Proof of Regularity of 2-Sum and 3-Sum of Matroids

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1 The 2-Sum of Regular Matroids Is Regular

Lemma 1. Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc}, the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic} \cdot a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \cdot \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column $a''_{\bullet k}$ with the pivot column $a_{\bullet c}$ without the pivot element a_{rc}.

By the cofactor expansion in A along row r, we have

$$\det A = \sum_{k=1}^{n} (-1)^{r+k} \cdot a_{rk} \cdot \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k. By swapping the order of columns in $B_{r,k}$ to match the form of B_k, we get

$$\det A = (-1)^{r+c} (a_{rc} \cdot \det A' - \sum_{k \in C} a_{rk} \cdot \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 2. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 3. Let B_l, B_r be matrices with $\{0, \pm 1\}$ entries expressed as $B_l = \begin{array}{|c|}\hline A_l \\ \hline x \\ \hline\end{array}$ and $B_r = \begin{array}{|c|c|}\hline y & A_r \\ \hline\end{array}$, where x is a row vector, y is a column vector, and A_l, A_r are matrices of appropriate dimensions. Let D be the outer product of y and x. The 2-sum of B_l and B_r is defined as

$$B_l \oplus_{2,x,y} B_r = \begin{array}{|c|c|}\hline A_l & 0 \\ \hline D & A_r \\ \hline\end{array}.$$

Definition 4. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 5. Note that a matrix is TU if and only if it is k-TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 6. Let B_l and B_r be TU matrices and let $B = B_l \oplus_{2,x,y} B_r$. Then B is 1-TU and 2-TU.

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B. If V is a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline D & A_r \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline A_l & 0 \\ \hline \end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_r \\ \hline \end{array}$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_l and A_r. Let i be the row shared by V and A_l and j be the column shared by V and A_r. Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij}. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. $\square$

Lemma 7. Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_l and B_r their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is ℓ-TU for every $\ell < k$. Then for any TU matrices B_l and B_r their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is also k-TU.

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_l and B_r such that their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is not k-TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline D & A_r \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline A_l & 0 \\ \hline \end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_r \\ \hline \end{array}$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_l and A_r each.

Consider the row of V whose index appears in A_l. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_l. Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc}. For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, $\tilde{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\begin{array}{|c|}\hline \tilde{A}_l \\ \hline \tilde{D} \\ \hline \end{array}$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_r$, i.e., A_r remains unchanged. This holds because of the 0 block in B.
- $\tilde{D}$ consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\begin{array}{|c|c|}\hline \tilde{D} & \tilde{A}_r \\ \hline \end{array}$ is TU, since this matrix consists of A_r and copies of y scaled by factors $\{0, \pm 1\}$.
- $\tilde{D}$ can be represented as an outer product of a column vector $\tilde{y}$ and a row vector $\tilde{x}$, and we can define $\tilde{B}_1 = \begin{array}{|c|}\hline \tilde{A}_l \\ \hline \tilde{x} \\ \hline \end{array}$ and $\tilde{B}_2 = \begin{array}{|c|c|}\hline \tilde{y} & \tilde{A}_r \\ \hline \end{array}$ similar to B_l and B_r, respectively. Note that $\tilde{B}_1$ and $\tilde{B}_2$ have the same size as B_l and B_r, respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- $\tilde{B}$ contains a square submatrix $\tilde{V}$ of size $k - 1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 2 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k-1) \times (k-1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B, the submatrix V'' corresponds to a submatrix $\tilde{V}$ of $\tilde{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\tilde{B}$ that represents a 2-sum of TU matrices $\tilde{B}_1$ and $\tilde{B}_2$ and contains a square submatrix of size $k - 1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k-1)$-TUness of $\tilde{B}$, which proves the lemma. $\square$

Lemma 8. Let B_l and B_r be TU matrices. Then $B_l \oplus_{2,x,y} B_r$ is also TU.

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_l and B_r, their 2-sum $B = B_l \oplus_{2,x,y} B_r$ is ℓ-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 6.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 7.

