Solution for HW 7

1. (a) and (b). Note that $\{\min(S,T) \le n\} = \{S \le n\} \cup \{T \le n\} \in \mathcal{F}_n$ and $\{\max(S,T) \le n\} = \{S \le n\} \cap \{T \le n\} \in \mathcal{F}_n$. (c). We have $\{S+T \le n\} = \cup_{k=0}^n \{S = k\} \cap \{T = n-k\} \in \mathcal{F}_n$.
2. (a).The identity is not true in general. Consider X_i i.i.d s.t. $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$ and the stopping time T s.t. $T = 1$ if $X_1 = 0$ and $T = 2$ if $X_1 = 1$. Simple computations lead to $VarX_1 = \frac{1}{4}$, $\mathbb{E}T = \frac{3}{2}$ and $VarS_T = \frac{8}{11}$. Observe that $VarS_T \neq VarX_1\mathbb{E}T$. (b). When $\mathbb{E}T = 0$, the identity holds true. Suppose that T is bounded by m. $VarS_T = \mathbb{E}S_T^2 = \mathbb{E}\sum_{n=1}^m S_n^2 1_{T=n} = \mathbb{E}\sum_{n=1}^m X_n^2 1_{T\ge n} + 2\mathbb{E}\sum_{1\le i<j\le n} X_iX_j1_{T\ge j}$. Note that $\{T \ge n\} \in \mathcal{F}_{n-1}$, which is independent of X_n. Thus $\mathbb{E}\sum_{n=1}^m X_n^2 1_{T\ge n} = \mathbb{E}X_n^2\mathbb{E}\sum_{n=1}^m 1_{T\ge n} = VarX_1\mathbb{E}T$. In addition, for $1 \le i < j \le n$, $\{T \ge j\} \in \mathcal{F}_{j-1}$ is independent of X_j. Therefore, $\mathbb{E}\sum_{1\le i\neq j\le n} X_iX_j1_{T=n} = 0$.
Remark : The identity in the question is known as Wald's identity. The result still holds if we only suppose that $\mathbb{E}T < \infty$ (by dominated convergence theorem).
3. (i). For fixed m, $\{X_n \to 0\} = \{X_{m+n} \to 0\} \in \mathcal{G}_m$. Thus $\{X_n \to 0\} \in \cap_{m=1}^\infty \mathcal{G}_m = \mathcal{T}$. (ii). For fixed m, $\{S_n$ converges$\} = \{S_n - S_{m-1}$ converges$\} \in \mathcal{G}_m$. Thus $\{S_n$ converges$\} \in \cap_{m=1}^\infty \mathcal{G}_m = \mathcal{T}$. (iii). For fixed m, $\{X_n > b_n \ i.o.\} = \{X_{m+n} > b_{m+n} \ i.o.\} \in \mathcal{G}_m$. Thus $\{X_n > b_n \ i.o.\} \in \cap_{m=1}^\infty \mathcal{G}_m = \mathcal{T}$. (iv). Consider $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = \frac{1}{2}$, $X_n = 1$ for $n \ge 2$ and $b_n = n$ for all n. Then $\mathbb{P}(S_n > b_n \ i.o.) = \frac{1}{2} \in \{0,1\}$. According to Kolmogorov's $0-1$ law, $\{S_n > b_n \ i.o.\} \notin \mathcal{T}$. (v). Remark that for fixed m, $\frac{\sqrt{\sum_{i=1}^n X_i^2}}{S_n} \to 0$ is equivalent to $\frac{\sqrt{\sum_{i=1}^n X_i^2 - \sum_{i=1}^{m-1} X_i^2}}{S_n - S_{m-1}} \to 0$. The event is in $\mathcal{G}_m$. Thus $\{\frac{\sqrt{\sum_{i=1}^n X_i^2}}{S_n} \to 0\} \in \mathcal{T}$.
4. (a). The moment generating function for exponential distributed random variable is $\phi(\theta) = \frac{1}{1-\theta}$. Note that $\frac{\phi'(\theta_a)}{\phi(\theta_a)} = a \Leftrightarrow \theta_a = 1 = \frac{1}{a}$. For $a > 1$, $\theta_a \in (0,1)$. By large deviation principle, $n^{-1}\log\mathbb{P}(n^{-1}S_n \ge a) \to \log\phi(\theta_a) - a\theta_a = \log a - a + 1$. (b). Observe that $\mathbb{P}(n^{-1}S_n \le a) = \mathbb{P}(-n^{-1}S_n \ge -a)$. A similar argument as in (a) permits to conclude that $\mathbb{P}(n^{-1}S_n \le a) \to \log a - a + 1$.
5. By definition of H_d, $H_d \le \sum_{i=1}^d \min(X_i^{up}, X_i^{right})$ where X_i^{up} and X_i^{right}'s are i.i.d. exponential distributed random variables. According to strong law of large numbers, $d^{-1}\sum_{i=1}^d \min$ $(X_i^{up}, X_i^{right}) \to \frac{1}{2}$ a.s. since the minimum of two independent exponential(1) random variables is exponential ($\frac{1}{2}$). Thus, $c \le \frac{1}{2}$. On the other hand, $\mathbb{P}(\frac{H_d}{d} \le a) = \mathbb{P}(\exists\pi^*, \frac{S_{\pi^*}}{d} \le a) \le 2^d\mathbb{P}(\frac{\sum_{i=1}^n X_i}{d} \le a)$. According to **Q4 (b)**, $d^{-1}\log\mathbb{P}(\frac{H_d}{d} \le a) \le d^{-1}\log 2^d\mathbb{P}(\frac{\sum_{i=1}^n X_i}{d} \le a) \to \log 2a - a + 1$. Therefore, $c > c_0$ where c_0 is the solution of $\log 2a - a + 1 = 0$. Numerically, $c_0 = 0.23$.