ST205A - Homework 5

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Problem 1. (X_n) i.i.d. $\mathbb{E}|X_i| < \infty$. $M_n = \max(X_1, ..., X_n)$. Prove that: $n^{-1}M_n \to 0$ a.s.

Proof. Let $\mu = \mathbb{E}|X_i| < \infty$. $P_n = \max(|X_1|, ..., |X_n|)$. Then $0 \le M_n \le P_n$. By SLLN, $\lim \frac{P_n}{n} = \mu$ a.s. Thus:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|X_n|}{n} &= \lim_{n\to\infty} \frac{P_n}{n} - \frac{P_{n-1}}{n} \\
&= \lim_{n\to\infty} \frac{P_n}{n} - \frac{n-1}{n}\frac{P_{n-1}}{n-1} \\
&= \mu - \mu = 0, \text{ a.s.}
\end{aligned}$$

By the deterministic lemma: If $x_n \ge 0$ and $0 < b_n \uparrow \infty$, then $\limsup \frac{\max(x_1,...,x_n)}{b_n} = \limsup \frac{x_n}{b_n}$, we have:

$$\begin{aligned}
\limsup n^{-1}P_n &= \limsup \frac{|X_n|}{n} = 0 \text{ a.s.} \\
\Rightarrow \limsup n^{-1}M_n &= 0 \text{ a.s.}
\end{aligned}$$

$\square$

Problem 2. Durrett 2.3.2

Proof. We have:

Let $\epsilon > 0$ be arbitrary. Since $\mathbb{E}X_n \sim an^\alpha \Leftrightarrow \lim \frac{\mathbb{E}X_n}{an^\alpha} = 1$, $\exists N \in \mathbb{N}, \forall n \ge N, \left|\frac{\mathbb{E}X_n}{an^\alpha} - 1\right| < \frac{\epsilon}{2}$. By the Chebysev inequality, and for $n \ge N$, we have:

$$\begin{aligned}
\mathbb{P}\left[|X_n - \mathbb{E}X_n| \ge \frac{\epsilon}{2}an^\alpha\right] &\le \frac{4\text{Var}X_n}{\epsilon^2a^2n^{2\alpha}} \text{ (Chebysev)} \\
\Leftrightarrow \mathbb{P}\left[\left|\frac{X_n}{an^\alpha} - \frac{\mathbb{E}X_n}{an^\alpha}\right| \ge \frac{\epsilon}{2}\right] &\le \frac{4\text{Var}X_n}{\epsilon^2a^2n^{2\alpha}} \\
\Leftrightarrow \mathbb{P}\left[\left|\frac{X_n}{an^\alpha} - 1 + 1 - \frac{\mathbb{E}X_n}{an^\alpha}\right| \ge \frac{\epsilon}{2}\right] &\le \frac{4\text{Var}X_n}{\epsilon^2a^2n^{2\alpha}} \text{ (1)}
\end{aligned}$$

We have for any real number x, having $|x| \ge \epsilon \Rightarrow |x + a| \ge \epsilon/2$ for any $a \in \mathbb{R}$ such that $a < \epsilon/2$. Thus having

$$\begin{aligned}
\left|\frac{X_n}{an^\alpha} - 1\right| \ge \epsilon &\Rightarrow \left|\frac{X_n}{an^\alpha} - 1 + 1 - \frac{\mathbb{E}X_n}{an^\alpha}\right| \ge \frac{\epsilon}{2} \\
\Rightarrow \mathbb{P}\left[\left|\frac{X_n}{an^\alpha} - 1\right| \ge \epsilon\right] &\le \mathbb{P}\left[\left|\frac{X_n}{an^\alpha} - 1 + 1 - \frac{\mathbb{E}X_n}{an^\alpha}\right| \ge \frac{\epsilon}{2}\right] \text{ (Prob of smaller set is smaller)} \\
&\le \frac{4\text{Var}X_n}{\epsilon^2a^2n^{2\alpha}} \text{ (From (1))} \\
&\le \frac{4B}{\epsilon^2a^2}\frac{1}{n^{2\alpha-\beta}} \text{ (2)}
\end{aligned}$$

Since $2\alpha > \beta$, $\frac{1}{n^{2\alpha-\beta}} \to 0$ as $n \to \infty$. So we have convergence in probability.