Conclusion: For any TU matrices B_l and B_r, their 2-sum $B_l \oplus_{2,x,y} B_r$ is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_l \oplus_{2,x,y} B_r$ is TU. $\square$

2 The 3-Sum of Regular Matroids Is Regular

2.1 Definition of 3-Sum

Definition 9. Let $B_l^{(0)} \in \mathbb{Z}_2^{(X_l \cup \{x_0, x_1\}) \times (Y_l \cup \{y_2\})}$, $B_r^{(0)} \in \mathbb{Z}_2^{(X_r \cup \{x_2\}) \times (Y_r \cup \{y_0, y_1\})}$ be matrices of the form

$$B_l^{(0)} = \begin{array}{|cc|c|}
\hline
\multicolumn{2}{|c|}{A_l^{(0)}} & 0 \\
 & 1 \quad 1 & 0 \\
\hline
D_l^{(0)} & D_0^{(0)} & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \\
\hline
\end{array}, \quad B_r^{(0)} = \begin{array}{|c|cc|}
\hline
1 \quad 1 & 0 & 0 \\
\hline
D_0^{(0)} & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & A_r^{(0)} \\
\hline
D_r^{(0)} & & \\
\hline
\end{array},$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Let $D_{lr}^{(0)} = D_r^{(0)} \cdot (D_0^{(0)})^{-1} \cdot D_l^{(0)}$ (note that $D_0^{(0)}$ is invertible by construction). Then the 3-sum of $B_l^{(0)}$ and $B_r^{(0)}$ is

$$B^{(0)} = B_l^{(0)} \oplus_3 B_r^{(0)} = \begin{array}{|cc|cc|}
\hline
\multicolumn{2}{|c|}{A_l^{(0)}} & 0 & \\
 & 1 \quad 1 & 0 & \\
\hline
D_l^{(0)} & D_0^{(0)} & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & A_r^{(0)} \\
D_{lr}^{(0)} & D_r^{(0)} & & \\
\hline
\end{array} \in \mathbb{Z}_2^{(X_l \cup X_r) \times (Y_l \cup Y_r)}.$$

Here $x_2 \in X_l$, $x_0, x_1 \in X_r$, $y_0, y_1 \in Y_l$, $y_2 \in Y_r$, $A_l^{(0)} \in \mathbb{Z}_2^{X_l \times Y_l}$, $A_r^{(0)} \in \mathbb{Z}_2^{X_r \times Y_r}$, $D_l^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times (Y_l \setminus \{y_0, y_1\})}$, $D_r^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times \{y_0, y_1\}}$, $D_0^{(0)} \in \mathbb{Z}_2^{\{x_0, x_1\} \times \{y_0, y_1\}}$, $D_{lr}^{(0)} \in \mathbb{Z}_2^{(X_r \setminus \{x_0, x_1\}) \times (Y_l \setminus \{y_0, y_1\})}$. The indexing is kept consistent between $B_l^{(0)}$, $B_r^{(0)}$, and $B^{(0)}$. To simplify notation, we use the following shorthands:

$$D_{l,lr}^{(0)} = \begin{array}{|c|} \hline D_l^{(0)} \\ \hline D_{lr}^{(0)} \\ \hline \end{array}, \quad D_{0,r}^{(0)} = \begin{array}{|c|} \hline D_0^{(0)} \\ \hline D_r^{(0)} \\ \hline \end{array}, \quad D_{l,0}^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline \end{array}, \quad D_{lr,r}^{(0)} = \begin{array}{|c|c|} \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}, \quad D^{(0)} = \begin{array}{|c|c|} \hline D_l^{(0)} & D_0^{(0)} \\ \hline D_{lr}^{(0)} & D_r^{(0)} \\ \hline \end{array}.$$

The following lemma justifies the additional assumption on the entries of $D_0^{(0)}$. can omit

Lemma 10. Let $D_0^{(0)} \in \mathbb{Z}_2^{2 \times 2}$ be non-singular. Then (up to row and column indices)

$$D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{or} \quad D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

Proof. Verify by complete enumeration. $\square$ need details?

