ST205A - Homework 6

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Problem 1. Let (X_i) be independent, $S_n = \sum_{i=1}^n X_i, S_n^* = \max_{i\leq n} |S_i|$. Prove that:

$$\mathbb{P}[S_n^* > 2a] \leq \frac{\mathbb{P}[|S_n| > a]}{\min_{j\leq n} \mathbb{P}[|S_n - S_j| \leq a]}, a > 0$$

Proof. We have:

$(|S_j| > 2a) \wedge (|S_n - S_j| \leq 2a) \Rightarrow (|S_n| > a)$

$\Rightarrow \left\{\omega \middle| (|S_j(\omega)| > 2a) \wedge (|S_n(\omega) - S_j(\omega)| \leq a)\right\} \subset \left\{\omega \middle| (|S_n(\omega)| > a)\right\}$ (*)

Let $A_j = \left\{\omega \middle| (|S_j(\omega)| > 2a) \wedge (|S_k(\omega)| \leq 2a, \forall k \in \{1, ..., j-1\})\right\}$, then A_j are disjoint.

Let $B_j = \left\{\omega \middle| |S_n(\omega) - S_j(\omega)| \leq a\right\}$. Since $(*)$ is true for all j, we have:

$\mathbb{P}\bigcup_{j=1}^n (A_j \cap B_j) \leq \mathbb{P}\left\{\omega \middle| (|S_n(\omega)| > a)\right\}$

$\Leftrightarrow \sum_{j=1}^n \mathbb{P}(A_j \cap B_j) \leq \mathbb{P}\left\{\omega \middle| (|S_n(\omega)| > a)\right\}$

$\Leftrightarrow \sum_{j=1}^n \mathbb{P}[A_j]\mathbb{P}[B_j]$ because $\sigma(A_j) \subset \sigma(X_1, ..., X_j)$ is independent with $\sigma(B_j) \subset \sigma(X_{j+1}, ..., X_n)$.

$\Rightarrow \min_{j\leq n} \mathbb{P}[B_j] \sum_{j=1}^n \mathbb{P}[A_j] \leq \mathbb{P}\left\{\omega \middle| (|S_n(\omega)| > a)\right\}$. Now let $k = \arg\max_{i\leq n} |S_i|$. Then $\mathbb{P}[S_n^* > 2a] = \mathbb{P}[|S_k| > 2a] = \mathbb{P}[A_k]$.

Thus $\mathbb{P}[S_n^* > 2a] \leq \sum_{j=1}^n \mathbb{P}[A_j]$. So:

$\mathbb{P}[S_n^* > 2a] \min_{j\leq n} \mathbb{P}[|S_n - S_j| \leq a] \leq \min_{j\leq n} \mathbb{P}[B_j] \sum_{j=1}^n \mathbb{P}[A_j] \leq \mathbb{P}[|S_n| > a]$. ☐

Problem 2. (i) If S_n converges in probability then S_n converges a.s.

(ii) If (X_i) are identically distributed and $n^{-1}S_n \to 0$ in probability then $n^{-1}\max_{m\leq n} S_m \to 0$ in probability,

Proof. (i) We state a lemma without proving, the proof is found in Chandra (2012) - The Borel-Cantelli Lemma book.

$X_n \to X$ a.s. iff $\mathbb{P}[|X_n - X| > \epsilon \text{ i.o.}] = 0, \forall \epsilon > 0$.

Assuming that S_n converges in probability, W.L.O.G, assume that $S_n \to 0$, otherwise we can change X_1 by an appropriate amount. Let $\epsilon > 0$ be arbitrary. We have:

$\mathbb{P}[|S_n| > \epsilon \text{ i.o.}] = \mathbb{P}\bigcap_{n=1}^\infty \bigcup_{m=n}^\infty \left\{\omega \middle| |S_m(\omega)| > \epsilon\right\} = \lim_{n\to\infty} \mathbb{P}\bigcup_{m=n}^\infty \left\{\omega \middle| |S_m(\omega)| > \epsilon\right\} =$

$= \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{P}\bigcup_{m=n}^N \left\{\omega \middle| |S_m(\omega)| > \epsilon\right\}$