Now we follow the method of the proof of Theorem 2.3.8 in Durrett to have convergence almost surely. Due to the convergence property of series $\sum_{n=1}^{\infty} \frac{1}{n^c}$, which converges iff $c > 1$. We consider two cases:

Case 1: $2\alpha - \beta > 1$. Then $\sum_{k=0}^{\infty} \mathbb{P}\left[\left|\frac{X_{n_k}}{an^{\alpha}} - 1\right| \geq \epsilon\right] < \infty$, and the First Borel-Cantell lemma implies that $\mathbb{P}\left[\left|\frac{X_n}{an^{\alpha}} - 1\right| \geq \epsilon \text{ i.o.}\right] = 0$. Since ϵ was arbitrary, it follows that $(X_{n_k}/an_k^{\alpha}) \to 1$ a.s..

Case 2: $2\alpha - \beta \leq 1$.

Let $n_k = \inf\{n : n^{2\alpha-\beta} \geq k^2\}$. Then we have: $k^2 \leq n_k^{2\alpha-\beta} \leq k^2 + 1$, (*) where the later inequality is true because $0 < 2\alpha - \beta < 1$. Since $2\alpha - \beta > 0$, it is obvious that n_k is an increasing sequence going to ∞.

By construction, we have:

$$\begin{aligned}\mathbb{P}\left[\left|\frac{X_{n_k}}{an_k^{\alpha}} - 1\right| \geq \epsilon\right] \leq & \frac{4B}{\epsilon^2 a^2}\frac{1}{n_k^{2\alpha-\beta}} \text{ (from (2))}\\ \leq & \frac{4B}{\epsilon^2 a^2}\frac{1}{k^2}\end{aligned}$$

So $\sum_{k=0}^{\infty} \mathbb{P}\left[\left|\frac{X_{n_k}}{an^{\alpha}} - 1\right| \geq \epsilon\right] < \infty$, and the First Borel-Cantelli lemma implies $\mathbb{P}\left[\left|\frac{X_{n_k}}{an_k^{\alpha}} - 1\right| \geq \epsilon \text{ i.o.}\right] = 0$. Since ϵ was arbitrary, it follows that $(X_{n_k}/an_k^{\alpha}) \to 1$ a.s. Now to show X_n/an^{α} a.s., pick an $\omega \in \Omega$ so that $X_{n_k}(\omega)/an_k^{\alpha} \to 1$, and observe that $\forall n \in \mathbb{N}$ sufficiently big, $\exists k : n_k \leq n < n_{k+1}$ since $n_k \uparrow \infty$. Using the fact that $0 \leq X_1 \leq X_2 \leq ...$, we have:

$$\begin{aligned}\frac{X_{n_k}(\omega)}{an_{k+1}^{\alpha}} \leq & \frac{X_n(\omega)}{an^{\alpha}} \leq \frac{X_{n_{k+1}}(\omega)}{an_k^{\alpha}}\\ \Rightarrow \frac{n_k^{\alpha}}{n_{k+1}^{\alpha}}\frac{X_{n_k}(\omega)}{an_k^{\alpha}} \leq & \frac{X_n(\omega)}{an^{\alpha}} \leq \frac{n_{k+1}^{\alpha}}{n_k^{\alpha}}\frac{X_{n_{k+1}}(\omega)}{an_{k+1}^{\alpha}}\end{aligned}$$

Now from (*):

$$k^2 \leq n_k^{2\alpha-\beta} \leq n_{k+1}^{2\alpha-\beta} \leq (k+1)^2 + 1$$

Abd since $\lim_{k\to\infty} \frac{(k+1)^2+1}{k^2} = 1$, we have $\lim_{k\to\infty} \frac{n_k^{\alpha}}{n_{k+1}^{\alpha}} = \lim_{k\to\infty} \frac{n_{k+1}^{\alpha}}{n_k^{\alpha}} = 1$.

Thus by the Sandwich Limit Theorem, $\lim_{n\to\infty} \frac{X_n(\omega)}{an^{\alpha}} = 1$. So we have $\frac{X_n}{an^{\alpha}} \to 1$ a.s. □

Problem 3. Prove that the following are equivalent

(i) $X_n \to X$ in probability ($\Leftrightarrow \forall \epsilon > 0, \lim_{n\to\infty} \mathbb{P}[|X_n - X| > \epsilon] = 0$)

(ii) $\exists \epsilon_n \downarrow 0$ such that $\mathbb{P}[|X_n - X| > \epsilon_n] \leq \epsilon_n$.

(iii) $\lim_{n\to\infty} \mathbb{E}\min(|X_n - X|, 1) = 0$.

