ST205 - Homework 11

Hoang Duong

November 24, 2014

Problem 1. Let (X_n) be a submartingale such that $\sup_n X_n < \infty$ a.s. and $\mathbb{E}\sup_n (X_n - X_{n-1})^+ < \infty$. Show that X_n converges a.s.

Proof. Let $N = \inf\{n \mid X_n > M\}$ for M fixed. By OST, we have $X_{n\wedge N}$ is also a submartingale. We have:

$$\begin{aligned} X_{n\wedge N}^+ &\leq M + \sup_n (X_n - X_{n-1})^+ \\ \Rightarrow \mathbb{E}X_{n\wedge N}^+ &\leq M + \mathbb{E}\sup_n (X_n - X_{n-1})^+ < \infty \end{aligned}$$

By Martingale Convergence Theorem, we have: $X_{n\wedge N}$ converges almost surely.

Let $M \to \infty$ and and since $\sup_n X_n < \infty$, $X_{n\wedge N} \to X_n$, thus we have $(X_n)_n$ converges *a.s.* □

Lemma 1. *For* $p_i \in [0,1], \forall i \in \mathbb{N}$, *prove that:*

$$\prod_{m=1}^{\infty}(1-p_m) = 0 \Leftrightarrow \sum_{m=1}^{\infty} p_m = \infty$$

Proof. $\Leftarrow$. Given $\sum_{i=1}^{\infty} p_m = \infty$. Otherwise, we have:

$$\begin{aligned} \log(1-x) =& -x - \frac{x^2}{2} - \frac{x^3}{3} - ..., \forall x \in [0,1) \\ \Rightarrow \log(1-x) \leq& -x \\ \Rightarrow \sum_{m=1}^{\infty} \log(1-p_m) \leq& \sum_{m=1}^{\infty} -p_m \to -\infty \\ \Rightarrow \prod_{m=1}^{\infty}(1-p_m) =& 0 \end{aligned}$$

$\Rightarrow$. Given $\prod_{m=1}^{\infty}(1-p_m) = 0$. (We don't need this direction for Problem 2.) □

Problem 2. For a sequence (A_n) of events, show that:

$$\sum_{n=2}^{\infty} \mathbb{P}\left[A_n \mid \bigcap_{m=1}^{n-1} A_m^c\right] = \infty \Rightarrow \mathbb{P}\left[\bigcup_{m=1}^{\infty} A_m\right] = 1$$

Proof. We have: $\mathbb{P}\left[\bigcup_{m=1}^{\infty}\right] = 1 \Leftrightarrow \mathbb{P}\left[\bigcap_{m=1}^{\infty} A_m^c\right] = 0$. Now set: $p_1 = \mathbb{P}[A_1], p_n = \left[A_n \mid \bigcap_{m=1}^{n-1} A_n^c\right]$. Then we have: $\mathbb{P}\left[\bigcap_{m=1}^{\infty} A_m^c\right] = \prod_{m=1}^{\infty}(1-p_m)$. Since we have $\sum_{m=1}^{\infty} p_m = \infty$, thus by Lemma 1, $\prod_{m=1}^{\infty}(1-p_m) = 0$. □

Problem 3. Let (X_n) be a martingale and write $\Delta_n = X_n - X_{n-1}$. Suppose that $b_m \uparrow \infty$ and $\sum_{m=1}^{\infty} b_m^{-2}\mathbb{E}\Delta_m^2 < \infty^{(*)}$. Prove that $X_n/b_n \to 0$ a.s.

Proof. Since X_n is a martingale, we have $Y_n = \sum_{m=1}^{n} \Delta_m/b_m$ is also a martingale as $\mathbb{E}\left[X_{n+1} - X_n \mid \mathcal{F}_n\right] = 0, \forall n \in \mathbb{N}$. Combining this fact with $(*)$, and applying Question 3 of Homework 10, we have:

$$\mathbb{E}Y_n^2 = \mathbb{E}Y_0^2 + \sum_{m=1}^{n} \frac{\mathbb{E}\Delta_m^2}{b_m^2}$$

Also because of $(*)$, we have $\sup_n \mathbb{E}Y_n^2 < \infty$. Applying the Martingale Convergence Theorem, we have: $Y_n \to Y_\infty$a.s. and in L^2.

