdot, y_1)$.

2. Let V be a square submatrix of $[A_2 \mid c_0 \mid c_1 \mid c_2]$. We will show that $\det V \in \{0, \pm 1\}$.

   Let z denote the index of the appended column c_2. Suppose that column z is not in V. Then V is a submatrix of B_2, which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

   Suppose that columns y_0 and y_1 are both in V. Then V contains columns z, y_0, and y_1, which are linearly dependent by construction of c_2. Thus, $\det V = 0$. Going forward we assume that at most one of the columns y_0 and y_2 is in V.

   Suppose that column y_0 is in V. Then V is a submatrix of $B_2^{(b)}$ from Lemma 8, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $B_2^{(a)}$ from Lemma 8, and so $\det V \in \{0, \pm 1\}$.

   Thus, every square submatrix V of $\tilde{T}$ has $\det V \in \{0, \pm 1\}$, and hence $\tilde{T}$ is TU.

□

Remark 2. Vectors c_0, c_1, and c_2 can be defined directly in terms of entries of B_2, e.g., c_2 is composed of entries of B_2 indexed by $(X_2 \setminus \{x_2\}) \times \{y_0\}$.

Lemma 10. *Let B_1 be a TU canonical signing of $B_1^{(0)}$. Let $d_0 = D_{1,0}(x_0, \cdot)$, $d_1 = D_{1,0}(x_1, \cdot)$, and $d_2 = d_0 - d_1$. Then the following properties hold.*

1. *d_2 is a signing of $D_{1,0}(x_0, \cdot) + D_{1,0}(x_1, \cdot)$.*
2. *$[A_1 / d_0 / d_1 / d_2]$ is TU.*

Proof. Apply Lemma 9 to B_1^T. Transposing preserves TUness. □ need details?

Lemma 11. *Let B_1 and B_2 be TU canonical signings of $B_1^{(0)}$ and $B_2^{(0)}$, respectively.*

- *Let $c_0 = D_{0,2}(\cdot, y_0)$, $c_1 = D_{0,2}(\cdot, y_1)$, and $c_2 = c_0 - c_1$.*
- *Let $d_0 = D_{1,0}(x_0, \cdot)$, $d_1 = D_{1,0}(x_1, \cdot)$, and $d_2 = d_0 - d_1$.*
- *If $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.*
- *Let D be the bottom-left block in the canonical signing B of $B^{(0)}$ corresponding to B_1 and B_2*

Then the following properties hold.

1. *$D = c_0 \cdot r_0 + c_1 \cdot r_1$.*
2. *Rows of D are in $[\pm r_0 / \pm r_1 / \pm r_2 / 0]$.*
3. *Columns of D are in $[\pm c_0 \mid \pm c_1 \mid \pm c_2 \mid 0]$.*
4. *$[A_2 \mid c_0 \mid c_1 \mid c_2]$ is TU.*
5. *$[A_2 \mid D]$ is TU.*
6. *$[A_1 / r_0 / r_1 / r_2]$ is TU.*
7. *$[A_1 / D]$ is TU.*

Proof. 1. Follows via a direct calculation. need details?

2. In the proof of Lemma 9 we showed that rows of $[c_0 \mid c_1]$ can only be of the following forms: $[0 \mid 0]$, $[0 \mid 1]$, $[1 \mid 0]$, $[1 \mid 1]$, $[0 \mid -1]$, $[-1 \mid 0]$, $[-1 \mid -1]$. Therefore, every row of D is equal to either 0, $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.
3. Similar to the argument for rows, every column of D is in $[\pm d_0 / \pm d_1 / \pm d_2 / 0]$, which has the same columns as $[\pm r_0 / \pm r_1 / \pm r_2 / 0]$ (up to reindexing). need details?
4. Holds, because columns of $[A_2 \mid D]$ are in $[A_2 \mid c_0 \mid c_1 \mid c_2]$, which is TU.
5. Holds by Lemma 9.2.
6. Follows from Lemma 10.2, as TUness is preserved by multiplying rows by ± 1 factors (and changing row order).
7. Holds, because rows of $[A_1/D]$ are in $[A_1/r_0/r_1/r_2]$, which is TU.

$\square$

Definition 5. Let $A_1 \in \mathbb{Q}^{X_1 \cup Y_1}$, $A_2 \in \mathbb{Q}^{X_2 \cup Y_2}$, $c_0, c_1 \in \mathbb{Q}^{X_2}$, $r_0, r_1 \in \mathbb{Q}^{Y_1}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–7 from the statement of Lemma 11 are satisfied for A_1, A_2, c_0, c_1, r_0, r_1. Given $k \in \mathbb{Z}_{>1}$, refactor define $\mathrm{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, k)$ to mean "$C = \begin{array}{|c|c|}\hline A_1 & 0 \\ \hline D & A_2 \\ \hline \end{array}$ is k-TU".