Fix an n here now consider:

$$\begin{aligned}\mathbb{P}\bigcup_{m=n}^{N}\left\{\omega\middle|\,|S_m(\omega)|>\epsilon\right\} =&\mathbb{P}\left\{\omega\middle|\max_{n\le m\le N}|S_m|>\epsilon\right\}\\ \le&\frac{\mathbb{P}\left[|S_N|>\epsilon/2\right]}{\min_{n\le m\le N}\mathbb{P}\left[|S_N-S_m|\le\epsilon/2\right]}\ (*)\\ =&\frac{\mathbb{P}\left[|S_N|>\epsilon/2\right]}{1-\max_{n\le m\le N}\mathbb{P}\left[|S_N-S_m|>\epsilon/2\right]}\ (**)\end{aligned}$$

Where $(*)$ is true by applying what we prove in question 1 to the sequence (Y_k) defined as $Y_0 = X_1 + ... + X_n, Y_1 = X_{n+1}, Y_2 = X_{n+2}$ and so on.

As $N \to \infty$, the numerator of (**) $\to 0$ because $S_n \to 0$ in probability (implies $|S_n| \to 0$ in probability). For the denominator, we can bound $\max_{n\le m\le N}\mathbb{P}\left[|S_N - S_m| > \epsilon/2\right]$ within any arbitrary range $[0, a), a > 0$, because $|S_n| \to 0$ in probability, thus $\exists N_0, \forall m, N > N_0, \mathbb{P}\left[|S_m| > \epsilon/4\right] < a, \mathbb{P}\left[|S_m| > \epsilon/4\right] < a$, which implies $\mathbb{P}\left[|S_N - S_m| > \epsilon/2\right] < a$, because the event $|S_n - S_m| > \epsilon/2$ implies $\left[|S_m| > \epsilon/4\right] \vee \mathbb{P}\left[|S_m| > \epsilon/4\right]$.

In conclusion, $\lim_{n\to\infty}\lim_{N\to\infty}\mathbb{P}\bigcup_{m=n}^{N}\left\{\omega\middle|\,|S_m(\omega)|>\epsilon\right\} = 0$ as the numerator of $(**)$ goes to zero and the denominator is bounded around 1.

(ii) Let $\epsilon > 0$ be arbitrary. Since (X_i) are i.i.d, we have:

$$\min_{j\le n}\mathbb{P}\left[\frac{|S_n - S_j|}{n}\le\epsilon\right] = \min_{j\le n-1}\mathbb{P}\left[\frac{|S_j|}{n}\le\epsilon\right] \ge \min_{j\le n-1}\mathbb{P}\left[\frac{|S_j|}{j}\le\epsilon\right]$$

Since we have: $\lim_{j\to\infty}\mathbb{P}\left[\frac{|S_j|}{j}>\epsilon\right] = 0$. Pick 1/2 as an arbitrary choice of number between 0 and 1, then $\exists N_0 \in \mathbb{N}$ such that: $\forall j \ge N_0, \mathbb{P}\left[\frac{|S_j|}{j}>\epsilon\right] < 1/2 \Rightarrow \mathbb{P}\left[\frac{|S_j|}{j}\le\epsilon\right] \ge \frac{1}{2}$.

We now use the previous observation that the inequality in question 1 also holds true when changing form to:

$$\mathbb{P}\left[S_N > 2a\right] \le\frac{\mathbb{P}\left[|S_N| > a\right]}{\min_{N_0\le j\le N}\mathbb{P}\left[|S_N - S_j|\le a\right]}, a > 0.$$

Thus we have $\forall N \ge N_0$:

$$\begin{aligned}\mathbb{P}\left[N^{-1}\max_{m\le N}S_m > \epsilon\right] \le&\mathbb{P}\left[S_N^* > N\epsilon\right]\\ \le&\frac{\mathbb{P}\left[|S_N| > N\epsilon/2\right]}{\min_{N_0\le j\le N}\mathbb{P}\left[|S_N - S_j|\le N\epsilon/2\right]}\end{aligned}$$

The denominator as proved above is $\ge 1/2$ for $\forall N \ge N_0$. And the numerator goes to zero as $n^{-1}S_n \to 0$ in probability implying $n^{-1}|S_n| \to 0$ in probability. □

Theorem 1. *Stolz-Cesaro Theorem for sequences going to infinity.*

Let (a_n) and (b_n) be two sequences of real numbers. Assuming that (b_n) is strictly increasing and divergent sequence. If:

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} =l$$

Then:

$$\lim_{n\to\infty}\frac{a_n}{b_n} =l$$

Stolz-Cesaro Theorem for sequences going to zero.

