ST210A - Homework 8

Hoang Duong

November 6, 2014

Problem 1. Binomial - Empirical Bayes

Proof. (a) We have the conditional density of X given Θ is:

$$p_{X|\Theta}(x) = \prod_{i=1}^{p} \binom{m}{x_i} \theta_i^{x_i} (1-\theta_i)^{m-x_i} \tag{1}$$

The marginal density of Θ is:

$$p_{\Theta}(\theta) = \prod_{i=1}^{p} \frac{1}{B(\alpha,\beta)} \theta_i^{\alpha-1} (1-\theta_i)^{\beta-1} \tag{2}$$

Multiplying these together, we have the joint density of X and Θ is:

$$p_{X,\Theta} = \prod_{i=1}^{p} \frac{\binom{m}{x_i}}{B(\alpha,\beta)} \theta_i^{x_i+\alpha-1} (1-\theta_i)^{\beta+m-x_i-1} \tag{3}$$

Integrating this against θ, the marginal density of X is:

$$p_X = \int \cdots \int \prod_{i=1}^{p} \frac{\binom{m}{x_i}}{B(\alpha,\beta)} \theta_i^{x_i+\alpha-1} (1-\theta_i)^{\beta+m-x_i-1} d\theta \tag{4}$$

$$= \prod_{i=1}^{p} \binom{m}{x_i} \frac{B(\alpha+x_i, \beta+m-x_i)}{B(\alpha,\beta)} \tag{5}$$

Thus we have the conditional density of Θ given X is:

$$\begin{aligned} p_{\Theta|X} =& \frac{p_{\Theta,X}}{p_X} \\ =& \prod_{i=1}^{p} \frac{1}{B(\alpha+x_i, \beta+m-x_i)} \theta_i^{x_i+\alpha-1} (1-\theta_i)^{\beta+m-x_i-1} \\ \Theta_i \mid X \sim& Beta(\alpha+x_i, \beta+m-x_i) \end{aligned}$$

Since the loss function is quadratic, the Bayes estimator is thus the posterior mean:

$$\hat{\Theta}_i = \frac{\alpha+x_i}{\alpha+\beta+m}$$

(b) For each X_i we have:

$$
\begin{aligned}
\mathbb{E}\Theta_i =& \frac{\alpha}{\alpha+\beta} \\
\mathbb{E}\Theta_i^2 =& \mathrm{Var}\Theta_i + (\mathbb{E}\Theta_i)^2 \\
=& \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2} \\
=& \frac{\alpha\beta+\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
=& \frac{(\alpha+\beta)(\alpha^2+\alpha)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
=& \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\
\mathbb{E}\left[X_i\right] =& \mathbb{E}\left[\mathbb{E}\left[X_i \mid \Theta_i\right]\right] \\
=& \mathbb{E}\left[\Theta_i m\right] = \frac{m\alpha}{\alpha+\beta} \\
\Rightarrow \frac{\beta}{\alpha} =& \frac{m}{\mathbb{E}X_i} - 1 = \frac{m-\mathbb{E}X_i}{\mathbb{E}X_i} \\
\mathbb{E}\left[X_i^2\right] =& \mathbb{E}\left[\mathbb{E}\left[X_i^2 \mid \Theta_i\right]\right] \\
=& \mathbb{E}\left[m\Theta_i - m\Theta_i^2 + m^2\Theta_i^2\right] \\
=& \frac{m\alpha}{\alpha+\beta} + \frac{m(m-1)\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\
=& \mathbb{E}X_i + \frac{\alpha+1}{\alpha+\beta+1}(m-1)\mathbb{E}X_i \\
\Rightarrow \frac{\beta}{\alpha+1} =& \frac{(m-1)\mathbb{E}X_i}{\mathbb{E}X_i^2 - \mathbb{E}X_i} - 1 \\
\Rightarrow \frac{\alpha+1}{\alpha} =& \frac{m-\mathbb{E}X_i}{\mathbb{E}X_i} \frac{\mathbb{E}X_i^2 - \mathbb{E}X_i}{m\mathbb{E}X_i - \mathbb{E}X_i^2} \\
\Rightarrow \frac{1}{\alpha} =& \frac{md - mc - cd + c^2 - mc^2 + cd}{c(mc-d)} \\
=& \frac{md - mc + c^2 - mc^2}{c(mc-d)} \\
\Rightarrow \alpha =& \frac{c(mc-d)}{md - mc + c^2 - mc^2} \\
\Rightarrow \beta =& \alpha\frac{m-c}{c} = \frac{c(mc-d)(m-c)}{c(md-mc+c^2-mc^2)} \\
=& \frac{(mc-d)(m-c)}{md-mc+c^2-mc^2}
\end{aligned}
$$

For $c = \mathbb{E}X_i, d = \mathbb{E}X_i^2$. So using method of moment, we can calculate the sample mean and sample $\frac{1}{p-1}\sum X_i^2$ to plug in the formula of α and β above, and we will get estimators for α and β.

