ST205A - HW2

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Problem 1. Let

$$\mu : \mathcal{B}(\mathbb{R}) \to \{0,1\}$$
$$\mu(B) \mapsto \begin{cases} 1 & \exists (0,\epsilon) \subset B \\ 0 & \nexists \end{cases}$$

Prove that μ is not finitely additive on $\mathcal{B}(\mathbb{R})$, and that μ is finitely additive but not countable additive on the field $\mathcal{B}_0$ of finite disjoint unions of intervals $(a,b]$.

Proof. a. Let $A = \mathbb{Q}, B = \mathbb{R}\backslash\mathbb{Q}$. Then $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$ since each set of one element in $\mathbb{R}$ is in $\mathcal{B}(\mathbb{R})$, and $\mathbb{Q}$ is the coutable union of sets of signal rational number. Thus $\mathbb{R}\backslash\mathbb{Q} \in \mathcal{B}(\mathbb{R})$. Now $\mu(\mathbb{Q}) = 0, \mu(\mathbb{R}\backslash\mathbb{Q}) = 0$ (due to the denseness of rational numbers), but $\mu(\mathbb{Q} \cup \mathbb{R}/\mathbb{Q}) = \mu(\mathbb{R}) = 1$ and $\mathbb{Q}$ and $\mathbb{R}/\mathbb{Q}$ are disjoint. Thus μ is not finitely additive on $\mathcal{B}(\mathbb{R})$.

b. Let $(a_i, b_i], i = \overline{1,n}$ be pairwise disjoint. Then there are at most one interval that contains $(0,\epsilon)$, since if there are two of them, then they are not disjoint.

Now if there is no interval $(a_i,b_i]$ that contains any $(0,\epsilon)$, then if $\exists \epsilon > 0, (0,\epsilon) \subset \bigcup(a_i,b_i] \Rightarrow \exists i, (0,\epsilon) \in (a_i,b_i]$ since $(a_i,b_i]$ are disjoint, which is a contradiction. Thus the union also does not contain any $(0,\epsilon)$. So $\mu(\bigcup(a_i,b_i]) = 0 = \sum \mu(a_i,b_i]$.

Secone case $\exists!(a_i,b_i] : \exists\epsilon, (0,\epsilon) \in (a_i,b_i]$. Then $\sum \mu(a_i,b_i] = 1$. And $(0,\epsilon) \in \bigcup(a_i,b_i] \Rightarrow \mu(\bigcup(a_i,b_i]) = 0$.

So in both case μ is additive.

Now we will prove that μ is not coutably additive.

$$\begin{aligned}
\forall i \in \mathbb{N}, \mu\Big((\frac{1}{i+1}, \frac{1}{i}] \Big) &= 0 \\
\Rightarrow \sum_{i=1}^{\infty} \mu\Big((\frac{1}{i+1}, \frac{1}{i}] \Big) &= 0 \\
\mu\Big(\bigcup \Big(\frac{1}{i+1}, \frac{1}{i}] \Big) &= \mu\big((0,1]\big) \\
&= 1
\end{aligned}$$

So μ is not countably additive. □

Problem 2. Prove that in the definition of probability measure, countably additive is equivalent to finitely additive and condition $A_n \downarrow \emptyset \Rightarrow \mu(A_n) \to 0$.

Proof. a. First we'll prove the $\Leftarrow$ direction. Let $A_1, A_2, ...$ be disjoint sets. Let

$$\begin{aligned}
A &= \bigcup_{i=1}^{\infty} A_i \\
B_n &= A \backslash \bigcup_{i=1}^{n} A_i
\end{aligned}$$

Then $B_{n+1} \subset B_n, \forall n \in \mathbb{N}$. Let $B = \bigcup_{n=1}^{\infty} B_n$. If $B \neq \emptyset \Rightarrow \exists b \in B \Rightarrow b \in B_n, \forall n$

$$\Rightarrow b \in A \wedge b \notin \bigcup_{i=1}^{n} A_n, \forall n$$

But since $b \in A \Rightarrow \exists n \in \mathbb{N}, b \in A_n \Rightarrow b \in \bigcup_{i=1}^{n} A_n$ which is a contradiction. So $B = \emptyset$ or $B_n \downarrow \emptyset$. Thus $\lim_{n\to+\infty} \mu(B_n) = 0$

$$\begin{aligned}
&\Rightarrow \lim_{n\to\infty} \mu(A \backslash \bigcup_{i=1}^{n} A_i) = 0 \\
&\Rightarrow \mu(A) - \sum_{i=1}^{\infty} \mu(A_n) \overset{(1)}{=} \lim_{n\to\infty} \Big(\mu(A) - \sum_{i=1}^{n} \mu(A_i) \Big) \\
&\qquad \overset{(2)}{=} \lim_{n\to\infty} \Big(\mu(A) - \mu(\bigcup_{i=1}^{n} A_i) \Big) \\
&\qquad \overset{(3)}{=} \lim_{n\to\infty} \Big(\mu(A \backslash \bigcup_{i=1}^{n} A_i) \Big) \\
&\qquad = 0
\end{aligned}$$