Proof. **"(ii) ⇒(i)"** Direction: Let $\epsilon > 0$ be arbitrary. Since $\epsilon_n \downarrow 0, \exists N, \forall n \geq N, \epsilon_n < \epsilon$. Now we have:

$$\begin{aligned}\forall n \geq N, 0 \leq \mathbb{P}[|X_n - X| > \epsilon] \leq & \mathbb{P}[|X_n - X| > \epsilon_n] \text{ (Since } \epsilon > \epsilon_n)\\ \leq & \epsilon_n \text{ (This is given)}\end{aligned}$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, by the Sandwich Theorem, $X_n \to X$ in probability.

"(i) ⇒(ii)" Direction:

Let $a_1 = 1$, then we have $\forall n, \mathbb{P}[|X_n - X| > a_1] \leq a_1$. Let $N_1 = 1$.

Let $a_2 = \frac{1}{2}$. Since $\lim_{n\to\infty} \mathbb{P}[|X_n - X| > 1/2] = 0$, $\exists N_2 > N_1, \forall n \geq N_2, \mathbb{P}[|X_n - X| > 1/2] < \frac{1}{2}$.

Let $a_3 = \frac{1}{3}$. $\exists N_3 > N_2, \forall n \geq N_3, \mathbb{P}[|X_n - X| > 1/3] < \frac{1}{3}$.

and so on for $a_k = \frac{1}{k}, \exists N_k > N_{k-1}, \forall n \geq N_{k-1}, \mathbb{P}[|X_n - X| > 1/k] < \frac{1}{k}$.

Thus if we construct $\epsilon_i = 1, N_1 \leq i < N_2, \epsilon_i = \frac{1}{2}, N_2 \leq i < N_3, ..., \epsilon_i = \frac{1}{k}, N_k \leq i < N_{k+1}$. Then $\epsilon_n \downarrow 0$ and $\mathbb{P}[|X_n - X| > \epsilon_n] \leq \epsilon_n$

"(iii) $\Rightarrow$ (i)" Direction: Let $\epsilon > 0$ be arbitrary, applying the general Markov inequality for $\phi(x) = \min(|x|, 1)$, which is non-decreasing, we have:

$$\mathbb{P}[|X_n - X| > \epsilon] \leq \frac{\mathbb{E}\phi(|X_n - X|)}{\phi(\epsilon)} = \frac{\mathbb{E}\min(|X_n - X|, 1)}{\min(1, \epsilon)}$$

Thus $\lim \mathbb{P}_{n\to\infty}[|X_n - X| > \epsilon] = 0$.

"(i) $\Rightarrow$ (iii)" Direction: For simplicity of notation let $Y_n = |X_n - X|$.

Let $\epsilon \in (0, 1)$. We have $\forall \epsilon > 0, \lim \mathbb{P}[Y_n > \epsilon] = 0 \Rightarrow \lim \mathbb{P}[Y_n \in [\epsilon_1, \epsilon_2)] = 0, \forall 0 < \epsilon_1 < \epsilon_2$ (*). Fix $k \in \mathbb{N}$, we have:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\min(Y_n, 1) &= \lim_{n\to\infty} \{\mathbb{P}[Y_n \leq 1]\mathbb{E}[Y_n \mid Y_n < 1] + \mathbb{P}[Y_n > 1]\} \text{ (Tower Property)} \\
&= \lim_{n\to\infty} \mathbb{P}[Y_n \leq 1]\mathbb{E}[Y_n \mid Y_n < 1] \\
&= \lim_{n\to\infty} \sum_{i=1}^{k} \mathbb{P}\left[Y_n \in [\frac{i-1}{k}, \frac{i}{k})\right]\mathbb{E}\left[Y_n \mid Y_n \in [\frac{i-1}{k}, \frac{i}{k})\right] \\
&:= \lim_{n\to\infty} \mathbb{P}\left[Y_n \in [0, \frac{1}{k})\right]\mathbb{E}\left[Y_n \mid Y_n \in [0, \frac{1}{k})\right]
\end{aligned}$$

Since all other term for $i \geq 2$ disappears because of (*). Now taking the limit of k and applying the Monotone Convergence Theorem we can interchenge the two limit operations, we have:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\min(Y_n, 1) &= \lim_{k\to\infty}\lim_{n\to\infty} \mathbb{P}\left[Y_n \in [0, \frac{1}{k})\right]\mathbb{E}\left[Y_n \mid Y_n \in [0, \frac{1}{k})\right] \\
&= \lim_{n\to\infty}\lim_{k\to\infty} \mathbb{P}\left[Y_n \in [0, \frac{1}{k})\right]\mathbb{E}\left[Y_n \mid Y_n \in [0, \frac{1}{k})\right] \text{ (Monotone Convergence)} \\
&\leq \lim_{n\to\infty}\lim_{k\to\infty} 1 \times \frac{1}{k} = 0
\end{aligned}$$

