Solution for HW 4

1. We show directly (ii). By pairwise independence, we get $\mathbb{E}D_n = 0$ and $\mathbb{E}D_n^2 = \frac{1}{n}[\int_0^1 f^2(x)dx - (\int_0^1 f(x)dx)^2] := \frac{\sigma^2}{n}$. Using Chebyshev's inequality, we obtain $\mathbb{P}(|D_n| > \epsilon) \leq \frac{VarD_n}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$.
2. Take S and T two measurable bounded functions, $\mathbb{E}[S(XY)T(\frac{X}{Y})] = \int_{x,y\geq 0} S(xy)T(\frac{x}{y})f(x) g(y)dxdy$ $(*)$. Set $u := xy$ and $v := \frac{x}{y}$ and note that $(\mathbb{R}^+)^2 \ni (x,y) \to (u,v) \in (\mathbb{R}^+)^2$ is $\mathcal{C}^1-$diffeomorphism with jacobian matrix $J := \begin{pmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}$. By change of variables, $(*) = \int_{u,v\geq 0} S(u)T(v)\frac{1}{|\det J|}f(\sqrt{uv})g(\sqrt{\frac{u}{v}})dudv = \int_{u,v\geq 0} S(u)T(v)\frac{1}{2v}f(\sqrt{uv})g(\sqrt{\frac{u}{v}})dudv$. Therefore, $(XY, \frac{X}{Y}) := (U,V)$ has joint distribution $\frac{1}{2v}f(\sqrt{uv})g(\sqrt{\frac{u}{v}})$. And the density of XY is $\int_{v\geq 0} \frac{1}{2v} f(\sqrt{uv})g(\sqrt{\frac{u}{v}})dv$ and that of $\frac{X}{Y}$ is $\int_{u\geq 0} \frac{1}{2v}f(\sqrt{uv})g(\sqrt{\frac{u}{v}})du$.
3. Let $\epsilon > 0$ and consider $K > 0$ s.t. for $k \geq K$, $r(k) \leq \epsilon$. According to Cauchy-Schwarz inequality, $\mathbb{E}X_iX_j \leq (\mathbb{E}X_i^2\mathbb{E}X_j^2)^{\frac{1}{2}} = r(0)$. Breaking the sum into $|i-j| \leq K$ and $|i-j| > K$, we have $\mathbb{E}S_n^2 \leq n(2K+1)r(0) + n^2\epsilon$. Thus, $\limsup_{n\to\infty} \frac{\mathbb{E}S_n^2}{n^2} \leq \epsilon$ for arbitary small ϵ. Therefore, $\frac{S_n}{n} \xrightarrow{\mathbb{L}^2} 0$, which implies convergence in probability.
Remark : The idea of splitting the sum into two parts goes back to Cesaro.
4. According to the first Borel-Cantelli lemma, $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$ implies that $\mathbb{P}(A_n^c \cap A_{n+1}\ i.o.) = 0$. This means that a.s. there are only a finite number of switches between $\{A_n\}$ and $\{A_n^c\}$. Thus, one of them occurs only a finite number of times after which the other one takes over forever. We have then $\mathbb{P}(A_n^c\ i.o.) = \lim_{n\to\infty} \mathbb{P}(\cup_{n\geq m} A_n^c) \geq \lim_{m\to\infty} \mathbb{P}(A_m^c) = 1$ where the last equality follows from the $\mathbb{P}(A_m) \to 0$ as $m \to \infty$.
Remark : This result is due to Barndorff and Nielsen.
5. (a). It is easy to check that $\frac{z}{\sqrt{2\pi}(1+z^2)} \exp(-\frac{z^2}{2}) \leq \mathbb{P}(Z > z) \leq \frac{1}{\sqrt{2\pi}z}\exp(-\frac{z^2}{2})$. We have then $\mathbb{P}(Z > z) \sim \frac{1}{\sqrt{2\pi}z}\exp(-\frac{z^2}{2})$ as $z \to \infty$. (b). Take $c_n = \sqrt{2\log n}$ and fix $\epsilon > 0$. According to (a), $\mathbb{P}(\frac{Z_n}{c_n} > 1+\epsilon) \sim \frac{1}{\sqrt{4\pi}(1+\epsilon)\log^{\frac{1}{2}} n} n^{-(1+\epsilon)^2}$ and thus $\sum_n \mathbb{P}(\frac{Z_n}{c_n} > 1+\epsilon) < \infty$. $\limsup_n \frac{Z_n}{c_n} \leq 1+\epsilon$ a.s. follows from the first Borel-Cantelli lemma. Similarly, we get $\sum_n \mathbb{P}(\frac{Z_n}{c_n} > 1-\epsilon) = \infty$ and $\limsup_n \frac{Z_n}{c_n} \geq 1-\epsilon$ a.s. follows from independence of Z_n and the second Borel-Cantelli lemma. Since ϵ is arbitary small, we have $\limsup \frac{Z_n}{c_n} = 1$ a.s.