MA205A - Homework 4

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Problem 1. Monte Carlo Integration

Proof. (i) First we will calculate the expectation of D_n :

$$\begin{aligned}\mathbb{E}D_n =&\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} f(U_i) - \int_0^1 f(x)dx\right]\\ =&\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}f(U_i) - \int_0^1 f(x)dx\\ =&\left[\frac{1}{n}\sum_{i=1}^{n}\int_0^1 f(x)dx\right] - \int_0^1 f(x)dx\\ =&0\end{aligned}$$

Now applying the Chebyshev inequality we have:

$$\begin{aligned}\mathbb{P}(|D_n - \mathbb{E}D_n| \geq \epsilon) \leq&\frac{\mathrm{Var}D_n}{\epsilon^2}\\ \Leftrightarrow \mathbb{P}(|D_n| \geq \epsilon) \leq&\frac{1}{\epsilon^2 n^2}\mathrm{Var}\sum_{i=1}^{n} f(U_i)\\ \overset{iid}{=}&\frac{1}{\epsilon^2 n^2}\sum_{1}^{n}\mathrm{Var}f(U_i)\\ =&\frac{1}{\epsilon^2 n^2}n\mathbb{E}f^2(U_i)\\ =&\frac{1}{n\epsilon^2}\int_0^1 f^2(x)dx\end{aligned}$$

(ii) The only part where we need iid are step 3 in the above arguments, but for pairwise independent U_i, we also have: $\mathrm{Var}\sum_{i=1}^n f(U_i) = \sum_{i=1}^n \mathrm{Var}f(U_i) + 2\sum_{i\neq j}\mathrm{Cov}(f(U_i), f(U_j)) = \sum_{i=1}^n \mathrm{Var}f(U_i)$. Thus the bound is still valid. □

Problem 2. Density of Product and Quotient

Proof. (i) Density of XY

We have:

$$
\begin{aligned}
\mathbb{P}[XY \leq a] =& \mathbb{E}[\mathbb{I}[XY \leq a]] \\
=& \mathbb{E}\left[\mathbb{E}[\mathbb{I}[XY \leq a] \mid Y]\right] \\
=& \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}[X \leq \frac{a}{Y} \mid Y\right]\right] \\
=& \mathbb{E}\left[\mathbb{E}\int_0^{a/y} f(x)dx\right] \\
=& \int_0^{\infty}\left\{\int_0^{a/y} f(x)dx\right\} f(y)dy \\
=& \int_0^{\infty}\int_0^{a/y} f(x)f(y)dxdy
\end{aligned}
$$

Applying the Diffentiating Under the Integral, with the regularity assumption on f, g, we have the density for XY is:

$$
\begin{aligned}
\frac{\partial \mathbb{P}[XY \leq a]}{\partial a} =& \frac{\partial}{\partial a}\int_0^{\infty}\int_0^{a/y} f(x)g(y)dxdy \\
=& \int_0^{\infty}\frac{\partial}{\partial a}\int_0^{a/y} f(x)g(y)dxdy \\
=& \int_0^{\infty}\frac{1}{y}f(\frac{a}{y})g(y)dy
\end{aligned}
$$

(ii) Similarly,

$$
\begin{aligned}
\mathbb{P}[\frac{X}{Y} \leq a] =& \mathbb{E}[\mathbb{I}[X \leq aY]] \\
=& \mathbb{E}\left[\mathbb{E}[\mathbb{I}[X \leq aY] \mid Y]\right] \\
=& \mathbb{E}\left[\mathbb{E}\int_0^{ay} f(x)dx\right] \\
=& \int_0^{\infty}\left\{\int_0^{ay} f(x)dx\right\} f(y)dy \\
=& \int_0^{\infty}\int_0^{ay} f(x)f(y)dxdy
\end{aligned}
$$

Then the density under regularity assumption on f, g is:

$$
\begin{aligned}
\frac{\partial \mathbb{P}[\frac{X}{Y} \leq a]}{\partial a} =& \frac{\partial}{\partial a}\int_0^{\infty}\int_0^{ay} f(x)g(y)dxdy \\
=& \int_0^{\infty}\frac{\partial}{\partial a}\int_0^{ay} f(x)g(y)dxdy \\
=& \int_0^{\infty} yf(ay)g(y)dy
\end{aligned}
$$

□

Problem 3. Law of Large Number

Proof. We need to prove:

$$\forall \epsilon > 0, \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \epsilon\right] \to 0$$