Where (1) is true by the definition of series (this series converges because it is increasing and bounded above by 1), (2) is true because of finite additivity, (3) is true because also of finite additivity as $A = \big(\bigcup_{i=1}^{n} A_i\big) \cup \big(A \backslash \bigcup_{i=1}^{n} A_i\big)$ and the two sets on the right are disjoint.

So we have finite additivity and $A_n \downarrow \emptyset \Rightarrow \lim \mu(A_n) = 0$ imply countable additivity.

b. Now we'll prove the $\Rightarrow$ direction. Obviously we have finite additivity from countable additivity. Now let $B_0, B_1, ...$ such that $B_n \downarrow \emptyset$, we need to prove that $\lim_{n\to\infty} \mu(B_n) = 0$. Let $A = B_0$. Let $A_n = B_{n-1} \backslash B_n$, then we have $A_n's$ are disjoint, $A = \bigcup_{i=1}^{\infty} A_i$, and $A = B_n \cup \bigcup_{i=1}^{n} A_i$

$$\begin{aligned}
\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) &= \sum_{i=1}^{\infty} \mu(A_n) \\
\Rightarrow \lim_{n\to\infty} \mu(B_n) &= \lim_{n\to\infty} \mu(A / \bigcup_{i=1}^{n-1} A_i) \\
&= \lim_{n\to\infty} (\mu(A) - \mu(\bigcup_{i=1}^{n-1})) \\
&= \mu(A) - \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) \\
&= 0
\end{aligned}$$

So we have countable additivity implies finite additivity and $A_n \downarrow \emptyset \Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$. $\square$

Problem 3. Two separated measures for a sigma algebra generated from a collection of sets that have measure agree on the collection of sets.

Proof. Let $S = \{1, 2, 3, 4\}, \mathcal{A} = \{\{1, 2\}, \{1, 3\}\}$. Then $\mathcal{S} = \sigma(\mathcal{A}) = 2^S$.

Define:

$$\begin{aligned}
\mu : \mathcal{S} &\to \mathbb{R} \\
\{1\} &\mapsto \frac{1}{2} \\
\{2\} &\mapsto 0 \\
\{3\} &\mapsto 0 \\
\{4\} &\mapsto \frac{1}{2}
\end{aligned}$$

Then μ is a probability measure and $\mu(\{1,2\}) = \mu(\{1,3\}) = \frac{1}{2}$. Define:

$$\begin{aligned}\nu : \mathcal{S} &\to \mathbb{R}\\ \{1\} &\mapsto 0\\ \{2\} &\mapsto \frac{1}{2}\\ \{3\} &\mapsto \frac{1}{2}\\ \{4\} &\mapsto 0\end{aligned}$$

Then ν is also a probability measure and $\nu(\{1,2\}) = \mu(\{1,3\}) = \frac{1}{2}$. But apparently $\mu \neq \nu$. $\square$

Lemma 1. *Let $(S, \mathcal{S}, \mu)$ be the probability measure triple. $\forall A, B \in \mathcal{S}, \mu(A \cup B) \leq \mu(A) + \mu(B)$.*

Proof. The proof follows easily from the countable additivity of probability measure:

$$\begin{aligned}\mu(A \cup B) &= \mu((A \backslash B) \cup (B \backslash A) \cup (A \cap B))\\ &= \mu(A \backslash B) + \mu(B \backslash A) + \mu(A \cap B)\\ &\leq \mu(A \backslash B) + \mu(B \backslash A) + 2\mu(A \cap B)\\ &= \mu(A) + \mu(B)\end{aligned}$$

The equality holds iff A, B are disjoint.
Similarly we have the property for finite number of set, as well as countable number of sets. $\square$

Lemma 2. *Let A, B, C, D be sets. Then $(A \cup B)\Delta(C \cup D) \subset (A\Delta C) \cup (B \Delta D)$.*

Proof. Let $x \in (A \cup B)\Delta(C \cup D)$ be arbitrary. W.L.O.G assume that $x \in A \cup B \wedge x \notin C \cup D$. Thus:

$$\begin{aligned}&x \in A \wedge x \notin C\\ \Rightarrow &x \in A\Delta C\\ &x \in B \wedge x \notin D\\ \Rightarrow &x \in B\Delta D\\ \Rightarrow &x \in (A\Delta C) \cup (B\Delta D)\end{aligned}$$

So $(A \cup B)\Delta(C \cup D) \subset (A\Delta C) \cup (B\Delta D)$.
Similarly we have the property for arbitrary number of sets: $\left(\bigcup A_i\right)\Delta\left(\bigcup B_i\right) \subset \bigcup(A_i \Delta B_i)$. $\square$

Problem 4. Measure on a field and $\sigma - field$.

