Matroid Decomposition Theorem Verification

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0.1 Basic Definitions

0.1.1 Matroid Structure

Definition 1 (matroid).

todo: add definition

Definition 2 (isomorphism). Two matroids are isomorphic if they become equal upon a suitable relabeling of the elements.

Definition 3 (loop).

todo: add definition

Definition 4 (coloop).

todo: add definition

Definition 5 (parallel elements).

todo: add definition

Definition 6 (series elements).

todo: add definition

0.1.2 Matroid Classes

Definition 7 (binary matroid).

todo: add definition

Definition 8 (regular matroid). A binary matroid M is regular if some binary representation matrix B of M has a totally unimodular signing (i.e., assignment of signs to the 1s in B that results in a TU matrix).

Definition 9 (graphic matroid).

todo: add definition

Definition 10 (cographic matroid).

todo: add definition

Definition 11 (planar matroid).

todo: add definition

Definition 12 (dual matroid).

todo: add definition

Definition 13 (self-dual matroid).

todo: add definition

0.1.3 Specific Matroids (Constructions)

Definition 14 (wheel).

todo: add definition

Definition 15 (W_3).

todo: add definition

Definition 16 (W_4).

todo: add definition

Definition 17 (R_{10}).

todo: add definition

Definition 18 (R_{12}).

todo: add definition

Definition 19 (F_7).

todo: add definition

Definition 20 (F_7^*).

todo: add definition

Definition 21 ($M(K_{3,3})$).

todo: add definition

Definition 22 ($M(K_{3,3})^*$).

todo: add definition

Definition 23 ($M(K_5)$).

todo: add definition

Definition 24 ($M(K_5)^*$).

todo: add definition

0.1.4 Connectivity and Separation

Definition 25 (k-separation). See text after Proposition 3.3.18.

Definition 26 (k-connectivity). See text after Proposition 3.3.18.

0.1.5 Reductions

Definition 27 (deletion).

todo: add definition

Definition 28 (contraction).

todo: add definition

Definition 29 (minor).

todo: add definition

0.1.6 Extensions

Definition 30 (1-element addition).

add name, label, uses, text

Definition 31 (1-element expansion).

add name, label, uses, text

Definition 32 (1-element extension).

todo: add definition

Definition 33 (2-element extension).

todo: add definition

Definition 34 (3-element extension).

todo: add definition

0.1.7 Sums

Definition 35 (1-sum).

todo: add definition

Definition 36 (2-sum).

todo: add definition

Definition 37 (3-sum).

todo: add definition

Definition 38 (Δ-sum).

todo: add definition

Definition 39 (Y-sum).

todo: add definition

0.1.8 Total Unimodularity

Definition 40 (TU matrix). A real matrix A is totally unimodular if every square submatrix D of A has $\det_{\mathbb{R}} D = 0$ or ± 1.

0.1.9 Auxiliary Results

Theorem 41 (Menger's theorem). *A connected graph G is vertex k-connected if and only if every two nodes are connected by k internally node-disjoint paths. Equivalent is the following statement. G is vertex k-connected if and only if any $m \leq k$ nodes are joined to any $n \leq k$ nodes by k internally node-disjoint paths. One may demand that the m nodes are disjoint from the n nodes, but need not do so. Also, the k paths can be so chosen that each of the specified nodes is an endpoint of at least one of the paths.*

Definition 42 (ΔY exchange)**.**

add

Definition 43 (gap)**.**

add

0.2 Chapter 2 from Truemper

Lemma 44 (2.3.14)**.** *Let A be a matrix over a field $\mathcal{F}$, with $\mathcal{F}$-rank $A = k$. If both a row submatrix and a column submatrix of A have $\mathcal{F}$-rank equal to k, then they intersect in a submatrix of A with the same $\mathcal{F}$-rank. In particular, any k $\mathcal{F}$-independent rows of A and any k $\mathcal{F}$-independent columns of A intersect in a $k \times k$ $\mathcal{F}$-nonsingular submatrix of A.*

Proof sketch. Result of linear algebra. Uses the submodularity of the $\mathcal{F}$-rank function. □

0.3 Chapter 3 from Truemper

0.3.1 Chapter 3.2

Theorem 45 (3.2.25.a)**.** *Let M be the graphic matroid of a connected graph G. Assume (E_1, E_2) is a k-separation of M with minimal $k \geq 1$. Define G_1 (resp. G_2) from G by removing the edges of E_2 (resp. E_1) from G. Let $R_1, \dots, R_g$ be the connected components of G_1, and $S_1, \dots, S_h$ be those of G_2.*

If $k = 1$, then the R_i and S_j are connected in tree fashion.

Proof sketch. Count edges and nodes. □

Theorem 46 (3.2.25.b)**.** *Same setting as Theorem 3.2.25.a. If $k = 2$, then the R_i and S_j are connected in cycle fashion.*

Proof sketch. Count edges and nodes. □

Definition 47 (switching operation from section 3)**.** A swap of identification of nodes between two subgraphs induced by a 2-separation of a graph. See description and illustration on page 45.