2.2 Construction of Canonical Signing

Definition 11. We call B_l and B_r canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively, if they have the form

$$B_l = \begin{array}{|c|cc|c|} \hline \multicolumn{3}{|c|}{A_l} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_l & \multicolumn{2}{c|}{D_0} & \begin{array}{c} 1 \\ \hline 1 \end{array} \\ \hline \end{array}\ , \quad B_r = \begin{array}{|cc|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & \begin{array}{c} 1 \\ \hline 1 \end{array} & \\ \cline{1-3} \multicolumn{2}{|c|}{D_r} & \multicolumn{2}{c|}{A_r} \\ \hline \end{array}$$

where every block in B_l and B_r is a signing of the corresponding block in $B_l^{(0)}$ and $B_r^{(0)}$, and D_0 is the canonical signing of $D_0^{(0)}$, which is defined as follows:

$$\text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & -1 \\ \hline \end{array}, \quad \text{if } D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \text{ then } D_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}.$$

Given canonical signings B_l and B_r, the corresponding canonical signing of $B^{(0)}$ is defined as

$$B = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{|c|}{A_l} & 0 \\ \cline{2-3} & 1\ 1 & 0 & \\ \hline D_l & D_0 & \begin{array}{c} 1 \\ \hline 1 \end{array} & \\ \cline{1-3} D_{lr} & D_r & \multicolumn{2}{c|}{A_r} \\ \hline \end{array}$$

where $D_{lr} = D_r \cdot (D_0)^{-1} \cdot D_l$ (calculated over $\mathbb{Q}$).

The following lemma helps construct canonical signings from arbitrary initial TU signings.

Lemma 12. Let Q' be a TU signing of the matrix

$$T = \begin{array}{|cc|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0^{(0)}} & \begin{array}{c} 1 \\ \hline 1 \end{array} \\ \hline \end{array} \in \mathbb{Z}_2^{(x_0,x_1,x_2)\times(y_0,y_1,y_2)}$$

where $D_0^{(0)}(x_0, y_0) = 1$, $D_0^{(0)}(x_0, y_1) \in \{0, 1\}$, $D_0^{(0)}(x_1, y_0) = 0$, and $D_0^{(0)}(x_1, y_1) = 1$. Define $u \in \{0, \pm 1\}^{\{x_0,x_1,x_2\}}$, $v \in \{0, \pm 1\}^{\{y_0,y_1,y_2\}}$, and Q as follows:

$$\begin{gathered} u(x_0) = Q'(x_2, y_0) \cdot Q'(x_0, y_0), \\ u(x_1) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), \\ u(x_2) = 1, \\ v(y_0) = Q'(x_2, y_0), \\ v(y_1) = Q'(x_2, y_1), \\ v(y_2) = Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), \\ \forall i \in \{x_0, x_1, x_2\},\ \forall j \in \{y_0, y_1, y_2\}, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j). \end{gathered}$$

Then Q is a TU signing of T and $Q = \begin{array}{|cc|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & \begin{array}{c} 1 \\ \hline 1 \end{array} \\ \hline \end{array}$ where D_0 is the respective canonical signing of $D_0^{(0)}$.

Proof. Since Q' is a TU signing of T and Q is obtained from Q' by multiplying rows and columns by ± 1 factors, Q is also a TU signing of T. By construction, we have

$$
\begin{aligned}
&Q(x_2, y_0) = Q'(x_2, y_0) \cdot 1 \cdot Q'(x_2, y_0) = 1,\\
&Q(x_2, y_1) = Q'(x_2, y_1) \cdot 1 \cdot Q'(x_2, y_1) = 1,\\
&Q(x_2, y_2) = 0,\\
&Q(x_0, y_0) = Q'(x_0, y_0) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_0) = 1,\\
&Q(x_0, y_1) = Q'(x_0, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot Q'(x_2, y_1),\\
&Q(x_0, y_2) = Q'(x_0, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1,\\
&Q(x_1, y_0) = 0,\\
&Q(x_1, y_1) = Q'(x_1, y_1) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_1)),\\
&Q(x_1, y_2) = Q'(x_1, y_2) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2)) \cdot (Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2)) = 1.
\end{aligned}
$$

Thus, it remains to check that $Q(x_0, y_1)$ and $Q(x_1, y_1)$ are correct.