Lemma 12. *Assume the notation of Definition 5. Then* $\mathrm{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, 1)$ *holds.*

Proof. Every entry of C is in one of four blocks: 0, A_1, D, A_2. By the assumptions of Definition 5, all of these blocks are TU. Thus, C is 1-TU. $\square$

Lemma 13. *Let $a \in \mathbb{Q}^{Y_1}$, $c_0, c_1 \in \mathbb{Q}^{X_2}$, and $D \in \mathbb{Q}^{X_2 \times Y_1}$. Suppose $[c_0 \mid c_1]$ contains a canonical signing of $D_0^{(0)}$ as a submatrix. Suppose that columns of D are in $[\pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1) \mid 0]$ and that $T = [a/D]$ is* refactor *TU. Suppose we pivot on entry $a_j \in \{\pm 1\}$ in T and obtain matrix $T' = [a' \mid D']$. Then columns of D' are in $[\pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1) \mid 0]$.*

Proof. Perform an exhaustive case distinction on what pivot column p in T could be and what another column q in T could be. After pivoting, T' must be TU. Every combination of p and q is either incompatible with TUness, or produces a column in T' with the desired property (i.e., its restriction to X_2 is in $[\pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1) \mid 0]$). This is tedious, but not difficult. $\square$ need details?

Lemma 14. *Assume the notation of Definition 5. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose* $\mathrm{Proposition}(A_1', A_2, c_0, c_1, r_0', r_1', k-1)$ *holds for all A_1', r_0', and r_1' satisfying the assumptions of Definition 5. Then* $\mathrm{Proposition}(A_1, A_2, c_0, c_1, r_0, r_1, k)$ *holds.*

Proof. Let V be a $k \times k$ submatrix of C. For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that V is a submatrix of $[A_1/D]$, $[A_1 \mid 0]$, $[D \mid A_2]$, or $[0/A_2]$. Since all of those four matrices are TU by the assumptions of Definition 5, we have $\det V \in \{0, \pm 1\}$. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row index shared by V and A_1. Note that this row in V cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a ± 1 entry shared by V and A_1. Let i and j denote the row and the column index of this entry, respectively.

Perform a pivot in C on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, B' denotes the entire matrix after the pivot. Note that the following statements hold.

- C' contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix V' of C' with the same row and column index sets as V minus the pivot row i and pivot column j.
- $C' = \begin{array}{|c|c|}\hline A_1' & 0 \\ \hline D' & A_2 \\ \hline \end{array}$, i.e., the 0 and the A_2 blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.

- $[A_1'/D']$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.
- The columns of D' are in $[0 \mid \pm c_0 \mid \pm c_1 \mid \pm(c_0 - c_1)]$. This holds by Lemma 13.
- There exist r_0' and r_1' such that $D' = c_0 \cdot r_0' + c_1 \cdot r_1'$ and the assumptions of Definition 5 are satisfied for A_1', A_2, c_0, c_1, r_0', r_1'. This follows from the previous bullet point by carefully checking assumptions. need details?
- C' is $(k-1)$-TU. This follows from the hypothesis: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k-1)$ holds.

To sum up, after pivoting we obtain a matrix C' (which can be obtained in the manner of Definition 5) that is $(k-1)$-TU and contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. □

Lemma 15. *Let B_1 and B_2 be TU canonical signings. Let $k \in \mathbb{Z}_{\geq 1}$. Then the corresponding canonical signing B is k-TU.*

Proof. Follows from Lemma 14. □ need details?

Lemma 16. *Let B_1 and B_2 be TU canonical signings. Then the corresponding canonical signing B is TU.*

Proof. Define A_1, A_2, c_0, c_1, r_0, r_1 as in Lemma 11. Note that canonical signing B has the form of C in the notation of Definition 5.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k)$ holds for all A_1', r_0', and r_1' satisfying the assumptions of Definition 5.

Base: The Proposition holds for $k = 1$ by Lemma 12.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 14.

Conclusion: Proposition$(A_1', A_2, c_0, c_1, r_0', r_1', k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$. In particular, canonical signing B is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, B is TU. □

Corollary 2. Suppose that $B_1^{(0)}$ and $B_2^{(0)}$ have TU signings. Then $B_1 \oplus_3 B_2$ has a TU signing.

Proof sketch. Start with some TU signings, obtain canonical signings, apply Lemma 16. □