Lemma 48 (3.2.48)**.** *The matroids $M(K_5)$ and $M(K_{3,3})$ are not graphic.*

Proof sketch. A short proof is given on page 51. A longer, but more general proof uses the graphicness testing subroutine described on page 47. □

0.3.2 Chapter 3.3

Lemma 49 (3.3.12)**.** *Let M be a binary matroid with a minor $\overline{M}$, and let $\overline{B}$ be a representation matrix of $\overline{M}$. Then M has a representation matrix B that displays $\overline{M}$ via $\overline{B}$ and thus makes the minor $\overline{M}$ visible.*

Proof sketch. Follows by the definition of minor via pivots and row/column deletions. □

Proposition 50 (3.3.17)**.** *Partitioned version of matrix B representing binary matroid M. (same as 3.3.3)*

Proposition 51 (3.3.18)**.** *If for some $k \geq 1$, $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k$,* GF(2)*-rank $D^1 +$* GF(2)*-rank $D^2 \leq k-1$, then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a (Tutte) k-separation of B and M. This separation is exact if the rank condition holds with equality. Both B and M are called (Tutte) k-separable if they have a k-separation. For $k \geq 2$, B and M are (Tutte) k-connected if they have no ℓ-separation for $1 \leq \ell < k$. When M is 2-connected, we also say that M is connected.*

Lemma 52 (3.3.19)**.** *Let M be a binary matroid with a representation matrix B. Then M is connected iff B is connected.*

Proof sketch. Check using (3.3.17) and (3.3.18) that B is connecte diff it is 2-connected. Thus M is 2-connected, and hence connected, iff B is connected. □

Lemma 53 (3.3.20)**.** *The following statements are equivalent for a binary matroid M with set E and a representation matrix B of M.*

- *M is 3-connected.*
- *B is connected, has no parallel or unit vector rows and columns, and has no partition as in (3.3.17) with* GF(2)*-rank $D^1 = 1$, $D^2 = 0$, and $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.*
- *Same as (ii), but $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 5$.*

Proof sketch.

- (i) is equivalent to (ii) by the definition of 3-connectivity.
- (iii) trivially implies (ii). (Typo in the book?)
- Assuming (ii), if the length of B^1 is 3 or 4, then B has a zero column or row, or parallel or unit vector rows or columns, which is excluded by the first part of (ii). Thus it suffices to require $|X_1 \cup Y_1| \geq 5$ and by duality $|X_2 \cup Y_2| \geq 5$.

□

Theorem 54 (census from Secion 3.3)**.** *A complete census of 3-connected binary matroids on ≤ 8 elements.*

Proof sketch. Verified by case enumeration. □

0.4 Chapter 4 from Truemper

Proposition 55 (4.4.5)**.** *ΔY exchange, case 1.*

Proposition 56 (4.4.6)**.** *ΔY exchange, case 2.*

0.5 Chapter 5 from Truemper

Lemma 57 (5.2.4)**.** *Let N be a connected minor of a connected binary matroid M. Let $z \in M \backslash N$. Then M has a connected minor N' that is a 1-element extension of N by z.*

Proof sketch.

- By Lemma 3.3.12, M has a representation matrix that displays N via a submatrix.

- Case distinction between z being represented by a nonzero or a zero vector.
- Nonzero case: immediately get submatrix representing N'.
- Zero case: take a shortest path in the matrix, perform pivots, in one subcase use duality.

$\square$

Proposition 58 (5.2.8). *Representation matrices for small wheels (from $M(W_1)$ to $M(W_4)$).*

Proposition 59 (5.2.9). *Representation matrix for $M(W_n)$, $n \geq 3$.*

Lemma 60 (5.2.10). *Let M be a binary matroid with a binary representation matrix B. Suppose the graph $BG(B)$ contains at least one cycle. Then M has an $M(W_2)$ minor.*

Proof sketch.

- $BG(B)$ is bipartite and has at least one cycle, so there is a cycle C without chords with at least 4 edges.
- Up to indices, the submatrix corresponding to C is either the matrix for $M(W_2)$ from (5.2.8) or the matrix for some $M(W_k)$, $k \geq 3$ from (5.2.9).
- In the latter case, use path shortening pivots on 1s to convert the submatrix to the former case.

$\square$

Lemma 61 (5.2.11). *Let M be a connected binary matroid with at least 4 elements. Then M has a 2-separation or an $M(W_3)$ minor.*

Proof sketch. Use Lemma 5.2.10 and apply path shortening technique. $\square$

Corollary 62 (5.2.15). *Every 3-connected binary matroid M with at least 6 elements has an $M(W_3)$ minor.*

Proof sketch. By Lemma 5.2.11, M has a 2-separation or an $M(W_3)$ minor. M is 3-connected, so the former case is impossible. $\square$

0.6 Chapter 6 from Truemper

0.6.1 Chapter 6.2

Goal of the chapter: separation algorithm for deciding if there exists a separation of a matroid induced by a separation of its minor.

Proposition 63 (6.2.1). *Partitioned version of matrix B^N representing a minor N of a binary matroid M, where N has an exact k-separation for some $k \geq 1$.*

Proposition 64 (6.2.3). *Matrix B for M displaying partitioned B^N*

Proposition 65 (6.2.5). *Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Lemma 66 (6.2.6). *Let N be a 3-connected binary matroid on at least 6 elements. Suppose a 1-, 2-, or 3-element binary extension of N, say M, has no loops, coloops, parallel elements, or series elements. Then M is 3-connected.*

Proof sketch.