First, consider the entry $Q(x_0, y_1)$. If $D_0^{(0)}(x_0, y_1) = 0$, then $Q(x_0, y_1) = 0$, as needed. Otherwise, if $D_0^{(0)}(x_0, y_1) = 1$, then $Q(x_0, y_1) \in \{\pm 1\}$, as Q is a signing of T. Our goal is to show that $Q(x_0, y_1) = 1$. For the sake of deriving a contradiction suppose that $Q(x_0, y_1) = -1$. Then the determinant of the submatrix of Q indexed by $\{x_0, x_2\} \times \{y_0, y_1\}$ is

$$\det \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q. Thus, $Q(x_0, y_1) = 1$, as needed.

Consider the entry $Q(x_1, y_1)$. Since Q is a signing of T, we have $Q(x_1, y_1) \in \{\pm 1\}$. Note that we know all the other entries of Q, so we can determine the sign of $Q(x_1, y_1)$ using TUness of Q. Consider two cases.

1. Suppose that $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q(x_1, y_1) = 1$, then $\det Q = \det \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q. Thus, $Q(x_1, y_1) = -1$, as needed.

2. Suppose that $D_0^{(0)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$. If $Q(x_1, y_1) = -1$, then $\det Q = \det \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & -1 & 1 \\ \hline \end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q. Thus, $Q(x_1, y_1) = 1$, as needed.

□

Definition 13. Let X and Y be sets with $\{x_0, x_1, x_2\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q' \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q^{(0)} \in \mathbb{Z}_2^{X \times Y}$. Let $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q be constructed as follows:

$$
u(i) = \begin{cases} Q'(x_2, y_0) \cdot Q'(x_0, y_0), & i = x_0,\\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2) \cdot Q'(x_1, y_2), & i = x_1,\\ 1, & i = x_2,\\ 1, & i \in X \setminus \{x_0, x_1, x_2\}, \end{cases}
$$

$$
v(j) = \begin{cases} Q'(x_2, y_0), & j = y_0,\\ Q'(x_2, y_1), & j = y_1,\\ Q'(x_2, y_0) \cdot Q'(x_0, y_0) \cdot Q'(x_0, y_2), & j = y_2,\\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}
$$

$$\forall i \in X,\ \forall j \in Y, \quad Q(i, j) = Q'(i, j) \cdot u(i) \cdot v(j).$$

We call Q a canonical resigning of Q'.

Lemma 14. Let B_l' be a TU signing of $B_l^{(0)}$. Let B_l be the canonical resigning (constructed following Definition 13) of B_l'. Then B_l is a canonical signing of $B_l^{(0)}$ (in the sense of Definition 11) and B_l is TU. Going forward, we refer to B_l as a TU canonical signing for short of $B_l^{(0)}$. A TU canonical signing B_r of $B_r^{(0)}$ is defined similarly (up to replacing subscripts 1 by 2).

Proof. This follows directly from Lemma 12. $\square$

Lemma 15. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = (D_{0,r})_{\bullet,y_0}$ and $c_1 = (D_{0,r})_{\bullet,y_1}$. Then the following matrices are TU:

$$B_r^{(a)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_0 & A_r \\ \hline\end{array}, \quad B_r^{(b)} = \begin{array}{|c|c|c|}\hline c_0 - c_1 & c_1 & A_r \\ \hline\end{array}.$$

Proof. Pivoting in B_r on (x_2, y_0) and (x_2, y_1) yields:

$$B_r = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_r \\ \hline\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -c_0 & c_1 - c_0 & A_r \\ \hline\end{array}$$

$$B_r = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 & c_1 & A_r \\ \hline\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline c_0 - c_1 & -c_1 & A_r \\ \hline\end{array}$$

By removing row x_2 from the resulting matrices and then multiplying columns y_0 and y_1 by $\{\pm 1\}$ factors, we obtain $B_r^{(a)}$ and $B_r^{(b)}$. By Lemma 14, B_r is TU. Since TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $B_r^{(a)}$ and $B_r^{(b)}$ are TU. $\square$

Lemma 16. Let B_r be a TU canonical signing of $B_r^{(0)}$. Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$. Then the following properties hold.