Let (a_b) and (b_n) be two sequences of real numbers. Assuming that (b_n) is strictly decreasing to zero. If:

$$\lim_{n\to\infty} \frac{a_n - a_{n+1}}{b_n - b_{n+1}} = l$$

Then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

Problem 3. (X_i) be i.i.d taking values in $\{-1, 1, 3, 7, 15, ...\}$. And

$$\mathbb{P}\left[X_1 = 2^k - 1\right] = \frac{1}{k(k+1)2^k}, k \geq 1$$

Proof. (a) We have:

$$\begin{aligned}
\mathbb{E}X_1 =& -1 \times \left(1 - \sum_{k=1}^{\infty} \mathbb{P}\left[X_1 = 2^k - 1\right]\right) + \sum_{k=1}^{\infty} \left(2^k - 1\right) \mathbb{P}\left[X_1 = 2^k - 1\right] \\
=& -1 + \sum_{k=1}^{\infty} \frac{2^k}{k(k+1)2^k} \\
=& -1 + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
=& -1 + 1 - \lim_{n\to\infty} \frac{1}{n+1} = 0
\end{aligned}$$

(b) We use the following Theorem, which is derived from Theorem 2.2.6 in Durrett:

Let (X_n) be i.i.d random variables, $b(n)$ be s.t. for $\bar{X}_n = X_n \mathbb{I}\{|X_n| \leq b_n\}$ and:

(i) $n\mathbb{P}\left[|X_n| > b_n\right] \to 0$

(ii) $\frac{n}{b_n^2} \mathbb{E}\left[\bar{X}_n^2\right] \to 0$

Then $\frac{S_n - a_n}{b_n} \to 0$ where $a_n = n\mathbb{E}\bar{X}_n$.

For our problem, let $b_n = 2^{m(n)}$ where $m(n) = \min\left\{m \in \mathbb{N} : 2^{-m}m^{-3/2} \leq n^{-1}\right\}$ as suggested in Durrett's book. For simplicity of notation, we denote $p_k = \mathbb{P}\left[X_1 = 2^k - 1\right], k \geq 1, p_0 = \mathbb{P}\left[X_1 = -1\right]$. We will check the first condition. For n big enough (so we don't have to worry about $X = -1$),

$$\begin{aligned}
n\mathbb{P}\left[|X_n| > b_n\right] =& n\mathbb{P}\left[|X_n| > 2^{m(n)}\right] \\
=& n\left(p_{m(n)+1} + p_{m(n)+2} + ...\right) \\
\leq& 2^{m(n)} m^{3/2}(n) \sum_{k=m(n)+1}^{\infty} \frac{1}{k(k+1)2^k} \\
=& 2^m m^{3/2} \sum_{k=m+1}^{\infty} \frac{1}{k(k+1)2^k} \text{ (For simplicity of notation)} \\
=& \frac{\sum_{k=m+1}^{\infty} \frac{1}{k(k+1)2^k}}{\frac{1}{2^m m^{3/2}}} := \frac{A_m}{B_m}
\end{aligned}$$

Applying the Stolz-Cesaro theorem for two sequences going to zero, and noticing that:

$$\begin{aligned}
\lim_{m\to\infty} \frac{A_m - A_{m+1}}{B_m - B_{m+1}} &= \lim_{m\to\infty} \frac{\frac{1}{(m+1)(m+2)2^{m+1}}}{\frac{1}{2^m m^{3/2}} - \frac{1}{2^{m+1}(m+1)^{3/2}}} \\
&= \lim_{m\to\infty} \frac{m^{3/2}(m+1)^{3/2}}{(m+1)(m+2)(2(m+1)^{3/2} - m^{3/2})} \\
&= \lim_{m\to\infty} \frac{\left(1+\frac{1}{m}\right)^{m/2} \Big/ m^{1/2}}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\left(2\left(1+\frac{1}{m}\right)^{3/2} - 1\right)} \text{ (Dividing by } m^{7/2}\text{)} \\
&= \lim_{m\to\infty} \frac{1}{m^{1/2}} = 0
\end{aligned}$$

