

Proof of Regularity of 2- and 3-Sum of Matroids

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1 2-Sum of Regular Matroids Is Regular

Lemma 1. *Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.*

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc} , the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic}a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column a''_k with the pivot column a_{rc} without the pivot element a_{rc} .

By the cofactor expansion in A along row r , we have

$$\det A = \sum_{k=1}^n (-1)^{r+k} a_{rk} \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k . By swapping the order of columns in $B_{r,k}$ to match the form of B_k , we get

$$\det A = (-1)^{r+c} (a_{rc} \det A' - \sum_{k \in C} a_{rk} \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 1. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc} . Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 1. Let B_1, B_2 be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = [A_1/x]$ and $B_2 = [y \mid A_2]$, where x is a row vector, y is a column vector, and A_1, A_2 are matrices of appropriate dimensions. Let D be the outer product of y and x . The 2-sum of B_1 and B_2 is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}.$$

Definition 2. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k -TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 1. Note that a matrix is TU if and only if it is k -TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 2. *Let B_1 and B_2 be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then B is 1-TU and 2-TU.*

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B . If V is a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_1 and A_2 . Let i be the row shared by V and A_1 and j be the column shared by V and A_2 . Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij} . Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. \square

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ -TU for every $\ell < k$. Then for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also k -TU.*

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_1 and B_2 such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not k -TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row of V whose index appears in A_1 . Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_1 . Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc} . For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, \tilde{B} denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $[\tilde{A}_1/\tilde{D}]$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., A_2 remains unchanged. This holds because of the 0 block in B .
- \tilde{D} consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $[\tilde{D} \mid \tilde{A}_2]$ is TU, since this matrix consists of A_2 and copies of y scaled by factors $\{0, \pm 1\}$.
- \tilde{D} can be represented as an outer product of a column vector \tilde{y} and a row vector \tilde{x} , and we can define $\tilde{B}_1 = [\tilde{A}_1/\tilde{x}]$ and $\tilde{B}_2 = [\tilde{y} \mid \tilde{A}_2]$ similar to B_1 and B_2 , respectively. Note that \tilde{B}_1 and \tilde{B}_2 have the same size as B_1 and B_2 , respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- \tilde{B} contains a square submatrix \tilde{V} of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k - 1) \times (k - 1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B , the submatrix V'' corresponds to a submatrix \tilde{V} of \tilde{B} with the same property.

To sum up, after pivoting we obtain a matrix \tilde{B} that represents a 2-sum of TU matrices \tilde{B}_1 and \tilde{B}_2 and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k - 1)$ -TUness of \tilde{B} , which proves the lemma. \square

Lemma 4. *Let B_1 and B_2 be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_1 and B_2 , their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ -TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 2.

Step: If the Proposition holds for some k , then it also holds for $k + 1$ by Lemma 3.

Conclusion: For any TU matrices B_1 and B_2 , their 2-sum $B_1 \oplus_{2,x,y} B_2$ is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. \square

2 Results Related to Regularity of 3-Sums

2.1 Resigning TU Matrices

Lemma 5. Suppose A and A' are TU signings of the same matrix $B \in \mathbb{Z}_2^{m \times n}$. Then there exist vectors $u \in \{\pm 1\}^m$ and $v \in \{\pm 1\}^n$ such that $a'_{ij} = u_i v_j a_{ij}$ for every $i \in [m]$, $j \in [n]$.

Proof.

adapt from Lemma 9.2.6 in Truemper

□

Corollary 2. Suppose A is a TU signing of $B \in \mathbb{Z}_2^{m \times n}$ and A' is a TU signing of a submatrix B' of B . Then there exists a TU signing \tilde{A} of B that coincides with A' when restricted to B' .

Proof.

proof

□

2.2 Delta-Wye Exchange

Delta-Wye Exchange or ΔY -exchange is an operation of replacing a triangle with a 3-star or vice versa.

Definition 3. The triangle to 3-star exchange for matrices is defined as follows.

1. Let $B \in \mathbb{Z}_2^{X \times (Y \cup \{e, f, g\})}$ be a binary matrix of the form

$$B = \begin{bmatrix} \overline{B} & a & b & c \end{bmatrix}, \quad \text{where } a + b + c = 0 \text{ in } \mathbb{Z}_2.$$

Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{y\}) \times (Y \cup \{x, z\})}$ where

$$B' = \begin{bmatrix} \overline{B} & a & b \\ 0 & 1 & 1 \end{bmatrix}.$$

2. Let $B \in \mathbb{Z}_2^{(X \cup \{f\}) \times (Y \cup \{e, g\})}$ be a binary matrix of the form

$$B = \begin{bmatrix} \overline{B} & b & b \\ a & 1 & 0 \end{bmatrix}.$$

Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{z, y\}) \times (Y \cup \{x\})}$ where

$$B' = \begin{bmatrix} \overline{B} & b \\ a & 1 \\ a & 0 \end{bmatrix}.$$

3. Let $B \in \mathbb{Z}_2^{(X \cup \{e, f\}) \times (Y \cup \{g\})}$ be a binary matrix of the form

$$B = \begin{bmatrix} \overline{B} & 0 \\ a & 1 \\ b & 1 \end{bmatrix}$$

Then the triangle to star exchange on B results in the binary matrix $B' \in \mathbb{Z}_2^{(X \cup \{x, y, z\}) \times Y}$ where

$$B' = \begin{bmatrix} \overline{B} \\ a \\ b \\ c \end{bmatrix}, \quad \text{where } a + b + c = 0 \text{ in } \mathbb{Z}_2.$$

The 3-star to triangle exchange is defined as the converse operation.

Remark 2. Note that in the case distinction \bar{B}, a, b, c refer to different matrices and vectors.

Definition 4. Let M be a binary matroid with the ground set E . Let $\{e, f, g\} \subseteq E$ be a triangle in M not containing a cocycle and let B be a standard binary representation matrix for M . The triangle to 3-star exchange on M results in a binary matroid M' with the ground set $E' = E \setminus \{e, f, g\} \cup \{x, y, z\}$ represented by the standard binary representation matrix B' obtained by the triangle to star exchange on B .

Conversely, let M' be a binary matroid with the ground set E' . Let $\{x, y, z\} \subseteq E'$ be a triad in M not containing a cycle and let B' be a standard binary representation matrix for M' . The 3-star to triangle exchange on M' results in a binary matroid M with the ground set $E = E' \setminus \{x, y, z\} \cup \{e, f, g\}$ represented by the standard binary representation matrix B obtained by the triangle to star exchange on B' .

Remark 3. Note that we may always choose B of the form from case 3. In this case, the condition that the triangle $\{e, f, g\}$ does not contain a cocycle is equivalent to the requirement that the row vectors a and b of B are non-zero and distinct. Hence, the row vectors a, b , and $c = a + b$ (in \mathbb{Z}_2) in B' are distinct, and $\{x, y, z\}$ is indeed a triad in M' .

Lemma 6. The triangle to triad exchange in M is a triad to triangle exchange in M^* .

Proof. By construction, if M has standard representation S , then $-S^T$ (and also S^T) is a standard representation of M^* . Plugging this into Definition 3 and reversing the operation shows the desired result. \square

2.3 Construction of 3-Sum and Delta-Sum

Definition 5. Let $B_1 \in \mathbb{Z}_2^{(X_1 \cup \{x_2, x_3\}) \times (Y_1 \cup \{y_3\})}$, $B_2 \in \mathbb{Z}_2^{(\{x_1\} \cup X_2) \times (\{y_1, y_2\} \cup Y_2)}$ be matrices of the form

$$B_1 = \begin{array}{|c|c|c|} \hline & A_1 & 0 \\ \hline & 1 & 1 \\ \hline D_1 & \bar{D} & \begin{array}{c} 1 \\ 1 \end{array} \\ \hline \end{array}, \quad B_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \bar{D} & \begin{array}{c} 1 \\ 1 \end{array} & & \\ \hline D_2 & & A_2 & \\ \hline \end{array},$$

where \bar{D} is a 2×2 matrix with \mathbb{Z}_2 rank 2 (i.e., \bar{D} is non-singular over \mathbb{Z}_2). Note that $x_1 \in X_1$, $x_2, x_3 \in X_2$, $y_1, y_2 \in Y_1$, $y_3 \in Y_2$, $A_1 \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2 \in \mathbb{Z}_2^{X_2 \times Y_2}$, $\bar{D} \in \mathbb{Z}_2^{(x_2, x_3) \times (y_1, y_2)}$, $D_1 \in \mathbb{Z}_2^{\{x_2, x_3\} \times (Y_1 \setminus \{y_1, y_2\})}$, $D_2 \in \mathbb{Z}_2^{(X_2 \setminus \{x_2, x_3\}) \times \{y_1, y_2\}}$. Then the 3-sum of B_1 and B_2 is defined as

$$B_1 \oplus_3 B_2 = \begin{array}{|c|c|c|} \hline & A_1 & 0 \\ \hline & 1 & 1 \\ \hline D_1 & \bar{D} & \begin{array}{c} 1 \\ 1 \end{array} \\ \hline D_{12} & D_2 & A_2 \\ \hline \end{array},$$

where $D_{12} = D_2 \cdot (\bar{D})^{-1} \cdot D_1$ and the indexing is preserved.