- Let C be a binary representation matrix of M that displays a binary representation matrix B for N.
- By assumption, B is 3-connected.
- C is connected, as otherwise by case analysis C contains a zero vector or unit vector, so M has a loop, coloop, parallel elements, or series elements, a contradiction.
- If C is not 3-connected, then by Lemma 3.3.20 there is a 2-separation of C with at least 5 rows/columns on each side. Then B has a 2-separation with at least 2 rows/columns on each side, a contradiction.
- Thus, C is 3-connected, so M is 3-connected.

$\square$

0.6.2 Chapter 6.3

Definition 67 (6.3.2)**.** M is called minimal if it satisfies the following conditions.

- M has an N minor.
- M has no k-separation induced by the exact k-separation (F_1, F_2) of N.
- The matroid M is minimal with respect to the above conditions.

Definition 68 (6.3.3)**.** M is called minimal under isomorphism if it satisfies the following conditions.

- M has at least one N minor.
- Some k-separation of at least one such minor corresponding to the exact k-separation (F_1, F_2) of N under one of the isomorphisms fails to induce a k-separation of M.
- The matroid M is minimal with respect to the above conditions.

Proposition 69 (6.3.11)**.** *Matrix B for M with partitioned B^N, row $x \in X_3$, and column $y \in Y_3$.*

Proposition 70 (6.3.12)**.** *Partitioned version of B^N: $B^N = \mathrm{diag}(A^1, A^2)$.*

Definition 71 (separation algorithm)**.** Polynomial-time recursive procedure to search for an induced partition. Described on pages 132–133 and again on pages 137–138.

Proposition 72 (6.3.13)**.** *Special case where B of a minimal M contains just one row x beyond B^N. This proposition gives properties of row subvectors of row x by step 1 of the separation algorithm.*

Proposition 73 (6.3.14)**.** *Special case where B of a minimal M contains just one column y beyond B^N. This proposition gives properties of column subvectors of column y by step 1 of the separation algorithm.*

Lemma 74 (6.3.15)**.** *Treats the case where B has at least two additional rows or columns beyond those of B^N.*

Proof sketch. Argue about the structure of the matrix, applying steps 1 and 2 of the separation algorithm. □

Lemma 75 (6.3.16)**.** *Expands case (i) of Lemma 6.3.15.*

Proof sketch. Further arguments about the structure of the matrix. □

Lemma 76 (6.3.17)**.** *Expands case (ii) of Lemma 6.3.15.*

Proof sketch. Further arguments about the structure of the matrix. □

Theorem 77 (6.3.18)**.** *Structural description of representation matrix (6.3.11) of a minimal M. Contains cases (a), (b), and (c) with sub-cases (c.1) and (c.2).*

Proof sketch.

- (6.3.13) and (6.3.14) establish (a) and (b).
- Lemmas 6.3.15, 6.3.16, and 6.3.17 prove (c.1) and (c.2).

□

Lemma 78 (6.3.19)**.** *Additional structural statements for cases (c.1) and (c.2) of Theorem 6.3.18.*

Proof sketch. Reason about representation matrices using Theorem 6.3.18, Lemma 6.3.15, minimality, isomorphisms, pivots, and so on. □

Proposition 79 (6.3.21)**.** *Matrix B for M minimal under isomorphism, case (a).*

Proposition 80 (6.3.22)**.** *Matrix B for M minimal under isomorphism, case (b).*

Proposition 81 (6.3.23)**.** *Matrix $\overline{B}$ for minor $\overline{M}$ of M minimal under isomorphism.*

Theorem 82 (6.3.20)**.** *Let M be minimal under isomorphism. Then one of 3 cases holds for matrix representation of M.*

Proof sketch. Follows directly from Theorem 6.3.18 and Lemma 6.3.19. □

Corollary 83 (6.3.24)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Suppose N given by B^N of (6.3.12) is in $\mathcal{M}$, but the 1- and 2-element extensions of N given by (6.3.21), (6.3.22), (6.3.23), and by the accompanying conditions are not in $\mathcal{M}$. Assume matroid $M \in \mathcal{M}$ has an N minor. Then any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

Proof sketch.

- Let $M \in \mathcal{M}$ satisfying the assumptions. Since $\mathcal{M}$ is closed under isomorphism, suppose that N itself is a minor of M.
- Suppose the k-separation of N does not induce one in M. Then M or a minor of M containing N is minimal under isomorphism.
- By Theorem 6.3.20, M has a minor represented by (6.3.21), (6.3.22), or (6.3.23). This minor is in $\mathcal{M}$, as $\mathcal{M}$ is closed under taking minors, but this contradicts our assumptions.

□

0.6.3 Chapter 6.4

Theorem 84 (6.4.1)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N. Suppose N has at least 6 elements. Then M has a 3-connected minor N' that is a 1- or 2-element extension of some N minor of M. In the 2-element case, N' is derived from the N minor by one addition and one expansion.*

Proof sketch.

- Let $z \in M \setminus N$. By Lemma 5.2.4, there is a connected minor N' that is a 1-element extension of N by z. Our theorem holds iff it holds for duals, so by duality, assume that the extension is an addition.
- Reason about a matrix representation of N and N' to get a 2-separation of N'. Since M is 3-connected, this 2-separation does not induce one in M. Let M' be a minor of M that proves this fact and is minimal under isomorphism. Additionally, M' has an N' minor, so we change the element labels in M' so that N' is a minor of M'.
- Apply Theorem 6.3.20 and perform case analysis, reaching either a contradiction or a desired extension.

