ST205 - Homework 10

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Problem 1. Let $S_n = \sum_{i=1}^{n} \xi_i$, where ξ_i are independent, $\mathbb{E}\xi_i = 0$ and $\text{Var}\xi_i < \infty$. Let $s_n^2 = \sum_{i=1}^{n} \text{Var}\xi_i$. We know that $S_n - s_n^2$ is a martingale. Suppose also that $|\xi_i| \le K$ for some constant K. Show that:

$$\mathbb{P}\left[\max_{m\le n} |S_m| < x\right] \le s_n^{-2}\left(K+x\right)^2, x > 0$$

Proof. Let $T = \min\{m \mid |S_m| \ge x\}$ then T is a stopping time, thus $T \wedge n$ is also a stopping time with $\mathbb{P}[T \wedge n \le n] = 1$. Thus we can apply the theorem 5.4.1 in Durrett for the martingale $S_n^2 - s_n^2$ and have:

$$\begin{aligned}
S_1^2 - s_1^2 \le& \mathbb{E}\left[S_{T\wedge n}^2 - s_{T\wedge n}^2\right] \\
\Leftrightarrow 0 \le& \mathbb{E}\left[\left(S_T^2 - s_T^2\right)\mathbb{I}_{\{T\le n\}}\right] + \mathbb{E}\left[\left(S_n^2 - s_n^2\right)\mathbb{I}_{\{T>n\}}\right] \\
\le& \mathbb{E}\left[\left((x+K)^2 - s_T^2\right)\mathbb{I}_{\{T\le n\}}\right] + \mathbb{E}\left[\left(x^2 - s_n^2\right)\mathbb{I}_{\{T>n\}}\right] \\
\le& \mathbb{E}\left[\left(x+K\right)^2\mathbb{I}_{\{T\le n\}}\right] + \mathbb{E}\left[\left(x^2 - s_n^2\right)\mathbb{I}_{\{T>n\}}\right] \\
=& \left(x+K\right)^2\left(1 - \mathbb{P}[T>n]\right) + \left(x^2 - s_n^2\right)\mathbb{P}[T>n] \\
=& (x+K)^2 - \left[\left(x+K\right)^2 - \left(x^2 - s_n^2\right)\right]\mathbb{P}[T>n] \\
\Rightarrow \mathbb{P}[T>n] \le& \frac{\left(x+K\right)^2}{K^2 + 2xK + s_n^2} \\
\le& \frac{(K+x)^2}{s_n^2}
\end{aligned}$$

And since $\mathbb{P}[T>n] = \mathbb{P}[\max_{m\le n}|S_m < x|]$, we have the proof. □

Problem 2. Let (X_n) be a martingale with $X_0 = 0$ and $\mathbb{E}X_n^2 < \infty$.Using the fact that $(X_n + c)^2$ is a submartingale, show that:

$$\mathbb{P}\left[\max_{m\le n} X_m \ge x\right] \le \frac{\mathbb{E}X_n^2}{x^2 + \mathbb{E}X_n^2}, x > 0$$

Proof. We will use Doob's inequality (or Theorem 5.4.1), we have $B := \{\max_{m\le n} X_m \ge x\} \subset \left\{\max_{m\le n}\left(X_m + c\right)^2 \ge (x+c)^2\right.$ A for $c \ge 0$, since $X_m \ge x \Leftrightarrow X_m + c \ge x + c \Rightarrow (X_m + c)^2 \ge (x+c)^2$. Thus applying the Doob's inequality we have:

$$
\begin{aligned}
\mathbb{P}[B] \leq & \mathbb{P}[A] && (1)\\
\leq & \frac{\mathbb{E}\left[(X_n+c)^2 \mathbb{I}_A\right]}{(x+c)^2} && (2)\\
\leq & \frac{\mathbb{E}\left[(X_n+c)^2\right]}{(x+c)^2} && (3)\\
= & \frac{\mathbb{E}X_n^2 + 2c\mathbb{E}X_n + c^2}{(x+c)^2} && (4)\\
= & \frac{\mathbb{E}X_n^2 + c^2}{(x+c)^2} && (5)
\end{aligned}
$$

