Proof of Regularity of 2- and 3-Sum of Matroids

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1 2-Sum of Regular Matroids Is Regular

Lemma 1. *Let A be a $k \times k$ matrix. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, such that $a_{rc} \neq 0$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $|\det A''| = \frac{|\det A|}{|a_{rc}|}$.*

Proof. Let A'' be the submatrix of A' given by row index set $R = \{1, \dots, k\} \setminus \{r\}$ and column index set $C = \{1, \dots, k\} \setminus \{c\}$. By the explicit formula for pivoting in A on a_{rc}, the entries of A'' are given by $a''_{ij} = a_{ij} - \frac{a_{ic}a_{rj}}{a_{rc}}$. Using the linearity of the determinant, we can express $\det A''$ as

$$\det A'' = \det A' - \sum_{k \in C} \frac{a_{rk}}{a_{rc}} \det B''_k$$

where B''_k is a matrix obtained from A'' by replacing column $a''_{\cdot k}$ with the pivot column $a_{\cdot c}$ without the pivot element a_{rc}.

By the cofactor expansion in A along row r, we have

$$\det A = \sum_{k=1}^{n} (-1)^{r+k} a_{rk} \det B_{r,k}$$

where $B_{r,k}$ is obtained from A by removing row r and column k. By swapping the order of columns in $B_{r,k}$ to match the form of B_k, we get

$$\det A = (-1)^{r+c}(a_{rc} \det A' - \sum_{k \in C} a_{rk} \det B''_k).$$

By combining the above results, we get $|\det A''| = \frac{|\det A|}{|a_{rc}|}$. □

Corollary 1. Let A be a $k \times k$ matrix with $\det A \notin \{0, \pm 1\}$. Let $r, c \in \{1, \dots, k\}$ be a row and column index, respectively, and suppose that $a_{rc} \in \{\pm 1\}$. Let A' denote the matrix obtained from A by performing a real pivot on a_{rc}. Then there exists a $(k-1) \times (k-1)$ submatrix A'' of A' with $\det A'' \notin \{0, \pm 1\}$.

Proof. Since $a_{rc} \in \{\pm 1\}$, by Lemma 1 there exists a $(k-1) \times (k-1)$ submatrix A'' with $|\det A| = |\det A''|$. Since $\det A \notin \{0, \pm 1\}$, we have $\det A'' \notin \{0, \pm 1\}$. □

Definition 1. Let B_1, B_2 be matrices with $\{0, \pm 1\}$ entries expressed as $B_1 = [A_1/x]$ and $B_2 = [y \mid A_2]$, where x is a row vector, y is a column vector, and A_1, A_2 are matrices of appropriate dimensions. Let D be the outer product of y and x. The 2-sum of B_1 and B_2 is defined as

$$B_1 \oplus_{2,x,y} B_2 = \begin{bmatrix} A_1 & 0 \\ D & A_2 \end{bmatrix}.$$

Definition 2. Given $k \in \mathbb{Z}_{\geq 1}$, we say that a matrix A is k-TU if every square submatrix of A of size k has determinant in $\{0, \pm 1\}$.

Remark 1. Note that a matrix is TU if and only if it is k-TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 2. *Let B_1 and B_2 be TU matrices and let $B = B_1 \oplus_{2,x,y} B_2$. Then B is 1-TU and 2-TU.*

Proof. To see that B is 1-TU, note that B is a $\{0, \pm 1\}$ matrix by construction.

To show that B is 2-TU, let V be a 2×2 submatrix V of B. If V is a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both A_1 and A_2. Let i be the row shared by V and A_1 and j be the column shared by V and A_2. Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij}. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. □

Lemma 3. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell < k$. Then for any TU matrices B_1 and B_2 their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is also k-TU.*

Proof. For the sake of deriving a contradiction, suppose there exist TU matrices B_1 and B_2 such that their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is not k-TU. Then B contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $[A_1/D]$, $[D \mid A_2]$, $[A_1 \mid 0]$, or $[0/A_2]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with A_1 and A_2 each.

Consider the row of V whose index appears in A_1. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and A_1. Let r and c denote the row and column index of this entry, respectively.

