Solution for HW 9

1. We first consider $h(x,y) = 1_{x\in A}1_{y\in B}$ where A, B are measurable sets in $\mathbb{R}$. On one hand, $\mathbb{E}(h(X,Y)|\mathcal{G})(w) \overset{(*)}{=} 1_{Y(w)\in B}\mathbb{P}(X\in A|\mathcal{G})(w) = 1_{Y(w)\in B}\mu(w,A)$, where $(*)$ is due to the fact that Y is $\mathcal{G}$-measurable. On the other hand, $\int h(x,Y(w))\mu(w,dx) = 1_{Y(w)\in B}\mu(w,A)$. Thus, $\mathbb{E}(1_{X\in A}1_{Y\in B}|\mathcal{G})(w) = \int 1_{x\in A}1_{Y(w)\in B}\mu(w,dx)$. Next by $\pi-\lambda$ lemma, the equality holds for all indicator functions in $\mathbb{R}^2$ (not necessarily decomposable). Finally the result also holds for bounded measurable functions by usual extension argument.

2. (b)$\Rightarrow$(a) by taking $h_1 = 1_{A_1}$ and $h_2 = 1_{A_2}$. (a)$\Rightarrow$(b) follows usual extension argument. Now we prove that (c)$\Rightarrow$(b). By tower property of conditional expectation, $\mathbb{E}(h_1(X_1)h_2(X_2)|\mathcal{G}) = \mathbb{E}[\mathbb{E}(h_1(X_1)h_2(X_2)|\mathcal{G},X_2)|\mathcal{G}] = \mathbb{E}[h_2(X_2)\mathbb{E}(h_1(X_1)|\mathcal{G},X_2)|\mathcal{G}] = \mathbb{E}[h_2(X_2)\mathbb{E}(h_1(X_1)|\mathcal{G})|\mathcal{G}] = \mathbb{E}(h_1(X_1)|\mathcal{G})\mathbb{E}(h_2(X_2)|\mathcal{G})$. Finally, we show that (b)$\Rightarrow$(c). Take Y a $\mathcal{G}$-measurable random variable. (b) implies that $\mathbb{E}(h_1(X_1)h_2(X_2)Y) = \mathbb{E}[\mathbb{E}(h_1(X_1)|\mathcal{G})\mathbb{E}(h_2(X_2)|\mathcal{G})Y] = \mathbb{E}[\mathbb{E}(h_1(X_1)|\mathcal{G})$ $h_2(X_2)Y]$ $(*)$, where the last equality is due to the fact that $\mathbb{E}(h_1(X_1)|\mathcal{G})Y$ is $\mathcal{G}$-measurable. To simplify the notation, denote $Z := \mathbb{E}(h_1(X_1)|\mathcal{G},X_2)$. Since $h_2(X_2)Y$ is $\sigma(\mathcal{G},X_2)$-measurable, $\mathbb{E}(Zh_2(X_2)Y) = \mathbb{E}(h_1(X_1)h_2(X_2)Y) \overset{(*)}{=} \mathbb{E}[\mathbb{E}(h_1(X_1)|\mathcal{G})h_2(X_2)Y]$. Thus, $\mathbb{E}[(Z-\mathbb{E}(h_1(X_1)|\mathcal{G}))$ $h_2(X_2)Y] = 0$ for all measurable functions h_2 and $\mathcal{G}$-measurable random variables Y. Again using $\pi-\lambda$ argument, $\mathbb{E}[(Z-\mathbb{E}(h_1(X_1)|\mathcal{G}))\mathcal{X}] = 0$ for all $\sigma(\mathcal{G},X_2)$-measurable random variable $\mathcal{X}$, which permits to conclude.

3. We use extensively the property (c) in **Q2**. Denote f a bounded measurable function. X and Y are conditionally independent given Z means that $\mathbb{E}(f(X)|Y,Z) \overset{(*)}{=} \mathbb{E}(f(X)|Z)$. X and Z are conditionally independent given $\mathcal{F}$ suggests that $\mathbb{E}(f(X)|Z) \overset{(**)}{=} \mathbb{E}(f(X)|\mathcal{F})$ since $\mathcal{F} \subset \sigma(Z)$. By tower property of conditional expectation, $\mathbb{E}(f(X)|Y,\mathcal{F}) = \mathbb{E}[\mathbb{E}(f(X)|Y,Z)|Y,\mathcal{F}] \overset{(\#)}{=} \mathbb{E}[\mathbb{E}(f(X)|\mathcal{F})|Y,\mathcal{F}] = \mathbb{E}(f(X)|\mathcal{F})$, where $(\#)$ follows $(*)$ and $(**)$. Therefore, X and Y are conditionally independent given $\mathcal{F}$.

4. Suppose that $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are submartingales with respect to $\mathcal{F}_n$. Then $\mathbb{E}(X_{n+1}+Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n)+\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \geq X_n+Y_n$: $(X_n+Y_n)_{n\in\mathbb{N}}$ is submartingale with respect to $\mathcal{F}_n$. In addition, observe that $(x,y)\to\max(x,y)$ is convex. Apply Jensen's inequality (for conditional expectation), we get $\mathbb{E}(\max(X_{n+1},Y_{n+1})|\mathcal{F}_n) \geq \max(\mathbb{E}(X_{n+1}|\mathcal{F}_n),\mathbb{E}(Y_{n+1}|\mathcal{F}_n)) \geq \max(X_n,Y_n)$: $(\max(X_n,Y_n))_{n\in\mathbb{N}}$ is submartingale with respect to $\mathcal{F}_n$.

5. Denote $(\xi_i)_{i\in\mathbb{N}}$ i.i.d such that $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$. Consider $X_n := \sum_{i=1}^{n}\xi_i$ and $Y_n := -\sum_{i=1}^{n+1}\xi_i$. It is immediate that $(X_n)_{n\in\mathbb{N}}$ is (sub)martingale with respect to filtration $\mathcal{F}_n := \sigma(\xi_1,\cdots\xi_n)$ and $(Y_n)_{n\in\mathbb{N}}$ is (sub)martingale with respect to filtration $\mathcal{G}_n := \sigma(\xi_1,\cdots\xi_{n+1})$. Suppose by contradiction that $X_n+Y_n = -\xi_{n+1}$ is submartingale with respect to some filtration. According to martingale convergence theorem, ξ_n converges to some random variable ξ_∞ a.s. This is impossible by the construction of $(\xi_n)_{n\in\mathbb{N}}$.