205A Homework #1, due Tuesday 10 September.

1. [Bill. 2.4] Let $\mathcal{F}_n$ be classes of subsets of S. Suppose each $\mathcal{F}_n$ is a field, and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for $n = 1, 2, \ldots$. Define $\mathcal{F} = \cup_{n=1}^{\infty} \mathcal{F}_n$. Show that $\mathcal{F}$ is a field. Give an example to show that $\mathcal{F}$ need not be a σ-field.

2. [Bill. 2.5(b)] Given a non-empty collection $\mathcal{A}$ of sets, we defined $\mathcal{F}(\mathcal{A})$ as the intersection of all fields containing $\mathcal{A}$. Show that $\mathcal{F}(\mathcal{A})$ is the class of sets of the form $\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{ij}$, where for each i and j either $A_{i,j} \in \mathcal{A}$ or $A_{ij}^c \in \mathcal{A}$, and where the m sets $\cap_{j=1}^{n_i} A_{ij}$, $1 \le i \le m$ are disjoint.

3. [Bill. 2.8] Suppose $B \in \sigma(\mathcal{A})$, for some collection $\mathcal{A}$ of subsets. Show there exists a countable subcollection $\mathcal{A}_B$ of $\mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$.

4. Show that the Borel σ-field on $\mathbb{R}^d$ is the smallest σ-field that makes all continuous functions $f : \mathbb{R}^d \to R$ measurable.

5. [Durr. 1.3.5] A function $f : \mathbb{R}^d \to R$ is *lower semicontinuous* (l.s.c.) if $\liminf_{y \to x} f(y) \ge f(x)$ for all x. A function is *upper semicontinuous* (u.s.c.) if $\limsup_{y \to x} f(y) \le f(x)$ for all x. Show that, if f is l.s.c. or u.s.c., then f is measurable.

205A Homework #2, due Tuesday 17 September.

1. [similar Bill. 2.15] Let $\mathcal{B}$ be the Borel subsets of $\mathbb{R}$. For $B \in \mathcal{B}$ define

$$\begin{aligned} \mu(B) \quad &= 1 \quad \text{if } (0,\varepsilon) \subset B \text{ for some } \varepsilon > 0 \\ &= 0 \quad \text{if not} \end{aligned}$$

(a) Show that μ is not finitely additive on $\mathcal{B}$.

(b) Show that μ is finitely additive but not countably additive on the field $\mathcal{B}_0$ of finite disjoint unions of intervals $(a, b]$.

2. Show that, in the definition of "a probability measure μ on a measurable space $(S, \mathcal{S})$", we may replace "countably additive" by "finitely additive, and satisfies

$$\text{if } A_n \downarrow \phi \text{ then } \mu(A_n) \to 0 \ . \ \text{"}$$

3. [similar Durr. A.1.1] Give an example of a measurable space $(S, \mathcal{S})$, a collection $\mathcal{A}$ and probability measures μ and ν such that
(i) $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$
(ii) $\mathcal{S} = \sigma(\mathcal{A})$
(iii) $\mu \neq \nu$.
Note: this can be done with $S = \{1, 2, 3, 4\}$

4. [similar Durr. Lemma A.2.1] Let μ be a probability measure on $(S, \mathcal{S})$, where $\mathcal{S} = \sigma(\mathcal{F})$ for a field $\mathcal{F}$. Show that for each $B \in \mathcal{S}$ and $\varepsilon > 0$ there exists $A \in \mathcal{F}$ such that $\mu(B \Delta A) < \varepsilon$.

5. Let $g : [0, 1] \to \mathbb{R}$ be integrable w.r.t. Lebesgue measure. Let $\varepsilon > 0$. Show that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $\int |f(x) - g(x)| \, dx \leq \varepsilon$.

205A Homework #3, due Tuesday 24 September.

1. Use the monotone convergence theorem to prove the following.
(i) If $X_n \geq 0$, $X_n \downarrow X$ a.s. and $EX_n < \infty$ for some n then $EX_n \to EX$.
(ii) If $E|X| < \infty$ then $E|X|1_{(|X|>n)} \to 0$ as $n \to \infty$.
(iii) If $E|X_1| < \infty$ and $X_n \uparrow X$ a.s. then either $EX_n \uparrow EX < \infty$ or else $EX_n \uparrow \infty$ and $E|X| = \infty$.
(iv) If X takes values in the non-negative integers then

$$EX = \sum_{n=1}^{\infty} P(X \geq n).$$

2. (i) For a counting r.v. $X = \sum_{i=1}^{n} 1_{A_i}$, give a formula for the variance of X in terms of the probabilities $P(A_i)$ and $P(A_i \cap A_j)$, $i \neq j$.