Perform a rational pivot in B on the element B_{rc}. For every object, its modified counterpart after pivoting is denoted by the same symbol with an added tilde; for example, $\tilde{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\left[\tilde{A}_1/\tilde{D}\right]$ is TU, since TUness is preserved by pivoting.
- $\tilde{A}_2 = A_2$, i.e., A_2 remains unchanged. This holds because of the 0 block in B.
- $\tilde{D}$ consists of copies of y scaled by factors in $\{0, \pm 1\}$. This can be verified via a case distinction and a simple calculation.
- $\left[\tilde{D} \mid \tilde{A}_2\right]$ is TU, since this matrix consists of A_2 and copies of y scaled by factors $\{0, \pm 1\}$.
- $\tilde{D}$ can be represented as an outer product of a column vector $\tilde{y}$ and a row vector $\tilde{x}$, and we can define $\tilde{B}_1 = \left[\tilde{A}_1/\tilde{x}\right]$ and $\tilde{B}_2 = \left[\tilde{y} \mid \tilde{A}_2\right]$ similar to B_1 and B_2, respectively. Note that $\tilde{B}_1$ and $\tilde{B}_2$ have the same size as B_1 and B_2, respectively, are both TU, and satisfy $\tilde{B} = \tilde{B}_1 \oplus_{2,\tilde{x},\tilde{y}} \tilde{B}_2$.
- $\tilde{B}$ contains a square submatrix $\tilde{V}$ of size $k-1$ with $\det \tilde{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element B_{rc} results in a matrix containing a $(k-1) \times (k-1)$ submatrix V'' with $\det V'' \in \{0, \pm 1\}$. Since V is a submatrix of B, the submatrix V'' corresponds to a submatrix $\tilde{V}$ of $\tilde{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\tilde{B}$ that represents a 2-sum of TU matrices $\tilde{B}_1$ and $\tilde{B}_2$ and contains a square submatrix of size $k-1$ with determinant not in $\{0, \pm 1\}$. This is a contradiction with $(k-1)$-TUness of $\tilde{B}$, which proves the lemma. □

Lemma 4. *Let B_1 and B_2 be TU matrices. Then $B_1 \oplus_{2,x,y} B_2$ is also TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any TU matrices B_1 and B_2, their 2-sum $B = B_1 \oplus_{2,x,y} B_2$ is ℓ-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 2.

Step: If the Proposition holds for some k, then it also holds for $k+1$ by Lemma 3.

Conclusion: For any TU matrices B_1 and B_2, their 2-sum $B_1 \oplus_{2,x,y} B_2$ is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $B_1 \oplus_{2,x,y} B_2$ is TU. □

2 3-Sum of Regular Matroids Is Regular: Streamlined

Definition 3. Let $B_1 \in \mathbb{Z}_2^{(X_1 \cup \{x_2,x_3\}) \times (Y_1 \cup \{y_3\})}$, $B_2 \in \mathbb{Z}_2^{(\{x_1\} \cup X_2) \times (\{y_1,y_2\} \cup Y_2)}$ be matrices of the form

$$B_1 = \left[\begin{array}{c|c|c} \multicolumn{2}{c|}{A_1} & 0 \\ & 1 \;\; 1 & 0 \\ \hline D_1 & \overline{D} & \begin{matrix} 1 \\ 1 \end{matrix} \end{array}\right], \quad B_2 = \left[\begin{array}{c|c|c} 1 \;\; 1 & 0 & 0 \\ \hline \overline{D} & \begin{matrix} 1 \\ 1 \end{matrix} & A_2 \\ \hline D_2 & \multicolumn{2}{c}{} \end{array}\right],$$

where $\overline{D}$ is a 2×2 matrix with $\mathbb{Z}_2$ rank 2 (i.e., $\overline{D}$ is non-singular over $\mathbb{Z}_2$). Note that $x_1 \in X_1$, $x_2, x_3 \in X_2$, $y_1, y_2 \in Y_1$, $y_3 \in Y_2$, $A_1 \in \mathbb{Z}_2^{X_1 \times Y_1}$, $A_2 \in \mathbb{Z}_2^{X_2 \times Y_2}$, $\overline{D} \in \mathbb{Z}_2^{(x_2,x_3) \times (y_1,y_2)}$, $D_1 \in \mathbb{Z}_2^{\{x_2,x_3\} \times (Y_1 \setminus \{y_1,y_2\})}$, $D_2 \in \mathbb{Z}_2^{(X_2 \setminus \{x_2,x_3\}) \times \{y_1,y_2\}}$. Then the 3-sum of B_1 and B_2 is defined as

$$B_1 \oplus_3 B_2 = \left[\begin{array}{c|c|c|c} \multicolumn{2}{c|}{A_1} & \multicolumn{2}{c}{0} \\ & 1 \;\; 1 & 0 & \\ \hline D_1 & \overline{D} & \begin{matrix} 1 \\ 1 \end{matrix} & A_2 \\ \hline D_{12} & D_2 & & \end{array}\right],$$

where $D_{12} = D_2 \cdot (\overline{D})^{-1} \cdot D_1$ and the indexing is preserved.

