STATISTICS 205A FALL 2013 TAKE-HOME FINAL

Deadline is Monday December 9 at 2.00 p.m. You can turn in a paper copy to my office, room 351 Evans (if I'm not there, put under the door) or you can email your solutions to aldous@stat.berkeley.edu.

Please start each answer on a new sheet of paper. More credit will be given for shorter arguments than for longer arguments. If you do not have a Stat Dept mailbox and want your exam returned to you after grading, write your campus mail address next to your name.

Rules. You may consult books but not people. You can quote results from the course or the parts of the textbooks dealing with course material. You may use results from homeworks we've assigned, but not (unless you give a proof) from other textbook exercises.

There are 6 questions, roughly ordered from easier to harder.

1. Let $X \geq 0$ have $\mathbb{E}X < \infty$, and consider x such that $0 < \mathbb{P}(X \leq x) < 1$. Prove

$$\mathbb{P}(X > x) \leq \frac{\mathbb{E}X - \mathbb{E}(X|X \leq x)}{x - \mathbb{E}(X|X \leq x)}.$$

2. Let $\psi(x) = \min(|x|, x^2)$. Suppose that $(X_i, 1 \leq i < \infty)$ are independent with $\mathbb{E}X_i = 0$ and $\sum_i \mathbb{E}\psi(X_i) < \infty$. Show that $\sum_{i=1}^{\infty} X_i$ converges a.s..

3. Let (B_t) be standard Brownian motion and, for $a > 0$, let $T = \inf\{t : |B_t| = a\}$. Show that

$$\mathbb{E}\exp(-\lambda T) = 1/\cosh(a\sqrt{2\lambda}), \quad \lambda > 0.$$

4. Let S be a finite set and consider a sequence $(X_1, \ldots, X_n)$, of finite length $3 \leq n < \infty$, of S-valued r.v.'s. Suppose the sequence is exchangeable; that is, $(X_1, \ldots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)})$ for all permutations π. So the distribution μ of X_i is the same for each $1 \leq i \leq n$. Let T be a stopping time with respect to the natural filtration, and suppose $T \leq n-1$. Prove that X_{T+1} has distribution μ.

5. In this question, consider a real-valued r.v. X with $\mathbb{E}X^4 = 1$.
(a) Show that $\mathbb{E}X^3 \leq 1$.
(b) Find an explicit constant $c < 1$ such that the assertion

$$\text{if also } \mathbb{E}X \leq 0 \text{ then } \mathbb{E}X^3 \leq c \tag{1}$$

is true.
(c) Find the smallest constant c for which (1) is true.

6. Let $(X_n, \mathcal{F}_n; n \geq 0)$ be a non-negative martingale such that $X_0 = 1$. Doob's inequality tells us that

$$\mathbb{P}(\sup_{n \geq 0} X_n \geq \lambda) \leq 1/\lambda, \quad \lambda > 1.$$

The ess. sup. of a probability distribution μ on R is defined to be $\sup\{x : \mu(x, \infty) > 0\}$. Let S_n be the ess. sup. of the conditional distribution of X_{n+1} given $\mathcal{F}_n$. For simplicity, you may suppose X_n has only finitely many possible values for each n, in which case S_n is the r.v. $\max\{x : P(X_{n+1} = x | \mathcal{F}_n) > 0\}$.

Under the additional hypothesis that $X_n \to 0$ a.s. as $n \to \infty$, show that "Doob's inequality goes the other way for S_n", that is

$$\mathbb{P}(\sup_{n \geq 0} S_n \geq \lambda) \geq 1/\lambda, \quad \lambda > 1.$$

Comment. In gambling terms, X_{n+1} was your actual fortune after seeing the result of the n'th bet, and S_n is what you would have had if the n'th bet had turned out in the best way possible for you.