Proof. Let $\mathcal{T} = \{B \in \mathcal{S} \mid \forall\epsilon > 0, \exists A \in \mathcal{F}, \mu(B\Delta A) < \epsilon\}$. We will prove that $\mathcal{T}$ is a $\sigma - algebra$. Given that $\mathcal{T}$ is a $\sigma - algebra$, it follows that $\mathcal{S} \subset \mathcal{T}$ and $\mathcal{T} \subset \mathcal{S}$ thus $\mathcal{S} = \mathcal{T}$ and so we have any $\forall B \in \mathcal{S}, \forall\epsilon > 0, \exists A \in \mathcal{F}, \mu(A\Delta B) = 0$. Now let's prove that $\mathcal{T}$ is a $\sigma - algebra$. We will use the two lemmas above throughout this proof. We'll check the three condition:
a. Since $\mathcal{F}$ is a field $\Rightarrow \mathcal{F} \neq \emptyset$. But $\mathcal{F} \subset \mathcal{T} \Rightarrow \mathcal{T} \neq \emptyset$.
b. Let $B \in \mathcal{T}$ be arbitrary. Then:

$$\begin{aligned}\forall\epsilon > 0, \exists A \in \mathcal{F}, &\mu(B\Delta A) < \epsilon\\ B\Delta A &= B^C \Delta A^C\\ \Rightarrow &\mu(B^C \Delta A^C) < \epsilon\end{aligned}$$

Since $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$. So we found $A^C \in \mathcal{F}$ such that $\mu(B^C \Delta A^C) < \epsilon$. Thus $B^C \in \mathcal{T}$.

c. Let $\mathcal{B} = \{B_i \in \mathcal{T} \mid i \in \mathbb{N}\}$ and $\epsilon > 0$ be arbitrary. Let $C_1 = B_1, C_n = B_n \backslash \bigcup_{i=1}^{n-1} B_i, \forall n > 1$. Then we have C_i are disjoint and $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} C_i$. We also have $C_i \in \mathcal{S}, \forall i \in \mathbb{N}$. Let $B = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} C_i$. Then we have $\sum_{i=1}^{n} \mu(C_i)$ is an increasing series with limit equal to $\mu(B)$. Thus $\exists N \in \mathbb{N}, \sum_{i=N+1}^{\infty} \mu(C_i) < \epsilon/2$. Now,

$$\begin{aligned}
\forall i \leq N, \exists A_i \in \mathcal{F}, \mu(A_i \Delta B_i) < \epsilon/(2N) & \\
\Rightarrow \mu(\bigcup_{i=1}^{N} B_i \Delta \bigcup_{i=1}^{N} A_i) &\leq \mu(\bigcup_{i=1}^{N} B_i \Delta A_i) \\
&\leq \sum_{i=1}^{N} \mu(B_i \Delta A_i) \\
&< N\frac{\epsilon}{2N} = \frac{\epsilon}{2}.
\end{aligned}$$

So let $A = \bigcup_{i=1}^{N} A_i$ then $A \in \mathcal{F}$ since $A_i \in \mathcal{F}, \forall i \leq N$. We have:

$$\begin{aligned}
\bigcup_{i=1}^{\infty} B_i &= \bigcup_{i=1}^{\infty} C_i \\
&= \Big(\bigcup_{i=1}^{N} C_i\Big) \cup \Big(\bigcup_{i=N+1}^{\infty} C_i\Big) \\
&= \Big(\bigcup_{i=1}^{N} B_i\Big) \cup \Big(\bigcup_{i=N+1}^{\infty} C_i\Big) \\
\Rightarrow \mu\Big((\bigcup_{i=1}^{\infty} B_i)\Delta A\Big) &\leq \mu\Big(\Big(\bigcup_{i=1}^{N} B_i \Delta A\Big) \cup \Big(\bigcup_{i=N+1}^{\infty} C_i \Delta \emptyset\Big)\Big) \\
&\leq \mu\Big(\bigcup_{i=1}^{N} B_i \Delta A\Big) + \mu\Big(\bigcup_{i=N+1}^{\infty} C_i \Delta \emptyset\Big) \\
&= \mu\Big(\bigcup_{i=1}^{N} B_i \Delta A\Big) + \mu\Big(\bigcup_{i=N+1}^{\infty} C_i\Big) \\
&\leq \mu\Big(\bigcup_{i=1}^{N} B_i \Delta A\Big) + \sum_{i=N+1}^{\infty} \mu(C_i) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus $\bigcup_{i=1}^{\infty} B_i \in \mathcal{T}$.

