ST205A - HW3

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Problem 1. Monotone Convergence Theorem

Proof. **(i)** We have $\mathbb{E}X_n < \infty$ for some n, since we are working with limit, we can assume $\mathbb{E}X_1 < \infty$ (reindex the sequence). Let $Y_n = X_1 - X_n, Y = X_1 - X$ we have: $0 = Y_1 \leq Y_2 \leq Y_3 \leq ...$, almost surely, and $\lim Y_n = \lim(X_1 - X_n) = X_1 - \lim X_n = X_1 - X = Y$. So $Y_n \uparrow Y$ and Y_n is bounded below by 0, thus $\lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E}Y \Rightarrow \lim_{n\to\infty} \mathbb{E}[X_1 - X_n] = \mathbb{E}[X_1 - X] \Rightarrow \lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$, since $\mathbb{E}X_1 < \infty$. (It seems we don't need condition $X_n \geq 0$??)

(ii) Let $X_n = |X|\mathbb{I}[|X| > n]$, then $X_1 \geq X_2 \geq X_3 \geq ...$ since $\mathbb{I}[|X| > m] \geq \mathbb{I}[|X| > n], \forall m < n$. Also $X_n \geq 0, \forall n$ and $\mathbb{E}X_1 < \mathbb{E}|X| < \infty$. Also, $\lim_{n\to\infty} X_n = 0$. Thus by (i) we have $\lim_{n\to\infty} \mathbb{E}X_n = 0$.

(iii) Let $Y_n = X_n - X_1$, then $0 \leq Y_n \uparrow X - X_1$ since $\mathbb{E}X_1 < \infty$. Thus $\lim \mathbb{E}Y_n = \mathbb{E}[X - X_1]$.

If $\mathbb{E}|X| = \infty \Rightarrow \mathbb{E}|X| - \mathbb{E}|X_1| = \infty \Rightarrow \mathbb{E}|X - X_1| \geq \mathbb{E}\left[|X| - |X_1|\right] = \mathbb{E}|X| - \mathbb{E}|X_1| = \infty$.

$\Rightarrow \mathbb{E}|X - X_1| = \infty \Rightarrow \mathbb{E}[X - X_1] = \mathbb{E}|X - X_1| = \infty$ since $X \geq X_1$ almost surely. So $\lim \mathbb{E}Y_n = \infty \Rightarrow \lim \mathbb{E}X_n = \infty$ as $\mathbb{E}X_1 < \infty$

Else if $\mathbb{E}|X| < \infty \Rightarrow \mathbb{E}X \leq \mathbb{E}|X| < \infty$. Thus $\lim \mathbb{E}Y_n = \mathbb{E}[X - X_1] = \mathbb{E}X - \mathbb{E}X_1 \Rightarrow \lim \mathbb{E}X_n = \mathbb{E}X$ since $\mathbb{E}X_1 < \infty$.

(iv) Let $X_n = \sum_{i=1}^{n} \mathbb{I}[X \geq i]$, then we have $0 \leq X_1 = \mathbb{I}[X \geq 1] \leq X_2 = \mathbb{I}[X \geq 1] + \mathbb{I}[X \geq 2] \leq X_3 \leq ...$ We need to prove that $\lim X_n = X$, which by definition means $\mathbb{P}\{\omega \mid X_n(\omega) \to X(\omega)\} = 1$.

Let $\omega \in \Omega$ be arbitrarym let $m = X(\omega)$ then $m \in \mathbb{N}^+$. We have:

$X(\omega) = 1, X_2(\omega) = 2, ..., X_{m-1}(\omega) = m-1, X_m(\omega) = m$, and $\forall n > m, X_n(\omega) = m$. Thus $\lim_{n\to\infty} X_n(\omega) = m = X(\omega)$. So $\lim X_n = X$ almost surely. Thus by the monotone convergence theorem, $\lim \mathbb{E}X_n = \mathbb{E}X \Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) = \mathbb{E}X$. □

Problem 2. Variance of simple function

Proof. **(i)** We have:

$$\begin{aligned}
\mathrm{Var}[X] =& \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 \\
=& \mathbb{E}\left[\left(\sum_{i=1}^{n} \mathbb{I}[A_i]\right)^2\right] - \left(\sum_{i=1}^{n} \mathbb{P}[A_i]\right)^2 \\
=& \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\mathbb{I}[A_i]\mathbb{I}[A_j]\right] - \left(\sum_{i=1}^{n} \mathbb{P}[A_i]\right)^2 \\
=& \sum_{i=1}^{n} \mathbb{P}[A_i] + 2\sum_{i\neq j} \mathbb{P}\left[A_i \cap A_j\right] - \left(\sum_{i=1}^{n} \mathbb{P}[A_i]\right)^2
\end{aligned}$$