(ii) If k balls are put at random into n boxes, what is the variance of $X =$ number of empty boxes?

3. (i) Suppose $EX = 0$ and $\text{var}(X) = \sigma^2 < \infty$. Prove

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \quad a > 0.$$

(ii) Suppose $X \geq 0$ and $EX^2 < \infty$. Prove

$$P(X > 0) \geq \frac{(EX)^2}{EX^2}.$$

4. Chebyshev's other inequality.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be bounded and increasing functions. Prove that, for any r.v. X,

$$E(f(X)g(X)) \geq (Ef(X))(Eg(X)).$$

[In other words, $f(X)$ and $g(X)$ are positively correlated. This is intuitively obvious, but a little tricky to prove. Hint: consider an independent copy Y of X. For this and the next question you may need the product rule for expectations of independent r.v.s]

5. Let X have Poisson(λ) distribution and let Y have Poisson(2λ) distribution.

(i) Prove $P(X \geq Y) \leq \exp(-(3 - \sqrt{8})\lambda)$ if X and Y are independent.

(ii) Find constants $A < \infty$, $c > 0$, not depending on λ, such that, without assuming independence, $P(X \geq Y) \leq A \exp(-c\lambda)$.

205A Homework #4, due Tuesday 1 October.

1. Monte Carlo integration [cf. Durr. 2.2.3] Let $f : [0,1] \to \mathbb{R}$ be such that $\int_0^1 f^2(x)\ dx < \infty$. Let (U_i) be i.i.d. Uniform$(0,1)$. Let

$$D_n := n^{-1} \sum_{i=1}^n f(U_i) \ - \int_0^1 f(x)\ dx.$$

(i) Use Chebyshev's inequality to bound $P(|D_n| > \varepsilon)$.
(ii) Show this bound remains true if the (U_i) are only *pairwise* independent.

2. Let $X \geq 0$ and $Y \geq 0$ be independent r.v.'s with densities f and g. Calculate the densities of XY and of X/Y.

Note: this is just to remind you of "undergraduate" results.

3. [Durr. 2.2.2.] Let (X_i) be r.v.'s with $EX_i = 0$ and $EX_iX_j \leq r(j-i)$, $1 \leq i \leq j < \infty$, where $r(n)$ is a deterministic sequence with $r(n) \to 0$ as $n \to \infty$. Prove that $n^{-1} \sum_{i=1}^n X_i \to 0$ in probability.

4. [Durr. 2.3.11] Suppose events A_n satisfy $P(A_n) \to 0$ and

$$\sum_{n=1}^\infty P(A_n^c \cap A_{n+1}) < \infty.$$

Prove that

$$P(A_n \text{ occurs infinitely often }) = 0.$$

5. (a) Let Z have standard Normal distribution. Show

$$P(Z > z) \sim z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \text{ as } z \to \infty.$$

(b) Let $(Z_1, Z_2, \ldots)$ be independent with standard Normal distribution. Find constants $c_n \to \infty$ such that

$$\limsup_n Z_n/c_n = 1 \text{ a.s.}$$

205A Homework #5, due Tuesday 8 October.

1. Let (X_n) be i.i.d. with $E|X_1| < \infty$. Let $M_n = \max(X_1, \ldots, X_n)$. Prove that $n^{-1}M_n \to 0$ a.s.

2. [Durr. 2.3.2] Let $0 \le X_1 \le X_2 \le \ldots$ be r.v.'s such that $EX_n \sim an^\alpha$ and $\mathrm{var}(X_n) \le Bn^\beta$, where $0 < a, B < \infty$ and $0 < \beta < 2\alpha < \infty$. Prove that $n^{-\alpha}X_n \to a$ a.s.