Thus $\lim \frac{A_m}{B_m} = 0$ by the Stolz-Cesaro Theorem. And it is obvious that $n \to \infty \Leftrightarrow m \to \infty$, also our original sequence which is always non-negative is bounded by a sequence that is going to zero. Thus we have $\lim_{n\to\infty} n\mathbb{P}\left[|X_n| > b_n\right] = 0$.

Now we check the second condition. We have:

$$\begin{aligned}
\frac{n}{b_n^2}\mathbb{E}\left[\bar{X}_n^2\right] \leq& \frac{n}{b_n^2} \sum_{k=1}^{m(n)} p_k(2^k - 1)^2 \\
\leq& \frac{n}{b_n^2} \sum_{k=1}^{m(n)} p_k 2^{2k} \\
\leq& \frac{2^{m(n)} m^{3/2}(n)}{2^{2m(n)}} \sum_{k=1}^{m(n)} p_k 2^{2k} \\
=& \frac{2^m m^{3/2}}{2^{2m}} \sum_{k=1}^{m} p_k 2^{2k} \text{ (Simplify notation)} \\
=& \frac{m^{3/2}}{2^m} \sum_{k=1}^{m} \frac{2^{2k}}{k(k+1)2^k} \\
=& \frac{\sum_{k=1}^m \frac{2^k}{k(k+1)}}{\frac{2^m}{m^{3/2}}} := \frac{C_m}{D_m}
\end{aligned}$$

Applying the Stolz-Cesaro for two sequences going to positive infinity, and noticing that:

$$\begin{aligned}
\lim_{m\to\infty} \frac{C_{m+1} - C_m}{D_{m+1} - D_m} &= \lim_{m\to\infty} \frac{\frac{2^{m+1}}{(m+1)(m+2)}}{\frac{2^{m+1}}{(m+1)^{3/2}} - \frac{2^m}{m^{3/2}}} \\
&= \lim_{m\to\infty} \frac{2(m+1)^{3/2} m^{3/2}}{(m+1)(m+2)\left(2m^{3/2} - (m+1)^{3/2}\right)} \\
&= \lim_{m\to\infty} \frac{1}{m^{1/2}} = 0
\end{aligned}$$

Thus $\lim_{m\to\infty} \frac{C_m}{D_m} = 0$. Again since $m \to \infty \Leftrightarrow n \to \infty$, and the sequence $a_n = \frac{n}{b_n^2}\mathbb{E}\left[\bar{X}_n^2\right]$ is always non-negative bounded above by a sequence going to zero, the sequence a_n is also going to zero as $n \to \infty$.

Now we calculate:

$$
\begin{aligned}
a_n =& n\mathbb{E}\bar{X}_n \\
=& n\left(\sum_{k=1}^{m(n)} p_k(2^k-1) + \sum_{k=1}^{\infty} p_k - 1\right) \\
=& n\left(\sum_{k=1}^{m(n)} p_k 2^k + \sum_{k=m(n)+1}^{\infty} p_k - 1\right) \\
=& n\left(1 - \frac{1}{m(n)+1} + \sum_{k=m(n)+1}^{\infty} p_k - 1\right) \\
=& n\left(\sum_{k=m(n)+1}^{\infty} p_k - \frac{1}{m(n)+1}\right)
\end{aligned}
$$

By Theorem 2.6.6 in Durrett, we have $\frac{S_n - a_n}{b_n} \to 0$ in probability. We will argue that of the two term in a_n, the term $n\sum_{k=m(n)+1}^{\infty}$ is relatively "insignificant". We consider the residual part:

$$
\begin{aligned}
\frac{n}{b_n}\sum_{k=m(n)+1}^{\infty} p_k \leq & \frac{2^{m(n)} m^{3/2}(n)}{2^{m(n)}} \sum_{k=m(n)+1}^{\infty} p_k \\
=& \frac{\sum_{k=m+1}^{\infty} p_k}{\frac{1}{m^{3/2}}} := \frac{E_m}{F_m}
\end{aligned}
$$

