Solution for HW 10

1. Consider $A := \{\max_{m\le n} |S_m| > x\}$ and $N := \inf\{m; |S_m| > x \text{ or } m = n\}$. Note that N is a stopping time which is a.s. bounded by n. Observe that $(S_n^2 - s_n^2)_{n\in\mathbb{N}}$ is martingale. Apply Theorem 5.4.1 in Durrett, $0 = \mathbb{E}(S_N^2 - s_N^s) \le (x+K)^2\mathbb{P}(A) + (x^2 - s_n^2)\mathbb{P}(A^c)$ since on A, $|S_N| \le x+K$ and on A^c, $S_N^2 = S_n^2 \le x^2$. Thus, $(x+K)^2 \ge [(x+K)^2 - x^2 + s_n^2]\mathbb{P}(A^c) \ge s_n^2\mathbb{P}(A^c)$. We have then $\mathbb{P}(\max_{m\le n} |S_m| < x) \le \mathbb{P}(A^c) \le s_n^{-2}(x+K)^2$.
2. According to Example 5.4.1 in Durrett, $\mathbb{P}(\max_{m\le n} X_m \ge \lambda) \le \mathbb{P}(\max_{m\le n}(X_n + c)^2 \ge (c+\lambda)^2) \le \frac{\mathbb{E}(X_n+c)^2}{(c+\lambda)^2} = \frac{\mathbb{E}X_n^2+c^2}{(c+\lambda)^2}$. Take now $c = \frac{\mathbb{E}X_n^2}{\lambda}$, we obtain the desired result.
3. By the orthogonality of martingale increments (Theorem 5.4.6 in Durrett), we have $\mathbb{E}(X_{m-1}Y_m) = \mathbb{E}(X_mY_{m-1}) = \mathbb{E}(X_{m-1}Y_{m-1})$ for $m \in \mathbb{N}$. Thus we have $\mathbb{E}(X_m - X_{m-1})(Y_m - Y_{m-1}) = \mathbb{E}X_mY_m = \mathbb{E}X_{m-1}Y_{m-1}$. We get the desired result by summing over m.
4. According to Theorem 5.2.5 in Durrett, $(\sum_{j=1}^n V_j(X_j - X_{j-1}))_{n\in\mathbb{N}}$ is submartingale. Note in addition that $\sum_{j=1}^n V_j(X_j - X_{j-1}) = V_nX_n + \sum_{j=1}^{n-1} X_j(V_j - V_{j+1}) \ge V_nX_n$. According to Doob's inequality (Theorem 5.2.5 in Durrett), $\mathbb{P}(\max_{m\le n} V_mX_m > \lambda) \le \mathbb{P}(\max_{m\le n} \sum_{j=1}^m V_j(X_j - X_{j-1}) > \lambda) \le \lambda^{-1}\sum_{j=1}^n \mathbb{E}[V_j(X_j - X_{j-1})]$.
5. We follow here the sketch in Durrett, i.e. Exercise 5.2.13 and Exercise 5.2.14 as well as the notations therein. We first prove the switching principle : for X_n^1, X_n^2 supermartingales with respect to $\mathcal{F}_n$ and N stopping time such that $X_N^1 \ge X_N^2$, $Y_n := X_n^1 1_{N>n} + X_n^2 1_{N\le n}$ is supermartingale. Note that $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}[1_{N>n}X_{n+1}^1 + 1_{N\le n}X_{n+1}^2 + 1_{N=n+1}(X_N^2 - X_N^1)|\mathcal{F}_n] \stackrel{(*)}{\le} \mathbb{E}[1_{N>n}X_{n+1}^1 + 1_{N\le n}X_{n+1}^2|\mathcal{F}_n] = 1_{N>n}\mathbb{E}[X_{n+1}^1|\mathcal{F}_n] + 1_{N\le n}\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] \le Y_n$, where the inequality $(*)$ is due to the fact that $X_N^1 \ge X_N^2$. Now we procede slightly different from Exercise 5.2.14 in Durrett. Set $Z_n^1 :\equiv 1$ and define inductively Z^k for $k \ge 2$ in terms of parity of k. For $k = 2j$, set $Z_n^{2j} := Z_n^{2j-1}1_{N_{2j-1}>n} + (\frac{b}{a})^{j-1}\frac{X_n}{a}1_{N_{2j-1}\le n}$. Observe that $Z_{N_{2j-1}}^{2j-1} = (\frac{b}{a})^{j-1} \ge (\frac{b}{a})^{j-1}\frac{X_{N_{2j-1}}}{a}$. The switching principle leads to that Z^{2j} is supermartingale. For $k = 2j+1$, set $Z_n^{2j+1} := Z_n^{2j}1_{N_{2j}>n} + (\frac{b}{a})^j 1_{N_{2j}\le n}$. Note that $Z_{N_{2j}}^{2j} = (\frac{b}{a})^{j-1}\frac{X_{N_{2j}}}{a} \ge (\frac{b}{a})^j$. Again by switching principle, Z^{2j+1} is supermartingale. Fix $k \in \mathbb{N}$. We have then $\mathbb{E}(\min(1, \frac{X_0}{a})) = \mathbb{E}Z_0^{2k+1} \ge \liminf_{n\to\infty}\mathbb{E}Z_n^{2k+1} \ge \mathbb{E}\liminf_{n\to\infty} Z_n^{2k+1} \ge (\frac{b}{a})^k\mathbb{E}(N_{2k} < \infty) = (\frac{b}{a})^k\mathbb{P}(U \ge k)$.