ST205A - HW1

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Problem 1. $\mathcal{F}_n \subset \mathcal{F}_{n+1}, \forall n \in \mathbb{N}$, prove $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field

Proof. 1. We will check the three condition of $\mathcal{F}$:
a. $\mathcal{F}_n \neq \emptyset \Rightarrow \mathcal{F} \neq \emptyset$
b. $\forall A \in \mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \Rightarrow \exists m \in \mathbb{N}, A \in \mathcal{F}_m \Rightarrow A^C \in \mathcal{F}_m \Rightarrow A^C \in \mathcal{F}$
c. $\forall A, B \in \mathcal{F} \Rightarrow \exists m, n \in \mathbb{N}, A \in \mathcal{F}_m, B \in \mathcal{F}_n$
Without loss of generality assume $m \leq n, \Rightarrow A, B \in \mathcal{F}_n \Rightarrow A \cup B \in \mathcal{F}_n \Rightarrow A \cup B \in \mathcal{F}$
Thus $\mathcal{F}$ is a field

2. However for the case of $\sigma - field$, $\mathcal{F}$ might not be the case. Counter example: Let $S = [0, 1)$ be the half open unit interval. Define:

$$\mathcal{F}_n = \sigma(\{[\frac{2^m}{2^n}, \frac{2^{m+1}}{2^n}) \mid m \in \{0, 1, ..., n-1\}\})$$

then $\mathcal{F}_n' s$ are $\sigma - field$. Assuming that $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$,we have:

$$\begin{aligned}
&[0, \frac{1}{2^n}) \in \mathcal{F}, \forall n \in \mathbb{N} \\
\Rightarrow &\bigcap_{n=0}^{\infty} [0, \frac{1}{2^n}) \in \mathcal{F} \\
\Rightarrow &\{0\} \in \mathcal{F} \\
\Rightarrow &\exists m \in \mathbb{N}, \{0\} \in \mathcal{F}_m
\end{aligned}$$

which is a contradiction because $\mathcal{F}_m$ is generated by m disjoint element each of them is countably infinite, so there is no boolean operation of them that could result in a finite nonempty set. □

Lemma 1. *Let Ω be a set, $\mathcal{A}$ is a collection of subsets of Ω. Define $U = \{\mathcal{F} \mid \mathcal{F} : field \wedge \mathcal{A} \subset \mathcal{F}\}$ and $f(\mathcal{A}) := \bigcap_{\mathcal{F} \in U} \mathcal{F}$, then $f(\mathcal{A})$ is a field.*

Proof. We check the three condition of field
a. Since the power set of Ω is a field that contains all elements in $\mathcal{A}$, $U \neq \emptyset$. Now $\forall \mathcal{F} \in U, \mathcal{A} \subset \mathcal{F} \Rightarrow \bigcap_{\mathcal{F} \in U} \mathcal{F} \neq \emptyset$

b. $\forall B \in \bigcap_{\mathcal{F} \in U} \mathcal{F}$

$$\begin{aligned} &\forall \mathcal{F} \in U, B \in \mathcal{F} \\ \Rightarrow &\forall \mathcal{F} \in U, B^C \in \mathcal{F} \\ \Rightarrow &B^C \in \bigcap_{\mathcal{F} \in U} \mathcal{F} \end{aligned}$$

c. $\forall B, C \in \bigcap_{\mathcal{F} \in U} \mathcal{F}$

$$\begin{aligned} &\forall \mathcal{F} \in U; B, C \in \mathcal{F} \\ \Rightarrow &\forall \mathcal{F} \in U; B \cup C \in \mathcal{F} \\ \Rightarrow &B \cup C \in \bigcap_{\mathcal{F} \in U} \mathcal{F} \end{aligned}$$

Thus $f(\mathcal{A})$ is a field

This result also holds for $\sigma - field$ □

Problem 2. Let Ω be a set, $\mathcal{A}$ is a collection of subsets of Ω, $f(\mathcal{A})$ is the field generated by $\mathcal{A}$ as defined in Lemma 1. Let $\mathcal{G} = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij}$ where for each i,j, either $A_{ij} \in \mathcal{A}$ or $A_{ij}^C \in \mathcal{A}$ and the m sets $\bigcap_{j=1}^{n_i}$ are disjoint. Prove that $f(\mathcal{A}) = \mathcal{G}$.