Again applying the Stolz-Cesaro Theorem for two sequences going down to zero, and notice that:

$$
\begin{aligned}
\frac{E_m - E_{m+1}}{F_m - F_{m+1}} =& \frac{p_{m+1}}{\frac{1}{m^{3/2}} - \frac{1}{(m+1)^{3/2}}} \\
=& \frac{m^{3/2}(m+1)^{3/2}}{(m+1)(m+2)2^m\left((m+1)^{3/2} - m^{3/2}\right)} \\
=& \frac{m^{3/2}(m+1)^{1/2}\left((m+1)^{3/2} + m^{3/2}\right)}{(m+2)2^m\left((m+1)^3 - m^3\right)} \text{ (Squareroot Conjugate)} \\
=& \frac{O(m^{7/2})}{O(m^3)2^m} \\
\Rightarrow \lim_{m\to\infty} \frac{E_m - E_{m+1}}{F_m - F_{m+1}} =& 0
\end{aligned}
$$

Because the term 2^m dominate all other term. Thus $\lim_{m\to\infty} \frac{E_m}{F_m} = 0$

So we have the remaining part in $\frac{S_n}{b_n} - \frac{a_n}{b_n}$ which is $\frac{S_n}{b_n} - \frac{n}{b_n}\left(-\frac{1}{m(n)+1}\right)$ goes to zero in probability.

This means $\frac{S_n}{b_n} + \frac{n}{b_n(m(n)+1)} \to 0$ in probability. By the construction of $m(n)$, we have: $2^{m(n)-1}\left(m(n)-1\right)^{3/2} < n \leq 2^{m(n)} m^{3/2}(n)$. So we can consider $n \approx 2^{m(n)} m^{3/2}(n)$. With this we have:

$$
\begin{aligned}
&\frac{S_n}{2^{m(n)}} + \frac{2^{m(n)} m^{3/2}(n)}{2^{m(n)}(m(n)+1)} \to^P 0 \\
&\Rightarrow \frac{S_n}{2^{m(n)}} + \frac{m^{3/2}(n)}{(m(n)+1)} \to^P 0 \\
&\Rightarrow \frac{S_n}{2^{m(n)}} + m^{1/2}(n) \to^P 0 \\
&\Rightarrow \frac{S_n}{2^{m(n)} m^{1/2}(n)} + 1 \to^P 0 \\
&\Rightarrow \frac{S_n}{2^{m(n)} m^{3/2}(n)/m(n)} + 1 \to^P 0 \\
&\Rightarrow \frac{S_n}{2^{m(n)} m^{3/2}(n) \Big/ \left(m(n) + \frac{3}{2}\log_2 m(n)\right)} + 1 \to^P 0 \\
&\Rightarrow \frac{S_n}{n/\log_2 n} + 1 \to^P 0
\end{aligned}
$$

So $\forall \epsilon > 0$, $\mathbb{P}\left[\frac{S_n}{n/\log_2 n} + 1 \geq \epsilon\right] \to 0 \Rightarrow \mathbb{P}\left[\frac{S_n}{n/\log_2 n} + 1 < \epsilon\right] \to 1$

$\Rightarrow \mathbb{P}\left[S_n < (\epsilon - 1)\frac{n}{\log_2 n}\right] \to 1$. Let $\alpha = 1 - \epsilon < 1$ then $\mathbb{P}\left[S_n < \alpha \frac{n}{\log_2 n}\right] \to 1$.

Remark: For the last part when dealing with convergence in probability, we assume without proving a "lemma" that for (X_n) sequence of random variable, a_n, b_n sequence of real numbers, then:

(i) If $X_n + a_n + b_n \to X$ in probability, and $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ then $X_n + a_n \to X$ in probability.

(ii) If $X_n(a_n + b_n) \to X$ in probability, and $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ then $X_n a_n \to X$ in probability

(iii) If $\frac{X_n}{a_n + b_n} \to X$ in probability, and $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ then $X_n/a_n \to X$ in probability. $\square$