ST205A - Homework 7

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Problem 1. Basic Stopping Time

Proof. (a) $\{\min(S,T)=n\} = A_1 \cup A_2 \cup A_3$, for:

$$\begin{aligned}
A_1 &= \{S=n\} \cap \{T=n\} \in \mathcal{F}_n \\
A_2 &= \{S=n\} \cap \{T>n\} = \{S=n\} \cap \{T \le n\}^C \in \mathcal{F}_n \\
A_3 &= \{T=n\} \cap \{S>n\} = \{T=n\} \cap \{S \le n\}^C \in \mathcal{F}_n
\end{aligned}$$

Thus $\{\min(S,T)=n\} \in \mathcal{F}_n$, so it is a stopping time.
(b) $\{\max(S,T)=n\} = B_1 \cup B_2 \cup B_3$, for:

$$\begin{aligned}
B_1 =& A_1 \in \mathcal{F}_n \\
B_2 =& \{S=n\} \cap \{T<n\} \in \mathcal{F}_n \\
B_3 =& \{T=n\} \cap \{S<n\} \in \mathcal{F}_n
\end{aligned}$$

Thus $\{\max(S,T)=n\} \in \mathcal{F}_n$, so it is a stopping time.
(c) $\{S+T=n\} = \bigcup_{i=1}^{n-1} A_i$, for:

$$A_i = \{S_i = i\} \cap \{T_i = n-i\} \in \mathcal{F}_n$$

since $i, n-i < n$.
Thus $\{S+T=n\} \in \mathcal{F}_n$, so it is a stopping time. □

Problem 2. Wald's Second Equation

Proof. (a) Counter example. Let $X_i = Bernoulli(0.5)$. $T=1$ if $X_1=0$, $T=2$ if $X_1=1$. Then:

$$\begin{aligned}
\mathbb{E}S_T =& \mathbb{E}X_1 \mathbb{E}T = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4} \\
\mathbb{E}S_T^2 =& \mathbb{E}\left[S_T^2 \mid X_1=0, T=1\right] \mathbb{P}\left[X_1=0,T=1\right] + \mathbb{E}\left[S_T^2 \mid X_1=1,T=2\right] \mathbb{P}\left[X_1=1,T=2\right] \\
=& 0 + \frac{1}{2}\mathbb{E}\left[(X_1+X_2)^2 \middle| X_1 = 1\right] = \frac{5}{4} \\
\Rightarrow \mathrm{Var}S_T =& \frac{5}{4} - \frac{9}{16} = \frac{11}{16} \\
\mathrm{Var}X_1 =& \mathbb{E}X_1^2 - (\mathbb{E}X_1)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
\mathbb{E}T =& \frac{3}{2}
\end{aligned}$$

So $\mathrm{Var}S_T = \frac{11}{16}$, and $\mathrm{Var}X_1 \mathbb{E}T = \frac{3}{8}$, and they are not equal.

(b) From Durrett's book
Let $\sigma^2 = \mathbb{E}X_i^2$. Denote $T \wedge n = \min\{T, n\}$. We have:

$$\begin{aligned} S^2_{T\wedge n} =& S^2_{T\wedge(n-1)} + \left(2X_n S_{n-1} + X_n^2\right)\mathbb{I}_{\{T\geq n\}} \\ \Rightarrow \mathbb{E}S^2_{T\wedge n} =& \mathbb{E}S^2_{T\wedge(n-1)} + \sigma^2 \mathbb{P}\left[T \geq n\right] \end{aligned}$$

Since X_n and S_{n-1} are independent, $\mathbb{E}X_n = 0$, and the expectation of $S_{n-1}X_n$ exists because both of them have finite second moment (so one can use Cauchy-Schwartz to bound the product). Using an induction argument we have:

$$\begin{aligned} \mathbb{E}S^2_{T\wedge n} =& \sigma^2 \sum_{m=1}^{n} \mathbb{P}\left[T \geq m\right] \;\; (1) \\ \Rightarrow \mathbb{E}\left[S^2_{T\wedge n} - S^2_{T\wedge m}\right] =& \sigma^2 \sum_{k=m+1}^{n} \mathbb{P}\left[T \geq k\right], \forall n > m \end{aligned}$$