(c) Combining (a) and (b) we get the empirical Bayes estimator for θ_i is:

$$
\begin{aligned}
&\frac{\hat{\alpha} + x_i}{\hat{\alpha} + \hat{\beta} + m} \\
\hat{\alpha} =& \frac{c(mc - d)}{md - mc + c^2 - mc^2} \\
\hat{\beta} =& \frac{(mc - d)(m - c)}{md - mc + c^2 - mc^2} \\
c =& \frac{1}{p} \sum_{i=1}^{p} X_i \\
d =& \frac{1}{p-1} \sum_{i=1}^{p} X_i^2
\end{aligned}
$$

□

Problem 2. UMP for Uniform Distribution $[0, \theta]$

Proof. (a) We use the order statistics notation of $x_{(1)}, ..., x_{(n)}$. The density p_θ for a uniform distribution on $[0, \theta]$ is:

$$
p_\theta(x) = \theta^{-n} \mathbb{I}_{\left\{x_{(n)} \in [0,\theta]\right\}}, x \in \mathbb{R}^n
$$

For δ as defined in the problem, we have for any $\theta_1 > \theta_0$,

$$
\begin{aligned}
p_{\theta_1}(x) >& \theta_0^n \theta_1^{-n} p_{\theta_0}(x) && (6) \\
\Leftrightarrow \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_1]\right\}} >& \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_0]\right\}} && (7) \\
\Leftrightarrow x_{(n)} \in& (\theta_0, \theta_1) && (8) \\
\Rightarrow \delta(x) =& 1 && (9) \\
p_{\theta_1}(x) <& \theta_0^n \theta_1^{-n} p_{\theta_0}(x) && (10) \\
\Leftrightarrow \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_1]\right\}} >& \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_0]\right\}} && (11)
\end{aligned}
$$

The last statement is never true when $\theta_1 > \theta_0$, so the set of x that sastifies (10) is empty. So with $k = \theta_0^n \theta_1^{-n}$, we have:

$$
\delta(x) = \begin{cases} 1 & \text{when } p_{\theta_1}(x) > k p_{\theta_0}(x) \\ 0 & \text{when } p_{\theta_1}(x) < k p_{\theta_0}(x) \end{cases}
$$

(The statement $\delta(x) = a, \forall x \in \emptyset$ is True for any a)

So by Simple vs. Simple Testing Theorem, we have $\delta(x)$ is MP at level α, for $k = \theta_0^n \theta_1^{-n}$.

Since δ was defined independently of θ_1, and we have $\mathbb{E}_{\theta_0} \delta(X) = \alpha$, and $\mathbb{E}_\theta \delta(X) \leq \alpha, \forall \theta \leq \theta_0$, the test δ is also UMP for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

(b) Let δ be as defined in the problem. We have:

$$
\begin{aligned}
\mathbb{E}_{\theta_0} \delta(X) =& \mathbb{P}_{\theta_0} \left[X_{(n)} \leq \theta \alpha^{1/n} \right] \\
=& \prod_{i=1}^{n} \mathbb{P}_{\theta_0} \left[X_i \leq \theta \alpha^{1/n} \right] \\
=& \left(\alpha^{1/n} \right)^n = \alpha
\end{aligned}
$$

For the case $\theta_1 > \theta_0$, choose $k = \theta_0^n \theta_1^{-n}$, we use the same reasoning as in (a), we have δ is MP for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

For the case $\theta_0 \alpha^{1/n} < \theta_1 < \theta_0$, we have:

$$\begin{aligned}
p_{\theta_1}(x) >& \theta_0^n \theta_1^{-n} p_{\theta_0}(x) \\
\Leftrightarrow \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_1]\right\}} >& \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_0]\right\}} \text{ which is always False} \\
p_{\theta_1}(x) <& \theta_0^n \theta_1^{-n} p_{\theta_0}(x) \\
\Leftrightarrow \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_1]\right\}} <& \mathbb{I}_{\left\{x_{(n)} \in [0,\theta_0]\right\}} \\
\Leftrightarrow x_{(n)} \in& [\theta_1, \theta] \\
\Rightarrow \delta(x) =& 0
\end{aligned}$$

So by the Simple vs. Simple Testing Theorem, we have $\delta(X)$ is MP at level α, for $k = \theta_0^n \theta_1^{-n}$.