To simplify notation, we write

$$D_{1,12} = [D_1/D_{12}], \quad D_{0,2} = [\bar{D}/D_2], \quad D_{1,0} = [D_1 \mid \bar{D}], \quad D_{12,2} = [D_{12} \mid D_2], \quad D = \begin{array}{|c|c|} \hline D_1 & \bar{D} \\ \hline D_{12} & D_2 \\ \hline \end{array}$$

Definition 6. Let B_1, B_2 satisfy the conditions of Definition 5. Let $B_{2\Delta} \in \mathbb{Z}_2^{X_2 \times (\{z, y_1, y_2\} \cup Y_2)}$ be the matrix obtained from B_2 via a triangle-star exchange from Definition 3:

$$B_{2\Delta} = \begin{array}{|c|c|c|} \hline & \bar{D} & \begin{array}{c} 1 \\ 1 \end{array} \\ \hline d & D_2 & A_2 \\ \hline \end{array}$$

where $d \in \mathbb{Z}_2^{Y_2}$ is such that $(D_{0,2})_{\cdot y_1} + (D_{0,2})_{\cdot y_2} + d = 0$.

Definition 7. Let B_1 , B_2 , and $B_{2\Delta}$ be matrices from Definitions 5 and 6. Then the Δ -sum of B_1 and $B_{2\Delta}$ is $B_1 \oplus_{\Delta} B_{2\Delta} = B_1 \oplus_3 B_2$.

2.4 From Regular Matroids to TU Matrices

Remark 4. We say that a matrix B is regular if B is a standard representation matrix of a regular matroid.

Lemma 7. Let B_2 be a matrix from Definition 5. If B_2 is regular, then it has a TU signing \tilde{B}_2 where all entries in columns y_1 and y_2 are in $\{0, 1\}$.

Proof. Since B_2 is regular, it has a TU signing B'_2 . Recall that multiplying rows and columns of a TU matrix by factors in $\{0, \pm 1\}$ preserves TUness.

If $B'_2(x_1, y_1) = -1$, multiply column y_1 by -1 . Similarly, if $B'_2(x_1, y_2) = -1$, multiply column y_2 by -1 . Thus, we may assume that B'_2 has $B'_2(x_1, y_1) = B'_2(x_1, y_2) = 1$.

Next, consider each row of B'_2 . It can have one of the following forms.

- $[0 \mid 0], [0 \mid 1], [1 \mid 0], [1 \mid 1]$. In this case, we do not need to modify the signing.
- $[0 \mid -1], [-1 \mid 0], [-1 \mid -1]$. In this case, we can multiply this row by -1 to make all its non-negative.
- $[1 \mid -1], [-1 \mid 1]$. This case leads to a contradiction, as the matrix composed of this row and row x_1 has

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \quad \text{or} \quad \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2,$$

which is impossible as B'_2 is a TU signing.

Thus, we can multiply columns and rows of B'_2 to obtain a TU signing \tilde{B}_2 where all entries in columns y_1 and y_2 are in $\{0, 1\}$, as desired. \square

Lemma 8. Let B_2 be a matrix from Definition 5 and let \tilde{B}_2 be a TU signing of B_2 from Lemma 7. To simplify notation, let $\tilde{a} = (\tilde{D}_{0,2})_{\cdot y_1}$ and $\tilde{b} = (\tilde{D}_{0,2})_{\cdot y_2}$. Then pivoting in \tilde{B}_2 on (x_1, y_1) and (x_1, y_2) yields:

$$\begin{array}{ccc} \tilde{B}_2 = \begin{array}{|c|c|c|} \hline \textcircled{1} & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline -\tilde{a} & \tilde{b} - \tilde{a} & \tilde{A}_2 \\ \hline \end{array} \\ \\ \tilde{B}_2 = \begin{array}{|c|c|c|} \hline 1 & \textcircled{1} & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \tilde{a} - \tilde{b} & -\tilde{b} & \tilde{A}_2 \\ \hline \end{array} \end{array}$$

Proof. Recall that a real pivot in matrix A on entry $a_{rc} \neq 0$ transforms the matrix as follows:

$$\begin{array}{|c|c|} \hline a_{rc} & a_{rj} \\ \hline a_{ic} & a_{ij} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \frac{1}{a_{rc}} & \frac{a_{rj}}{a_{rc}} \\ \hline -\frac{a_{ic}}{a_{rc}} & a_{ij} - \frac{a_{rj}a_{ic}}{a_{rc}} \\ \hline \end{array}$$

A direct calculation proves the claim. \square

Corollary 3. Let B_2 be a matrix from Definition 5 and let \tilde{B}_2 be a TU signing of B_2 from Lemma 7. Then the following matrices are TU:

$$\tilde{B}_2^{(a)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{a} & \tilde{A}_2 \\ \hline \end{array}, \quad \tilde{B}_2^{(b)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array}.$$

