STAT 205A: 2013 Final Exam

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December 2, 2014

1 Problem 1

Let $X \geq 0$ have $\mathbb{E}X < \infty$, and consider x such that $0 < \mathbb{P}(X \leq x) < 1$. Prove

$$\mathbb{P}(X > x) \leq \frac{\mathbb{E}X - \mathbb{E}(X|X \leq x)}{x - \mathbb{E}(X|X < x)}.$$

$$\begin{aligned} x\mathbb{P}(X > x) - \mathbb{E}(X|X \leq x)\mathbb{P}(X > x) &= x\mathbb{P}(X > x) - \mathbb{E}(X|X \leq x) + \mathbb{E}(X1_{(X \leq x)}) \\ &\leq \mathbb{E}(X1_{(X \leq x)}) - \mathbb{E}(X|X \leq x) + \mathbb{E}(X1_{(X \leq x)}) \\ &= \mathbb{E}(X) - \mathbb{E}(X|X \leq x). \end{aligned}$$

Divide both sides by $x - \mathbb{E}(X|X \leq x)$. ($x - \mathbb{E}(X|X \leq x)$ is strictly greater than 0 because $0 < \mathbb{P}(X \leq x) < 1$.)□

2 Problem 2

Let $\phi(x) = \min(|x|, x^2)$. Suppose that $(X_i, 1 \leq i < \infty)$ are independent with $\mathbb{E}X_i = 0$ and $\sum_i \mathbb{E}\phi(X_i) < \infty$. Show that $\sum_{i=1}^{\infty} X_i$ converges a.s..

Use the proof of Theorem 2.5.3 in Durrett. □
(Key idea: Cauchy's criterion; Kolmogorov's Maximal Inequality)

3 Problem 3

Let (B_t) be the standard Brownian motion and, for $a > 0$, let $T = \inf\{t : |B_t| = a\}$. Show that

$$\mathbb{E}\exp(-\lambda T) = 1/\cosh(a\sqrt{2\lambda}), \lambda > 0.$$

Recall that $\exp(\theta B_t - \theta^2 t/2)$ is a martingale (proven in class) for $\theta > 0$. Applying Optional Sampling Theorem with $T \wedge t$, we have

$$1 = \mathbb{E}\exp\left(\theta B_{T\wedge t} - \frac{\theta^2}{2} T \wedge t\right).$$

Because $T < \infty$ a.s. ($B_t \sim N(0,t)$), $\exp[\theta B_{T\wedge t}] \to \exp[\theta B_T]$ a.s. and $\exp(-\frac{\theta^2}{2}T \wedge t) \downarrow \exp(-\theta T)$ a.s.. Since $|\exp[\theta B_{T\wedge t}]| \leq \exp(\theta a)$, by Bounded Convergence Theorem, $\mathbb{E}\exp[\theta B_{T\wedge t}] \to \mathbb{E}\exp[\theta B_T]$. By Monotone Convergence Theorem, $\mathbb{E}\exp(-\frac{\theta^2}{2}T \wedge t) \downarrow \mathbb{E}\exp(-\frac{\theta^2}{2}T)$. Since $B_T \in \{-a, a\}$, by symmetry, $\mathbb{P}(B_T = a) = \mathbb{P}(B_T = -a) = 1/2$. We conclude that

$$\mathbb{E}(-\frac{\theta^2}{2}T) = \frac{1}{\mathbb{E}\exp(\theta B_T)} = \frac{1}{\frac{1}{2}(e^{-\theta a} + e^{\theta a})} = \frac{1}{\cosh(a\theta)}.$$

Now let $\theta = \sqrt{2\lambda}$. □

4 Problem 4

Let S be a finite set and consider a sequence $(X_1, ..., X_n)$, of finite length $3 \leq n < \infty$, of S-valued r.v.'s. Suppose the sequence is exchangeable. Let T be a stopping time with repsect to the natural filtration, and suppose $T \leq n-1$. Prove that X_{T+1} has distribution μ.

Let $B \subset S$. Define a filtration $\{\mathcal{F}_k\}$ by $\mathcal{F}_k = \sigma(X_1, ..., X_k)$ and random variables $Y_k = \mathbb{P}(X_{k+1} \in B | X_1, ..., X_k)$. One can show that Y_k is a martingale with respect to $\{\mathcal{F}_k\}$. Since T is bounded, we can apply Optional Sampling Theorem to conclude that

$$\begin{aligned} \mathbb{P}(X_{T+1} \in B) &= \mathbb{E}\mathbb{P}(X_{T+1} \in B | X_1, ..., X_T) \\ &= \mathbb{E}\mathbb{P}(X_n \in B | X_1, ..., X_T) \\ &= \mathbb{P}(X_n \in B). \end{aligned}$$

Since $B \subset S$ is arbitrary, this proves that X_{T+1} and X_n have the same distribution μ.