Durret Probability
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0.1 Property of Integral

Theorem 0.1. ***Jensen's inequality***. *Suppose φ is convex, that is,*

$$\lambda\varphi(x)+(1-\lambda)\varphi(y) \geq \varphi(\lambda x+(1-\lambda)y), \forall\lambda \in (0,1), x, y \in \mathbb{R}.$$

If μ is a probability measure, and f and $\varphi(f)$ are integrable, then:

$$\varphi\left(\int f d\mu\right) \leq \int \varphi(f) d\mu$$

Theorem 0.2. ***Holder's inequality***. *If $p, q \in (1, \infty)$ with $1/p+1/q=1$. Then:*

$$\int |fg|\, d\mu \leq \|f\|_p \|g\|_q$$

The special case $p=q=2$ is called ***Cauchy-Schwarz*** *inequality*

Theorem 0.3. ***Bounded Convergenge Theorem***. *Let E be a set, $\mu(E) < \infty$. Suppose $f_n$vanishes on $E^c, |f_n(x)| \leq M$, and $f_n \to f$ in measure. Then:*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Theorem 0.4. ***Fatou's Lemma***. *If $f_n \geq 0$, then:*

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \left(\liminf_{n\to\infty} f_n\right) d\mu$$

Theorem 0.5. ***Monotone Convergence Theorem***. *If $f_n \geq 0$, and $f_n \uparrow f$, then*

$$\int f_n d\mu \uparrow \int f d\mu$$

Theorem 0.6. ***Dominated Convergence Theorem***. *If $f_n \to f$ a.e., $|f_n| \leq g, \forall n$, and g is integrable, then:*

$$\int f_n d\mu \to \int f d\mu$$

Let $(X, \mathcal{A}, \mu_1), (Y, \mathcal{B}, \mu_2)$ be two $\sigma-finite$ measure spaces. Let

$$\begin{aligned}\Omega =& X \times y\\ \mathcal{S} =& \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}\\ \mathcal{F} =& \mathcal{A} \times \mathcal{B} = \sigma(\mathcal{S}).\end{aligned}$$

Theorem 0.7. *There is a uniqye measure μ on $\mathcal{F}$ with:*

$$\mu(A \times B) = \mu_1(A)\mu_2(B)$$

Theorem 0.8. *Fubini's Theorem. If $f \geq 0$ or $\int |f|\, d\mu < \infty$ then:*

$$\int_X \int_Y f(x,y)\mu_2(dy)\mu_1(dx) = \int_{X\times Y} f d\mu = \int_Y \int_X f(x,y)\mu_1 d(x)\mu_2(dx)$$

1 Laws of Large Number

1.1 Independence

Definition 1.1. $(\Omega, \mathcal{F}, \mathbb{P})$; $A, B \in \mathcal{F}$ are called independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)]\mathbb{P}(B)$

Two $\sigma - fields$ $\mathcal{G}, \mathcal{H}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)]\mathbb{P}(B), \forall A \in \mathcal{G}, B \in \mathcal{H}$.

Two random variable X, Y are independent iff $\sigma(X), \sigma(Y)$ are independent.

Definition 1.2. $\mathcal{A}$ is a $\pi - system$ if it is closed under intersection. $\mathcal{L}$ is a $\lambda - system$ if (i) $\Omega \in \mathcal{L}$, (ii) $\forall A, B \in \mathcal{L}, A \subset B$ then $B - A \in \mathcal{L}$, and (iii) If $A_n \in \mathcal{L}, A_n \uparrow A$ then $A \in \mathcal{L}$.