3. Prove that the following are equivalent.
(i) $X_n \to X$ in probability.
(ii) There exist $\varepsilon_n \downarrow 0$ such that $P(|X_n - X| > \varepsilon_n) \le \varepsilon_n$.
(iii) $E\min(|X_n - X|, 1) \to 0$.

4. Durr. exercise 2.4.4 (*An Investment Problem*).

5. Prove the deterministic lemma we used in the proof of the Glivenko-Cantelli Theorem.

Lemma. *If* $F_1, F_2, \ldots, F$ *are distribution functions and*
(i) $F_n(x) \to F(x)$ *for each rational* x
(ii) $F_n(x) \to F(x)$ *and* $F_n(x-) \to F(x-)$ *for each atom* x *of* F
then $\sup_x |F_n(x) - F(x)| \to 0$.

205A Homework #6, due Tuesday 15 October.

1. [Durr. 2.5.9] Let (X_i) be independent, $S_n = \sum_{i=1}^n X_i$, $S_n^* = \max_{i \le n} |S_i|$. Prove that

$$P(S_n^* > 2a) \le \frac{P(|S_n| > a)}{\min_{j \le n} P(|S_n - S_j| \le a)} \quad , \ a > 0.$$

[Hint. If $|S_j| > 2a$ and $|S_n - S_j| \le a$ then $|S_n| > a$.]

2. [Durr. 2.5.10 and 11] In the setting of the previous question, prove
(i) if $\lim_{n\to\infty} S_n$ exists in probability then the limit exists a.s.
(ii) if the (X_i) are identically distributed and if $n^{-1}S_n \to 0$ in probability then $n^{-1}\max_{m \le n} S_m \to 0$ in probability.

3. [cf. Durr 2.2.8] Let (X_i) be i.i.d. taking values in $\{-1, 1, 3, 7, 15, \ldots\}$, such that

$$P(X_1 = 2^k - 1) = \frac{1}{k(k+1)2^k}, \ k \ge 1$$

(which implicitly specifies $P(X_1 = -1)$).
(a) Show $EX_1 = 0$.
(b) Show that for all $\alpha < 1$,

$$P\left(S_n < -\frac{\alpha n}{\log_2 n}\right) \to 1.$$

Comment. This is sometimes described as "an unfair, fair game". It shows that the conclusions of the SLLN and the "recurrence of sums" theorem can't be strengthened much.

205A Homework #7, due Tuesday 22 October.

1. Suppose S and T are stopping times. Are the following necessarily stopping times? Give proof or counter-example.

(a) $\min(S,T)$

(b) $\max(S,T)$

(c) $S+T$.

2. Let (X_i) be i.i.d. with $EX_i^2 < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Let T be a bounded stopping time. Is it true in general that

$$\mathrm{var}(S_T) = (\mathrm{var}(X_1))(ET)?$$

If not, is it true in the special case $EX_1 = 0$?

3. Let (X_i) be a sequence of random variables, and let $\mathcal{T}$ be its tail σ-field. Let $S_n = \sum_{i=1}^n X_i$. Let $b_n \uparrow \infty$ be constants. Which of the following events must be in $\mathcal{T}$? Give proof or counter-example.

(i) $\{X_n \to 0\}$

(ii) $\{S_n \text{ converges }\}$

(iii) $\{X_n > b_n \text{ infinitely often }\}$

(iv) $\{S_n > b_n \text{ infinitely often }\}$

(v) $\{\frac{\sqrt{\sum_{i=1}^n X_i^2}}{S_n} \to 0\}$.

4. Let $S_n = \sum_{i=1}^n X_i$, where (X_i) are i.i.d. with exponential(1) distribution. Use the large deviation theorem to get explicit limits for
$n^{-1}\log P(n^{-1}S_n \geq a),\ a > 1$ and $n^{-1}\log P(n^{-1}S_n \leq a),\ a < 1$.

5. Oriented first passage percolation. Consider the lattice quadrant $\{(i,j) : i,j \geq 0\}$ with directed edges $(i,j) \to (i+1,j)$ and $(i,j) \to (i,j+1)$. Associate to each edge e an exponential(1) r.v. X_e, independent for different edges. For each directed path π of length d started at $(0,0)$, let $S_\pi = \sum_{\text{edges } e \text{ in path}} X_e$. Let H_d be the minimum of S_π over all such paths π of length d. It can be shown that $d^{-1}H_d \to c$ a.s., for some constant c. Give explicit upper and lower bounds on c.