For a fixed arbitrary $\epsilon > 0$, applying the Chebysev inequality we have:

$$\begin{aligned}
\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \epsilon\right] \leq & \frac{1}{\epsilon^2}\text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
= & \frac{1}{n^2\epsilon^2}\left\{\sum_{i=1}^{n}\text{Var}X_i + \sum_{i\neq j}\text{Cov}(X_i, X_j)\right\} \\
= & \frac{1}{n^2\epsilon^2}\left\{nr(0) + 2(n-1)r(1) + 2(n-2)r(2) + ... + 4r(n-2) + 2r(n-1)\right\} \\
\leq & \frac{1}{n^2\epsilon^2}\left\{2nr(0) + 2(n-1)r(1) + 2(n-2)r(2) + ... + 4r(n-2) + 2r(n-1)\right\} \\
\leq & \frac{2}{n^2\epsilon^2}\left\{|nr(0)| + |(n-1)r(1)| + |(n-2)r(2)| + ... + |2r(n-2)| + |r(n-1)|\right\} \\
\leq & \frac{2}{n^2\epsilon^2}\left\{n\,|r(0)| + n\,|r(1)| + n\,|r(2)| + ... + n\,|r(n-2)| + n\,|r(n-1)|\right\} \\
= & \frac{2}{\epsilon^2}\frac{1}{n}\left\{|r(0)| + |r(1)| + |r(2)| + ... + |r(n-2)| + |r(n-1)|\right\}
\end{aligned}$$

Now since: $\lim_{n\to\infty} r(n) = 0 \Rightarrow \lim_{n\to\infty} |r(n)| = 0$. According to the Cesaro theorem, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} r(i) \to 0$.

Thus $\lim_{n\to\infty}\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \epsilon\right] = 0$. □

Lemma 1. *Prove that:*

$$\limsup A_n \cap B_n \subset \limsup A_n \cap \limsup B_n$$

Proof. We have:

$$\begin{aligned}
A_n \cap B_n \subset & A_n, \forall n \\
\Rightarrow \bigcup_{k=n}^{\infty} A_k \cap B_k \subset & \bigcup_{k=n}^{\infty} A_k \\
\Rightarrow \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \cap B_k \subset & \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k
\end{aligned}$$

Similarly:

$$\begin{aligned}
\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \cap B_k \subset & \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k \\
\Rightarrow \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \cap B_k \subset & \left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right] \cup \left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k\right]
\end{aligned}$$

□

Lemma 2. *Prove that:*

$$\limsup A_n \cap \limsup A_{n+1} = \limsup A_n \cap A_{n+1}^C \tag{1}$$

$$\limsup A_n \cap \limsup A_{n+1} = \limsup A_n^C \cap A_{n+1} \tag{2}$$

Proof. This proof follows directly from the book The Borel Cantelli Lemma by Tapas Kumar Chandra.

Note that it suffices to show just one of the equality, as the second follows by replacing A_n by A_n^C.

$\supset$ Direction follows from lemma 1.

$\subset$ Direction. Let $\omega \in LHS$ of (1). We will need to prove that $\omega \in RHS$ of (1), which means ω happens infinitely often in $A_n \cap A_{n+1}^C$. This is equivalent to proving $\forall n \in \mathbb{N}, \exists k > n : \omega \in A_k \cap A_{k+1}^C$. Let $n \in \mathbb{N}$ be arbitrary. Let $m \geq n$ be such that $\omega \in A_m$. m exists because ω happens infinitely often in A_n. Now let $k = \inf\{j > m : \omega \in A_j^C\}$.

First case: $k = m + 1$. Then $\omega \in A_k^C \cap A_m = A_k^C \cap A_{k-1}$

Second case: $k \geq m+2$. Then $\omega \notin A_{k-1}^C$ since k is the smallest such index bigger than m. $\Rightarrow \omega \in A_{k-1} \Rightarrow \omega \in A_k^C \cap A_{k-1}$.

So in both case: $\omega \in A_k^C \cap A_{k-1}$. Thus we have the $\subset$ direction. □

Lemma 3. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $A_n \in \mathcal{A}, \forall n \geq 1$. Then:*

$$\mathbb{P}\left[\liminf A_n\right] \leq \liminf \mathbb{P}\left[A_n\right] \leq \limsup \mathbb{P}\left[A_n\right] \leq \mathbb{P}\left[\limsup A_n\right] \tag{3}$$

Proof. We have:

$$\mathbb{P}\left[\limsup A_n\right] = \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{k=n}^{\infty} A_k\right] \geq \limsup_{n\to} \mathbb{P}\left[A_n\right]$$