To simplify notation, we write

$$D_{1,12} = [D_1 / D_{12}], \quad D_{0,2} = [\overline{D} / D_2], \quad D_{1,0} = [D_1 \mid \overline{D}], \quad D_{12,2} = [D_{12} \mid D_2], \quad D = \begin{array}{|c|c|} \hline D_1 & \overline{D} \\ \hline D_{12} & D_2 \\ \hline \end{array}$$

Lemma 5. *Suppose B_2 from Definition 3 has a TU signing. Then we can construct a TU signing $\tilde{B}_2$ of B_2 where all entries in columns y_1 and y_2 are in $\{0,1\}$.*

Proof. Since B_2 is regular, it has a TU signing B_2'. Recall that multiplying rows and columns of a TU matrix by factors in $\{0, \pm 1\}$ preserves TUness.

If $B_2'(x_1, y_1) = -1$, multiply column y_1 by -1. Similarly, if $B_2'(x_1, y_2) = -1$, multiply column y_2 by -1. Thus, we may assume that B_2' has $B_2'(x_1, y_1) = B_2'(x_1, y_2) = 1$.

Next, consider each row of B_2'. It can have one of the following forms.

- $[0 \mid 0]$, $[0 \mid 1]$, $[1 \mid 0]$, $[1 \mid 1]$. In this case, we do not need to modify the signing.
- $[0 \mid -1]$, $[-1 \mid 0]$, $[-1 \mid -1]$. In this case, we can multiply this row by -1 to make all its entries non-negative.
- $[1 \mid -1]$, $[-1 \mid 1]$. This case leads to a contradiction, as the matrix composed of this row and row x_1 has
$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \quad \text{or} \quad \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2,$$
which is impossible as B_2' is a TU signing.

Thus, we can multiply columns and rows of B_2' to obtain a TU signing $\tilde{B}_2$ where all entries in columns y_1 and y_2 are in $\{0,1\}$, as desired. $\square$

Lemma 6. *Suppose B_2 from Definition 3 has a TU signing. Let $\tilde{B}_2$ be a TU signing of B_2 from Lemma 5. To simplify notation, let $\tilde{a} = (\tilde{D}_{0,2})_{\cdot y_1}$ and $\tilde{b} = (\tilde{D}_{0,2})_{\cdot y_2}$. Then the following matrices are TU:*

$$\tilde{B}_2^{(a)} = \left[\begin{array}{c|c|c} \tilde{a} - \tilde{b} & \tilde{a} & \tilde{A}_2 \end{array}\right], \quad \tilde{B}_2^{(b)} = \left[\begin{array}{c|c|c} \tilde{a} - \tilde{b} & \tilde{b} & \tilde{A}_2 \end{array}\right].$$

Proof. Recall that pivoting in matrix A on entry $a_{rc} \neq 0$ transforms the matrix as follows:

$$\begin{array}{|c|c|}\hline a_{rc} & a_{rj} \\ \hline a_{ic} & a_{ij} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|}\hline \frac{1}{a_{rc}} & \frac{a_{rj}}{a_{rc}} \\ \hline -\frac{a_{ic}}{a_{rc}} & a_{ij} - \frac{a_{rj}a_{ic}}{a_{rc}} \\ \hline \end{array}$$

Pivoting in $\tilde{B}_2$ on (x_1, y_1) and (x_1, y_2) yields:

$$\tilde{B}_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline -\tilde{a} & \tilde{b} - \tilde{a} & \tilde{A}_2 \\ \hline \end{array}$$

$$\tilde{B}_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \tilde{a} & \tilde{b} & \tilde{A}_2 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline \tilde{a} - \tilde{b} & -\tilde{b} & \tilde{A}_2 \\ \hline \end{array}$$

By removing row x_1 from the resulting matrices and then multiplying columns y_1 and y_2 by $\{\pm 1\}$ factors, we obtain $\tilde{B}_2^{(a)}$ and $\tilde{B}_2^{(b)}$. Since $\tilde{B}_2$ is TU and TUness is preserved under pivoting, taking submatrices, and multiplying columns by ± 1 factors, we conclude that $\tilde{B}_2^{(a)}$ and $\tilde{B}_2^{(b)}$ are TU. $\square$