Thus by Kronecker's lemma:

$$\frac{X_n}{b_n} = \frac{\sum_{m=1}^{n} \Delta_m}{b_n} \to^{a.s.} 0$$

□

Problem 4. Let (X_n) be a martingale with $\sup_n \mathbb{E}\left|Y_n\right| < \infty$. Show that there is a representation $X_n = Y_n - Z_n$ where (Y_n) and (Z_n) are non-negative martingale such that $\sup_n \mathbb{E}Y_n < \infty$ and $\sup_n \mathbb{E}Z_n < \infty$.

Proof. Assuming uniform integrability. X_n is a martingale and $\sup_n \mathbb{E}\left|X_n\right| < \infty$, by the Martingale Convergence Theorem, we have: $X_n \to^{a.s.} X_\infty$. Define:

$$Y_n = \mathbb{E}\left[X_\infty^+ \mid \mathcal{F}_n\right]$$
$$Z_n = \mathbb{E}\left[X_\infty^- \mid \mathcal{F}_n\right]$$

then we have Y_n, Z_n are non-negative martingale (by uniform integrability), and $X_n = Y_n - Z_n$. □

Problem 5. Let (X_n) be adapted to $(\mathcal{F}_n)$ with $0 \le X_n \le 1$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Suppose $X_0 = x_0$ and:

$$\mathbb{P}\left[X_{n+1} = \alpha + \beta X_n \mid \mathcal{F}_n\right] = X_n$$
$$\mathbb{P}\left[X_{n+1} = \beta X_n \mid \mathcal{F}_n\right] = X_n$$

Show that $X_n \to X_\infty$ a.s., where $\mathbb{P}\left[X_\infty = 1\right] = x_0$ and $\mathbb{P}\left[X_\infty = 0\right] = 1 - x_0$.

Proof. We already have X_n is $\mathcal{F}_n-$adapted, and finite expectation. We now check:

$$\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = X_n(\alpha + \beta X_n) + (1 - X_n)\beta X_n = X_n$$

Thus X_n is a martingale. Also $X_n \in [0, 1]$, thus $\sup_n \mathbb{E}X_n^+ < \infty$. Thus by the Martingale Convergence Theorem, we have $X_n \to X_\infty$ a.s.

Given $X_n = x, X_{n+1} = \alpha + \beta x$ or βx for $\alpha, \beta > 0$. Consider $x = \alpha + \beta x$ or $x = \beta x$, which are true iff $x = 1$ or $x = 0$. So $X_\infty \in \{0, 1\}$.

And since $\mathbb{E}X_\infty = \mathbb{E}X_0 = x_0 \Rightarrow \mathbb{P}\left[X_\infty = 1\right] = x_0$, and $\mathbb{E}\left[X_\infty = 0\right] = 1 - x_0$ □

Problem 6. Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y_\infty$ in L^1. Show that $\mathbb{E}\left[Y_n \mid \mathcal{F}_n\right] \to \mathbb{E}\left[Y_\infty \mid \mathcal{F}_\infty\right]$ in L^1.

Proof. By triangle inequality we have:

$$\mathbb{E}\left|\mathbb{E}\left[Y_n \mid \mathcal{F}_n\right] - \mathbb{E}\left[Y_\infty \mid \mathcal{F}_\infty\right]\right| \tag{1}$$
$$\le \mathbb{E}\left|\mathbb{E}\left[Y_n \mid \mathcal{F}_n\right] - \mathbb{E}\left[Y_\infty \mid \mathcal{F}_n\right]\right| + \tag{2}$$
$$+ \mathbb{E}\left|\mathbb{E}\left[Y_\infty \mid \mathcal{F}_n\right] - \mathbb{E}\left[Y_\infty \mid \mathcal{F}_\infty\right]\right| \tag{3}$$

By Jensen inequality (or just the fact that taking absolute value makes real number bigger), we have:

$$(2) \leq \mathbb{E}\left[\mathbb{E}\left|Y_n - Y_\infty\right| \mid \mathcal{F}_n\right] = \mathbb{E}\left[Y_n - Y_\infty\right] \to 0$$

By theorem 5.5.7, (3) $\to 0$.