[Hint: use result of previous question for lower bound.]

205A Homework #8, due Tuesday 5 November.

[Theorem 7 and Corollary 8 refer to the notes linked from the "week 8" row of the schedule.]

1. Suppose probability measures satisfy $\pi \ll \nu \ll \mu$. Show that

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \times \frac{d\nu}{d\mu}.$$

2. In the setting of Theorem 7 [hard part], where S_2 is nice, show that Q is unique in the following sense. If Q^* is another conditional probability kernel for μ, then

$$\mu_1\{x : Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2\} = 1.$$

3. Let F be a distribution function. Let $c > 0$. Find a simple formula for

$$\int_{-\infty}^{\infty} (F(x + c) - F(x))\, dx.$$

4. In the proof of Corollary 8 we used the inverse distribution function

$$f(x, u) = \inf\{y : u \leq Q(x, (-\infty, y])\}$$

associated with the kernel Q. Show that f is product measurable.

5. Given a triple (X_1, X_2, X_3), we can define 3 p.m.'s $\mu_{12}, \mu_{13}, \mu_{23}$ on $\mathbb{R}^2$ by

$$\mu_{ij} \text{ is the distribution of } (X_i, X_j). \tag{1}$$

These p.m.'s satisfy a consistency condition:

$$\begin{gathered}\text{the marginal distribution } \mu_1 \text{ obtained from } \mu_{12} \text{ must coincide} \\ \text{with the marginal obtained from } \mu_{13}, \text{ and similarly for } \mu_2 \text{ and } \mu_3.\end{gathered} \tag{2}$$

Give an example to show that the converse is false. That is, give an example of $\mu_{12}, \mu_{13}, \mu_{23}$ satisfying (2) but for which there does not exist a triple (X_1, X_2, X_3) satisfying (1).

205A Homework #9, due Tuesday 12 November

1. Let X, Y be random variables, and suppose Y is measurable with respect to some sub-σ-field $\mathcal{G}$. Let $\mu(\omega, \cdot)$ be a regular conditional distribution for X given $\mathcal{G}$. Prove that, for bounded measurable h,

$$E(h(X,Y)|\mathcal{G})(\omega) = \int h(x, Y(\omega))\mu(\omega, dx) \ a.s.$$

2. For $i = 1, 2$ let X_i be a r.v. defined on $(\Omega, \mathcal{F}, P)$ taking values in $(S_i, \mathcal{S}_i)$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Prove that assertions (a),(b) and (c) below are equivalent. When these assertions hold, we say call X_1 and X_2 are conditionally independent given $\mathcal{G}$.

(a) $P(X_1 \in A_1, X_2 \in A_2|\mathcal{G}) = P(X_1 \in A_2|\mathcal{G})P(X_2 \in A_2|\mathcal{G})$ for all $A_i \in \mathcal{S}_i$.

(b) $E(h_1(X_1)h_2(X_2)|\mathcal{G}) = E(h_1(X_1)|\mathcal{G}) \ E(h_2(X_2)|\mathcal{G})$ for all bounded measurable $h_i : S_i \to \mathbb{R}$.

(c) $E(h_1(X_1)|\mathcal{G}, X_2) = E(h_1(X_1)|\mathcal{G})$ for all bounded measurable $h_1 : S_1 \to \mathbb{R}$.

3. Suppose X and Y are conditionally independent given Z. Suppose X and Z are conditionally independent given $\mathcal{F}$, where $\mathcal{F} \subseteq \sigma(Z)$. Prove that X and Y are conditionally independent given $\mathcal{F}$.

4. Let (X_n) and (Y_n) be submartingales w.r.t. $(\mathcal{F}_n)$. Show that $(X_n + Y_n)$ and that $(\max(X_n, Y_n))$ are also submartingales w.r.t. $(\mathcal{F}_n)$.

5. Give an example where

(X_n) is a submartingale w.r.t. $(\mathcal{F}_n)$

(Y_n) is a submartingale w.r.t. $(\mathcal{G}_n)$

$(X_n + Y_n)$ is not a submartingale w.r.t. any filtration.

205A Homework #10, due Tuesday 19 November.