Lemma 7. *Suppose B_2 from Definition 3 has a TU signing. Then we can construct signings $\tilde{A}_2$, $\tilde{D}_{0,2}$, and $\tilde{d}$ of A_2, $D_{0,2}$, and $d = D_{\cdot y_1} + D_{\cdot y_2}$ respectively, such that $[\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2]$ is TU.*

Proof. Let $\tilde{B}_2$ be a TU signing of B_2 from Lemma 5. Let $\tilde{D}_{0,2}$ and $\tilde{A}_2$ denote the corresponding signings of $D_{0,2}$ and A_2. Let $\tilde{d} = (\tilde{D}_{0,2})_{\cdot y_1} - (\tilde{D}_{0,2})_{\cdot y_2}$. Since $\tilde{D}_{0,2} \in \{0,1\}^{X_2 \times \{y_1, y_2\}}$ by Lemma 5, we have

$$\tilde{d} = (\tilde{D}_{0,2})_{\cdot y_1} - (\tilde{D}_{0,2})_{\cdot y_2} = (D_{0,2})_{\cdot y_1} - (D_{0,2})_{\cdot y_2} = D_{\cdot y_1} - D_{\cdot y_2}.$$

Thus, $\tilde{d} \in \{0, \pm 1\}^{X_2}$ and $\tilde{d}$ is a signing of d. Our goal is to prove that $\tilde{T} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2] \in \{0, \pm 1\}^{X_2 \times (\{z, y_1, y_2\} \cup Y_2)}$ is TU. To this end, let V be a square submatrix of $\tilde{T}$. We will show that $\det V \in \{0, \pm 1\}$.

Suppose that column (with index) z (i.e., corresponding to $\tilde{d}$) is not in V. Then V is a submatrix of $\tilde{B}_2$, which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column (with index) z is in V.

Suppose that columns (with indices) y_1 and y_2 are both in V. Then V contains columns (with indices) z, y_1, and y_2, which are linearly dependent by construction of $\tilde{d}$. Thus, $\det V = 0$. Going forward we assume that at most one of the columns (with indices) y_1 and y_2 is in V.

Suppose that column (with index) y_1 is in V. Then V is a submatrix of $\tilde{B}_2^{(b)}$ from Corollary 6, and thus $\det V \in \{0, \pm 1\}$. Otherwise, V is a submatrix of $\tilde{B}_2^{(a)}$ from Corollary 6, and so $\det V \in \{0, \pm 1\}$.

Thus, every square submatrix V of $\tilde{T}$ has $\det V \in \{0, \pm 1\}$, and hence $\tilde{T}$ is TU. $\square$

Lemma 8. *Assume the notation of Definition 3 and let $d = D_{\cdot y_1} + D_{\cdot y_2} \in \mathbb{Z}_2^{X_2}$. Then the columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$.*

Proof. Columns of $[d \mid D_{0,2}]$ trivially satisfy the claim, so it only remains to show that columns of $D_{1,12}$ are in $[d \mid D_{0,2} \mid 0]$. Note that $D_{1,12} = D_{0,2} \cdot ((\overline{D})^{-1} \cdot D_1)$, i.e., every column of $D_{1,12}$ can be expressed as a linear combination of the columns of $D_{0,2}$ (over $\mathbb{Z}_2$). In particular, every column of $D_{1,12}$ is either zero, one of the columns of $D_{0,2}$, or their sum. By construction, $(D_{0,2})_{\cdot y_1} + (D_{0,2})_{\cdot y_2} = d$. Thus, the desired result holds. $\square$

Lemma 9. *Suppose B_2 from Definition 3 has a TU signing. Then we can construct signings $\tilde{A}_2$, $\tilde{D}$, and $\tilde{d}$ of A_2, D, and $d = D_{\cdot y_1} + D_{\cdot y_2}$, respectively, such that $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ is TU.*

Proof. By Lemma 8, columns of $[d \mid D]$ are in $[d \mid D_{0,2} \mid 0]$. Thus, columns of $U = [d \mid D \mid A_2]$ are in $T = [d \mid D_{0,2} \mid A_2 \mid 0]$. Let $\tilde{A}_2$, $\tilde{D}_{0,2}$, and $\tilde{d}$ be the signings from Lemma 7. Since adjoining zero columns does not affect TUness, $\tilde{T} = [\tilde{d} \mid \tilde{D}_{0,2} \mid \tilde{A}_2 \mid 0]$ is a TU signing of T.