Proof. Recall that pivoting, taking submatrices, and multiplying columns by ± 1 factors preserves TUness. Combining these facts with Lemma 8 gives the corollary. \square

Lemma 9. *Let B_2 and $B_{2\Delta}$ be matrices from Definitions 5 and 6. If B_2 is regular, then $B_{2\Delta}$ is regular.*

Proof. Let \tilde{B}_2 be a TU signing of B_2 from Lemma 7. Let \tilde{D} , \tilde{D}_2 , and \tilde{A}_2 be the signings of \bar{D} , D_2 , and A_2 , respectively, etc. Let $\tilde{d} = (\tilde{D}_{0,2})_{y_1} - (\tilde{D}_{0,2})_{y_2}$ and $\tilde{B}_{2\Delta} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2]$. Since $\tilde{D}_{0,2} \in \{0, 1\}^{X_2 \times \{y_1, y_2\}}$ by Lemma 7, we have $\tilde{d} \in \{0, \pm 1\}^{X_2}$, so $\tilde{B}_{2\Delta}$ is a signing of $B_{2\Delta}$. Our goal is to prove that $\tilde{B}_{2\Delta}$ is TU. To this end, let V be a square submatrix of $\tilde{B}_{2\Delta}$. We will show that $\det V \in \{0, \pm 1\}$.

Suppose that column \tilde{d} (with index z) is not in V . Then V is a submatrix of $[\tilde{D}_{0,2} \mid \tilde{A}_2]$ and hence a submatrix of \tilde{B}_2 . Since \tilde{B}_2 is TU, we have $\det V \in \{0, \pm 1\}$. Going forward we assume that column \tilde{d} (with index z) is in V .

Suppose that columns $(\tilde{D}_{0,2})_{y_1}$ and $(\tilde{D}_{0,2})_{y_2}$ (with indices y_1 and y_2 , respectively) are both in V . Then V contains three linearly dependent columns: \tilde{d} , $(\tilde{D}_{0,2})_{y_1}$, and $(\tilde{D}_{0,2})_{y_2}$ (with indices z , y_1 , and y_2 , respectively). Thus, $\det V = 0$. Going forward we assume that at most one of the columns $(\tilde{D}_{0,2})_{y_1}$ and $(\tilde{D}_{0,2})_{y_2}$ is in V .

Suppose that column $(\tilde{D}_{0,2})_{y_1}$ (with index y_1) is in V . Then V is a submatrix of $\tilde{B}_2^{(b)}$ from Corollary 3, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $\tilde{B}_2^{(a)}$ from Corollary 3, and so $\det V \in \{0, \pm 1\}$.

Since our case distinction is exhaustive, we showed that every square submatrix V of $\tilde{B}_{2\Delta}$ has $\det V \in \{0, \pm 1\}$. Thus, $\tilde{B}_{2\Delta}$ is TU, and so $B_{2\Delta}$ is regular. \square

Lemma 10. *Let B_2 and $B_{2\Delta}$ be matrices from Definitions 5 and 6. If $B_{2\Delta}$ is regular, then B_2 is regular.*

Proof. Since $B_{2\Delta}$ is regular, $B_{2\Delta}^*$ is also regular. Since $B_{2\Delta}$ is obtained from B_2 via a ΔY -exchange, B_2^* can be obtained from $B_{2\Delta}^*$ via the same operation. Therefore, B_2^* is regular by Lemma 9. Thus, B_2 is regular. \square

Corollary 4. B_2 from Definition 5 is regular if and only if $B_{2\Delta}$ from Definition 6 is regular.

Proof. Combine the results of Lemmas 9 and 10. \square

Lemma 11. *Assume the notation of Definitions 5 and 6. Then the columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$.*

Proof. Columns of $[d \mid D_{0,2}]$ trivially satisfy the claim, so it only remains to show that columns of $D_{1,12}$ are in $[d \mid D_{0,2} \mid 0]$. Note that $D_{1,12} = D_{0,2} \cdot ((\bar{D})^{-1} \cdot D_1)$, i.e., every column of $D_{1,12}$ can be expressed as a linear combination of the columns of $D_{0,2}$ (over \mathbb{Z}_2). In particular, every column of $D_{1,12}$ is either zero, one of the columns of $D_{0,2}$, or their sum. By construction, $(D_{0,2})_{y_1} + (D_{0,2})_{y_2} = d$. Thus, the desired result holds. \square

Corollary 5. As a direct corollary of Lemma 19, columns of $[d \mid D \mid A_2]$ are in $[d \mid D_{0,2} \mid A_2 \mid 0]$.

Lemma 12. *Let B_1 be a matrix from Definition 5. If B_1 is regular, then $[A_1/D]$ is regular.*

Proof.

Truemper: "By the regularity of B_1 and duality". Note: doable without the general resigning lemma by manual sign rigging.