From a., b., and c., we have $\mathcal{T}$ is a $\sigma - algebra$. □

Lemma 3. *Any open subset of* $\mathbb{R}$ *is a countable union of disjoint open intervals.*

Proof. Let $U \subset \mathbb{R}$ be an arbitrary open set. Define a relation in U: $\forall a, b \in U, a \sim b \Leftrightarrow (a, b) \subset U \vee (b, a) \subset U$. It follows that this relation is an equivalence relation, and the equivalent classes are pairwise disjoint. Let $\mathcal{A}$ be the set of equivalence classes. Then $U = \bigcup \mathcal{A}$ and $\mathcal{A}$ is countable (since each element in $\mathcal{A}$ contains at least one rational number, and there are only countable number of rational number). Also, each equivalent class in $\mathcal{A}$ is an open interval. Thus we have the proof. □

Problem 5. Approximate Lebesgue integrable function with continuous function

Proof. We will prove through three steps. Let $\epsilon > 0$ be arbitrary.

a. By the construction of Lebesgue integration, integration of a (non-negative) function f on $[0, 1]$ w.r.t. Lebesgue measure dL is defined as $\sup\{\int s dL \mid s \leq f, s = \sum_{i=1}^{n} a_i 1_{A_i}, Y_i \in \mathcal{B}([0, 1]), a_i \in \mathbb{R}^+\}$. As such $\exists s = \sum_{i=1}^{n} a_i 1_{A_i}, \int \mid f(x) - s(x) \mid dx < \epsilon/2$. For a general function (not neccessary non-negative), we can use the same trick in defining integration of f^-, f^+, and we'll also have a simple function $g(x) = \sum_{i=1}^{n} b_i 1_{B_i}, B_i \in \mathcal{B}([0,1]), b_i \in \mathbb{R}$ such that $\int \mid f(x) - g(x) \mid dx < \epsilon/3$.

b. Now for each $B_i \in \mathcal{B}([0,1])$, by the regularity of Lebesque measure, $\exists C_i$ open, $B_i \subset C_i, \mu(C_i \backslash B_i) < \epsilon/(3|b_i|)$. Let $h(x) = \sum_{i=1}^{n} b_i 1_{C_i}$, then $\int \mid g(x) - h(x) \mid dx < \epsilon/3$.

c. For each C_i open, $C_i = \bigcup_{i=1}^{\infty} D_i$ such that D_i pairwise disjoint and open interval (Lemma 3). Now $\mu(C_i) = \mu(\bigcup_{i=1}^{\infty} D_i) = \sum_{i=1}^{\infty} \mu(D_i)$, for μ is the Lebesque measure. By the definition of limit, $\exists N_i, \forall n \geq N_i, |\mu(C_i) - \sum_{i=1}^{n} \mu(D_i)| < \epsilon/(3|b_i|)$. Let $k(x) = \sum_{i=1}^{n} b_i \sum_{j=1}^{N_i} 1_{D_{ij}}$ for $D_{ij}'s$ are open intervals. Then $\int \mid h(x) - k(x) \mid dx < \epsilon/3$.

d. $k(x)$ can be rewritten as $\sum_{i=1}^{N} e_i 1_{E_i}$ for $E_i's$ are disjoint open interval. Define $l(x)$ such as $l(x) = 0, \forall x \notin \bigcup_{i=1}^{N} E_i$. For each $D_i = (a_i, b_i)$, define $l(a_i) = l(b_i) = 0, l(x) = 2 - \frac{2}{b_i - a_i} \mid 2x - a_i - b_i \mid$, then $l(x)$ has an isosceles triangle shape with area equal to $|b_i - a_i|$. Then we have $\int |k(x) - l(x)| dx = 0$ and $l(x)$ is continuous.

From a., b., c., and d., apply the triangle inequality for the absolute value, we have

$\int \mid f(x) - l(x) \mid dx < 3\epsilon/3 = \epsilon$ and $l(x)$ is continuous. □