Solution for HW 1

1. (a). It is obvious that $\emptyset \in \mathcal{F}$. Take $A \in \mathcal{F}$, then $\exists n \in \mathbb{N}$ s.t. $A \in \mathcal{F}_n$. This implies $A^c \in \mathcal{F}_n \subset \mathcal{F}$. Now take $B \in \mathcal{F}$, then $\exists m \in \mathbb{N}$ s.t. $B \in \mathcal{F}_m$. We have $A \cup B \in \mathcal{F}_{m\vee n}$ since $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is increasing. Therefore, $\mathcal{F}$ is a field. (b). Consider $\Omega = \mathbb{N}$ and $\mathcal{F}_n = \sigma(\{k\}; k \le n)$ (i.e. all the subsets of $\{1, \cdots, n\}$ and their complements) : $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is increasing. Observe that $\{2n\} \in \mathcal{F}_{2n} \subset \mathcal{F}$ but $2\mathbb{N} \notin \mathcal{F}$.

Remark : If you feel this question boring, think of a more challenging one : If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is strictly increasing σ-fields, then $\cup\mathcal{F}_n$ is necessarily not a $\sigma-$field.

2. Denote $\mathcal{G}$ the class of the sets mentioned in the problem. We would like to show that $\mathcal{F}(\mathcal{A}) = \mathcal{G}$. It's straightforward that any field containing $\mathcal{A}$ contains $\mathcal{G}$ (closed under complement, finite intersection and union). Thus $\mathcal{G} \subset \mathcal{F}(\mathcal{A})$. It is enough to prove that $\mathcal{G}$ itself is a field. Observe that $(\cup_{i=1}^m \cap_{j=1}^{n_i} A_{ij}) \cap (\cup_{k=1}^p \cap_{l=1}^{q_k} A_{kl}) = \cup_{i\le m, k\le p}(\cap_{j=1}^{n_i} A_{ij} \bigcap \cap_{l=1}^{q_k} A_{kl})$: closed under finite intersection. In addition, $(\cup_{i=1}^m \cap_{j=1}^{n_i} A_{ij})^c = \cap_{i=1}^m \cup_{j=1}^{n_i} A_{ij}^c$ and $\cup_{j=1}^{n_i} A_{ij}^c = \cup_{j=1}^{n_i}(A_{ij}^c \bigcap \cap_{k=1}^{j-1} A_{ik})$. We have thus proved the desired result.

3. Denote $\mathcal{F} = \cup\sigma(\mathcal{C})$ where $\mathcal{C}$ runs over the set of all countable subsets of $\mathcal{A}$. The question consists in proving that $\mathcal{F} = \sigma(\mathcal{A})$. It is obvious that $\mathcal{F} \subset \sigma(\mathcal{A})$. Thus it suffices to show that $\mathcal{F}$ is a $\sigma-$field. Take $A \in \mathcal{F}$: $\exists$ countable subset $\mathcal{C}$ s.t. $A \in \sigma(\mathcal{C})$. Then $A^c \in \sigma(\mathcal{C}) \subset \mathcal{F}$. Now take $(A_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$: $\exists \mathcal{C}_n$ s.t. $A_n \in \sigma(\mathcal{C}_n)$. We have then $\cup_n A_n \in \sigma(\cup\mathcal{C}_n) \subset \mathcal{F}$. Hence $\mathcal{F}$ is a $\sigma-$field.

4. Note that $\mathcal{B}(\mathbb{R}^d) = \sigma(F; F$ closed in $\mathbb{R}^d)$. For $f : \mathbb{R}^d \to \mathbb{R}$ continuous, $f^{-1}([a,\infty])$ is closed in $\mathbb{R}^d$. It follows that $\mathcal{B}(\mathbb{R}^d)$ makes all continuous functions measurable. Now denote $\mathcal{F}$ $\sigma-$field on $\mathbb{R}^d$ which makes all continuous functions measurable. For F closed set in $\mathbb{R}^d$, take $d_F(x) := \inf\{|x-y|; y \in F\}$: d_F is $1-$Lipschitz and thus continuous. Observe that $d_F^{-1}(\{0\}) = F \subset \mathcal{F}$. Thus, $\mathcal{F} \supset \sigma(F; F$ closed in $\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ is the smallest $\sigma-$field that makes all continuous function measurable.

5. We will prove the result for l.s.c. functions and the proof for u.s.c. functions is similar. It is enough to prove that $\{f(x) > t\}$ is open for $\forall t \in \mathbb{R}$. Take $x_0 \in \{f(x) > t\}$. By the definition of *l.s.c.*, $\forall\epsilon > 0, \exists\delta > 0$ s.t. $f(x_0) - \epsilon < f(y)$ provided $|x_0 - y| < \delta$. In particular, take $\epsilon = f(x_0) - t$, we have $\exists\delta > 0$ s.t. $f(y) > f(x_0) - \epsilon = t$ for all $|x - y| < \delta$, which permits to conclude.