for the last statement is true because X_n is a martingale, thus $\mathbb{E}X_n = \mathbb{E}X_0 = 0$. Now let $\mathbb{E}X_n^2 = a$, we want:

$$
\begin{aligned}
\frac{a+c^2}{(x+c)^2} \leq & \frac{a}{x^2+a}\\
\Leftrightarrow ax^2 + a^2 + c^2x^2 + ac^2 \leq & ax^2 + 2axc + ac^2\\
\Leftrightarrow a^2 + c^2x^2 \leq & 2axc\\
\Leftrightarrow (a-cx)^2 \leq & 0\\
\Leftrightarrow a = & cx\\
\Leftrightarrow c = & \frac{a}{x} = \frac{\mathbb{E}X_n^2}{x}
\end{aligned}
$$

Thus if we apply (5) for $c = \frac{\mathbb{E}X_n^2}{x}$ we have the proof. □

Problem 3. X_n, Y_n martingale with $\mathbb{E}\left(X_n^2 + Y_n^2\right) < \infty$. Show that:

$$
\mathbb{E}[X_n Y_n] - \mathbb{E}[X_0 Y_0] = \sum_{m=1}^{n} \mathbb{E}\left[(X_m - X_{m-1})(Y_m - Y_{m-1})\right] \tag{6}
$$

Proof. From $\mathbb{E}\left(X_n^2 + Y_n^2\right) < \infty$ we also have X_n, Y_n are bounded. So we can use the property of conditional expectation $\mathbb{E}[X_{n+1}Y_n \mid \mathcal{F}_n] = Y_n \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = Y_n X_n$.

We will prove (6) by induction.

(i) For $n = 1$. We need to prove:

$$
\begin{aligned}
\mathbb{E}[X_1Y_1 - X_0Y_0] = & \mathbb{E}\left[(X_1 - X_0)(Y_1 - Y_0)\right]\\
\Leftrightarrow \mathbb{E}[X_1Y_0] + \mathbb{E}[X_0Y_1] = & 2\mathbb{E}[X_0Y_0]
\end{aligned}
$$

Using tower property we have:

$$
\begin{aligned}
\mathbb{E}[X_1Y_0] = & \mathbb{E}\left[\mathbb{E}[X_1Y_0 \mid \mathcal{F}_0]\right]\\
= & \mathbb{E}\left[Y_0 \mathbb{E}[X_1 \mid \mathcal{F}_0]\right]\\
= & \mathbb{E}[Y_0X_0]
\end{aligned}
$$

Similarly:

$$
\mathbb{E}[X_0Y_1] = \mathbb{E}[X_0Y_0]
$$

Thus we have statement (6) is true for $n = 1$

(ii) Assuming statement (6) is true for case $n = k$ for $k \geq 1$, which means we have:

$$\mathbb{E}\left[X_k Y_k\right] - \mathbb{E}\left[X_0 Y_0\right] = \sum_{m=1}^{k} \mathbb{E}\left[\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right)\right] \tag{7}$$

(iii) We need to show that statement (6) is also true for $n = k+1$. Indeed using the same argument as (i) we have:

$$\mathbb{E}\left[X_{k+1}Y_{k+1} - X_k Y_k\right] = \mathbb{E}\left[\left(X_{k+1} - X_k\right)\left(Y_{k+1} - Y_k\right)\right]$$

Combining this face with (7) we have:

$$\begin{aligned}
\mathbb{E}\left[X_{k+1}Y_{k+1} - X_0 Y_0\right] =& \mathbb{E}\left[X_{k+1}Y_{k+1} - X_k Y_k\right] + \mathbb{E}\left[X_k Y_k - X_0 Y_0\right] \\
=& \mathbb{E}\left[\left(X_{k+1} - X_k\right)\left(Y_{k+1} - Y_k\right)\right] + \sum_{m=1}^{k} \mathbb{E}\left[\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right)\right] \\
=& \sum_{m=1}^{k+1} \mathbb{E}\left[\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right)\right]
\end{aligned}$$