Proof. We will prove two sides

a. To prove$\mathcal{G} \subset f(\mathcal{A})$. Since $\forall A_{ij}, A_{ij} \in f(\mathcal{A}) \Rightarrow \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij} \in f(\mathcal{A}) \Rightarrow \mathcal{G} \subset f(\mathcal{A})$

b. To prove $f(A) \subset \mathcal{G}$. First we will prove that $\mathcal{G}$ is a field.

i. $\forall A \in \mathcal{A}, A \in \mathcal{G} \Rightarrow \mathcal{G} \neq \emptyset$

ii. $\forall A = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij}, B = \bigcup_{h=1}^{p} \bigcap_{k=1}^{q_i} B_{ij}$, where A_{ij}'s, B_{ij}'s satisfy the condition mentioned above, we have:

$$A \cap B = \bigcup_{i=1\ldots,m;h=1,\ldots,p} (\bigcap_{j=1}^{n_i} A_{ij}) \cap (\bigcap_{k=1}^{q_h} B_{hk})$$

Now consider any two term of the form inside the union, $(\bigcap_{j=1}^{n_i} A_{ij}) \cap (\bigcap_{k=1}^{q_h} B_{hk})$, they must be disjoint because if they have any term in common, that implies two term $(\bigcap_{j=1}^{n_i} A_{ij})$ have element in common which is a contradiction. So $A \cap B \in \mathcal{G}$. Thus a $\mathcal{G}$ is also closed under finite intersection.

iii. $\forall A = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij}$, we have:

$$\begin{aligned}
A^C =&(\bigcup_{i=1}^{m}\bigcap_{j=1}^{n_i} A_{ij})^C\\
=&\bigcap_{i=1}^{m}\bigcup_{j=1}^{n_i} A_{ij}^C\\
=&\bigcap_{i=1}^{m}\bigcup_{j=1}^{n_i}[A_{ij}^C \cap \bigcap_{k=1}^{j-1} A_{ik}]
\end{aligned}$$

The terms inside each union operand are disjoint, thus each union term is in $\mathcal{G}$, and from ii., $\mathcal{G}$ is closed under finite intersection, thus $A^C \in \mathcal{G}$.

So $\mathcal{G}$ is a field. Since $\forall A \in \mathcal{A}, A \in \mathcal{G} \Rightarrow f(\mathcal{A}) \subset \mathcal{G}$ since $f(A)$ is the intersection of all field containing $\mathcal{A}$. From a. and b., we have $f(\mathcal{A}) = \mathcal{G}$. □

Problem 3. Let Ω be a set, $\mathcal{A}$ is a collection of subsets of Ω, $B \in \sigma(\mathcal{A})$. Prove that $\exists \mathcal{A}_B \subset \mathcal{A}, |\mathcal{A}_B| = |\mathbb{N}|, B \in \sigma(\mathcal{A}_B)$

Proof. Let $\mathcal{G} = \bigcup_{\mathcal{C}\subset\mathcal{A}\wedge|\mathcal{C}|=|\mathbb{N}|}\sigma(\mathcal{C})$. It is obvious that $\mathcal{G} \subset \sigma(\mathcal{A})$. We will prove that $\sigma(\mathcal{A}) \subset \mathcal{G}$. First we will prove that $\mathcal{G}$ is a $\sigma - field$.

a. We can assume that $\mathcal{A} \neq \emptyset \Rightarrow \exists A \in \mathcal{A} \Rightarrow \sigma(\{A\}) = \{\emptyset, A, A^C, \Omega\} \neq \emptyset \Rightarrow \mathcal{G} \neq \emptyset$

b. $\forall A \in \mathcal{G}, \exists \mathcal{C} \subset \mathcal{A}, |\mathcal{C}| = |\mathbb{N}|, A \in \sigma(\mathcal{C}) \Rightarrow A^C \in \sigma(\mathcal{C}) \Rightarrow A^C \in \mathcal{G}$

c. $\forall A_1, A_2, ..., A_n, ... \in \mathcal{G}, \exists \mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n, ... \subset \mathcal{A}, |\mathcal{C}_i| = |\mathbb{N}|, A_i \in \sigma(\mathcal{C}_i)$. Now consider

$$\mathcal{C} = \bigcup_{i=1}^{\infty}\mathcal{C}_i$$

Since countable union of countable sets is countable, $\mathcal{C}$ is countable. Thus:

$$\begin{aligned}
&\forall i \in \mathbb{N}, A_i \in \sigma(C)\\
\Rightarrow &\bigcup_{i=1}^{\infty} A_i \in \sigma(\mathcal{C})\\
\Rightarrow &\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}
\end{aligned}$$