1. For every $i \in X_r$, we have $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}\}$.

2. $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU.

Proof. 1. Let $i \in X_r$. If $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has det $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & -1 \\ \hline\end{array} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r (which holds by Lemma 14). Similarly, if $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}$, then the 2×2 submatrix of B_r indexed by $\{x_2, i\} \times \{y_0, y_1\}$ has det $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline -1 & 1 \\ \hline\end{array} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of B_r.

2. Let V be a square submatrix of $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$. We will show that $\det V \in \{0, \pm 1\}$.

   Let z denote the index of the appended column c_2. Suppose that column z is not in V. Then V is a submatrix of B_r, which is TU by Lemma 14. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

   Suppose that columns y_0 and y_1 are both in V. Then V contains columns z, y_0, and y_1, which are linearly dependent by construction of c_2. Thus, $\det V = 0$. Going forward we assume that at most one of the columns y_0 and y_2 is in V.

   Suppose that column y_0 is in V. Then V is a submatrix of $B_r^{(b)}$ from Lemma 15, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $B_r^{(a)}$ from Lemma 15, and so $\det V \in \{0, \pm 1\}$.

   Thus, every square submatrix V of $\tilde{T}$ has $\det V \in \{0, \pm 1\}$, and hence $\tilde{T}$ is TU.

$\square$

Remark 17. Vectors c_0, c_1, and c_2 can be defined directly in terms of entries of B_r, e.g., c_2 consists of entries of B_r indexed by $(X_r \setminus \{x_2\}) \times \{y_0\}$.

Lemma 18. Let B_l be a TU canonical signing of $B_l^{(0)}$. Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$. Then the following properties hold.

1. For every $j \in Y_r$, we have $\begin{array}{|c|}\hline d_0(i) \\ \hline d_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{x_0, x_1\}} \setminus \{\begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}, \begin{array}{|c|}\hline -1 \\ \hline 1 \\ \hline\end{array}\}$.

2. $\begin{array}{|c|}\hline A_l \\ \hline d_0 \\ \hline d_1 \\ \hline d_2 \\ \hline\end{array}$ is TU.

Proof. Apply Lemma 16 to $B_l^\top$, or repeat the same argument up to interchanging rows and columns. □

Lemma 19. Let B_l and B_r be TU canonical signings of $B_l^{(0)}$ and $B_r^{(0)}$, respectively.

- Let $c_0 = D_{0,r}(\bullet, y_0)$, $c_1 = D_{0,r}(\bullet, y_1)$, and $c_2 = c_0 - c_1$.
- Let $d_0 = D_{l,0}(x_0, \bullet)$, $d_1 = D_{l,0}(x_1, \bullet)$, and $d_2 = d_0 - d_1$.
- If $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$, let $r_0 = d_0$, $r_1 = -d_1$, $r_2 = d_2$. If $D_0^{(0)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array}$, let $r_0 = d_2$, $r_1 = d_1$, $r_2 = d_0$.
- Let D be the bottom-left block in the canonical signing B of $B^{(0)}$ corresponding to B_l and B_r

Then the following properties hold.

1. $D = c_0 \cdot r_0 + c_1 \cdot r_1$.

2. Rows of D are in $\begin{array}{|c|}\hline \pm r_0 \\ \hline \pm r_1 \\ \hline \pm r_2 \\ \hline 0 \\ \hline\end{array}$.

3. Columns of D are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm c_2 & 0 \\ \hline\end{array}$.

4. $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU.

5. $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ is TU.

6. $\begin{array}{|c|}\hline A_l \\ \hline r_0 \\ \hline r_1 \\ \hline r_2 \\ \hline\end{array}$ is TU.

7. $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$ is TU.

8. $\begin{array}{|c|c|}\hline c_0 & c_1 \\ \hline\end{array}$ contains D_0 (the canonical signing of $D_0^{(0)}$) as a submatrix.