\square

Lemma 13. *Let B_1 , B_2 , and $B_{2\Delta}$ be matrices from Definitions 5 and 6. If B_1 and B_2 are regular, then there are signings \tilde{A}_1 , \tilde{A}_2 , \tilde{D} , and \tilde{d} of A_1 , A_2 , D , and d , respectively, such that $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ and $[\tilde{A}_1/\tilde{D}]$ are TU.*

Proof. By Lemma 9, since B_2 is regular, $B_{2\Delta}$ is also regular. By Corollary 5, columns of $[d \mid D \mid A_2]$ are in $B_{2\Delta}$. Thus, there is a TU signing $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ of $[d \mid D \mid A_2]$.

By Lemma 12, since B_1 is regular, $[A_1/D]$ is also regular. Note that \tilde{D} is TU a submatrix of a TU matrix $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$. Thus, by Corollary 2, there is a TU signing $[\tilde{A}_1/\tilde{D}]$ of $[A_1/D]$.

Note: doable without the resigning lemma.

\square

2.5 3-Sum of Regular Matroids Is Regular

Definition 8 (Repeats Definition 2). Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k -TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 5. Note that a matrix is TU if and only if it is k -TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 14. *If B_1 and B_2 from Definition 5 are regular, then $B = B_1 \oplus_3 B_2$ has a signing \tilde{B} that is 1-TU and 2-TU.*

Proof.

sketch: combine signings of B_2 and B_1 using lemmas above; use the same argument as for 2-sum

□

Lemma 15. *If B_1 and B_2 from Definition 5 are regular, then $B_1 \oplus_3 B_2$ is regular.*

Proof. Let $B_{2\Delta}$ be the matrix from Definition 6. By Lemma 9, $B_{2\Delta}$ is regular. Since $B_1 \oplus_{\Delta} B_{2\Delta} = B_1 \oplus_3 B_2$, to prove the desired result it suffices to show that $B_1 \oplus_{\Delta} B_{2\Delta}$ is regular.

similar argument as for 2-sums; keep track of the form of D , the form is preserved under pivoting

□

3 3-Sum of Regular Matroids Is Regular: Streamlined

Definition 9. Let $B_1 \in \mathbb{Z}_2^{(X_1 \cup \{x_2, x_3\}) \times (Y_1 \cup \{y_3\})}$, $B_2 \in \mathbb{Z}_2^{(\{x_1\} \cup X_2) \times (\{y_1, y_2\} \cup Y_2)}$ be matrices of the form

$$B_1 = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_1 & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_1 & \overline{D} & 1 \\ \hline & & 1 \\ \hline \end{array}, \quad B_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \overline{D} & 1 & & \\ \hline & 1 & & A_2 \\ \hline D_2 & & & \\ \hline \end{array},$$

where \overline{D} is a 2×2 matrix with \mathbb{Z}_2 rank 2 (i.e., \overline{D} is non-singular over \mathbb{Z}_2). Note that $x_1 \in X_1$, $x_2, x_3 \in X_2$, $y_1, y_2 \in Y_1$, $y_3 \in Y_2$, $A_1 \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2 \in \mathbb{Z}_2^{X_2 \times Y_2}$, $\overline{D} \in \mathbb{Z}_2^{(x_2, x_3) \times (y_1, y_2)}$, $D_1 \in \mathbb{Z}_2^{\{x_2, x_3\} \times (Y_1 \setminus \{y_1, y_2\})}$, $D_2 \in \mathbb{Z}_2^{(X_2 \setminus \{x_2, x_3\}) \times \{y_1, y_2\}}$. Then the 3-sum of B_1 and B_2 is defined as

$$B_1 \oplus_3 B_2 = \begin{array}{|c|c|c|} \hline & & \\ \hline & A_1 & 0 \\ \hline & 1 & 1 & 0 \\ \hline D_1 & \overline{D} & 1 \\ \hline & & 1 \\ \hline D_{12} & D_2 & \\ \hline \end{array},$$

where $D_{12} = D_2 \cdot (\overline{D})^{-1} \cdot D_1$ and the indexing is preserved.

To simplify notation, we write

$$D_{1,12} = [D_1/D_{12}], \quad D_{0,2} = [\overline{D}/D_2], \quad D_{1,0} = [D_1 \mid \overline{D}], \quad D_{12,2} = [D_{12} \mid D_2], \quad D = \begin{array}{|c|c|} \hline D_1 & \overline{D} \\ \hline D_{12} & D_2 \\ \hline \end{array}$$

Lemma 16. Suppose B_2 from Definition 9 has a TU signing. Then we can construct a TU signing \tilde{B}_2 of B_2 where all entries in columns y_1 and y_2 are in $\{0, 1\}$.

