Solution for HW 11

1. Define $N := \inf\{n; X_n > M\}$ for $M > 0$. According to Theorem 5.2.6, $(X_{n\wedge N})_{n\in\mathbb{N}}$ is also submartingale. Observe that $X^+_{n\wedge N} \le M + \sup_n (X_n - X_{n-1})^+$ and thus $\sup_n \mathbb{E}X^+_{n\wedge N} \le M + \mathbb{E}\sup_n (X_n - X_{n-1})^+ < \infty$. By Theorem 5.2.8 (martingale convergence theorem), $X_{n\wedge N}$ converges a.s. Let $M \to \infty$, since $\sup_n X_n < \infty$ a.s., we have X_n converges a.s.

2. Denote $p_1 := \mathbb{P}(A_1)$ and $p_n := \mathbb{P}(A_n | \cap_{m=1}^{n-1} A^c_m)$ for $n \ge 2$. Note that $\prod_{n=1}^\infty (1-p_n) = \mathbb{P}(\cap_{n=1}^\infty A^c_m)$. It is well-known that for $p_n \in [0,1)$, $\prod_{n=1}^\infty (1-p_n) = 0 \Leftrightarrow \sum_{n=1}^\infty p_n = \infty$ $(*)$. According to the assumption, $\mathbb{P}(\cap_{n=1}^\infty A^c_m) = 0$ and thus $\mathbb{P}(\cup_{n=1}^\infty A_m) = 1$.

Remark : we provide an probabilistic argument to $(*)$. Consider $(X_n)_{n\in\mathbb{N}}$ i.i.d. with $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = p_n$. Then $\prod_{n=1}^\infty (1-p_n) = \mathbb{P}(X_n = 0 \text{ for all } n \ge 1)$. If $\sum_n p_n = \infty$, $\mathbb{P}(X_n = 1\ i.o.) = 1$ by Borel-Cantelli lemma and $\prod_{n=1}^\infty (1-p_n) = 0$. If $\sum_n p_n < \infty$, $\sum_{n>N} p_n < 1$ for N large enough. Thus $\mathbb{P}(X_n = 0 \text{ for } n > N) > 0$ and $\mathbb{P}(X_n = 0 \text{ for } n \ge 1) = \prod_{n=1}^N (1-p_n) \times \mathbb{P}(X_n = 0 \text{ for } n > N) > 0$.

3. Consider the martingale $Y_n := \sum_{m=1}^n \frac{\Delta_m}{b_m}$. According to **Q3** in **HW10**, $\mathbb{E}Y_n^2 = \mathbb{E}Y_0^2 + \sum_{m=1}^n \frac{\mathbb{E}\Delta_m^2}{b_m^2}$. By assumption, $\sup_n \mathbb{E}Y_n^2 < \infty$. According to Theorem 5.4.5, Y_n converges a.s. and in L^2. Finally, by Kronecker's lemma (Theorem 2.5.5), $\frac{X_n}{b_n} = \frac{\sum_{m=1}^n \Delta_m}{b_n} \to 0$ a.s.

4. Since X_n is martingale with $\sup_n \mathbb{E}|X_n| < \infty$, $X_n \to X_\infty$ a.s. by martingale convergence theorem. Define $Y_n := \mathbb{E}(X^+_\infty|\mathcal{F}_n)$ and $Z_n := \mathbb{E}(X^-_\infty|\mathcal{F}_n)$, which obvious;y satisfy the conditions in the question.

Remark : The result is known as Krickeberg's decomposition for martingales. In fact, the martingale $(X_n)_{n\in\mathbb{N}}$ has such decomposition if and only if $\lim_{n\to\infty} \mathbb{E}|X_n| < \infty$.

5. Observe that $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n(\alpha + \beta X_n) + (1 - X_n)\beta X_n = X_n$. Thus $(X_n)_{n\in\mathbb{N}}$ is martingale taking values in $[0,1]$. By martingale convergence theorem, $X_n \to X_\infty$ a.s. Note that given $X_n = x$, $X_{n+1} = \alpha + \beta x$ or βx for $\alpha, \beta > 0$. This implies that $X_\infty \in \{0,1\}$. In addition, $\mathbb{E}X_\infty = EX_0 = x_0$, which permits to conclude.

6. By triangle inequality, $\mathbb{E}|\mathbb{E}(Y_n|\mathcal{F}_n) - \mathbb{E}(Y_\infty|\mathcal{F}_\infty)| \le \mathbb{E}|\mathbb{E}(Y_n|\mathcal{F}_n) - \mathbb{E}(Y_\infty|\mathcal{F}_n)| + \mathbb{E}|\mathbb{E}(Y_\infty|\mathcal{F}_n) - \mathbb{E}(Y_\infty|\mathcal{F}_\infty)| \overset{(*)}{\le} \mathbb{E}\mathbb{E}(|Y_n - Y_\infty||\mathcal{F}_n) + \mathbb{E}|\mathbb{E}(Y_\infty|\mathcal{F}_n) - \mathbb{E}(Y_\infty|\mathcal{F}_\infty)| = \mathbb{E}|Y_n - Y_\infty| + \mathbb{E}|\mathbb{E}(Y_\infty|\mathcal{F}_n) - \mathbb{E}(Y_\infty|\mathcal{F}_\infty)|$ $(**)$, where $(*)$ follows Jensen's inequality for conditional expectation. The first term in $(**)$ converges to 0 since $Y_n \to Y_\infty$ in L^1 and the second one in $(**)$ goes to 0 by Theorem 5.5.7.

7. Write $S_n - S_0 = \sum_{i=1}^n \zeta_i$, where $\zeta_i := c - \xi_i$ are i.i.d $\mathcal{N}(c-\mu, \sigma^2)$. Denote $\theta := \frac{2(\mu - c)}{\sigma^2}$ and it is easy to check that $\mathbb{E}e^{\theta\zeta_1} = 1$. Thus, $X_n := e^{\theta(S_n - S_0)}$ is martingale. Define $T := \inf\{n; S_n \le 0\}$. Then $X_{n\wedge T}$ is also martingale and monotone convergence theorem leads to $\mathbb{E}X_T = 1$. By Chebyshev inequality, $\mathbb{P}(\text{ruin}) \le e^{-\theta S_0}\mathbb{E}X_T = \exp(-2(c-\mu)S_0/\sigma^2)$.