Proof. 1. Follows via a direct calculation. need details?

2. By item 1, for every $i \in X_r$ we have $D(i, \bullet) = c_0(i) \cdot r_0 + c_1(i) \cdot r_1$. By Lemma 15.1, we know that $\begin{array}{|c|c|}\hline c_0(i) & c_1(i) \\ \hline\end{array} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}\}$. Therefore, $D(i, \bullet)$ is equal to either 0, $\pm r_0$, $\pm r_1$, or $\pm(r_0 + r_1) = \pm r_2$.

3. Holds by the same argument as item 2 up to interchanging rows and columns.

4. Holds by Lemma 16.2.
5. By item 3, columns of $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ are in $\begin{array}{|c|c|c|c|c|}\hline A_r & \pm c_0 & \pm c_1 & \pm c_2 & 0 \\ \hline\end{array}$. Since $\begin{array}{|c|c|c|c|}\hline A_r & c_0 & c_1 & c_2 \\ \hline\end{array}$ is TU and since adding zero columns and copies of columns multiplied by ± 1 factors preserves TUness, $\begin{array}{|c|c|}\hline A_r & D \\ \hline\end{array}$ is also TU.
6. By Lemma 18.2 (or by the same argument as item 4 up to interchanging rows and columns), $\begin{array}{|c|}\hline A_l \\ \hline d_0 \\ \hline d_1 \\ \hline d_2 \\ \hline\end{array}$ is TU. Since TUness is preserved under multiplication of rows by ± 1 and exchanging rows, $\begin{array}{|c|}\hline A_l \\ \hline r_0 \\ \hline r_1 \\ \hline r_2 \\ \hline\end{array}$ is also TU.
7. Holds by the same argument as item 5 up to interchanging rows and columns.
8. Holds by construction.

$\square$

Definition 20. Let $A_l \in \mathbb{Q}^{X_l \cup Y_l}$, $A_r \in \mathbb{Q}^{X_r \cup Y_r}$, $c_0, c_1 \in \mathbb{Q}^{X_r}$, $r_0, r_1 \in \mathbb{Q}^{Y_l}$. Let $D = c_0 \cdot r_0 + c_1 \cdot r_1$. Suppose that properties 2–8 from the statement of Lemma 19 are satisfied for A_l, A_r, c_0, c_1, r_0, r_1. Given $k \in \mathbb{Z}_{\geq 1}$, define Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, k)$ to mean "$C = \begin{array}{|c|c|}\hline A_l & 0 \\ \hline D & A_r \\ \hline\end{array}$ is k-TU".

Lemma 21. Assume the notation of Definition 20. Then Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, 1)$ holds.

Proof. Every entry of C is in one of four blocks: 0, A_l, D, A_r. By the assumptions of Definition 20, all of these blocks are TU. Thus, C is 1-TU. $\square$

Lemma 22. Assume the notation of Definition 20. Let $i \in X_l$, let $T = \begin{array}{|c|}\hline A_l(i, \bullet) \\ \hline D \\ \hline\end{array}$. Suppose we pivot on entry $T(i, j) \in \{\pm 1\}$ in T and obtain matrix $T' = \begin{array}{|c|}\hline a' \\ \hline D' \\ \hline\end{array}$. Then columns of D' are in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$.

Proof. Since T is a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$, which is TU by assumptions of Definition 20, we have that T is TU. Since pivoting preserves TUness, T' is also TU. To prove the claim, perform an exhaustive case distinction on what pivot column p in T could be and what another column q in T could be. This uniquely determines the resulting columns p' and q' in T' by the pivot formula. In every case, either $\begin{array}{|c|c|}\hline p' & q' \\ \hline\end{array}$ contains a submatrix with determinant not in $\{0, \pm 1\}$, which contradicts TUness of T', or the restriction of p' and q' to X_r is in $\begin{array}{|c|c|c|c|}\hline \pm c_0 & \pm c_1 & \pm(c_0 - c_1) & 0 \\ \hline\end{array}$). $\square$ need details?