Proof. Since B_2 is regular, it has a TU signing B'_2 . Recall that multiplying rows and columns of a TU matrix by factors in $\{0, \pm 1\}$ preserves TUness.

If $B'_2(x_1, y_1) = -1$, multiply column y_1 by -1 . Similarly, if $B'_2(x_1, y_2) = -1$, multiply column y_2 by -1 . Thus, we may assume that B'_2 has $B'_2(x_1, y_1) = B'_2(x_1, y_2) = 1$.

Next, consider each row of B'_2 . It can have one of the following forms.

- $[0 \mid 0]$, $[0 \mid 1]$, $[1 \mid 0]$, $[1 \mid 1]$. In this case, we do not need to modify the signing.
- $[0 \mid -1]$, $[-1 \mid 0]$, $[-1 \mid -1]$. In this case, we can multiply this row by -1 to make all its entries non-negative.
- $[1 \mid -1]$, $[-1 \mid 1]$. This case leads to a contradiction, as the matrix composed of this row and row x_1 has

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \quad \text{or} \quad \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2,$$

which is impossible as B'_2 is a TU signing.

Thus, we can multiply columns and rows of B'_2 to obtain a TU signing \tilde{B}_2 where all entries in columns y_1 and y_2 are in $\{0, 1\}$, as desired. \square

Lemma 17. Suppose B_2 from Definition 9 has a TU signing. Let \tilde{B}_2 be a TU signing of B_2 from Lemma 16. To simplify notation, let $\tilde{a} = (\tilde{D}_{0,2})_{y_1}$ and $\tilde{b} = (\tilde{D}_{0,2})_{y_2}$. Then the following matrices are TU:

$$\tilde{B}_2^{(a)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{a} & \tilde{A}_2 \\ \hline \end{array}, \quad \tilde{B}_2^{(b)} = \begin{array}{|c|c|c|} \hline \tilde{a} - \tilde{b} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array}.$$

Proof. Recall that pivoting in matrix A on entry $a_{rc} \neq 0$ transforms the matrix as follows:

$$\begin{array}{|c|c|} \hline a_{rc} & a_{rj} \\ \hline a_{ic} & a_{ij} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \frac{1}{a_{rc}} & \frac{a_{rj}}{a_{rc}} \\ \hline -\frac{a_{ic}}{a_{rc}} & a_{ij} - \frac{a_{rj}a_{ic}}{a_{rc}} \\ \hline \end{array}$$

Pivoting in \tilde{B}_2 on (x_1, y_1) and (x_1, y_2) yields:

$$\begin{array}{ccc} \tilde{B}_2 = \begin{array}{|c|c|c|} \hline \textcircled{1} & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline -\tilde{a} & \tilde{b} - \tilde{a} & \tilde{A}_2 \\ \hline \end{array} \\ \\ \tilde{B}_2 = \begin{array}{|c|c|c|} \hline 1 & \textcircled{1} & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \tilde{a} - \tilde{b} & -\tilde{b} & \tilde{A}_2 \\ \hline \end{array} \end{array}$$

By removing row x_1 from the resulting matrices and then multiplying columns y_1 and y_2 by $\{\pm 1\}$ factors, we obtain $\tilde{B}_2^{(a)}$ and $\tilde{B}_2^{(b)}$. Since \tilde{B}_2 is TU and TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $\tilde{B}_2^{(a)}$ and $\tilde{B}_2^{(b)}$ are TU. \square

Lemma 18. *Suppose B_2 from Definition 9 has a TU signing. Then we can construct signings \tilde{A}_2 , $\tilde{D}_{0,2}$, and \tilde{d} of A_2 , $D_{0,2}$, and $d = D_{\cdot y_1} + D_{\cdot y_2}$ respectively, such that $[\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2]$ is TU.*

Proof. Let \tilde{B}_2 be a TU signing of B_2 from Lemma 16. Let $\tilde{D}_{0,2}$ and \tilde{A}_2 denote the corresponding signings of $D_{0,2}$ and A_2 . Let $\tilde{d} = (\tilde{D}_{0,2})_{\cdot y_1} - (\tilde{D}_{0,2})_{\cdot y_2}$. Since $\tilde{D}_{0,2} \in \{0, 1\}^{X_2 \times \{y_1, y_2\}}$ by Lemma 16, we have

$$\tilde{d} = (\tilde{D}_{0,2})_{\cdot y_1} - (\tilde{D}_{0,2})_{\cdot y_2} = (D_{0,2})_{\cdot y_1} - (D_{0,2})_{\cdot y_2} = D_{\cdot y_1} - D_{\cdot y_2}.$$

Thus, $\tilde{d} \in \{0, \pm 1\}^{X_2}$ and \tilde{d} is a signing of d . Our goal is to prove that $\tilde{T} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2] \in \{0, \pm 1\}^{X_2 \times (\{z, y_1, y_2\} \cup Y_2)}$ is TU. To this end, let V be a square submatrix of \tilde{T} . We will show that $\det V \in \{0, \pm 1\}$.