Theorem 1.1. *$\pi - \lambda$ Theorem. If $\mathcal{P}$ is a $\pi - system$ and $\mathcal{L}$ is a $\lambda - system$ that contains $\mathcal{P}$ then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*

Theorem 1.2. *Suppose $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ are independent and each A_i is a $\pi-system$, then $\sigma(A_1), \sigma(A_2), ..., \sigma(A_n)$ are independent.*

Theorem 1.3. *In order for $X_1, X_2, ..., X_n$ to be independent, it is sufficient that for all $x_1, ..., x_n \in \mathbb{R}$,*

$$\mathbb{P}\left[X_1 \leq 1, ..., X_n \leq x_n\right] = \prod_{i=1}^{n} \mathbb{P}\left[X_i \leq x_i\right]$$

Theorem 1.4. *Suppose $X_1, ..., X_n$ are independent random variables and X_i has distribution μ_i, then $(X_1, ..., X_n)$ has distribution $\mu_1 \times \mu_2 ... \times \mu_n$.*

Theorem 1.5. *If X and Y are independent, then:*

$$\mathbb{P}\left[X + Y \leq z\right] = \int F(z - y) dG(y)$$

1.2 Weak Laws of Large Number

Definition 1.3. We say Y_n converges to Y in probability if $\forall \epsilon > 0$, $\lim_{n \to \infty} \mathbb{P}\left[|Y_n - Y| < \epsilon\right] = 0$.

Lemma 1.1. *If $p > 0$ and $\mathbb{E}\left|Z_n\right|^p \to 0$ then $Z_n \to 0$ in probability.*

Theorem 1.6. *L^2 weak law. Let $X_1, X_2, ...$ be uncorrelated random variables with $\mathbb{E}X_i = \mu$ and $\mathrm{Var}(X_i) \leq C < \infty$. If $S_n = X_1 + ... + X_n$ then $S_n/n \to \mu$ in L^2 and in probability.*

Theorem 1.7. *L^1 weak law. Let $X_1, X_2, ...$ be i.i.d with $\mathbb{E}\left|X_i\right| < \infty$. Then $S_n/n \to \mathbb{E}X_1$ in probability*

1.3 Borel-Cantelli Lemmas

Definition 1.4. $A_n \subset \Omega$.

$$\limsup A_n = \lim_{m \to \infty} \bigcup_{n=m}^{\infty} A_n = \{\omega \text{ that are in infinitely many } A_n\}$$
$$\liminf A_n = \lim_{m \to \infty} \bigcap_{n=m}^{\infty} A_n = \{\omega \text{ that are in all but finitely many } A_n\}$$

Theorem 1.8. *The First Borel-cantelli Lemma. If $\sum_{n=1}^{\infty} \mathbb{P}A_n < \infty$ then:*

$$\mathbb{P}\left[A_n \ i.o.\right] = 0$$

Theorem 1.9. *Relation between Convergence in Probability and Almose Surely.*

$X_n \to X$ in probability iff for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges almost surely to X.

Theorem 1.10. *If f is continuous and $X_n \to X$ in probability then $f(X_n) \to f(X)$ in probability. If, in addition, f is bounded then $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$.*

Theorem 1.11. *L^4 Strong Law of Large Number 1. Let $X_1, X_2, ...$ be i.i.d with $\mathbb{E}X_i = \mu$ and $\mathbb{E}X_i^4 < \infty$. Then $S_n/n \to \mu$ a.s.*

Theorem 1.12. *The Second Borel-Cantelli Lemma. If A_n are independent then $\sum \mathbb{P}A_n = \infty$ implies $\mathbb{P}[A_n \text{ i.o.}] = 1$.*

Theorem 1.13. *"Anti" LLN. If X_i are i.i.d with $\mathbb{E}|X_i| = \infty$, then $\mathbb{P}[|X_n| \geq n \text{ i.o.}] = 1$. So $\mathbb{P}[\lim S_n/n = a \in (-\infty, \infty)] = 0$*

Theorem 1.14. *If $A_1, A_2, ...$ are pairwise independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ then as $n \to \infty$*

$$\sum_{m=1}^{n} \mathbb{I}[A_m] \Big/ \sum_{m=1}^{n} \mathbb{P}[A_m] \to 1 \text{ a.s.}$$

1.4 Strong Law of Large Numbers

Theorem 1.15. *SLLN. Let $X_1, X_2, ...$ be pairwise independent identically distributed random variables with $\mathbb{E}|X_i| = \mu < \infty$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.*