Lemma 23. Assume the notation of Definition 20. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose Proposition$(A_l', A_r, c_0, c_1, r_0', r_1', k-1)$ holds for all A_l', r_0', and r_1' satisfying the assumptions of Definition 20 (together with A_r, c_0, and c_1). Then Proposition$(A_l, A_r, c_0, c_1, r_0, r_1, k)$ holds.

Proof. Let V be a $k \times k$ submatrix of C. For the sake of deriving a contradiction assume that $\det V \notin \{0, \pm 1\}$.

Suppose that V is a submatrix of $\begin{array}{|c|}\hline A_l \\ \hline D \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline A_l & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline D & A_r \\ \hline\end{array}$, or $\begin{array}{|c|}\hline 0 \\ \hline A_r \\ \hline\end{array}$. Since all of those four matrices are TU by the assumptions of Definition 20, we have $\det V \in \{0, \pm 1\}$. Thus, V shares at least one row and one column index with A_l and A_r each.

Consider the row index shared by V and A_l. Note that this row in V cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus, there exists a ± 1 entry shared by V and A_l. Let i and j denote the row and the column index of this entry, respectively.

Perform a pivot in C on the element $C(i, j)$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a prime; for example, B' denotes the entire matrix after the pivot. Note that the following statements hold.

- C' contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This holds by the same argument as for the 2-sum: look at the submatrix V' of C' with the same row and column index sets as V minus the pivot row i and pivot column j.

- $C' = \begin{array}{|c|c|} \hline A'_l & 0 \\ \hline D' & A_r \\ \hline \end{array}$, i.e., the 0 and the A_r blocks remain unchanged. This holds by the same argument as for the 2-sum: the pivot row is in the 0 block.

- $\begin{array}{|c|} \hline A'_l \\ \hline D' \\ \hline \end{array}$ is TU. This holds by the same argument as for the 2-sum: TUness is preserved under pivoting.

- The columns of D' are in $\begin{array}{|c|c|c|c|} \hline 0 & \pm c_0 & \pm c_1 & \pm(c_0 - c_1) \\ \hline \end{array}$. This holds by Lemma 22.

- There exist r'_0 and r'_1 such that $D' = c_0 \cdot r'_0 + c_1 \cdot r'_1$ and the assumptions of Definition 20 are satisfied for A'_l, A_r, c_0, c_1, r'_0, r'_1. This follows from the previous bullet point by carefully checking assumptions. need details?

- C' is $(k-1)$-TU. This follows from the hypothesis: Proposition$(A'_l, A_r, c_0, c_1, r'_0, r'_1, k-1)$ holds.

To sum up, after pivoting we obtain a matrix C' (which can be obtained in the manner of Definition 20) that is $(k-1)$-TU and contains a $(k-1) \times (k-1)$ submatrix V' with $\det V' \notin \{0, \pm 1\}$. This contradiction proves the lemma. □

Lemma 24. Let B_l and B_r be TU canonical signings. Then the corresponding canonical signing B is TU.

Proof. Define A_l, A_r, c_0, c_1, r_0, r_1 as in Lemma 19. Note that canonical signing B has the form of C in the notation of Definition 20.

Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: Proposition$(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$ holds for all A'_l, r'_0, and r'_1 satisfying the assumptions of Definition 20.

Base: The Proposition holds for $k = 1$ by Lemma 21.

Step: If the Proposition holds for some k, then it also holds for $k+1$ by Lemma 23.

Conclusion: Proposition$(A'_l, A_r, c_0, c_1, r'_0, r'_1, k)$ holds for all $k \in \mathbb{Z}_{\geq 1}$.

Specializing the conclusion to A_l, A_r, c_0, c_1, r_0, r_1 (obtained from B_l and B_r as described in the statement of Lemma 19) shows that canonical signing B is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, B is TU. □

Corollary 25. Suppose that $B_l^{(0)}$ and $B_r^{(0)}$ have TU signings. Then $B_l \oplus_3 B_r$ has a TU signing.

Proof sketch. Start with some TU signings, obtain canonical signings, apply Lemma 24. □