And

$$\begin{aligned}
\left(\liminf A_n\right)^C =& \limsup A_n^C \\
\Rightarrow \mathbb{P}\left[\liminf A_n\right] =& 1 - \mathbb{P}\left[\limsup A_n^C\right] \\
\leq& 1 - \limsup \mathbb{P}\left[A_n^C\right] \\
=& \liminf \mathbb{P}\left[A_n\right]
\end{aligned}$$

And obviously, $\liminf \mathbb{P}\left[A_n\right] \leq \limsup \mathbb{P}\left[A_n\right]$. □

Problem 4. Borel Cantelli. Given:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}A_n =& 0 \\
\sum_{n=1}^{\infty} \mathbb{P}\left(A_n^c \cap A_{n+1}\right) <& \infty
\end{aligned}$$

Prove: $\mathbb{P}(\limsup A_n) = 0$.

Proof. We have:

$$\begin{aligned}
\mathbb{P}\left[\limsup A_n^C \cap A_{n+1}\right] \leq& \mathbb{P}\left[\limsup A_n^C\right] \\
=& \mathbb{P}\left[\liminf A_n\right] \\
\leq& \liminf \mathbb{P}A_n = 0
\end{aligned}$$

For arbitrary set A, B, we have $A \subset \left[(A \cap B) \cup B^C\right]$. Thus:

$$
\begin{aligned}
\mathbb{P}\left[\limsup A_n\right] \leq & \mathbb{P}\left[\limsup A_n \cap \limsup A_n^C\right] + \mathbb{P}\left[\left(\limsup A_n^C\right)^C\right] \\
= & \mathbb{P}\left[\limsup A_n^C \cap A_{n+1}\right] + \mathbb{P}\left[\liminf A_n\right]
\end{aligned}
$$

$\mathbb{P}\left[\limsup A_n^C \cap A_{n+1}\right] = 0$ according to Borel Cantelli Lemma 1.
$\mathbb{P}\left[\liminf A_n\right] \leq \liminf \mathbb{P}A_n = 0$. So $\mathbb{P}\left[\limsup A_n\right] = 0$. $\square$

Problem 5. Normal Dist

Proof. (a) We have:

$$
\begin{aligned}
\lim_{z\to\infty} \frac{\mathbb{P}[Z > z]}{\frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)} &= \lim_{z\to\infty} \frac{\int_z^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}{\frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)} \\
&\overset{L'Hopital}{=} \lim_{z\to\infty} \frac{-\exp\left(-\frac{1}{2}z^2\right)}{-\frac{1}{z^2}\exp\left(-\frac{1}{2}z^2\right) - \frac{1}{z} z \exp\left(-\frac{1}{2}z^2\right)} \\
&= \lim_{z\to\infty} \frac{1}{\frac{1}{z^2} + 1} = 1
\end{aligned}
$$

(b) For a constant $a > 0$, from (a) we have:

$$
\begin{aligned}
\mathbb{P}\left[X_n > a\sqrt{2\log n}\right] \sim & \frac{1}{\sqrt{2\pi} a \sqrt{2\log n}} \exp\left(-\frac{1}{2}a^2 2\log n\right) \\
= & \frac{1}{2a\sqrt{\pi}\sqrt{\log n} n^{a^2}}
\end{aligned}
$$

From calculus, we have $\sum \frac{1}{n^a \log n}$ converges for $a > 1$, diverges for $0 \leq a \leq 1$. For $a = 1$. From the comparision test for series, we have: $\sum_{i=1}^n \mathbb{P}\left[X_n > a\sqrt{2\log n}\right]$ also converges for $a > 1$, diverges for $0 \leq a \leq 1$. Let $\epsilon > 0$ be arbitrary, ($\epsilon < 1$). Then:

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}\left[X_n > (1+\epsilon)\sqrt{2\log n}\right] = & b < \infty \\
\overset{BC1}{\Rightarrow} \mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} > 1 + \epsilon\right] = & 0 \\
\lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}\left[X_n > (1-\epsilon)\sqrt{2\log n}\right] = & \infty \\
\overset{BC2}{\Rightarrow} \mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} > 1 - \epsilon\right] = & 1 \\
\Rightarrow \mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} \leq 1 - \epsilon\right] = & 0
\end{aligned}
$$

Thus for arbitrary $\epsilon > 0$, $\Rightarrow \mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} \leq 1 - \epsilon\right] = \mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} > 1 + \epsilon\right] = 0$
So $\mathbb{P}\left[\limsup \frac{X_n}{\sqrt{2\log n}} = 1\right] = 1$ $\square$