Solution for HW 5

1. Fix $\epsilon > 0$. Using the argument in **HW3, Q1(iv)** we get that $\sum_{n=1}^{\infty} \mathbb{P}(X_n \geq n\epsilon) \leq \frac{\mathbb{E}X_1}{\epsilon}$. The first Borel-Cantelli lemma gives that $\mathbb{P}(X_n \geq n\epsilon \ i.o.) = 0$, which implies that $\limsup_n \frac{M_n}{n} \leq \epsilon$ a.s. Since ϵ is arbitary small, we have $\limsup_n \frac{M_n}{n} = 0$. Similarly, $\liminf_n \frac{M_n}{n} = 0$ a.s. and we conclude that $\frac{M_n}{n} \to 0$ a.s.
2. According to Chebyshev inequality, $\mathbb{P}[\frac{X_n - \mathbb{E}X_n}{n^\alpha} \geq \epsilon] \leq \frac{VarX_n}{\epsilon^2 n^{2\alpha}} \leq \frac{B}{\epsilon^2} n^{\beta-2\alpha}$. Consider a fixed sequence $\phi(n) \sim n^{\frac{2}{2\alpha-\beta}}$ and apply the first Borel-Cantelli lemma, we obtain $\phi(n)^{-\alpha} X_{\phi(n)} \to a$ a.s. Take k_n s.t. $\phi(k_n) \leq n < \phi(k_n+1)$. Since X_n is increasing, we get $\frac{X_{\phi(k_n)}}{\phi(k_n)^\alpha} \frac{\phi(k_n)^\alpha}{\phi(k_n+1)^\alpha} \leq \frac{X_n}{n^\alpha} \leq \frac{X_{\phi(k_n+1)}}{\phi(k_n+1)^\alpha} \frac{\phi(k_n+1)^\alpha}{\phi(k_n)^\alpha}$. Then we conclude by observing that $\frac{\phi(k_n)}{\phi(k_n+1)} \to 1$ as $n \to \infty$.
3. (i) $\to$ (ii) : By definition of convergence in probability, $\exists \phi : \mathbb{N} \to \mathbb{N}$ strictly increasing s.t. $\mathbb{P}(|X_n - X| \geq \frac{1}{m}) \leq \frac{1}{m}$ for $\forall n \geq \phi(m)$. Take $\epsilon_n = \frac{1}{m}$ when $\phi(m) \leq n < \phi(m+1)$. Clearly $\epsilon_n \downarrow 0$. (ii) $\to$ (iii) : Note that $\min(|X_n - X|, 1) \leq \epsilon_n 1_{|X_n - X| \leq \epsilon} + 1_{|X_n - X| > \epsilon}$. Then $\mathbb{E}\min(|X_n - X|, 1) \leq \epsilon_n \mathbb{P}(|X_n - X| \leq \epsilon) + \mathbb{P}(|X_n - X| > \epsilon) \leq 2\epsilon_n \to 0$ as $n \to \infty$. (iii) $\to$ (i) : Apply Markov inequality to $f(x) := \min(|x|, 1)$, we get $\mathbb{P}(|X_n - X| > \epsilon) \leq \frac{\mathbb{E}f(X_n - X)}{f(\epsilon)} \to 0$.
Remark : Convergence in probability defines a topology on the space of random variables with metrics $d(X,Y) := \inf\{\epsilon > 0; \mathbb{P}(|X - Y| > \epsilon) < \epsilon\}$ or equivalently $d'(X,Y) := \mathbb{E}\min(|X - Y|, 1)$, named after Ky fan.
4. (i). The SLLN guarantees that $\frac{1}{n}\log W_n \to c(p) := \mathbb{E}\log(ap + (1-p)V_n)$. (ii). By Theorem A1.5 in Durrett's book, we can exchange differentiation and expectation : $c'(p) = \mathbb{E}\left(\frac{a - V_n}{ap + (1-p)V_n}\right)$ and $c''(p) = -\mathbb{E}\left[\left(\frac{a - V_n}{ap + (1-p)V_n}\right)^2\right]$. (iii). In order to have a maximum in $(0,1)$, we need $c'(0) > 0$ and $c'(1) < 0$, i.e. $\mathbb{E}\left(\frac{1}{V_n}\right) > \frac{1}{a}$ and $\mathbb{E}V_n > a$. (iv). In this case, $\mathbb{E}\left(\frac{1}{V}\right) = \frac{5}{8}$ and $\mathbb{E}V = \frac{5}{2}$. Thus, when $a > \frac{5}{2}$, the maximum is at 1 and when $a < \frac{5}{8}$, the maximum is at 0. In between, the maximum occurs at p for which $\frac{1}{2}\frac{a-1}{ap+(1-p)} + \frac{1}{2}\frac{a-4}{ap+4(1-p)} = 0 \Leftrightarrow p = \frac{5a-8}{2(4-a)(a-1)}$.
5. Let $x \in \mathbb{R}$, we first show that $F_n(x) \to F(x)$ $(*)$. The atom case is guaranteed by (ii). For x a continuity point of F, let $s, t \in \mathbb{Q}$ s.t. $s < x < t$: $\lim_{s \to x^-} F(s) = F(x) = \lim_{t \to x^+} F(t)$. Since $F_n(s) \leq F_n(x) \leq F_n(t)$, $F(s) \leq \liminf F_n(x) \leq \limsup F_n(x) \leq F(t)$ by (i), from which the desired result follows. Now we prove that $F_n(x^-) \to F(x^-)$ $(**)$. It also suffices to prove for x continuity point of F. Since $F_n(s) \leq F_n(x^-) \leq F_n(x)$, $F(x^-) \leq \liminf F_n(x^-) \leq \limsup F_n(x^-) \leq \limsup F_n(x) \overset{(*)}{=} F(x) = F(x^-)$. Finally, we deal with the uniform convergence. Fix $\epsilon > 0$, consider $-\infty = t_0 < t_1 < \cdots < t_k = \infty$ s.t. $F(t_{i+1}^-) - F(t_i) \leq \epsilon$ for $0 \leq i \leq k-1$. For any $x \in \mathbb{R}$, $\exists i$ s.t. $t_i \leq x < t_{i+1}$ and thus $F_n(t_i) \leq F_n(x) \leq F_n(t_{i+1}^-)$, $F(t_i) \leq F(x) \leq F(t_{i+1}^-)$. We then obtain $F_n(t_i) - F(t_{i+1}^-) \leq F_n(x) - F(x) \leq F_n(t_{i+1}^-) - F(t_i)$, from which $F_n(t_i) - F(t_i) - \epsilon \leq F_n(x) - F(x) \leq F_n(t_{i+1}^-) - F(t_{i+1}^-) + \epsilon$. For each i, take N_j s.t. $\forall n \geq N_j$, $F_n(t_i) - F(t_i) > -\epsilon$ by $(*)$ and M_j s.t. $\forall n \geq N_j$, $F_n(t_i^-) - F(t_i^-) < \epsilon$ by $(**)$. Take $N := \max_{1 \leq i \leq k} \max(N_i, M_i)$, then for $n \geq N$ and $\forall x \in \mathbb{R}$, $|F_n(x) - F(x)| \leq 2\epsilon$ and we obtain the desired result.