This equality implies that $S^2_{T\wedge n}$ is a Cauchy sequence in L1. Thus the limit of $S^2_{T\wedge n}$ exists. If we let $n \to \infty$ in (1), we have $\mathbb{E}S_T^2 = \sigma^2\mathbb{E}T$. $\square$

Problem 3. Let $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, ...)$. Tail $\sigma - field$ is $\mathcal{T} = \bigcap_{n=1}^{\infty}\mathcal{F}'_n$.

Proof. (i) $\{X_n \to 0\}$. We have $\{X_n \to 0\} \in \mathcal{F}'_m, \forall m \in \mathbb{N}$, since $X_1, ..., X_{m-1}$ does not affect whether $X_n \to 0$ or not. So $\{X_n \to 0\} \in \bigcap_{m=1}^{\infty}\mathcal{F}'_m = \mathcal{T}$

(ii) $\{S_n \text{ converges}\}$. We have $\{S_n \text{ converges}\} \in \mathcal{F}'_m, \forall m \in \mathbb{N}$, since $X_1, ..., X_m$ does not affect whether S_n converges or not. Put it another way, $S_n = \sum_{i=1}^{n} X_i$ converges iff $S'_n = \sum_{i=m}^{n} X_i$ converges.

So $\{S_n \text{ converges}\} \in \bigcap_{m=1}^{\infty}\mathcal{F}'_n = \mathcal{T}$.

(iii) Let $m \in \mathbb{N}$ be fixed and arbitrary, then:

$$\begin{aligned} \{X_n > b_n \text{ i.o.}\} &= \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{\omega \mid X_i(\omega) > b_i\} \\ &= \bigcap_{n=m}^{\infty}\bigcup_{i=n}^{\infty}\{\omega \mid X_i(\omega) > b_i\} \\ &\in \mathcal{F}'_m \\ \Rightarrow \{X_n > b_n \text{ i.o.}\} &\in \bigcap_{m=1}^{\infty}\mathcal{F}'_m = \mathcal{T} \end{aligned}$$

(iv) This statement is not true. Counter example: $X_1 = Bernoulli(0.5)$. $X_2 = X_3 = ... = 1$ constant. $b_n = n-1$. Then $\mathbb{P}\left[S_n > b_n \text{ i.o.}\right] = \frac{1}{2}$. But if we take out X_1 then $\mathbb{P}\left[S_n > b_n \text{ i.o.}\right] = 1$. Thus this event depends on X_1. So it does not belong to the tail $\sigma - field$. Another way to note this is that X_i are independent, thus those event that belongs to tail $\sigma - field$ has probability of either 0 or 1 according to the Komogorov theorem. But our event has probability 1/2, so it does not belong to the tail $\sigma - field$.

(v) Fix $m \in \mathbb{N}$. Consider the two set $A = \left\{\omega \middle| \lim_{n\to\infty}\frac{\sqrt{\sum_{i=1}^{n}X_i^2}}{\sum_{i=1}^{n}X} = 0\right\}$, $B = \left\{\omega \middle| \lim_{n\to\infty}\frac{\sqrt{\sum_{i=1}^{n}X_i^2}}{\sum_{i=m}^{n}X_i} = 0\right\}$. We will prove that the two sets are equal.

$\sqrt{\sum_{i=1}^{n}X_i^2}$ is an non-negative increasing sequence. Thus there are exactly three cases:

(a) First, for those ω such that $\lim\sum_{i=1}^{n}X_i^2 = 0$, then both limits in A and B do not exist since all $X_i = 0$. So $\omega \notin A, \omega \notin B$.