(ii) Let A_i be the event that box $i'th$ is empty. We need to $\mathrm{Var}[X] = \mathrm{Var}\left[\sum_{i=1}^{n} \mathbb{I}[A_i]\right]$. From (i), we have:

$$
\begin{aligned}
\mathrm{Var}[X] =& \sum_{i=1}^{n} \mathbb{P}[A_i] + 2\sum \mathbb{P}\left[A_i \cap A_j\right] - \left(\sum_{i=1}^{n} \mathbb{P}[A_i]\right)^2 \\
=& n\left(\frac{n-1}{n}\right)^k + 2\frac{n(n-1)}{2}\left(\frac{n-2}{n}\right)^k - \left(n\left(\frac{n-1}{n}\right)^k\right)^2 \\
=& \frac{(n-1)^k}{n^{k-1}} + \frac{(n-1)(n-2)^k}{n^{k-1}} - \frac{(n-1)^{2k}}{n^{2k-2}}
\end{aligned}
$$

□

Problem 3. Markov Inequality

Proof. **(i)** Consider $\phi(x) = (x+b)^2$. According to the General Markov Inequality,

$$
\begin{aligned}
\mathbb{P}[X \geq a] \leq& \frac{\mathbb{E}\phi(X)}{\phi(a)}, \forall b \\
\Rightarrow \mathbb{P}[X \geq a] \leq& \frac{\sigma^2 + b^2}{(a+b)^2}, \forall b \\
=& \frac{\sigma^2 + b^2}{a^2 + 2ab + b^2}
\end{aligned}
$$

We need to find b such that:

$$
\begin{aligned}
\frac{\sigma^2 + b^2}{a^2 + 2ab + b^2} \leq& \frac{\sigma^2}{\sigma^2 + a^2} \\
\Leftrightarrow \sigma^4 + \sigma^2 a^2 + \sigma^2 b^2 + a^2 b^2 \leq& \sigma^2 a^2 + 2ab\sigma^2 + b^2\sigma^2 \\
\Leftrightarrow \sigma^4 + a^2 b^2 \leq& 2ab\sigma^2
\end{aligned}
$$

But with A.C. inequality we have: $\sigma^4 + a^2b^2 \geq 2ab\sigma^2$, the equality hold iff $\sigma^4 = a^2b^2 \Leftrightarrow b = \sigma^2/|a| = \sigma^2/a$ since $a > 0$. So if we pick $b = \sigma^2/a$, then we have the inequality that we need to prove.

(ii) We need to prove:

$$
\begin{aligned}
\mathbb{P}[X > 0] \geq& \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} \\
\Leftrightarrow (\mathbb{E}X^2)\mathbb{P}[X > 0] \geq& (\mathbb{E}X)^2
\end{aligned}
$$

Let $Y = \mathbb{I}[X > 0]$ then Y is a random variable. According to the Cauchy-Schwarz inequality:

$$
\begin{aligned}
(\mathbb{E}X^2)(\mathbb{E}Y^2) \geq& (\mathbb{E}[XY])^2 \\
\Leftrightarrow (\mathbb{E}X^2)\mathbb{E}Y \geq& (\mathbb{E}[X\mathbb{I}[X > 0]])^2 \\
\Leftrightarrow \mathbb{E}X^2\mathbb{P}[X > 0] \geq& (\mathbb{E}X)^2
\end{aligned}
$$

□

Problem 4. Chebyshev's other inequality

Proof. Let Y be an independent copy of X. Since $f(x), g(x)$ is an increasing bounded function, and X, Y independent, we have:

