Solution for HW 3

1. (1). Suppose $\mathbb{E}X_{n_0} < \infty$. This implies that a.s. X_{n_0} is finite. Since $X_n \downarrow X$ a.s., we have for $n \geq n_0$, $X_{n_0} - X_n \uparrow X_{n_0} - X$ a.s. and is positive. By monotone convergence theorem, $\mathbb{E}(X_{n_0} - X_n) \uparrow \mathbb{E}(X_{n_0} - X)$, which implies $\mathbb{E}X_n \downarrow \mathbb{E}X$. (2). Note that $|X|1_{|X|>n} \downarrow 0$ a.s. and $\mathbb{E}(|X|1_{|X|>n}) < \mathbb{E}|X| < \infty$ for all n. Apply the result in (1), we get $\mathbb{E}(|X|1_{|X|>n}) \downarrow 0$. (3). From $\mathbb{E}|X_1| < \infty$, we have a.s. X_1 is finite. Then the sequence $X_n - X_1 \uparrow X - X_1$ a.s. and is positive. Apply monotone convergence theorem, $\mathbb{E}X_n \uparrow \mathbb{E}X$. Moreover, in case where $\mathbb{E}X_n \uparrow \infty$, $\mathbb{E}|X| \geq \mathbb{E}X = \infty$. (4). Note that $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \mathbb{P}(X = m) \overset{(*)}{=} \sum_{m=1}^{\infty} \sum_{n=1}^{m} \mathbb{P}(X = m) = \sum_{m=1}^{\infty} m\mathbb{P}(X = m) = \mathbb{E}X$ where $(*)$ is justified by Fubini-Tonelli's theorem.

2. (1). $VarX = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_j) - (\sum_{i=1}^{n} \mathbb{P}(A_i))^2 = \sum_{i \neq j} \mathbb{P}(A_i \cap A_j) + \sum_{i=1}^{n} \mathbb{P}(A_i) - (\sum_{i=1}^{n} \mathbb{P}(A_i))^2$. (2). Take $A_i = \{\text{ith box is empty}\}$. Note that for all i, $\mathbb{P}(A_i) = (1 - \frac{1}{n})^k$ and for all $i \neq j$, $\mathbb{P}(A_i \cap A_j) = (1 - \frac{2}{n})^k$. Inject these terms in the expression obtained in (1), we have $VarX = n(1 - \frac{1}{n})^k + n(n-1)(1 - \frac{2}{n})^k - n^2(1 - \frac{1}{n})^{2k}$.

3. (1). Note that for every $t > 0$, we have $\mathbb{P}(X \geq a) = \mathbb{P}(\frac{X+t}{a+t} \geq 1) \overset{(*)}{\leq} \frac{\sigma^2 + t^2}{(a+t)^2}$ where $(*)$ follows from Chebyshev's inequality. In particular, pick $t = \frac{\sigma^2}{a}$, we get the desired inequality. (2). According to Cauchy-Schwarz inequality, $\mathbb{E}X = \mathbb{E}[X1_{X>0}] \leq \mathbb{E}[X^2]^{\frac{1}{2}} \mathbb{P}(X > 0)^{\frac{1}{2}}$, which permits to conclude.

Remark : (1) is known as Cantelli's inequality and (2) is the Paley-Zygmund inequality.

4. Since f and g are increasing, we have $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in \mathbb{R}$. Now take Y an independent random variable with the same distribution as X, then $0 \leq \mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))] = 2\mathbb{E}[f(X)g(X)] - 2\mathbb{E}[f(X)]\mathbb{E}[g(X)]$, which leads to the disired result.

Remark : The condition of boundedness in the statement can be removed by monotone convergence theorem. This inequality is known as Harris' inequality.

5. (1). Using Markov inequality, for all $t > 0$, we have $\mathbb{P}(X \geq Y) \leq \mathbb{E}e^{t(X-Y)} = e^{\lambda e^t + 2\lambda e^{-t} - 3\lambda}$ by independence. In particular, take $t = \frac{\ln 2}{2}$, we obtain $\mathbb{P}(X \geq Y) \leq \exp(-(3 - \sqrt{8})\lambda)$. (2). Following from Cauchy-Schwarz inequality, $\mathbb{P}(X \geq Y) \leq \mathbb{E}[e^{2tX}]^{\frac{1}{2}} \mathbb{E}[e^{-2tY}]^{\frac{1}{2}} = \exp(\frac{\lambda}{2}e^{2t} - \lambda e^{-2t} - \frac{3}{2}\lambda)$. In particular, take $t = \frac{\ln 2}{2}$, we get the desired bound with $A = 1$ and $c = \frac{3}{2} - \sqrt{2}$.