□

0.7 Chapter 7 from Truemper

0.7.1 Chapter 7.2

Definition 85 (splitter). Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If every $M \in \mathcal{M}$ with a proper N minor has a 2-separation, then N is called a splitter of $\mathcal{M}$.

Theorem 86 (7.2.1.a splitter for nonwheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If N is not a wheel, then N is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.*

Proof sketch.

- If N is a splitter of $\mathcal{M}$, then clearly $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.
- Prove the converse by contradiction. To this end, suppose that $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and that N is not a splitter of $\mathcal{M}$.
- Thus, $\mathcal{M}$ contains a 3-connected matroid M with a proper N minor and no 2-separation.
- Since $\mathcal{M}$ is closed under isomorphism, we may assume N itself to be that N minor.
- By Theorem 6.4.1 (applied to M and N), M has a 3-connected minor N' that is a 3-connected 1- or 2-element extension of an N minor.
- The 1-extension case has been ruled out.

- In the 2-element extension case, N' is derived from the N minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take N itself to be that N minor. Thus, N' is derived from N by one addition and one expansion.
- Let C be a binary matrix representing N' and displaying N. By investigating the structure of C, one can show that N' contains a 3-connected 1-element extension of an N minor, which has been ruled out.

□

Theorem 87 (7.2.1.b splitter for wheels)**.** *Let $\mathcal{M}$ be a class of binary matroids closed under isomorphism and under taking minors. Let N be a 3-connected minor of $\mathcal{M}$ on at least 6 elements. If N is a wheel, then N is a splitter of $\mathcal{M}$ iff $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and does not contain the next larger wheel.*

Proof sketch. Similar to proof of Theorem 7.2.1.a. The analysis of the matrix C can be done in one go for both cases. □

Corollary 88 (7.2.10.a)**.** *Theorem 7.2.1.a specialized to graphs.*

Proof sketch. Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Corollary 89 (7.2.10.b)**.** *Theorem 7.2.1.b specialized to graphs.*

Proof sketch. Consider the corresponding graphic matroids, apply splitter theorem, extensions in graphic matroids correspond to extensions in graphs. □

Theorem 90 (7.2.11.a)**.** *K_5 is a splitter of the graphs without $K_{3,3}$ minors.*

Proof sketch. Up to isomorphism, there is just one 3-connected 1-edge extension of K_5. To obtain it, one partitions one vertex of K_5 into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph has a $K_{3,3}$ minor. Thus, the theorem follows from Corollary 7.2.10.a. □

Theorem 91 (7.2.11.b)**.** *W_3 is a splitter of the graphs without W_4 minors.*

Proof sketch. There is no 3-connected 1-edge extension of W_3, so the theorem follows from Corollary 7.2.10.b. □

0.7.2 Chapter 7.3

Theorem 92 (7.3.1.a)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. Assume N is not a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where M_0 is isomorphic to N and where the gap is 1.*

Proof sketch.

- Inductively for $i \geq 0$ assume the existence of a sequence $M_0, \ldots, M_i$ of 3-connected minors where M_0 is isomorphic to N, M_i is not a wheel, and the gap is 1.
- If $M_i = M$, we are done, so assume that M_i is a proper minor of M.
- Use the contrapositive of the splitter Theorem 7.2.1.a to find a larger sequence.

  - Let $\mathcal{M}$ be the collection of all matroids isomorphic to a (not necessarily proper) minor of M.
  - Since M_i is a 3-connected proper minor of the 3-conected $M \in \mathcal{M}$, it cannot be a splitter of $\mathcal{M}$. By Theorem 7.2.1.a, $\mathcal{M}$ contains a matroid M_{i+1} that is a 3-connected 1-element extension of a matroid isomorphic to M_i.
  - Since every 1-element reduction of a wheel with at least 6 elements is 2-separable, M_{i+1} is not a wheel, as otherwise M_i is 2-separable, which is a contradiction.

- If necessary, relabel $M_0, \ldots, M_i$ so that they consistute a sequence of nested minors of M_{i+1}. This sequence satisfies the induction hypothesis.

- By induction, the claimed sequence exists for M.

□

Theorem 93 (7.3.1.b)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. Assume N is a wheel. Then for some $t \geq 1$, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where:*

- *M_0 is isomorphic to N,*
- *for some $0 \leq s \leq t$ the subsequence $M_0, \ldots, M_s$ consists of wheels and has gap 2,*
- *the subsequence $M_s, \ldots, M_t$ has gap 1.*

Proof sketch. Same as the proof of Theorem 7.3.1.a, but uses Theorem 7.2.1.b instead of 7.2.1.a to extend the sequence of minors. □

Proposition 94 (7.2.1 from 7.3.1)**.** *Theorem 7.3.1 implies Theorem 7.2.1.*

Proof sketch.

- Let $\mathcal{M}$ and N be as specified in Theorem 7.2.1. Suppose N is not a wheel.
- Prove the nontrivial "if" part by contradiction: let M be a 3-connected matroid of $\mathcal{M}$ with N as a proper minor.
- By Theorem 7.3.1, there is a sequence $M_0, \ldots, M_t = M$ of nested 3-connected minors where M_0 is isomorphic to N and where the gap is 1.
- Since $\mathcal{M}$ is closed under isomorphism, we may assume that M is chosen such that $M_0 = N$.
- Then $M_1 \in \mathcal{M}$ is a 3-connected 1-element extension of N, which contradicts the assumed absence of such extensions.
- If N is a wheel, the proof is analogous.