Suppose that column (with index) z (i.e., corresponding to \tilde{d}) is not in V . Then V is a submatrix of \tilde{B}_2 , which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column (with index) z is in V .

Suppose that columns (with indices) y_1 and y_2 are both in V . Then V contains columns (with indices) z , y_1 , and y_2 , which are linearly dependent by construction of \tilde{d} . Thus, $\det V = 0$. Going forward we assume that at most one of the columns (with indices) y_1 and y_2 is in V .

Suppose that column (with index) y_1 is in V . Then V is a submatrix of $\tilde{B}_2^{(b)}$ from Corollary 17, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $\tilde{B}_2^{(a)}$ from Corollary 17, and so $\det V \in \{0, \pm 1\}$.

Thus, every square submatrix V of \tilde{T} has $\det V \in \{0, \pm 1\}$, and hence \tilde{T} is TU. \square

Lemma 19. *Assume the notation of Definition 9 and let $d = D_{\cdot y_1} + D_{\cdot y_2} \in \mathbb{Z}_2^{X_2}$. Then the columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$.*

Proof. Columns of $[d \mid D_{0,2}]$ trivially satisfy the claim, so it only remains to show that columns of $D_{1,12}$ are in $[d \mid D_{0,2} \mid 0]$. Note that $D_{1,12} = D_{0,2} \cdot ((\overline{D})^{-1} \cdot D_1)$, i.e., every column of $D_{1,12}$ can be expressed as a linear combination of the columns of $D_{0,2}$ (over \mathbb{Z}_2). In particular, every column of $D_{1,12}$ is either zero, one of the columns of $D_{0,2}$, or their sum. By construction, $(D_{0,2})_{\cdot y_1} + (D_{0,2})_{\cdot y_2} = d$. Thus, the desired result holds. \square

Lemma 20. *Suppose B_2 from Definition 9 has a TU signing. Then we can construct signings \tilde{A}_2 , \tilde{D} , and \tilde{d} of A_2 , D , and $d = D_{\cdot y_1} + D_{\cdot y_2}$, respectively, such that $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ is TU.*

Proof. By Lemma 19, columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$. Thus, columns of $U = [d \mid D \mid A_2]$ are in $T = [d \mid D_{0,2} \mid A_2 \mid 0]$. Let \tilde{A}_2 , $\tilde{D}_{0,2}$, and \tilde{d} be the signings from Lemma 18. Since adjoining zero columns does not affect TUness, $\tilde{T} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2 \mid 0]$ is a TU signing of T .

We construct signing \tilde{U} of U as follows. Let i be a column index in U . Then $U_{\cdot i} = T_{\cdot j}$ for some column index j in T , and we set $\tilde{U}_{\cdot i} = \tilde{T}_{\cdot j}$ with that j . By construction, \tilde{U} consists of columns of \tilde{T} , so \tilde{U} is a submatrix of \tilde{T} . Since \tilde{T} is TU, \tilde{U} is also TU. \square

Remark 6. Note that if a column of $[d \mid D \mid A_2]$ appears in $[d \mid D_{0,2} \mid A_2 \mid 0]$ multiple times, we may choose any of its occurrences when defining the signing.

Lemma 21. Suppose B_1 and B_2 from Definition 9 have TU signings. Then we can construct signings \tilde{B} and \tilde{d} of $B = B_1 \oplus_3 B_2$ and $d = D_{\cdot y_1} + D_{\cdot y_2}$, respectively, such that $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ and $[\tilde{A}_1/\tilde{D}]$ are both TU.

Proof. Let \tilde{A}_2 , \tilde{D} , and \tilde{d} be signings from Lemma 20. Let \tilde{B}_1 be a TU signing of B_1 . Use the procedure from the proof of Lemma 16 to multiply some rows and columns of \tilde{B}_1 and $\tilde{T} = [\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ by $\{\pm 1\}$ factors so that all entries in the submatrix $\{x_2, x_3\} \times (Y_1 \cup \{y_3\})$ are non-negative in both \tilde{B}_1 and \tilde{T} . This procedure preserves the validity and TUness of both signings. Thus, by restricting \tilde{B}_1 we get a signing \tilde{A}_1 of A_1 . By construction, $[\tilde{A}_1/\tilde{D}_{1,0}]$ is a TU signing of $[A_1/D_{1,0}]$.

