ST205A - Homework 8

Hoang Duong

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Problem 1. Radon Nikodym

Proof. Since $\pi \ll \mu$, by the Radon-Nykodym theorem, $\forall A$ measurable,

$$\pi(A) = \int_A \frac{d\pi}{d\mu} d\mu \tag{1}$$

Next, $\nu \ll \mu$, by the Radon-Nykodym theorem, $\forall B$ measurable,

$$\int_B d\nu = \nu(B) = \int_B \frac{d\nu}{d\mu} d\mu$$

And thus for any f positive and measurable,

$$\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu \tag{2}$$

Since $\pi \ll \nu$, take $f = \mathbb{I}_A \frac{d\pi}{d\nu}$, positive and measurable, plug this to (2), we have:

$$\pi(A) = \int \mathbb{I}_A \frac{d\pi}{d\nu} d\nu = \int \mathbb{I}_A \frac{d\pi}{d\nu} \frac{d\nu}{d\mu} d\mu = \int_A \frac{d\pi}{d\nu} \frac{d\nu}{d\mu} d\mu \tag{3}$$

(1) and (3) are true for all measurable A, thus:

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \frac{d\nu}{d\mu}$$

□

Lemma 1. $A \coloneqq \sigma\left(\{(-\infty, r] \mid r \in \mathbb{Q}\}\right) = \mathcal{B}(\mathbb{R})$

Proof. First, the Borel set of $\mathbb{R}$ is the $\sigma - algebra$ generated by all open set in $\mathbb{R}$, and $(-\infty, r]^C = (r, \infty)$, so $A \subset \mathcal{B}(\mathbb{R})$

Second, by the dense property of rational numbers, $\sigma\left(\{(-\infty, r] \mid r \in \mathbb{Q}\}\right) = \sigma\left(\{(-\infty, r] \mid r \in \mathbb{R}\}\right) = \sigma\left(\{(a, b] \mid a, b \in \mathbb{R}\}\right) = \sigma\left(\{(a, b) \mid a, b \in \mathbb{R}\}\right)$

Now we use the Lemma 2, proved in Homework 2, that any open set in $\mathbb{R}$ is a countable union of open interval, we have the last set in the previous chain of equality to be equal to $\mathcal{B}(\mathbb{R})$ □

Problem 2. Uniqueness of Conditional Probability Kernel

Proof. S_2 is nice, thus S_2 is homeomorphic to $\mathbb{R}$. So we can reduce our proof to $\mathbb{R}$.

$$A = \{x \mid \mathbb{Q}^*(x,B) = \mathbb{Q}(x,B), \forall B \in \mathcal{B}(\mathbb{R})\}$$
$$A_r = \{x \mid \mathbb{Q}^*(x,(-\infty,r]) = \mathbb{Q}(x,(-\infty,r])\}, \forall r \in \mathbb{Q}$$

Claim: $A = \bigcap_{r\in\mathbb{Q}} A_r$.
First, $A \subset A_r, \forall r \in \mathbb{Q} \Rightarrow A \subset \bigcap_{r\in\mathbb{Q}} A_r$
Second, by Lemma 1, we have $\sigma(\{(-\infty,r] \mid r \in \mathbb{Q}\}) = \mathcal{B}(\mathbb{R})$, thus $\bigcap_{r\in\mathbb{Q}} A_r \subset A$.
So it follows that $A = \bigcap_{r\in\mathbb{Q}} A_r$.
Now, $\forall A, B \in \mathcal{B}(\mathbb{R})$, we have:

$$\begin{aligned}\mu(A \times B) &= \int_A Q(x,B)\mu_1(dx)\\ &= \int_A Q^*(x,B)\mu_1(dx)\end{aligned}$$

So $Q^*(x,B) = Q(x,B)$ a.s. for every fixed B, in particular $B^r = (-\infty,r]$
$\mu_1(A^C) = \mu_1(\bigcup A_r^C) \leq \sum_{r\in\mathbb{Q}} \mu_1(A_r^C) = 0. \Rightarrow \mu_1(A) = 1$ ☐

Problem 3. Fubini's Theorem

Proof. Let X be a random variable with dist F. Applying the Fubini theorem for two $\sigma - finite$ measure: Lebesque and the measure of F, for the measurable, and positive indicator function, we have:

$$\begin{aligned}\int_{\mathbb{R}} [F(x+c) - F(x)]\,dx &= \int \mathbb{E}\mathbb{I}_{x\leq z\leq x+c} dx\\ &= \mathbb{E}\int \mathbb{I}_{x\leq z\leq x+c} dx \text{ (Fubini)}\\ &= \mathbb{E}c = c\end{aligned}$$

☐

Problem 4. Inverse Distribution Function

Proof. Fix $a \in \mathbb{R}$, we have:

$$\begin{aligned}f(x,u) \leq a &\Leftrightarrow \mu \leq Q(x,(-\infty,a])\\ &\Leftrightarrow g(u,x) = u - Q(x,(-\infty,a]) \leq 0\end{aligned}$$

g is the sum of 2 measurable functions, thus it is (product) measurable. So the inverse of $(-\infty,a]$ under f is equal to the inverse of $(-\infty,0]$ under g, which is measurable. And as we proved before, the $\sigma - algebra$ generated by the set of form $(-\infty,a]$ is the Borel set of $\mathbb{R}$, it follows that f is measurable. ☐

Problem 5. Marginal might not imply joint

Proof. Consider the Gaussian random variable. We have a normal distribution $\mathcal{N}(\mu,\Sigma)$ is valid iff Σ is positive definite. Consider the pair marginal distribution:

$$\begin{aligned}f(X_1,X_2) =& \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & c\\ c & 1\end{bmatrix}\right)\\ f(X_1,X_3) =& \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & b\\ b & 1\end{bmatrix}\right)\\ f(X_2,X_3) =& \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & a\\ a & 1\end{bmatrix}\right)\end{aligned}$$

For $a, b, c \in [-1, 1]$, these are valid distribution, and the marginal of X_i obtained from each of the pair marginal are all standard normal $\mathcal{N}(0, 1)$.

Now the joint distribution of X_1, X_2, X_3 if existed must have the form:

$$f(X_1, X_2, X_3) = \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & c & b \\ c & 1 & a \\ b & a & 1 \end{bmatrix}\right)$$

If we can find $a, b, c \in [-1, 1]$ such that the covariance matrix is not positive definite, then there is a contradiction. In fact consider:

$$\det \Sigma = 1 + 2abc - a^2 - b^2 - c^2$$

We want:

$$\begin{aligned} \det \Sigma &< 0 \\ \Leftrightarrow a^2 + b^2 + c^2 &> 1 + 2abc \end{aligned}$$

For example if we pick $a = b = 1, c = 0$, then $\det \Sigma < 0$, thus Σ is not positive definite. Thus there does not exist a triple (X_1, X_2, X_3) corresponding to the pairwise distribution of $(X_1, X_2), (X_2, X_3), (X_1, X_3)$ above. $\square$