$$
\begin{aligned}
\big(f(X)-f(Y)\big)\big(g(X)-g(Y)\big) &\geq 0\\
\Rightarrow f(X)g(X)+f(Y)g(Y) &\geq f(X)g(Y)+f(Y)g(X)\\
\Rightarrow \mathbb{E}\left[f(X)g(X)+f(Y)g(Y)\right] &\geq \mathbb{E}\left[f(X)g(Y)+f(Y)g(X)\right]\\
\Rightarrow \mathbb{E}\left[f(X)g(X)\right]+\mathbb{E}\left[f(Y)g(Y)\right] &\geq \mathbb{E}\left[f(X)g(Y)\right]+\mathbb{E}\left[f(Y)g(X)\right]\\
\Rightarrow 2\mathbb{E}\left[f(X)g(X)\right] &\geq \mathbb{E}[f(X)]\mathbb{E}[g(Y)]+\mathbb{E}[f(Y)]\mathbb{E}[g(X)]\\
\Rightarrow 2\mathbb{E}[f(X)g(X)] &\geq 2\mathbb{E}[f(X)]\mathbb{E}[g(X)]\\
\Rightarrow \mathbb{E}[f(X)g(X)] &\geq \mathbb{E}[f(X)]\mathbb{E}[g(X)]
\end{aligned}
$$

□

Lemma 1. *Moment Generating Function for $X \sim Poisson(\lambda)$ is*

Proof. We have

$$
\begin{aligned}
\mathbb{E}\left[\exp(uX)\right] &= \sum_{x=0}^{\infty} \exp(ux)\exp(-\lambda)\frac{\lambda^x}{x!}\\
&= \sum_{x=0}^{\infty} \exp(-\lambda)\frac{(\lambda\exp(u))^x}{x!}\\
&= \sum_{x=0}^{\infty} \exp(-\lambda+\lambda\exp u)\exp(-\lambda\exp u)\frac{(\lambda\exp(u))^x}{x!}\\
&= \exp(\lambda(\exp u-1))\sum_{x=0}^{\infty} \exp(-\lambda\exp u)\frac{(\lambda\exp(u))^x}{x!}\\
&= \exp(\lambda(\exp u-1))
\end{aligned}
$$

□

Problem 5. Difference of Poisson random variable

Proof. (i) Applying the general Markov Inequality (special version Elementary Large Deviation inequality) we have:

$$
\begin{aligned}
\mathbb{P}[X \geq Y] =& \mathbb{P}[X-Y \geq 0]\\
&\leq \inf_{\theta} \exp(-\theta\times 0)\mathbb{E}[\exp\{\theta(X-Y)\}]\\
&= \inf_{\theta} \mathbb{E}[\exp(\theta X)\exp(-\theta Y)]\\
&= \inf_{\theta} \mathbb{E}[\exp(\theta X)]\mathbb{E}[\exp(-\theta Y)]\\
&= \inf_{\theta} \exp(\lambda(\exp\theta-1)+2\lambda(\exp-\theta-1))\\
&= \inf_{\theta} \exp(-3\lambda+\lambda\exp\theta+2\lambda\exp(-\theta))
\end{aligned}
$$

Applying the A.C. inequality we have:

$$
\begin{aligned}
\exp\theta+2\exp(-\theta) &\geq 2\sqrt{2\exp(\theta)\exp(-\theta)} = 2\sqrt{2}\\
\Rightarrow \mathbb{P}[X\geq Y] &\leq \exp((-3+\sqrt{8})\lambda)
\end{aligned}
$$

(Equality for A.C. hold iff θ=0)
(ii) Applying the Large Deviation inequality and Cauchy-Schwarz inequality we have:

$$
\begin{aligned}
\mathbb{P}[X \geq Y] &\leq \inf_{\theta} \mathbb{E}[\exp(\theta X) \exp(-\theta Y)] \\
&\leq \inf_{\theta} \left(\left(\mathbb{E}[\exp^2(\theta X)] \right) \left(\mathbb{E}[\exp^2(\theta Y)] \right) \right)^{1/2} \\
&= \inf_{\theta} \left((\mathbb{E}[\exp(2\theta X)]) (\mathbb{E}[\exp(2\theta Y)]) \right)^{1/2} \\
&= \inf_{\theta} \left(\exp(\lambda(\exp(2\theta) - 1) + 2\lambda(\exp(-2\theta) - 1)) \right)^{1/2} \\
&= \inf_{\theta} \exp\left(-\frac{3\lambda}{2} + \frac{\lambda}{2} \exp(2\theta) + \lambda \exp(-2\theta) \right)
\end{aligned}
$$

Applying the A.C. inequality we have:

$$
\begin{aligned}
\frac{1}{2} \exp(2\theta) + \exp(-2\theta) &\geq \sqrt{2} \\
\Rightarrow \mathbb{P}[X \geq Y] &\leq \exp((-\frac{3}{2} + \sqrt{2})\lambda).
\end{aligned}
$$

$\square$