□

Corollary 95 (7.3.2.a)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H, and where each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. Translate Theorem 7.3.1.a directly into graph language. □

Corollary 96 (7.3.2.b)**.** *Let G be a 3-connected graph with a 3-connected proper minor H with at least 6 edges. Assume H is a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where:*

- *G_0 is isomorphic to H,*
- *for some $0 \leq s \leq t$ the subsequence $G_0, \ldots, G_t$ consists of wheels where each G_{i+1} has exactly one additional spoke beyond those of G_i,*
- *in the subsequence $G_s, \ldots, G_t$ each G_{i+1} has exactly one edge beyond those of G_i.*

Proof sketch. Translate Theorem 7.3.1.b directly into graph language. □

Theorem 97 (7.3.3, wheel theorem)**.** *Let G be a 3-connected graph on at least 6 edges. If G is not a wheel, then G has some edge z such that at least one of the minors G/z and $G \setminus z$ is 3-connected.*

Proof sketch.

- By Corollary 5.2.15, G has a W_3 minor.
- Let H be a largest wheel minor of G. Since G is not a wheel, H is a proper minor of G.
- Apply Corollary 7.3.2.b to G and H to get a sequence of nested 3-connected minors $G_0, \ldots, G_t = G$ where G_0 is isomorphic to H.
- Since H is the largest wheel minor and G is not a wheel, Corollary 7.3.2.b shows that $s = 0$ and $t \geq 1$.
- Additionally, from corollary we know that $G = G_t$ has exactly one extra edge compared to G_{t-1}. In other words, $G_{t-1} = G/z$ or $G \setminus z$ for some edge z.

□

Theorem 98 (7.3.3 for binary matroids)**.** *Theorem 7.3.3 can be rewritten for binary matroids instead of graphs.*

Proof sketch. Similar to the proof of Theorem 7.3.3, but use Theorem 7.3.1 instead of Corollary 7.3.2. □

Proposition 99 (7.3.4.observation)**.** *Oservation in text on pages 160–161.*

Theorem 100 (7.3.4)**.** *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least 6 elements. If M does not contain a 3-connected 1-element expansion (resp. addition) of any N minor, then M has a sequence of nested 3-connected minors $M_0, \ldots, M_t = M$ where M_0 is an N minor of M and where each M_{i+1} is obtained from M_i by expansions (resp. additions) involving some series (resp. parallel) elements, possibly none, followed by a 1-element addition (resp. expansion).*

Proof sketch.

- The case in parenthesis is dual to the normally stated one. Thus, only consider expansions below.
- Apply construction from observation before Theorem 7.3.4 to the sequence of minors from Theorem 7.3.1 to get the desired sequence.

□

Corollary 101 (7.3.5)**.** *Specializes Theorem 7.3.4 to graphs.*

0.7.3 Chapter 7.4

Theorem 102 (7.4.1 planarity characterization)**.** *A graph is planar if and only if it has no $K_{3,3}$ or K_5 minors.*

Proof sketch.

- "Only if": planarity is preserved by taking minors, and by Lemma 3.2.48 both $K_{3,3}$ and K_5 are not planar.
- Let G be a connected nonplanar graph with all proper minors planar. Goal: show that G is isomorphic to $K_{3,3}$ or K_5.
- Prove that G cannot be 1- or 2-separable. Thus G is 3-connected.
- By Corollary 5.2.15, G has a W_3 minor, say H. Note: no H minor of G can be extended to a minor of G by addition of an edge that connects two nonadjacent nodes.
- Then by Corollary 7.3.5.b, there exists a sequence $G_0, \ldots, G_t = G$ of 3-connected minors where G_0 is an H minor and G_{i+1} is constructed from G_i following very specific steps.
- By minimality, G_{t-1} is planar and G is not. Argue about a planar drawing of G_{t-1} and how G can be derived from it. Show that this must result in a subdivision of $K_{3,3}$ or K_5.

□

Theorem 103 (Kuratowski)**.** *A graph is planar if and only if it has no subdivision of $K_{3,3}$ or K_5.*

Proof. Note: Theorem 7.4.1 is equivalent to Kuratowski's theorem: a $K_{3,3}$ minor induces a subdivision of $K_{3,3}$ and a K_5 minor also leads to a subdivision of K_5 or $K_{3,3}$ (the latter in the case when an expansion step splits a vertex of degree 4 into two vertices of degree 3 after the new edge is inserted). □

0.8 Chapter 8 from Truemper

0.8.1 Chapter 8.2

This chapter is about deducing and manipulating 1- and 2-sum decompositions and compositions.