It remains to prove that $[\tilde{A}_1/\tilde{D}]$ is TU. Let V be a square submatrix of $[\tilde{A}_1/\tilde{D}]$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

If V is a submatrix of only $[\tilde{A}_1/\tilde{D}_{1,0}]$ or \tilde{D} , then $\det V \in \{0, \pm 1\}$, as those matrices are TU. Going forward, assume that V intersects \tilde{A}_1 and $\tilde{D}_{12,2}$.

finish

\square

Definition 10 (Repeats Definition 2). We say that a matrix A is k -TU for $k \in \mathbb{Z}_{\geq 1}$ if every square submatrix T of A of size k has $\det T \in \{0, \pm 1\}$.

Remark 7. Note that a matrix is TU if and only if it is k -TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 22. Suppose B_1 and B_2 from Definition 9 have TU signings. Let \tilde{B} be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 21. Then \tilde{B} is 1-TU and 2-TU.

Proof. By construction,

$$\tilde{B} = \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{D} & \tilde{A}_2 \end{bmatrix}$$

where $[\tilde{A}_1/\tilde{D}]$, $[\tilde{D} \mid \tilde{A}_2]$, \tilde{A}_1 , \tilde{D} , and \tilde{A}_2 are all TU (by Lemma 21 and as submatrices of TU matrices). In particular, all entries of \tilde{B} are in $\{0, \pm 1\}$, so \tilde{B} is 1-TU.

To show that \tilde{B} is 2-TU, let V be a 2×2 submatrix of \tilde{B} . If V is a submatrix of $[\tilde{A}_1/\tilde{D}]$, $[\tilde{D} \mid \tilde{A}_2]$, $[\tilde{A}_1 \mid 0]$, or $[0/\tilde{A}_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both \tilde{A}_1 and \tilde{A}_2 . Let i be the row shared by V and \tilde{A}_1 and j be the column shared by V and \tilde{A}_2 . Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij} . Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. \square

Lemma 23. Suppose B_1 and B_2 from Definition 9 have TU signings. Let \tilde{B} be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 21. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose \tilde{B} is ℓ -TU for every $\ell < k$. Then \tilde{B} is also k -TU.

Proof. For the sake of deriving a contradiction, suppose there exist B_1 and B_2 such that \tilde{B} is not k -TU. Then \tilde{B} contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[\tilde{A}_1/\tilde{D}]$, $[\tilde{D} \mid \tilde{A}_2]$, $[\tilde{A}_1 \mid 0]$, or $[0/\tilde{A}_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with \tilde{A}_1 and \tilde{A}_2 each.

Consider the row of V whose index appears in \tilde{A}_1 . Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and \tilde{A}_1 . Let r and c denote the row and column index of this entry, respectively.

Perform a pivot in \tilde{B} on the element \tilde{B}_{rc} . For every object, its modified counterpart after pivoting is denoted by the same symbol with a hat; for example, \hat{B} denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $[\hat{A}_1/\hat{D}]$ is TU, since TUness is preserved by pivoting.
- $\hat{A}_2 = \tilde{A}_2$, i.e., \tilde{A}_2 remains unchanged. This holds because of the 0 block in \tilde{B} .
- The columns of \hat{D} are scaled (by $\{\pm 1\}$ factors) versions of columns of $[\tilde{d} \mid \tilde{D}]$. This can be verified via a direct calculation.

need more details

- $[\hat{D} \mid \hat{A}_2]$ is TU, since up to scaling (i.e., multiplying columns by $\{\pm 1\}$ factors) (and taking parallel columns) it is a submatrix of $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$.
- \hat{B} is $(k-1)$ -TU.

need to adjust assumptions of this lemma to be able to use $(k-1)$ -TUness here

- \hat{B} contains a square submatrix \hat{V} of size $k-1$ with $\det \hat{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element \tilde{B}_{rc} results in a matrix containing a $(k-1) \times (k-1)$ submatrix V' with $\det V' \in \{0, \pm 1\}$. Since V is a submatrix of \tilde{B} , the submatrix V' corresponds to a submatrix \hat{V} of \hat{B} with the same property.

To sum up, after pivoting we obtain a matrix \hat{B} that is $(k-1)$ -TU, but contains a square submatrix of size $k-1$ with determinant not in $\{0, \pm 1\}$. This contradiction proves the lemma. \square

Lemma 24. *Suppose B_1 and B_2 from Definition 9 have TU signings. Let \tilde{B} be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 21. Then \tilde{B} is TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any matrices B_1 and B_2 that have TU signings, the signing \tilde{B} of $B = B_1 \oplus_3 B_2$ from Lemma 21 is ℓ -TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 22.

Step: If the Proposition holds for some k , then it also holds for $k+1$ by Lemma 23.

Conclusion: For any matrices B_1 and B_2 that have TU signings, the signing \tilde{B} of $B = B_1 \oplus_3 B_2$ from Lemma 21 is k -TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, \tilde{B} is TU. \square