(b) Second for those ω such that $\lim\sum_{i=1}^{n}X_i^2 = a, a \in (0,\infty)$. Thus $\lim_{n\to\infty}\sum_{i=m}^{n}X_i^2 = b, b \in [0,a]$. If $b = 0$. Then the limit in A goes to some number not zero, and the limit in B does not exist. So $\omega \notin A$ and $\omega \notin B$. If $b > 0$. Then $\lim_{n\to\infty}\frac{\sqrt{\sum_{i=1}^{n}X_i^2}}{\sqrt{\sum_{i=m}^{n}X_i^2}} = \frac{\sqrt{a}}{\sqrt{b}} > 1$ and is finite. Thus:

$$\lim_{n\to\infty}\frac{\sqrt{\sum_{i=1}^n X_i^2}}{\sum_{i=1}^n X} = 0 \Leftrightarrow \lim_{n\to\infty}\frac{\sqrt{\sum_{i=m}^n X_i^2}}{\sum_{i=1}^n X}\frac{\sqrt{\sum_{i=1}^n X_i^2}}{\sqrt{\sum_{i=m}^n X_i^2}} = 0 \tag{1}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sqrt{\sum_{i=m}^n X_i^2}}{\sum_{i=1}^n X_i} = 0 \tag{2}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=m}^n X_i^2}} = \pm\infty \tag{3}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sum_{i=m}^n X_i}{\sqrt{\sum_{i=m}^n X_i^2}} = \pm\infty \tag{4}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sqrt{\sum_{i=m}^n X_i^2}}{\sum_{i=m}^n X_i} = 0 \tag{5}$$

For (4) is true because the expression is different from that in (3) by a finite amount.
So $\omega \in A \Leftrightarrow \omega \in B$ in this case.
(c) Third, for those ω such that $\lim \sum_{i=1}^n X_i^2 = \infty$.
Then:

$$\lim_{n\to\infty}\frac{\sqrt{\sum_{i=1}^n X_i^2}}{\sum_{i=1}^n X} = 0 \Leftrightarrow \lim_{n\to\infty}\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} = \pm\infty \tag{6}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sum_{i=m}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} = \pm\infty \tag{7}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sum_{i=m}^n X_i}{\sqrt{\sum_{i=m}^n X_i^2}} = \pm\infty \tag{8}$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{\sqrt{\sum_{i=m}^n X_i^2}}{\sum_{i=m}^n X_i} = 0 \tag{9}$$

For (6) $\Leftrightarrow$ (7) is true because the different between the expression in (7) and (6) is a finite amount. (7) $\Rightarrow$ (8) is true because the expression in (8) is larger in magnitude as the denominator is smaller in magnitude. (8) $\Rightarrow$ 7 is true because the ratio $\lim_{n\to\infty}\frac{\sum_{i=1}^{\infty} X_i^2}{\sum_{i=m}^{\infty} Xi^2} = 1$.

In conclusion, $\omega \in A \Leftrightarrow \omega \in B$. Thus $A = B$. Thus $A \in \mathcal{F}'_m$ for arbitrary $m \in \mathbb{N}$. Thus A is in the tail $\sigma - field$. □

Problem 4. Large Deviation Theorem

Proof. (a) $a > 1$. We check the two condition:
(H1)

$$\begin{aligned}
\varphi(\theta) =& \mathbb{E}\exp(\theta X_i)\\
=& \int_0^\infty \exp(\theta x)\exp(-x)dx\\
=& \int_0^\theta \exp((\theta-1)x)dx\\
=& \frac{\exp((\theta-1)x)}{\theta-1}\bigg|_0^\infty\\
=& \frac{1}{1-\theta}\frac{1}{\exp((1-\theta)x)}\bigg|_\infty^0 \text{ for } \theta¡1\\
=& \frac{1}{1-\theta}
\end{aligned}$$

So $\theta_- = -\infty, \theta_+ = \infty$. And $\varphi(\theta) < \infty, \forall\theta \in (-\infty, 1)$.
(H2) The exponential distribution is a continuous distribution, and it obviously is not a point mass at 1.
Now we find the solution to:

$$\begin{aligned} a =& \frac{1}{(1-\theta)^2}(1-\theta) \\ \Leftrightarrow 1-\theta =& \frac{1}{a} \\ \Leftrightarrow \theta =& 1 - \frac{1}{a} \end{aligned}$$