Proposition 104 (8.2.1)**.** *Matrix of* 1*-separation.*

Lemma 105 (8.2.2)**.** *Let M be a binary matroid. Assume M to be a* 1*-sum of two matroids M_1 and M_2.*

- *If M is graphic, then there eixst graphs G, G_1, G_2 for M, M_1, M_2, respectively, such that identification of a node of G_1 with one of G_2 creates G.*
- *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Proof sketch. Elementary application of Theorem 3.2.25.a. □

Proposition 106 (8.2.3)**.** *Matrix of exact* 2*-separation.*

Proposition 107 (8.2.4)**.** *Matrices B^1 and B^2 of* 2*-sum.*

Lemma 108 (8.2.6)**.** *Any 2-separation of a connected binary matroid M produces a 2-sum with connected components M_1 and M_2. Conversely, any 2-sum of two connected binary matroids M_1 and M_2 is a connected binary matroid M.*

Proof sketch.

- Definitions imply everything except connectedness.
- It is easy to check that connectedness of (8.2.3) implies connectedness of (8.2.4) and vice versa.
- By Lemma 3.3.19, connectedness of representation matrices is equivalent to connectedness of the corresponding matroids.

□

Lemma 109 (8.2.7)**.** *Let M be a connected binary matroid that is a 2-sum of M_1 and M_2, as given via B, B_1, and B_2 of (8.2.3) and (8.2.4).*

- *If M is graphic, then there exist 2-connected graphs G, G_1, and G_2 for M, M_1, and M_2, respectively, with the following feature. The graph G is produced when one identifies the edge x of G_1 with the edge y of G_2, and when subsequently the edge so created is deleted.*
- *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Proof sketch.

- Ingredients: look at a 2-separation and the corresponding subgraphs, use Theorem 3.2.25.b, use the switching operation of Section 3.2, use Lemma 8.2.6 and representations (8.2.3) and (8.2.4).
- Use the construction from the drawing, check that fundamental circuits match, conclude that M is graphic. For planar graphs, the edge identification can be done in a planar way.

□

0.8.2 Chapter 8.3

Proposition 110 (8.3.1)**.** *Matrix B with exact k-separation.*

Proposition 111 (8.3.2)**.** *Partition of B displaying k-sum.*

Proposition 112 (8.3.9)**.** *The (well-chosen) matrix $\overline{B}$ representing the connecting minor $\overline{M}$ of a 3-sum.*

Proposition 113 (8.3.10)**.** *The matrix B representing a 3-sum (after reasoning).*

Proposition 114 (8.3.11)**.** *Representation matrices B^1 and B^2 of the components M_1 and M_2 of a 3-sum (after reasoning).*

Lemma 115 (8.3.12)**.** *Let M be a 3-connected binary matroid on a set E. Then any 3-separation (E_1, E_2) of M with $|E_1|, |E_2| \geq 4$ produces a 3-sum, and vice versa.*

Proof.

- The converse easily follows from (8.3.10), which directly produces a desired 3-separation.

- Take a 3-separation. Since M is 3-connected, it must be exact. Consider the representation matrix (8.3.11). Reason about that matrix.
- Analyse shortest paths in a bipartite graph based on the matrix.
- Apply path shortening technique from Chapter 5 to reduce a shortest path by pivots to one with exactly two arcs.
- Reason about the corresponding entries and about the effects of the pivots on the matrix.
- Apply Lemma 2.3.14. Eventually get an instance of (8.3.10) with (8.3.9). Thus, M is a 3-sum.

□

0.8.3 Chapter 8.5

Proposition 116 (8.5.3)**.** *Matrix $B^{2\Delta}$ for $M_{2\Delta}$.*

0.9 Chapter 9 from Truemper

Proposition 117 (9.2.14)**.** *Matrix B^{12} of regular matroid R_{12}.*

0.10 Chapter 10 from Truemper

Proposition 118 (10.2.4)**.** *Derivation of a graph with T nodes for F_7.*

Proposition 119 (10.2.6)**.** *Derivation of a graph with T nodes for $M(K_{3,3})^*$.*

Proposition 120 (10.2.8)**.** *Derivation of a graph with T nodes for R_{10}.*

Proposition 121 (10.2.9)**.** *Derivation of a graph with T nodes for R_{12}.*

Theorem 122 (10.2.11 only if)**.** *If a regular matroid is planar, then it has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors.*

Proof sketch.

- Planarity is preserved under taking minors.
- The listed matroids are not planar.

□

Theorem 123 (10.2.11 if)**.** *If a regular matroid has no $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$ minors, then it is planar.*

Proof sketch.

- Let M be minimally nonplanar with respect to taking minors, i.e., regular nonplanar, but with all proper minors planar.
- Goal: show that M is isomorphic to one of the listed matroids.
- By Theorem 7.4.1, M is not graphic or cographic.

- By Lemmas 8.2.2, 8.2.6, and 8.2.7, if M has a 1- or 2-separation, then M is a 1- or 2-sum. But then the components of the sum are planar, so M is also planar. Therefore, M is 3-connected.
- By the census of Section 3.3, every 3-connected ≤ 8-element matroid is planar, so $|M| \geq 9$.
- By the binary matroid version of the wheel Theorem 7.3.3, there exists an element z such that $M \setminus z$ or M/z is 3-connected. Dualizing does not afect the assumptions, so we may assume that $M \setminus z$ is 3-connected.
- Let G be a planar graph representing $M \setminus z$. Extend G to a representation of M as follows:
  - If G is a wheel, invoke (10.2.6) or (10.2.4). The latter contracdicts regularity of M, the former shows what we need.
  - If G is not a wheel, use Theorem 7.3.3 and Menger's theorem. Use a path argument and edge contraction to reduce to (10.2.6) and conclude the proof.

