Solution for HW 6

1. Let $A_j := \{|S_j| > 2a$ and $|S_k| \leq 2a$ for $k < j\}$ and $B_j := \{|S_n - S_j| \leq a\}$. Note that $A_j \cap B_j \subset \{|S_n| > a\}$ for $j \leq n$. Moreover, $(A_j)_{j\leq n}$ are pairwise disjoint $(*)$; A_j and B_j are independent for $j \leq n$ $(**)$, we have then $\mathbb{P}(|S_n| > a) \geq \mathbb{P}(\cup_{j=1}^n (A_j \cap B_j)) \overset{(*)}{=} \sum_{j=1}^n \mathbb{P}(A_j \cap B_j) \overset{(**)}{=} \sum_{j=1}^n \mathbb{P}(A_j)\mathbb{P}(B_j) \geq \min_{j\leq n} \mathbb{P}(B_j) \sum_{j=1}^n \mathbb{P}(A_j) = \min_{j\leq n} \mathbb{P}(|S_n - S_j| \leq a)\mathbb{P}(S_n^* > 2a)$.
2. (i). Fix $a > 0$. Using **Q1**, for $n \geq m$, $\mathbb{P}(\max_{m<j\leq n} |S_j - S_m| > 2a) \leq \frac{\mathbb{P}(|S_n - S_m| > a)}{\min_{m<j\leq n} \mathbb{P}(|S_n - S_j| \leq a)}$. Since $(S_m)_{m\in\mathbb{N}}$ converges in probability, $\mathbb{P}(|S_n - S_m| > a) \to 0$ and $\min_{m<j\leq n} \mathbb{P}(|S_n - S_j| \leq a) \to 1$ as $m \to \infty$. Therefore, $\mathbb{P}(\max_{m<j\leq n} |S_j - S_m| > 2a) \to 0$ as $m \to 0$. This implies that a.s. $(S_m)_{m\in\mathbb{N}}$ is a Cauchy sequence and thus it converges a.s. (ii). Again by **Q1**, $\mathbb{P}(S_n^* > 2na) \leq \frac{\mathbb{P}(|S_n| > na)}{\min_{1\leq j\leq n} \mathbb{P}(|S_n - S_j| \leq na)}$. Note that $(\frac{S_n}{n})_{n\in\mathbb{N}}$ converges in probability to 0 : $\mathbb{P}(|S_n| > na) \to 0$ and $\min_{1\leq j\leq n-1} \mathbb{P}(|S_j| \leq ja)$ is bounded away from 0. In addition, $\min_{1\leq j\leq n} \mathbb{P}(|S_n - S_j| \leq na) \overset{(*)}{=} \min_{1\leq j\leq n-1} \mathbb{P}(|S_j| \leq na) \geq \min_{1\leq j\leq n-1} \mathbb{P}(|S_j| \leq ja)$ where $(*)$ is due to the fact that $(X_i)_{i\in\mathbb{N}}$ is i.i.d. Thus we prove the desired result.
3. (a) By definition, $\mathbb{E}X_1 = \sum_{k=1}^\infty (2^k - 1)\frac{1}{k(k+1)2^k} + (-1) \times (1 - \sum_{k=1}^\infty \frac{1}{k(k+1)2^k}) = -1 + \sum_{k=1}^\infty \frac{1}{k(k+1)} = 0$. (b). We apply **Thm 2.2.6** to $b_n := 2^{m(n)}$ where $m(n) := \inf\{m; 2^{-m}m^{-\frac{3}{2}} \leq \frac{1}{n}\}$. To this end, we need to check the hypotheses (i) and (ii) in the theorem. Observe that for $m \in \mathbb{N}$, $\mathbb{P}(X_1 > 2^m) \leq \sum_{k=m+1}^\infty \frac{1}{2^k m(m+1)} = \frac{1}{2^m m(m+1)}$. We have then $n\mathbb{P}(X_1 > b_n) \leq \frac{n2^{-m(n)}}{m(n)(m(n)+1)} \leq \frac{1}{\sqrt{m(n)+1}} \to 0$ as $n \to \infty$. Thus (i) is satisfied. Now consider $\bar{X} := X1_{|X|\leq b_n}$. We have, $\mathbb{E}\bar{X}^2 \leq 1 + \sum_{k=1}^{m(n)} \frac{2^{2k}}{2^k k(k+1)} \leq 1 + \sum_{k=1}^{\frac{m(n)}{2}} 2^k + \frac{4}{m(n)^2} \sum_{k=\frac{m(n)}{2}}^{m(n)} 2^k \leq \frac{C2^{m(n)}}{m(n)^2}$ for some $C > 0$. Therefore, $\frac{n\mathbb{E}\bar{X}^2}{b_n} \leq \frac{C2^{m(n)}}{m(n)^2} \frac{n}{2^{2m(n)}} \leq \frac{C}{\sqrt{m(n)}} \to 0$ as $n \to \infty$: we have checked (ii). We now compute $a_n := n\mathbb{E}\bar{X}$. Observe that $a_n = -n\sum_{k=m(n)+1}^\infty \frac{2^k - 1}{2^k k(k+1)} = -\frac{1}{m(n+1)} + \sum_{k=m(n)+1}^\infty \frac{1}{2^k k(k+1)} \sim -\frac{1}{m(n)} \sim -\frac{1}{\log_2 n}$. Therefore, $\frac{S_n + n/\log_2 n}{n/(\log_2 n)^{\frac{3}{2}}} \to 0$ as $n \to \infty$. This implies that for $\alpha < 1$, $\mathbb{P}(S_n < -\frac{\alpha n}{\log_2 n}) \to 0$.