By the Large Deviation Theorem, we have:

$$\begin{aligned} n^{-1}\log \mathbb{P}\left[S_n \geq na\right] \to& -a(1-\frac{1}{a}) + \log\frac{1}{1-(1-\frac{1}{a})} \\ =& 1 - a + \log a \end{aligned}$$

(b) The Large Deviation Theorem is similar for $a < 1$ except that $\theta_a \in (\theta_-, 0)$. Following the same step we have $\theta_a = 1 - \frac{1}{a}$. And

$$n^{-1}\log \mathbb{P}\left[S_n \leq na\right] \to 1 - a + \log a, \text{ for a ¡ 1.}$$

□

Lemma 1. *Statement 2.6.1 in Durrett's book. Let $X_1, X_2, ...$ be i.i.d and $S_n = X_1 + ... + X_n$. Then $\mathbb{P}\left[S_n \geq na\right] \leq \exp(n\gamma(\alpha))$.*

Proof. From the first part of Section 2.6 in Durrett's book, denote $\pi_n = \mathbb{P}\left[S_n \geq na\right]$, we have: $\pi_{m+n} \geq \pi_m\pi_n$. Thus $\pi_{mn} \geq \pi_n^m$. Fix n. We have:

$$\begin{aligned} m\log\pi_n \leq& \log\pi_{mn} \\ \Rightarrow \log\pi_n \leq& n\frac{1}{mn}\log\pi_{mn}, \forall m \end{aligned}$$

Let the $m \to \infty$ we have, and using the fact that $\lim_{i\to\infty}\frac{1}{i}\log\pi_i = \gamma(a)$ (as derived from Lemma 2.6.1 in Durrett's book), we have:

$$\begin{aligned} \log\pi_n \leq& n\gamma(a) \\ \Rightarrow \pi_n \leq& \exp(n\gamma(a)) \end{aligned}$$

□

Problem 5. Oriented First Passage Percolution

Proof. Upper Bound. Consider a path where at each point, we choose the smaller edge. Then each of the chosen edge has the distribution of $\min(X, X^{'})$ for X, X' iid Exponential(1). We have $\min(X, X') \sim Exponential(1/2)$. Thus the sample mean according to the Law of Large Number $S_\pi/d = \frac{1}{d}\sum\min(X_i, X_i') \to \frac{1}{2}$ a.s. So an upper bound is 1/2.
Lower Bound. We have:

$$
\begin{aligned}
\mathbb{P}\left[\frac{H_d}{d} \le a\right] \le & \mathbb{P}\left[\frac{S_\pi}{d} \le a \text{ for all possible path} \pi\right] \\
=& \mathbb{P}\left[\bigcup_{\pi_0}\left(\frac{S_{\pi_0}}{d} \le a\right)\right] \\
\le & \sum\left(\mathbb{P}\left[\frac{S_{\pi_0}}{d} \le a\right]\right) \\
\le & 2^d \mathbb{P}\left[\frac{S_{\pi_0}}{d} \le a\right] \\
\le & 2^d \exp(d\gamma(a)) \text{ (From Lemma 1)} \\
\le & \exp\left(d \log 2 + d\gamma(a)\right) \\
= & \exp\left(d\left(\log 2 + 1 - a + \log a\right)\right) \text{ (From Q.4)} \\
= & \exp\left(d(\log(2a) + 1 - a)\right)
\end{aligned}
$$

The last expression goes to infinity as $d \to \infty$ iff $\log(2a)+1-a < 0$. Solving the equation $\log(2a)+1-a = 0$ we have one root $a^* \approx 0.231$, and $\forall a < a^*, \log(2a) + 1 - a < 0$. So picking any $a < a^*$ we have: $\lim_{d\to\infty} \mathbb{P}\left[\frac{H_d}{d} \le a\right] \to 0$ So a lower bound would be 0.23. $\square$