□

Lemma 124 (10.3.1)**.** *$M(K_5)$ is a splitter of the regular matroids with no $M(K_{3,3})$ minors.*

Proof.

- By Theorem 7.2.1.a, we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor.
- Then case analysis. (The book sketches one way of checking.)

□

Lemma 125 (10.3.6)**.** *Every 3-connected binary 1-element expansion of $M(K_{3,3})$ is nonregular.*

Proof sketch. By case analysis via graphs plus T sets. □

Theorem 126 (10.3.11)**.** *Let M be a 3-connected regular matroid with an $M(K_{3,3})$ minor. Assume that M is not graphic and not cographic, but that each proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof. This proof is extremely long and technical. It involves case distinctions and graph constructions. □

Theorem 127 (10.4.1 only if)**.** *If 3-connected regular matroid is graphic or cographic, then it has no R_{10} or R_{12} minors.*

Proof sketch. Representations (10.2.8) and (10.2.9) for R_{10} and R_{12} show that these are nongraphic and isomorphic to their duals, hence also noncographic, so we are done. □

Theorem 128 (10.4.1 if)**.** *If a 3-connected regular matroid has no R_{10} or R_{12} minors, then it is graphic or cographic.*

Proof sketch.

- Let M be 3-connected, regular, nongraphic, and noncographic matroid.
- Thus M is not planar, so by Theorem 10.2.11 it has a minor isomorphic to $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, or $M(K_{3,3})^*$.

- By Lemma 10.3.1, $M(K_5)$ is a splitter for the regular matroids with no $M(K_{3,3})$ minors.
- These results imply that M has a minor isomorphic to $M(K_{3,3})$, or $M(K_{3,3})^*$, or M is isomorphic to $M(K_5)$ or $M(K_5)^*$.
- The latter is a contradiction, so M or M^* has an $M(K_{3,3})$ minor.
- Theorem 10.3.11 implies that M or M^* has R_{10} or R_{12} as a minor.
- Since R_{10} and R_{12} are self-dual, M has R_{10} or R_{12} as a minor.

□

Note: Truemper's proof of **??** and **??** relies on representing matroids via graphs plus T sets. An alternative proof, which utilizes the notion of graph signings, can be found in J. Geelen, B. Gerards - Regular matroid decomposition via signed graphs. Although the proof appears shorter than Truemper's, it heavily relies certain relatively advanced graph-theoretic results.

Bonus: Whitney's characterization of planar graphs (Corollary 10.2.13).

0.11 Chapter 11 from Truemper

0.11.1 Chapter 11.2

The goal of this chapter is to prove the "simple" direction of the regular matroid decomposition theorem.

Ingredients from Section 9.2:

- A matrix is TU if all its subdeterminants are 0, ± 1.
- A binary matroid is regular if it has a signing that is TU.
- By Lemma 9.2.6 and Corollary 9.2.7, this signing is unique up to scaling by ± 1 factors.
- The signing can be accomplished by signing one arbitrarily selected row or column at a time.

Ingredients from minimal violation matrices:

- Definition: a minimal violation matrix of total unimodularity (minimal violation matrix, MVM) is a $\{0, \pm 1\}$ matrix that is not TU, but all its submatrices are TU.
- MVMs are square and have determinant not equal to $0, \pm 1$.
- In particular, a 2×2 violation matrix has four ± 1's.
- Cosider a MVM of order ≥ 3. Perform a pivot in it, then delete the pivot row and column. Then the resulting matrix is also MVM ("by a simple cofactor argument").

Lemma 129 (11.2.1)**.** *Any* 1- *or* 2-*sum of two regular matroids is also regular.*

Proof sketch.

- 1-sum case: $M_1 \oplus_1 M_2$ is represented by a matrix $B = \operatorname{diag}(A_1, A_2)$ where A_1 and A_2 represent M_1 and M_2. Use the same signings for A_1 and A_2 in B to prove that B is TU and hence the 1-sum is regular.

- 2-sum case: Slightly more complicated signing process. Similarly, reuse signings from M_1 and M_2, define signing on remaining nonzero elements via a concrete formula, then prove that the resulting matrix is TU.

□

Lemma 130 (11.2.7)**.** *M_2 of (8.3.10) and (8.3.11) is regular iff $M_{2\Delta}$ of (8.5.3) (M_2 converted by a ΔY exchange) is regular.*

Proof sketch. Utilize signings, minimal violation matrices, intersections (inside matrices), column dependence, pivot, duality. □

Corollary 131 (11.2.8)**.** *ΔY exchanges maintain regularity.*

Proof. Follows by Lemma 11.2.7. □

Lemma 132 (11.2.9)**.** *Any* 3*-sum of two regular matroids is also regular.*

Proof sketch. Yet more complicated, but similar. Uses the result that "ΔY exchanges maintain regularity" (Corollary 11.2.8 of Lemma 11.2.7). The rest of the arguments are similar to the 2-sum case: prove that submatrices are TU, then prove that the whole matrix is TU. □

Theorem 133 (11.2.10)**.** *Any* 1-*,* 2-*, or* 3*-sum of two regular matroids is regular.*

Proof sketch. Combine Lemmas 11.2.1 and 11.2.9. □

Corollary 134 (11.2.12)**.** *Any Δ-sum or Y-sum of two regular matroids is also regular.*

Proof sketch. Follows from definitions of Δ-sums and Y-sum, together with Theorem 11.2.10 and Corollary 11.2.8. □

0.11.2 Chapter 11.3

Proposition 135 (11.3.3)**.** *Graph plus T set representing R_{10}*

Proposition 136 (11.3.5)**.** *Graph plus T set representing F_7.*

Proposition 137 (11.3.11)**.** *The binary representation matrix B^{12} for R_{12}.*

The goal of the chapter is to prove the "hard" direction of the regular matroid decomposition theorem.

