STAT 205A - Homework 9

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Problem 1. Conditional Expectation

Proof. Let $(\Omega_1, \mathcal{F}, \nu_x)$ be the probability space for X, $(\Omega_2, \mathcal{G}, \nu_y)$ be the probability space for Y. Let the product:

$$\begin{aligned}
\Omega = \Omega_1 \times \Omega_2 &= \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \\
\mathcal{S} &= \sigma\left(\{S_1 \times S_2 \mid S_1 \in \mathcal{F}, S_2 \in \mathcal{G}\}\right) \\
\nu &= \nu_x \times \nu_y \\
\nu(S_1 \times S_2) &= \nu_x(S_1)\nu_y(S_2), \forall S_1 \in \mathcal{F}, S_2 \in \mathcal{G}
\end{aligned}$$

Let $Z = \mathbb{E}\left[h(X,Y) \mid \mathcal{G}\right]$, then by definition of conditional expectation, Z is $\mathcal{G}-$measurable. Since h is bounded measurable, it is absolutely integrable so we can interchange integrals. By the definition of conditional expectation we have:

$$\begin{aligned}
\mathbb{E}\left[Z\mathbb{I}_G\right] &= \mathbb{E}\left[h(X,Y)\mathbb{I}_G\right], \forall G \in \mathcal{G} \\
&= \int h(X,Y)\mathbb{I}_G \nu(dx, dy) \\
&= \int \left(\int h(x,Y)\mathbb{I}_G \mu(Y, dx)\right) \nu_y(dy) \\
&= \int \mathbb{I}_G \left(\int h(x,Y)\mu(Y, dx)\right) \nu_y(dy)
\end{aligned}$$

So Z and $\int h(x,Y)\mu(Y,dx)$ are $\mathcal{G}$ measurable function, that have the same expectation on all measurable set in $\mathcal{G}$. Thus they are equal almost surely. □

Problem 2. Conditional Independence Definition

Proof. $(b) \Rightarrow (a)$ Given (b), (a) is true as we can apply (b) for the case $h_1(X) = \mathbb{I}_{A_1}, h_2(X) = \mathbb{I}_{A_2}$.

$(a) \Rightarrow (b)$ Given (a) is true.

First step, since (a) is true, (b) is true for any simple function $h_1(X_1), h_2(X_2)$ each of the form $\sum a_i \mathbb{I}_{A_i}$, by the linearlity of expectation.

Second step, from the first step, we have (b) is true for any bounded positive measurable function $h_i(X_i), i = 1, 2$ by the Monotone Convergence Theorem

Third step, from the second step, for any bounded measurable function, define h_i^+ as the positive part of h_i, and h_i^- as the negative part, then from h_i^+ and $-h_i^-$ are positive bounded measurable function, so (b) is true for both of these function as by second step. Thus (b) is also true for h.

$(c) \Rightarrow (b)$ Given (c) we have:

$$\begin{aligned}
\mathbb{E}\left[h_1(X_1)h_2(X_2) \mid \mathcal{G}\right] &= \mathbb{E}\left[\mathbb{E}\left[h_1(X_1)h_2(X_2) \mid \mathcal{G}, X_2\right] \mid \mathcal{G}\right] \\
&= \mathbb{E}\left[h_2(X_2)\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}, X_2\right]\right];\ h_2(X_2) \text{ is } \sigma(X_2) - measurable \\
&= \mathbb{E}\left[h_2(X_2)\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right] \mid \mathcal{G}\right];\ \text{because of (c)} \\
&= \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]\mathbb{E}\left[h_2(X_2) \mid \mathcal{G}\right];\ \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right] \text{ is } \mathcal{G} - measurable
\end{aligned}$$

$(b) \Rightarrow (c)$ Let Y be a $\mathcal{G}$-measurable r.v. From (b) we have:

$$\begin{aligned}
&\mathbb{E}\left[h_1(X_1)h_2(X_2)Y\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1)h_2(X_2)Y \mid \mathcal{G}\right]\right]; \text{ Tower Property} \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1)h_2(X_2) \mid \mathcal{G}\right]Y\right]; \text{ Y is } \mathcal{G}\textit{-measurable} \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]\mathbb{E}\left[h_2(X_2) \mid \mathcal{G}\right]Y\right]; \text{ from (b)} \\
=^{(*)}&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]h_2(X_2)Y\right]; \text{ why?}
\end{aligned}$$

The last statement is true for only $h_2(X_2)$ that is independent of $\mathcal{G}$. So we have to limit the scope of $h_2(X_2)$.