Since $0 \leq \lim_{n\to\infty} \mathbb{E}\min(Y_n, 1)$, by the Sandwich Theorem, we have $\lim_{n\to\infty} \mathbb{E}\min(Y_n, 1) = 0$. $\square$

Problem 4. Investment Problem

Proof. (i) We have:

$$\begin{aligned}
W_{n+1} &= (ap + (1-p)V_n)W_n \\
\Rightarrow \log W_{n+1} &= \log(ap + (1-p)V_n) + \log W_n \\
&= \log(ap + (1-p)V_n) + \log(ap + (1-p)V_{n-1}) + \log W_{n-1} \\
&= \ldots \\
&= \sum_{i=0}^{n} \log(ap + (1-p)V_i) \\
\Rightarrow n^{-1}\log W_n &= \frac{1}{n}\sum_{i=0}^{n-1} \log(ap + (1-p)V_i)
\end{aligned}$$

We want to apply SLLN, thus we need $\mathbb{E}\log(ap + (1-p)V_i) < \infty$. Consider $a + bV_i, a \geq 0, b > 0$. We have $\mathbb{E}\left[V_n^{-2}\right] < \infty \Rightarrow \mathbb{E}\left[\frac{1}{(V+a/b)^2}\right] \leq \mathbb{E}\left[\frac{1}{V^2}\right] < \infty \Rightarrow \mathbb{E}\left[\frac{1}{(bV+a)^2}\right] < \infty$. We also have $\mathbb{E}\left[(bV+a)^2\right] < \infty$. Also, $\varphi(x) = \frac{1}{4}x^2 + \log(x) + \frac{1}{12x^2}, x > 0$.

$$
\begin{aligned}
\varphi'(x) =& x/2 + \frac{1}{x} - \frac{1}{6x^3} \\
\Rightarrow \varphi''(x) =& 1/2 - \frac{1}{x^2} + \frac{1}{2x^4} \geq 0, \forall x > 0
\end{aligned}
$$

So $\varphi(x)$ is convex, and thus $\varphi(ap + (1-p)x)$ is also convex since $1-p > 0$. Applying Jensen theorem we have, and let $\mathbb{E}V = \mu < \infty$ (since $\mathbb{E}V^2 < \infty$):

$$
\begin{aligned}
\mathbb{E}\varphi(ap + (1-p)V_i) \leq& \varphi(\mathbb{E}(ap + (1-p)V_i)) \text{ (Jensen)} \\
\Leftrightarrow \mathbb{E}\varphi(ap + (1-p)V_i) \leq& \varphi(ap + (1-p)\mu) < \infty \\
\Leftrightarrow \frac{1}{4}\mathbb{E}(ap + (1-p)V_i)^2 +& \mathbb{E}\log(ap + (1-p)V_i) + \frac{1}{12}\mathbb{E}(ap + (1-p)V_i)^{-2} < \infty \\
\Rightarrow \mathbb{E}\log(ap + (1-p)V_i) <& \infty
\end{aligned}
$$

Thus we can apply SLLN, and have $n^{-1}\log W_n \to \mathbb{E}\log(ap + (1-p)V_i) := c(p)$ a.s.

(ii) We make the assumption that we can interchange the differential and expectation (as in Durrett Theorem A.5.1) sign, then we will have:

$$
\begin{aligned}
\frac{\partial^2}{\partial p^2}c(p) =& \frac{\partial^2}{\partial p^2}\mathbb{E}\log(pa + (1-p)V_n) \\
=& \mathbb{E}\frac{\partial^2}{\partial p^2}\log(pa + (1-p)V_n) \text{ (Interchange differential and expectation)} \\
=& \mathbb{E}\frac{\partial}{\partial p}\frac{a - V_n}{(pa + (1-p)V_n)} \\
=& \mathbb{E} - \frac{(a - V_n)^2}{(pa + (1-p)V_n)^2} \leq 0
\end{aligned}
$$

Thus $c(p)$ is concave.