Thus $\mathbb{E}\left|\mathbb{E}\left[Y_n \mid \mathcal{F}_n\right] - \mathbb{E}\left[Y_\infty \mid \mathcal{F}_\infty\right]\right| \to 0$. □

Problem 7. Let S_n be the total assets of an insurance company at the end of year n. Suppose that in year n the company receives premium of c and pays claims totaling ξ_n, where ξ_n are independent with $\mathcal{N}\left(\mu, \sigma^2\right)$ distribution, where $0 < \mu < c$. The company is ruined if its assets fall to 0 or below. Show:

$$\mathbb{P}\left[\text{ruin}\right] \leq \exp\left\{-2\left(c-\mu\right)S_0/\sigma^2\right\}$$

Proof. (a) We have $S_n = S_0 + \sum_{i=1}^n \left(c - \xi_i\right)$. Let:

$$\begin{aligned}
Y_n =& \exp\left\{\frac{2\left(\mu - c\right)}{\sigma^2}\left(S_n - S_0\right)\right\} \\
\Rightarrow \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] =& Y_n \mathbb{E}\left[\exp\left\{\frac{2\left(\mu - c\right)}{\sigma^2}\left(c - \xi_{n+1}\right)\right\} \mid \mathcal{F}_n\right] \\
=& Y_n \mathbb{E}\left[\exp\left\{\frac{2\left(\mu - c\right)}{\sigma^2}\left(c - \xi_{n+1}\right)\right\}\right]
\end{aligned}$$

For $X_n \sim \mathcal{N}(a, b^2)$, we have:

$$\mathbb{E}\left[\exp\left\{\lambda X_n\right\}\right] = \exp\left\{\lambda a + \lambda^2 b^2/2\right\}$$

Thus for $X_n = c - \xi_{n+1} \sim \mathcal{N}\left(c - \mu, \sigma^2\right)$, we have:

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\frac{2\left(\mu - c\right)}{\sigma^2} X_n\right\}\right] =& \exp\left\{\frac{2\left(\mu - c\right)}{\sigma^2}\left(c - \mu\right) + \frac{4\left(\mu - c\right)^2}{\sigma^4}\frac{\sigma^2}{2}\right\} = 1 \\
\Rightarrow \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] =& Y_n
\end{aligned}$$

It is obvious that Y_n is adapted to $\mathcal{F}_n$, and have finite expectation (since expectation of Gaussian is finite), thus Y_n is a martingale.

(b) Let $T = \inf\left\{n, S_n \leq 0\right\}$. Applying the Optional Sampling Theorem, we have $Y_{n\wedge T}$ is a martingale. Now we have $\mathbb{E}Y_{n\wedge T} = \mathbb{E}Y_0 < \infty$, and since $Y_{n\wedge T} > 0$, thus $\sup_n \mathbb{E}Y_n^+ < \infty$. Thus $Y_{n\wedge T}$ converges to a limit Y_∞ by Martingale Convergence Theorem, and $\mathbb{E}Y_\infty = \mathbb{E}Y_0 = 1$.

Now we have $Y_{n\wedge T} = Y_T \mathbb{I}_{\{T<n\}} + Y_n \mathbb{I}_{\{T\geq n\}}$. Thus:

$$\begin{aligned}
\mathbb{E}Y_{n\wedge T} =& \mathbb{E}\left[Y_T \mathbb{I}_{\{T<n\}} + Y_n \mathbb{I}_{\{T\geq n\}}\right] && (4) \\
\geq& \mathbb{E}\left[\exp\left\{-\frac{2\left(\mu - c\right)}{\sigma^2} S_0\right\} \mathbb{I}_{\{T<n\}}\right] + \mathbb{E}\left[Y_n \mathbb{I}_{\{T\geq n\}}\right] && (5) \\
\geq& \exp\left\{-\frac{2\left(\mu - c\right)}{\sigma^2} S_0\right\} \mathbb{P}\left[T < n\right] && (6)
\end{aligned}$$

For (5) is true because $S_T \leq 0 \Leftrightarrow \left(\mu - c\right) S_T \geq 0 \Leftrightarrow \left(\mu - c\right)\left(S_T - S_0\right) \geq \left(\mu - c\right)\left(0 - S_0\right)$.

In (6) taking $n \to \infty$, we have:

$$\mathbb{P}\left[\text{ruin}\right] \leq \exp\left\{-\frac{2\left(c - \mu\right)}{\sigma^2} S_0\right\}$$

□