Solution for HW 8

1. Assume that π, ν, μ are probability measures on $(S, \mathcal{S})$ satisfying $\pi \ll \nu \ll \mu$. On one hand, $\pi \ll \mu$ implies that $\pi(A) = \int_A \frac{d\pi}{d\mu} d\mu$ for all $A \in \mathcal{S}$. On the other hand, $\nu \ll \mu$ gives that $\int_S f d\nu = \int_S f \frac{d\nu}{d\mu} d\mu$ for all bounded measurable $f \geq 0$. In particular, take $f = 1_A \frac{d\pi}{d\nu}$ for some $A \in \mathcal{S}$, we obtain $\pi(A) = \int_A \frac{d\pi}{d\nu}\frac{d\nu}{d\mu} d\mu$. By disintegration, we have $\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu}\frac{d\nu}{d\mu}$.
2. Since S_2 is nice, we can reduce the problem to the case where $S_2 = \mathbb{R}$. Denote $\mathcal{A} := \{x; Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2\}$ and $\mathcal{A}_r := \{x; Q^*(x, (-\infty, r)) = Q(x, (-\infty, r))\}$ for all $r \in \mathbb{Q}$. We have then $\mathcal{A} = \cap_{r \in \mathbb{Q}} \mathcal{A}_r$ by appealing to $\pi - \lambda$ argument. Note that for fixed $B \in \mathcal{S}_2$, $\mu(A \times B) = \int_A Q(x, B) d\mu_1(x) = \int_A Q^*(x, B) d\mu_1(x)$ for all $A \in \mathcal{S}_1$. It is immediate that $Q(x, B) = Q^*(x, B)$ μ_1 a.e. In particular, $\mu_1(\mathcal{A}^c) = \mu_1(\cup_{r \in \mathbb{Q}} \mathcal{A}_r^c) \leq \sum_{r \in \mathbb{Q}} \mu_1(\mathcal{A}_r^c) = 0$. Therefore, $\mu_1(x; Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2) = 1$.
3. Denote X the random variable with the distribution function F. The we have $\int_{\mathbb{R}} F(x + c) - F(x) dx = \int_{\mathbb{R}} \mathbb{E} 1_{x < X \leq x + c} dx \overset{(*)}{=} \mathbb{E} \int_{\mathbb{R}} 1_{X - c \leq x < X} dx = c$, where $(*)$ is due to Tonelli-Fubini theorem.
4. Fix $a \in \mathbb{R}$. Observe that $f(x, u) \leq a \iff g(u, x) := u - Q(x, (-\infty, a]) \leq 0$. Remark that g is product measurable as the sum of two (product) measurable functions. Thus, the inverse distribution function is product measurable.
5. For $1 \leq i < j \leq 3$, define the probability measures μ_{ij} on $\{0, 1\}^2$ such that $\mu_{ij}(\{0\} \times \{1\}) = \mu_{ij}(\{1\} \times \{0\}) = \frac{1}{2}$ and $\mu_{ij}(\{0\} \times \{0\}) = \mu_{ij}(\{1\} \times \{1\}) = 0$. Obviously, the consistency condition (2) is satisfied with $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$ for $i = 1, 2, 3$. However, there is no triple (X_1, X_2, X_3) such that μ_{ij} is the joint distribution of (X_i, X_j) since with probablity 1, X_1, X_2 and X_3 are pairwise different. This contradicts with the fact that only 2 values (0 and 1) are accessible for three random variables.