From a., b., and c., we have $\mathcal{G}$ is a $\sigma - field$. And since $\mathcal{G}$ is a field that contains $\mathcal{A}$, we have $\sigma(\mathcal{A}) \subset \mathcal{G}$. (From Lemma 2, where we prove that the union of all $\sigma - field$ containing $\mathcal{A}$ is a field denoted $\sigma(\mathcal{A})$). So $\sigma(\mathcal{A}) = \mathcal{G} \Rightarrow \forall A \in \sigma(\mathcal{A}), \exists \mathcal{C} \subset \mathcal{A} \wedge |\mathcal{C}| = |\mathbb{N}|, A \in \sigma(\mathcal{C})$. □

Problem 4. Let $\mathbb{R}$ be equipped with the Borel sigma algebra $\mathcal{B}(\mathbb{R})$. Show that of all the $\sigma - field$ in $\mathbb{R}^d$ that satisfy all continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is measurable, the Borel $\sigma - field$ of $\mathbb{R}^d$ is the smallest such $\sigma - field$

Proof. We will prove two direction.

a. $\mathcal{B}(\mathbb{R}^d)$ satisfies the condition that all continuous function f is measurable. This is true because for any open set on $\mathbb{R}$, its pre-image w.r.t continuous f is also an open set and thus measurable. Any measurable set on $\mathbb{R}$ (with respect to $\mathcal{B}(\mathbb{R})$) is the result of boolean operation on countable number of open set (Result from Problem 3), so the pre-image of any measurable set the result of boolean operation on countable number of open set on $\mathbb{R}^d$, and so belong to $\mathcal{B}(\mathbb{R}^d)$, thus measurable. So f is measurable with respect to $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R})$.

b. Now we need to prove that $\mathcal{B}(\mathbb{R}^d)$ is the smallest such $\sigma - field$, by proving that any $\sigma - field$ $\mathcal{A}$ that satisfies the condition must contain $\mathcal{B}(\mathbb{R}^d)$. Let E be an arbitrary non-empty closed set in $\mathbb{R}$. Consider the function:

$$
\begin{aligned}
g : \mathbb{R}^d &\to \mathbb{R} \\
x &\mapsto \inf\{||y - x||, \forall y \in E\}
\end{aligned}
$$

It follows from the triangle inequality that $\forall x, y \in \mathbb{R}^d, d(x, E) \leq d(x, y) + d(y, E) \Rightarrow d(x, y) \geq d(x, E) - d(y, E)$. Similarly $d(x, y) \geq d(y, E) - d(x, E)$. So $d(x, y) \geq |d(x, E) - d(y, E)|$ or $g(x, y) \leq ||x - y||$. So g is 1-Lipschitz function, thus it is continuous. Since we need g to be measurable, and $\{0\} \in \mathcal{B}(\mathbb{R})$ is measurable, thus $g^{-1}(\{0\}) = E$ is measurable. So any closed set in $\mathbb{R}^d$ must be measurable w.r.t this $\sigma - algebra$ $\mathcal{A}$. Thus $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{A}$.

From a., and b., we have $\mathcal{B}(\mathbb{R}^d)$ is the smallest $\sigma - algebra$ that satisfies the condition. □

Problem 5. Upper semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$ is measurable.

Proof. Let $U_t = \{x \in \mathbb{R}^d \mid f(x) < t\}$. Let $x_0 \in U_t$ be fixed, $\epsilon = t - f(x_0)$. From the definition of upper semicontinuous function, $\exists \delta \in \mathbb{R}^+, \forall y, ||y - x_0|| < \delta \Rightarrow f(y) < f(x) + \epsilon = f(x_0) + t - f(x_0) = t$. So the ball around x_0 radius δ is in U_t. This is true for all x_0 in U_t. So U_t is open and thus measurable. So the pre-image of $(-\infty, a)$ is U_a and is measurable. Since the set of $\{(-\infty, a) \mid a \in \mathbb{R}\}$ generates the Borel set $\mathcal{B}(\mathbb{R})$, we have the pre-image of any measurable set in $\mathcal{B}(\mathbb{R})$ is also measurable. So f is measurable.

The proof is analogous for lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$. □