Solution for HW 2

1. (a). Note that $\mu(\mathbb{Q}\cap(0,1)) = \mu(\mathbb{Q}^c\cap(0,1)) = 0$ but $\mu((0,1)) = 1$. Thus μ is not finitely additive on $\mathcal{B}$. (b). Consider $B = (a_1, b_1]\cup\cdots\cup(a_n, b_n]$ and $B' = (a_1', b_1']\cup\cdots\cup(a_n', b_n']$ disjoint. We can check that at most one of them is equal to 1 (otherwise $\exists\epsilon > 0$ s.t. $(0,\epsilon)\in B\cap B'$, which contradicts disjointness). This leads to $\mu(B)+\mu(B') = \mu(B\cup B')$: μ is finitely additive on $\mathcal{B}_0$. In addition, for $k\in\mathbb{N}$, $\mu((\frac{1}{k+1},\frac{1}{k}]) = 0$ but $\mu(\cup_{k\in\mathbb{N}}(\frac{1}{k+1},\frac{1}{k}]) = \mu((0,1)) = 1$: μ is not countably additive on $\mathcal{B}_0$.

2. It is straightforward that countable additivity implies finite additivity and continuity from above (in particular, if $A_n\downarrow\emptyset$ then $\mu(A_n) = 0$). Conversely, consider $(A_n)_{n\in\mathbb{N}}$ disjoint. We have $\mu(\cup_{n\in\mathbb{N}}A_n) = \mu(\cup_{n\in\mathbb{N}}A_n\setminus\cup_{n\leq N}A_n) + \sum_{n\leq N}\mu(A_n)$ by finite additivity. Note that the first term goes to 0 and the second converges to $\sum_{n\in\mathbb{N}}\mu(A_n)$ by hypotheses. Thus we obtain countable additivity.

3. Take $S = \{1,2,3,4\}$ and $\mathcal{S} = \mathcal{P}(S)$. Consider $\mathcal{A} = \{\{1,2\},\{1,3\}\}$, we have $\mathcal{S} = \sigma(\mathcal{A})$. Set $\mu(\{1\}) = \mu(\{4\}) = \frac{1}{6}$, $\mu(\{2\}) = \mu(\{3\}) = \frac{1}{3}$ and $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{4\}) = \frac{1}{4}$ s.t. $\mu\neq\nu$ but $\mu = \nu$ on $\mathcal{A}$.

4. Denote $\mathcal{T} = \{B\in\mathcal{S}$ s.t. $\forall\epsilon > 0, \exists A\in\mathcal{F}, \mu(B\triangle A) < \epsilon\}$. Clearly, $\emptyset\in\mathcal{T}$. Moreover, $\mathcal{T}$ is closed under complement since $B^c\triangle A^c = B\triangle A$. Now consider $B = \cup_n B_n$ where $B_n\in\mathcal{T}$ for $\forall n$. Given $\epsilon > 0$, take $N\in\mathbb{N}$ s.t. $\mu(B\setminus\cup_{n\leq N}B_n)\leq\frac{\epsilon}{2}$. Then for $n\leq N$, take A_n s.t. $\mu(B_n\triangle A_n) < \frac{\epsilon}{2N}$. Since $\cup_{n<N}B_n\triangle\cup_{n<N}A_n\subset\cup_{n\leq N}(B_n\triangle A_n)$. We have then $\mu(B\triangle\cup_{n\leq N}A_n) < \epsilon$. Therefore, $\mathcal{T}$ is a $\sigma-$field containing $\mathcal{F}$, which permits to conclude.

Remark : For those who have already read the **Appendix A.1** of Durrett's book, it is also possible to appeal to the notion of outer measure to solve the problem.

5. First we approximate g by simple functions with bounded intervals. Note that $\exists\sum_{n=1}^N x_n 1_{A_n}$ where $\mu(A_n) < \infty$ s.t. $\int_0^1|g-\sum_{n=1}^N x_n 1_{A_n}|dx < \frac{\epsilon}{4}$. According to **Q4**, there exists finite disjoint union B_n s.t. $\mu(A_n\triangle B_n)\leq\frac{\epsilon}{4Nx_n}$. Thus, $\int_0^1|g-\sum_{n=1}^N x_n 1_{B_n}|dx < \frac{\epsilon}{2}$. WLOG, we suppose that $B_n = (a_n, b_n]$. Then take f_δ equal to 1 on $(a_n, b_n]$, 0 outside $(a_n-\delta, b_n+\delta]$ and piecewise linear elsewhere. Remark that $\int_0^1|f_\delta-\sum_{n=1}^N x_n 1_{B_n}|dx\to 0$ as $\delta\to 0$. Take $f = f_{\delta_\epsilon}$ s.t. $\int_0^1|f_{\delta_\epsilon}-\sum_{n=1}^N x_n 1_{B_n}|dx < \frac{\epsilon}{2}$. Then $\int_0^1|g-f|dx < \epsilon$.

Remark : Some powerful measure theory (or functional analysis) theorems can also be applied to conclude : Urysohn's lemma and Luzin's theorem among others.