We construct signing $\tilde{U}$ of U as follows. Let i be a column index in U. Then $U_{\cdot i} = T_{\cdot j}$ for some column index j in T, and we set $\tilde{U}_{\cdot i} = \tilde{T}_{\cdot j}$ with that j. By construction, $\tilde{U}$ consists of columns of $\tilde{T}$, so $\tilde{U}$ is a submatrix of $\tilde{T}$. Since $\tilde{T}$ is TU, $\tilde{U}$ is also TU. $\square$

Remark 2. Note that if a column of $[d \mid D \mid A_2]$ appears in $[d \mid D_{0,2} \mid A_2 \mid 0]$ multiple times, we may choose any of its occurrences when defining the signing.

Lemma 10. *Suppose B_1 and B_2 from Definition 3 have TU signings. Then we can construct signings $\tilde{B}$ and $\tilde{d}$ of $B = B_1 \oplus_3 B_2$ and $d = D_{\cdot y_1} + D_{\cdot y_2}$, respectively, such that $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ and $[\tilde{A}_1 / \tilde{D}]$ are both TU.*

Proof. Let $\tilde{A}_2$, $\tilde{D}$, and $\tilde{d}$ be signings from Lemma 9. Let $\tilde{B}_1$ be a TU signing of B_1. Use the procedure from the proof of Lemma 5 to multiply some rows and columns of $\tilde{B}_1$ and $\tilde{T} = [\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$ by $\{\pm 1\}$ factors so that all entries in the submatrix $\{x_2, x_3\} \times (Y_1 \cup \{y_3\})$ are non-negative in both $\tilde{B}_1$ and $\tilde{T}$. This procedure preserves the validity and TUness of both signings. Thus, by restricting $\tilde{B}_1$ we get a signing $\tilde{A}_1$ of A_1. By construction, $[\tilde{A}_1 / \tilde{D}_{1,0}]$ is a TU signing of $[A_1 / D_{1,0}]$.

It remains to prove that $[\tilde{A}_1 / \tilde{D}]$ is TU. Let V be a square submatrix of $[\tilde{A}_1 / \tilde{D}]$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

If V is a submatrix of only $[\tilde{A}_1 / \tilde{D}_{1,0}]$ or $\tilde{D}$, then $\det V \in \{0, \pm 1\}$, as those matrices are TU. Going forward, assume that V intersects $\tilde{A}_1$ and $\tilde{D}_{12,2}$.

finish

$\square$

Definition 4 (Repeats Definition 2)**.** We say that a matrix A is k-TU for $k \in \mathbb{Z}_{\geq 1}$ if every square submatrix T of A of size k has $\det T \in \{0, \pm 1\}$.

Remark 3. Note that a matrix is TU if and only if it is k-TU for every $k \in \mathbb{Z}_{\geq 1}$.

Lemma 11. *Suppose B_1 and B_2 from Definition 3 have TU signings. Let $\tilde{B}$ be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 10. Then $\tilde{B}$ is 1-TU and 2-TU.*

Proof. By construction,

$$\tilde{B} = \begin{array}{|c|c|} \hline \tilde{A}_1 & 0 \\ \hline \tilde{D} & \tilde{A}_2 \\ \hline \end{array}$$

where $[\tilde{A}_1 / \tilde{D}]$, $[\tilde{D} \mid \tilde{A}_2]$, $\tilde{A}_1$, $\tilde{D}$, and $\tilde{A}_2$ are all TU (by Lemma 10 and as submatrices of TU matrices). In particular, all entries of $\tilde{B}$ are in $\{0, \pm 1\}$, so $\tilde{B}$ is 1-TU.

To show that $\tilde{B}$ is 2-TU, let V be a 2×2 submatrix of $\tilde{B}$. If V is a submatrix of $[\tilde{A}_1 / \tilde{D}]$, $[\tilde{D} \mid \tilde{A}_2]$, $[\tilde{A}_1 \mid 0]$, or $[0 / \tilde{A}_2]$, then $\det V \in \{0, \pm 1\}$, as all of those four matrices are TU. Otherwise V shares exactly one row and one column index with both $\tilde{A}_1$ and $\tilde{A}_2$. Let i be the row shared by V and $\tilde{A}_1$ and j be the column shared by V and $\tilde{A}_2$. Note that $V_{ij} = 0$. Thus, up to sign, $\det V$ equals the product of the entries on the diagonal not containing V_{ij}. Since both of those entries are in $\{0, \pm 1\}$, we have $\det V \in \{0, \pm 1\}$. $\square$