1. Let $S_n = \sum_{i=1}^n \xi_i$, where the (ξ_i) are independent, $E\xi_i = 0$ and var $\xi_i < \infty$. Let $s_n^2 = \sum_{i=1}^n \text{var } \xi_i$. So we know that $(S_n^2 - s_n^2)$ is a martingale. Suppose also that $|\xi_i| \leq K$ for some constant K. Show that

$$P\left(\max_{m\leq n} |S_m| < x\right) \leq s_n^{-2}(K+x)^2, \qquad x > 0.$$

2. Let (X_n) be a martingale with $X_0 = 0$ and $EX_n^2 < \infty$. Using the fact that $(X_n + c)^2$ is a submartingale, show that

$$P\left(\max_{m\leq n} X_m \geq x\right) \leq \frac{EX_n^2}{x^2 + EX_n^2}, \qquad x > 0.$$

3. Let (X_n) and (Y_n) be martingales with $E(X_n^2 + Y_n^2) < \infty$. Show that

$$EX_nY_n - EX_0Y_0 = \sum_{m=1}^n E(X_m - X_{m-1})(Y_m - Y_{m-1}).$$

4. Let $(X_n, \mathcal{F}_n), n \geq 0$ be a positive submartingale with $X_0 = 0$. Let V_n be random variables such that
(i) $V_n \in \mathcal{F}_{n-1},\ n \geq 1$
(ii) $B \geq V_1 \geq V_2 \geq \ldots \geq 0$, for some constant B.

Prove that for $\lambda > 0$

$$P(\max_{1\leq j\leq n} V_jX_j > \lambda) \leq \lambda^{-1} \sum_{j=1}^n E[V_j(X_j - X_{j-1})].$$

5. Prove *Dubins' inequality.* If (X_n) is a positive martingale then the number U of upcrossings of $[a,b]$ satisfies

$$P(U \geq k) \leq (a/b)^k E \min(X_0/a, 1).$$

[if you follow sketch in Durrett then prove the quoted exercise]

205A Homework #11, due Tuesday 26 November.

In each question, there is some fixed filtration $(\mathcal{F}_n)$ with respect to which martingales are defined.

1. Let (X_n) be a submartingale such that $\sup_n X_n < \infty$ a.s. and $E \sup_n (X_n - X_{n-1})^+ < \infty$. Show that X_n converges a.s.

2. For a sequence (A_n) of events, show that

$$\sum_{n=2}^{\infty} P(A_n | \cap_{m=1}^{n-1} A_m^c) = \infty \text{ implies } P(\cup_{m=1}^{\infty} A_m) = 1.$$

3. Let (X_n) be a martingale and write $\Delta_n = X_n - X_{n-1}$, Suppose that $b_m \uparrow \infty$ and $\sum_{m=1}^{\infty} b_m^{-2} E\Delta_m^2 < \infty$. Prove that $X_n/b_n \to 0$ a.s.

4. Let (X_n) be a martingale with $\sup_n E|X_n| < \infty$. Show that there is a representation $X_n = Y_n - Z_n$ where (Y_n) and (Z_n) are non-negative martingales such that $\sup_n EY_n < \infty$ and $\sup_n EZ_n < \infty$.

5. Let (X_n) be adapted to $(\mathcal{F}_n)$ with $0 \le X_n \le 1$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Suppose $X_0 = x_0$ and

$$P(X_{n+1} = \alpha + \beta X_n | \mathcal{F}_n) = X_n, \ P(X_{n+1} = \beta X_n | \mathcal{F}_n) = 1 - X_n.$$

Show that $X_n \to X_\infty$ a.s., where $P(X_\infty = 1) = x_0$ and $P(X_\infty = 0) = 1 - x_0$.

6. Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y_\infty$ in L^1. Show that $E(Y_n|\mathcal{F}_n) \to E(Y_\infty|\mathcal{F}_\infty)$ in L^1.

7. Let S_n be the total assets of an insurance company at the end of year n. Suppose that in year n the company receives premiums of c and pays claims totaling ξ_n, where ξ_n are independent with Normal(μ, σ^2) distribution, where $0 < \mu < c$. The company is ruined if its assets fall to 0 or below. Show

$$P(\text{ruin}) \le \exp(-2(c-\mu)S_0/\sigma^2).$$