Let $Z = \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}, X_2\right]$, since $h_2(X_2)Y$ is $\sigma(\mathcal{G}, X_2) - \textit{measurable}$, we have:

$$\begin{aligned}
&\mathbb{E}\left[Zh_2(X_2)Y\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}, X_2\right]h_2(X_2)Y\right]; \text{ definition of } Z \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1)h_2(X_2)Y \mid \mathcal{G}, X_2\right]\right]; \; h_2(X_2)Y \text{ is } \mathcal{G} - \textit{measurable} \\
=&\mathbb{E}\left[h_1(X_1)h_2(X_2)Y\right]; \text{ Tower Property} \\
=&\mathbb{E}\left[\mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]h_2(X_2)Y\right]; \text{ because of (*)}
\end{aligned}$$

Thus $\mathbb{E}\left[\left(Z - \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]\right)h_2(X_2)Y\right] = 0$, by the $\pi - \lambda$ theorem, we have:

$$\mathbb{E}\left[\left(Z - \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]\right)X\right] = 0$$

for all X that is $\sigma(\mathcal{G}, X_2)-$measurable. Take $X = Z - \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]$ then we have $Z = \mathbb{E}\left[h_1(X_1) \mid \mathcal{G}\right]$. □

Problem 3. Conditional Independence with respect to different σ-algebra.

(a) X, Y conditional independent given Z

(b) X, Z conditional independent given $\mathcal{F}$

Proof. For h bounded and measurable, we have:

$$\begin{aligned}
\mathbb{E}\left[h(X) \mid Y, Z\right] =&\mathbb{E}\left[h(X) \mid Z\right] \text{ by (a) and 2(c)} \quad (1) \\
\mathbb{E}\left[h(Y) \mid X, Z\right] =&\mathbb{E}\left[h(Y) \mid Z\right] \text{ by (a) 2(c)} \quad (2) \\
\mathbb{E}\left[h(X) \mid Z, \mathcal{F}\right] =&\mathbb{E}\left[h(X) \mid \mathcal{F}\right] \text{ by (b) and 2(c)} \quad (3) \\
\mathbb{E}\left[h(X) \mid Z, \mathcal{F}\right] =&\mathbb{E}\left[h(X) \mid Z\right] \text{ because } \mathcal{F} \subset \sigma(Z) \quad (4) \\
\Rightarrow \mathbb{E}\left[h(X) \mid \mathcal{F}\right] =&\mathbb{E}\left[h(X) \mid Z\right] \quad (5)
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{E}\left[h(X) \mid Y, \mathcal{F}\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[h(X) \mid Y, Z\right] \mid Y, \mathcal{F}\right] \text{ Tower Property} \\
=&\mathbb{E}\left[\mathbb{E}\left[h(X) \mid Z\right] \mid Y, \mathcal{F}\right] \text{ by (1)} \\
=&\mathbb{E}\left[\mathbb{E}\left[h(X) \mid \mathcal{F}\right] \mid Y, \mathcal{F}\right] \text{ by (5)} \\
=&\mathbb{E}\left[h(X) \mid \mathcal{F}\right] \text{ Tower Property}
\end{aligned}$$

So by 2(c) we have X, Y are conditionally independent given $\mathcal{F}$. □

Problem 4. Super martingale

Proof. Given X_n, Y_n sunmartingale, we have:

$$\begin{aligned}\mathbb{E}\left[X_{n+1}+Y_{n+1} \mid \mathcal{F}_n\right] =& \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right]+\mathbb{E}\left[Y_n+1 \mid \mathcal{F}_n\right] \\ \geq & X_n+Y_n\end{aligned}$$

So $X_n + Y_n$ is a submargingale.

$$\begin{aligned}\mathbb{E}\left[\max(X_{n+1}, Y_{n+1}) \mid \mathcal{F}_n\right] \geq & \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] \geq X_n \\ \mathbb{E}\left[\max(X_{n+1}, Y_{n+1}) \mid \mathcal{F}_n\right] \geq & \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] \geq Y_n \\ \Rightarrow \mathbb{E}\left[\max(X_{n+1}, Y_{n+1}) \mid \mathcal{F}_n\right] \geq & \max(X_n+Y_n)\end{aligned}$$

So $\max(X_n, Y_n)$ is a submartingale. □

Problem 5. Counter Example

Proof. Let W_i be i.i.d with $\mathbb{P}\left[W_i = 1\right] = \mathbb{P}\left[W_i = -1\right] = 1/2$. Let $X_n = -\sum_{i=1}^{n} W_i, Y_n = \sum_{i=1}^{n+1} W_i$. Then X_n is a martingale w.r.t. $\sigma(W_1, ..., W_n)$, thus it is a submartingale. Y_n is a martingale w.r.t. $\sigma(W_1, ..., W_{n+1})$, thus it is a submartingale.

We will prove by contradiction, assuming that $X_n + Y_n = W_{n+1}$ is a submartingale. Since $\sup_{n>0} \mathbb{E}\left[W_n^+\right] = 1 < \infty$, by the Doob's first martingale convergence theorem, we have W_n converges pointwise to a random variable. But W_n are i.i.d. non constant. Thus we have contradiction. So W_{n+1} is not a submartingale. □