Lemma 12. *Suppose B_1 and B_2 from Definition 3 have TU signings. Let $\tilde{B}$ be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 10. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose $\tilde{B}$ is ℓ-TU for every $\ell < k$. Then $\tilde{B}$ is also k-TU.*

Proof. For the sake of deriving a contradiction, suppose there exist B_1 and B_2 such that $\tilde{B}$ is not k-TU. Then $\tilde{B}$ contains a $k \times k$ submatrix V with $\det V \notin \{0, \pm 1\}$.

Note that V cannot be a submatrix of $\left[\tilde{A}_1 / \tilde{D}\right]$, $\left[\tilde{D} \mid \tilde{A}_2\right]$, $\left[\tilde{A}_1 \mid 0\right]$, or $\left[0 / \tilde{A}_2\right]$, as all of those four matrices are TU. Thus, V shares at least one row and one column index with $\tilde{A}_1$ and $\tilde{A}_2$ each.

Consider the row of V whose index appears in $\tilde{A}_1$. Note that it cannot consist of only 0 entries, as otherwise $\det V = 0$. Thus there exists a ± 1 entry shared by V and $\tilde{A}_1$. Let r and c denote the row and column index of this entry, respectively.

Perform a pivot in $\tilde{B}$ on the element $\tilde{B}_{rc}$. For every object, its modified counterpart after pivoting is denoted by the same symbol with a hat; for example, $\hat{B}$ denotes the entire matrix after the pivot. Note that after pivoting the following statements hold:

- $\left[\hat{A}_1 / \hat{D}\right]$ is TU, since TUness is preserved by pivoting.
- $\hat{A}_2 = \tilde{A}_2$, i.e., $\tilde{A}_2$ remains unchanged. This holds because of the 0 block in $\tilde{B}$.
- The columns of $\hat{D}$ are scaled (by $\{\pm 1\}$ factors) versions of columns of $[\tilde{d} \mid \tilde{D}]$. This can be verified via a direct calculation.

  need more details
- $\left[\hat{D} \mid \hat{A}_2\right]$ is TU, since up to scaling (i.e., multiplying columns by $\{\pm 1\}$ factors) (and taking parallel columns) it is a submatrix of $[\tilde{d} \mid \tilde{D} \mid \tilde{A}_2]$.
- $\hat{B}$ is $(k-1)$-TU.

  need to adjust assumptions of this lemma to be able to use $(k-1)$-TUness here
- $\hat{B}$ contains a square submatrix $\hat{V}$ of size $k-1$ with $\det \hat{V} \notin \{0, \pm 1\}$. Indeed, by Corollary 1 from Lemma 1, pivoting in V on the element $\tilde{B}_{rc}$ results in a matrix containing a $(k-1) \times (k-1)$ submatrix V' with $\det V' \in \{0, \pm 1\}$. Since V is a submatrix of $\tilde{B}$, the submatrix V' corresponds to a submatrix $\hat{V}$ of $\hat{B}$ with the same property.

To sum up, after pivoting we obtain a matrix $\hat{B}$ that is $(k-1)$-TU, but contains a square submatrix of size $k-1$ with determinant not in $\{0, \pm 1\}$. This contradiction proves the lemma. □

Lemma 13. *Suppose B_1 and B_2 from Definition 3 have TU signings. Let $\tilde{B}$ be a signing of $B = B_1 \oplus_3 B_2$ from Lemma 10. Then $\tilde{B}$ is TU.*

Proof. Proof by induction.

Proposition for any $k \in \mathbb{Z}_{\geq 1}$: For any matrices B_1 and B_2 that have TU signings, the signing $\tilde{B}$ of $B = B_1 \oplus_3 B_2$ from Lemma 10 is ℓ-TU for every $\ell \leq k$.

Base: The Proposition holds for $k = 1$ and $k = 2$ by Lemma 11.

Step: If the Proposition holds for some k, then it also holds for $k + 1$ by Lemma 12.

Conclusion: For any matrices B_1 and B_2 that have TU signings, the signing $\tilde{B}$ of $B = B_1 \oplus_3 B_2$ from Lemma 10 is k-TU for every $k \in \mathbb{Z}_{\geq 1}$. Thus, $\tilde{B}$ is TU. □