(iii) For a concave function f on a open interval (a, b), it attains a maximum on the open interval (a, b) iff $f'(a) > 0$ and $f'(b) < 0$. Again, assuming the four condition for interchanging expectation and derivative hold, we have :

$$
\begin{aligned}
c'(p) =& \mathbb{E}\frac{a - V_n}{pa + (1-p)V_n} \text{ (Interchange } \partial \text{ and } \mathbb{E}) \\
\Rightarrow c'(0) =& \mathbb{E}\frac{a - V_n}{V_n} = \mathbb{E}\frac{a}{V_n} - 1 \\
c'(0) >& 0 \\
\Leftrightarrow \mathbb{E}\frac{a}{V_n} >& 1 \\
\Leftrightarrow \mathbb{E}\frac{1}{V_n} >& \frac{1}{a} \\
c'(1) =& \mathbb{E}\frac{a - V_n}{a} = 1 - \mathbb{E}\frac{V_n}{a} \\
c'(1) <& 0 \\
\Leftrightarrow 1 <& \mathbb{E}\frac{V_n}{a} \\
\Leftrightarrow \mathbb{E}V_n >& a
\end{aligned}
$$

So the condition is $\mathbb{E}V_n > a$, meaning stock on average has higher return than bond (otherwise one will put all money on bond), and $\mathbb{E}\frac{1}{V_n} > \frac{1}{a}$.

(iv) First we need the condition, $\mathbb{E}V_n = 2.5 > a, \mathbb{E}\frac{1}{V_n} = \frac{1}{2} + \frac{1}{8} = \frac{1}{1.6}$. So $1.6 < a < 2.5$. When $1.6 < a < 2.5$, then the optimal p is attained when $c'(p) = 0$:

$$
\begin{aligned}
\mathbb{E}\frac{a - V_n}{pa + (1-p)V_n} &= 0 \\
\Leftrightarrow \frac{1}{2}\frac{a-1}{pa + (1-p)} + \frac{1}{2}\frac{a-4}{pa + 4(1-p)} &= 0 \\
\Leftrightarrow \frac{a-1}{pa + 1 - p} &= \frac{4-a}{pa + 4 - 4p} \\
\Leftrightarrow (a-1)(pa + 4 - 4p) &= (4-a)(pa + 1 - p) \\
\Leftrightarrow pa^2 + 4a - 4pa - pa - 4 + 4p &= 4pa + 4 - 4p - a^2p - a + ap \\
\Leftrightarrow pa^2 - 5pa + 4a + 4p - 4 &= -pa^2 + 5ap - 4p - a + 4 \\
\Leftrightarrow 2pa^2 - 10ap + 5a + 8p - 8 &= 0 \\
\Leftrightarrow p(2a^2 - 10a + 8) &= 8 - 5a \\
\Leftrightarrow p &= \frac{8 - 5a}{2(a-1)(a-4)}
\end{aligned}
$$

We can see that p is between 0 and 1 iff $1.6 < a < 2.5$. And $p = 0$ if $a = 1.6$ or $p = 2.5$. □

Problem 5. Glivenko-Cantelli Theorem

Proof. Fix $k \in \mathbb{N}$. For $1 \le j \le k-1$, let $x_{j,k} = \inf\{y : F(y) \ge j/k\}$. And let $x_{0,k} = -\infty, x_{k,k} = \infty$. The pointwise convergence of $F_n(x)$ and $F_n(x-)$ imply that we can pick N_k so that if $n \ge N_k$, then for $0 \le j \le k$:

$$|F_n(x_{j,k}) - F(x_{j,k})| < k^{-1} \text{ and } |F_n(x_{j,k}-) - F(x_{j,k}-)| < k^{-1}$$

If $x \in (x_{j-1,k}, x_{j,k})$ with $1 \le j \le k$ and $n \ge N_k$ then using the monotonicity of F_n and F, and $F(x_{j,k}-) - F(x_{j-1,k}) \le k^{-1}$, we have:

$$
\begin{aligned}
F_n(x) \le& F_n(x_{j,k}-) \le F(x_{j,k}-) + k^{-1} \le F(x_{j-1,k}) + 2k^{-1} \le F(x) + 2k^{-1} \\
F_n(x) \ge& F_n(x_{j-1,k}) \ge F(x_{j-1,k}) - k^{-1} \ge F(x_{j,k}-) - 2k^{-1} \ge F(x) - 2k^{-1}
\end{aligned}
$$

So $\sup_x |F_n(x) - F(x)| \le 2k^{-1}$, thus $\lim_{n\to\infty} \sup_x |F_n(x) - F(x)| = 0$. □