By the induction principle, we have (6) is true for all $n \in \mathbb{N}$. □

Problem 4. Let $\left(X_n, \mathcal{F}_n\right), n \geq 0$ be a positive submartingale with $X_0 = 0$. Let V_n be random variables such that

(i) $V_n \in \mathcal{F}_{n-1}, n \geq 1$

(ii) $B \geq V_1 \geq V_2 \geq ... \geq 0$, for some constant B

Prove that for $\lambda > 0$

$$\mathbb{P}\left[\max_{i \leq j \leq n} V_j X_j > \lambda\right] \leq \lambda^{-1} \sum_{j=1}^{n} \mathbb{E}\left[V_j\left(X_j - X_{j-1}\right)\right]$$

Proof. We will prove that $Y_n \coloneqq \sum_{j=1}^{n} V_j\left(X_j - X_{j-1}\right)$ is a submartingale.

(i) First $\mathbb{E}\left[V_j\left(X_j - X_{j-1}\right)\right] \leq B\mathbb{E}\left[X_j - X_j - 1\right]$. Thus $\mathbb{E}Y_n \leq B\mathbb{E}X_n < \infty$

(ii) Y_n is adapted to $\mathcal{F}_n$

(iii) We have

$$\mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}\left[V_{n+1}\left(X_{n+1} - X_n\right) \mid \mathcal{F}_n\right] + \mathbb{E}Y_n$$

Applying theorem 5.2.5 for X_{n+1} and V_{n+1} we have $X_{n+1}V_{n+1}$ is a submartingale (since V_{n+1} is adapted to $\mathcal{F}_n$), thus we have:

$$\mathbb{E}\left[V_{n+1}X_{n+1} \mid \mathcal{F}_n\right] \geq \mathbb{E}\left[V_n X_n \mid \mathcal{F}_n\right] = V_n X_n \geq V_{n+1} X_n$$

Thus

$$\begin{aligned}
\mathbb{E}\left[V_{n+1}\left(X_{n+1} - X_n\right) \mid \mathcal{F}_n\right] \geq& 0 \\
\Rightarrow \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] \geq& \mathbb{E}Y_n
\end{aligned}$$

So $Y_n$is a submartingale.

We also have:

$$\sum_{j=1}^{n} V_j\left(X_j - X_{j-1}\right) = V_n X_n + \sum_{j=1}^{n-1} X_j\left(V_j - V_{j+1}\right) \geq V_n X_n$$

Combining this fact with Doob's inequality we have:

$$\begin{aligned}\mathbb{P}\left[\max V_j X_j \geq \lambda\right] \leq& \mathbb{P}\left[\max_m \sum_{j=1}^{m} V_j\left(X_j - X_{j-1}\right) > \lambda\right]\\ \leq& \lambda^{-1}\mathbb{E}\left[\sum_{j=1}^{m} V_j\left(V_j - V_{j-1}\right)\right]\end{aligned}$$

□

Lemma 1. *The Switching Principle. Suppose X_n^1 and X_n^2 are supermartingale with respect to $\mathcal{F}_n$, and N is a stopping time so that $X_N^1 \geq X_N^2$. Then:*

$$\begin{aligned}Y_n =& X_n^1\mathbb{I}_{\{N>n\}} + X_n^2\mathbb{I}_{\{N\leq n\}} \text{ is a supermartingale}\\ Z_n =& X_n^1\mathbb{I}_{\{N\geq n\}} + X_n^2\mathbb{I}_{\{N<n\}} \text{ is a supermartingale}\end{aligned}$$

Proof. The first two condition for a supermartingale of finite expectation and adaptiveness are met. We check the final condition for Y_n, we have:

$$\begin{aligned}&\mathbb{E}\left[Y_{n+1} - Y_n \mid \mathcal{F}_n\right]\\ =&\mathbb{E}\left[X_{n+1}^1\mathbb{I}_{\{N>n+1\}} - X_n^1\mathbb{I}_{\{N>n\}} + X_{n+1}^2\mathbb{I}_{\{N\leq n+1\}} - X_n^2\mathbb{I}_{\{N\leq n\}} \mid \mathcal{F}_n\right]\\ =&\mathbb{E}\left[\left(X_{n+1}^1 - X_n^1\right)\mathbb{I}_{\{N>n\}} - X_{n+1}^1\mathbb{I}_{\{N=n+1\}} + \left(X_{n+1}^2 - X_n^2\right)\mathbb{I}_{\{N\leq n\}} + X_{n+1}^2\mathbb{I}_{\{N=n+1\}} \mid \mathcal{F}_n\right]\\ =&\mathbb{I}_{\{N>n\}}\mathbb{E}\left[\left(X_{n+1}^1 - X_n^1\right) \mid \mathcal{F}_n\right] + \mathbb{I}_{\{N\leq n\}}\mathbb{E}\left[\left(X_{n+1}^2 - X_n^2\right)\right] + \mathbb{E}\left[\left(X_N^2 - X_N^1\right)\mathbb{I}_{\{N=n+1\}} \mid \mathcal{F}_n\right] \qquad (8)\end{aligned}$$

Now we have X_n^1 and X_n^2 are supermartingale, and $X_N^2 \geq X_N^1$. Thus we have $\mathbb{E}\left[Y_{n+1} - Y_n \mid \mathcal{F}_n\right] \leq 0$. So Y_n is a supermartingale.

Similarly we have Z_n is a supermartingale. □

Problem 5. Dubins' inequality. If (X_n) is a positive martingale then the number U of upcrossings of $[a,b]$ sastifies:

$$\mathbb{P}\left[U \geq k\right] \leq \left(\frac{a}{b}\right)^k \mathbb{E}\min\left\{X_0/a, 1\right\}$$

Proof. Let $N_0 = -1$ and for $j \geq 1$ let:

$$\begin{aligned}N_{2j-1} =& \inf\left\{m > N_{2j-2} \mid X_m \leq a\right\}\\ N_{2j} =& \inf\left\{m > N_{2j-1} \mid X_m \geq b\right\}\end{aligned}$$

Let $Y_n = 1$ for $0 \leq n < N_1$ and for $j \geq 1$

$$Y_n = \begin{cases}(b/a)^{j-1}\left(X_n/a\right) & \text{for } N_{2j-1} \leq n < N_{2j}\\ (b/a)^j & \text{for } N_{2j} \leq n < N_{2j+1}\end{cases}$$

(i) From the switching principle we proved above, and using the induction argument we have Y_n is a supermartingale. Thus by theorem 5.2.6 in Durrett, we have $Y_{N_{2k}\wedge n}$ is a supermartingale.

(ii) Now since $Y_{n\wedge N_{2k}}$ is a supermartigale, we have $\mathbb{E}Y_{n\wedge N_{2k}} \leq \mathbb{E}Y_0$. Now we have:

$$
\begin{aligned}
X_0 \leq a \Leftrightarrow N_1 = 0 \Leftrightarrow Y_0 = X_0/a \\
X_0 > a \Leftrightarrow N_1 > 0 \Leftrightarrow Y_0 = 1
\end{aligned}
$$

Thus:

$$
\mathbb{E}Y_0 = \mathbb{E}\left[\frac{X_0}{a}\mathbb{I}_{\{X_0/a\leq 1\}} + \mathbb{I}_{\{X_0/a>1\}}\right] = \mathbb{E}\min\{X_0/a, 1\}
$$

Let $n \to \infty$ we have:

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}Y_{n\wedge N_{2k}} &\leq \mathbb{E}Y_0 \\
\Rightarrow \lim_{n\to\infty} \mathbb{E}\left[Y_n\mathbb{I}_{\{N_{2k}>n\}} + Y_{N_{2k}}\mathbb{I}_{\{N_{2k}\leq n\}}\right] &\leq \mathbb{E}\min\{X_0/a, 1\} \\
\Rightarrow \lim_{n\to\infty}\left(\frac{b}{a}\right)^k \mathbb{E}\mathbb{I}_{\{N_{2k}\leq n\}} &\leq \mathbb{E}\min\{X_0/a, 1\} \\
\Rightarrow \mathbb{P}[U \geq k] \leq \left(\frac{a}{b}\right)^k \mathbb{E}\min\{X_0/a, 1\}
\end{aligned}
$$

$\square$