Theorem 138 (11.3.2)**.** *R_{10} is a splitter of the class of regular matroids.*
In short: up to isomorphism, the only 3*-connected regular matroid with R_{10} minor is R_{10}.*

Proof sketch.

- Splitter theorem case (a)
- R_{10} is self-dual, so it suffices to consider 1-element additions.
- Represent R_{10} by (11.3.3)
- Up to isomorphism, there are only 3 distinct 3-connected 1-element extensions.
- Case 1 (graphic): contract a certain edge, the resulting graph contains a subdivision of (11.3.5), which represents F_7. Thus, this extension is nonregular.

- Cases 2, 3 (nongraphic): reduce instances to (11.3.5), same conclusion.

□

Theorem 139 (11.3.10)**.** *In short: Restatement of* **??** *for* R_{12}*. Replacements:* $\mathcal{M}$ *is the class of regular matroids,* N *is* R_{12}*, (6.3.12) is (11.3.6), (6.3.21-23) are (11.3.7-9).*

Theorem 140 (11.3.12)**.** *Let* M *be a regular matroid with* R_{12} *minor. Then any* 3*-separation of that minor corresponding to the* 3*-separation* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* R_{12} *(see (11.3.11) – matrix* B^{12} *for* R_{12} *defining the* 3*-separation) under one of the isomorphisms induces a* 3*-separation of* M*.*

In short: every regular matroid with R_{12} *minor is a* 3*-sum of two proper minors.*

Proof sketch.

- Preparation: calculate all 3-connected regular 1-element additions of R_{12}. This involves somewhat tedious case checking. (Representation of R_{12} in (10.2.9) helps a lot.) By the symmetry of B^{12} and thus by duality, this effectively gives all 3-connected 1-element extensions as well.
- Verify conditions of theorem 11.3.10 (which implies the result).
- (11.3.7) and (11.3.9) are ruled out immediately from preparatory calculations.
- The rest is case checking ((c.1) and (c.2)), simpified by preparatory calculations.

□

Theorem 141 (11.3.14 regular matroid decomposition, easy direction)**.** *Every binary matroid produced from graphic, cographic, and matroids isomorphic to* R_{10} *by repeated* 1-, 2-, *and* 3*-sum compositions is regular.*

Proof sketch. Follows from theorem 11.2.10. □

Theorem 142 (11.3.14 regular matroid decomposition, hard direction)**.** *Every regular matroid* M *can be decomposed into graphic and cographic matroids and matroids isomorphic to* R_{10} *by repeated* 1-, 2-, *and* 3- *sum decompositions. Specifically: If* M *is a regular* 3*-connected matroid that is not graphic and not cographic, then* M *is isomorphic to* R_{10} *or has an* R_{12} *minor. In the latter case, any* 3*-separation of that minor corresponding to the 3-separation* $(X_1 \cup Y_1, X_2 \cup Y_2)$ *of* R_{12} *((11.3.11)) under one of the isomorphisms induces a* 3*-separation of* M*.*

Proof sketch.

- Let M be a regular matroid. Assume M is not graphic and not cographic.
- If M is 1-separable, then it is a 1-sum. If M is 2-separable, then it is a 2-sum. Thus assume M is 3-connected.
- By theorem 10.4.1, M has an R_{10} or an R_{12} minor.
- R_{10} case: by theorem 11.3.2, M is isomorphic to R_{10}.
- R_{12} case: by theorem 11.3.12, M has an induced by 3-separation, so by lemma 8.3.12, M is a 3-sum.

□

0.11.3 Extensions of Regular Matroid Decomposition

- Theorem 11.3.14 remains valid when 3-sums are replaced by Δ- and Y-sums (Theorem 11.3.16).
- Theorem 11.3.14 (and 11.3.16) can also be proved for matroids with no F_7 minors or with no F_7^* minors. (Uses Lemma 11.3.19: F_7 (F_7^*) is a splitter of the binary matroids with no F_7^* (F_7) minors.)

0.11.4 Applications of Regular Matroid Decomposition

- Efficient algorithm:for.testing.if a binary matroid is regular (Section 11.4).
- Efficient algorithm:for.deciding.if a real matrix is TU (Section 11.4).
- Constructing TU matrices (Theorem 11.5.9). (Translate 3-sum version of theorem 11.3.16 into matrix language.)
- Constructing $0, 1$ TU matrices (Theorem 11.5.13).
- Characterization of the cycle polytope (theorem 11.5.17). (Problem: let M be a connected binary matroid with ground set E and element weighs w_e for all $e \in E$. Find a disjoint union C of circuits of M such that $\sum_{e \in C} w_e$ is maximized.)
- Number of nonzeros in TU matrices (Theorem 11.5.18).
- Triples in circuits (Theorem 11.5.18).
- Odd cycles (Theorem 11.5.20).