Finally, for the case $\theta_1 \leq \theta_0 \alpha^{1/n}$, with $k = 0$,we have:

$$\begin{aligned}
p_{\theta_1}(x) >& k p_{\theta_0}(x) \\
\Leftrightarrow p_{\theta_1}(x) >& 0 \\
\Leftrightarrow x_{(n)} \in& [0, \theta_1] \\
\Rightarrow \delta(x) =& 1 \\
p_{\theta_1}(x) <& k p_{\theta_0}(x) \\
\Leftrightarrow p_{\theta_1}(x) <& 0
\end{aligned}$$

The last statement is never true. So also by the Simple vs. Simple Testing Theorem, we have $\delta(X)$ is MP at level α, for $k = 0$.

From the three cases, since $\delta(X)$ was defined independently of θ_1, we have $\delta(X)$ is UMP.

Now we need to show that it is unique. Let λ be the Lebesque measure in $\mathbb{R}^n$, we have the uniform distribution in $\mathbb{R}^n$ is absolutely continuous w.r.t λ. Let δ^* be any UMP test for the population. Define: $D = \{x \mid \delta(x) \neq \delta^*(x)\}, D_1 = D$

$$\begin{aligned}
D =& \{x \mid \delta(x) \neq \delta^*(x)\} \\
D_1 =& D \cap \left\{x \mid x_{(n)} \leq \theta_0 \alpha^{1/n}\right\} \\
D_2 =& D \cap \left\{x \mid \theta_0 \alpha^{1/n} < x_{(n)} \leq \theta_0\right\} \\
D_3 =& D \cap \left\{x \mid \theta_0 < x_{(n)} < K\right\}
\end{aligned}$$

for some $K > \theta_0$. Because δ and δ^* are both UMP, we have:

$$\mathbb{E}_\theta \delta(X) = \mathbb{E}_\theta \delta^*(X), \forall \theta > 0 \tag{12}$$

$$\Rightarrow \int_{x_{(n)} < \theta} [\delta(x) - \delta^*(x)]\, d\lambda(x) = 0, \forall \theta > 0 \tag{13}$$

$$\int_{D_1} [1 - \delta^*(x)]\, d\lambda(x) = \int_{x_{(n)} < \theta_0 \alpha^{1/n}} [\delta(x) - \delta^*(x)]\, d\lambda(x) \tag{14}$$

For RHS of (14), apply what we have in (13) for $\theta = \theta_0 \alpha^{1/n}$, we have RHS(14) = 0. So $\lambda(D_1) = 0$ because otherwise, $\int_{D_1} [1 - \delta^*(x)]\, d\lambda(x) > 0$ as $1 > \delta^*(x)$ over the region D_1 of integration.

For D_2, we also have:

$$\int_{D_2} [-\delta^*(x)]\, d\lambda(x) = \int_{x_{(n)}<\theta_0} [\delta(x) - \delta^*(x)]\, d\lambda(x) - \int_{x_{(n)}<\theta_0\alpha^{1/n}} [\delta(x) - \delta^*(x)]\, d\lambda(x)$$
$$=0$$

So we also have $\lambda(D_2) = 0$.
Finally for D_3, we have:

$$\int_{D_3} [1 - \delta^*(x)]\, d\lambda(x) = \int_{x_{(n)}<K} [\delta(x) - \delta^*(x)]\, d\lambda(x) - \int_{x_{(n)}<\theta_0} [\delta(x) - \delta^*(x)]\, d\lambda(x)$$
$$=0$$

Thus $\lambda(D_3) = 0$.
Hence, for all $K > \theta_0$, we have $\lambda(D \cap \{x \mid x_{(n)} < K\}) = 0$, this implies $\lambda(D) = 0$.
So $\delta(X)$ is unique (any other UMP disagree with $\delta(X)$ in a set of Lebesque measure 0) $\square$

Problem 3. Exponential Distribution

Proof. (a) We have:

$$\begin{aligned}
p_{X_i|\beta} =& \frac{1}{\beta t_i} \exp\left\{-\frac{x_i}{\beta t_i}\right\} \\
p_{X|\beta} =& \prod \frac{1}{\beta t_i} \exp\left\{-\frac{x_i}{\beta t_i}\right\} \\
\log p_{X|\beta} =& -n\log\beta - \sum_{i=1}^{n} \log t_i - \frac{1}{\beta}\sum_{i=1}^{n}\frac{x_i}{t_i} \\
\frac{\partial \log p_{X|\beta}}{\partial\beta} =& -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}\frac{x_i}{t_i} = 0 \\
\Leftrightarrow \hat{\beta}_{MLE} =& \frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{t_i}
\end{aligned}$$

(b) We have $Y_i = X_i/t_i$ are i.i.d with exponential distribution mean parameter β.

$$\begin{aligned}
\frac{\partial^2 \log p_{Y_i|\beta}}{\partial\beta^2} =& \frac{1}{\beta^2} - \frac{2}{\beta^3}Y_i \\
I(\beta) =& \mathbb{E}\left[-\frac{\partial^2 \log p_{Y_i|\beta}}{\partial\beta^2}\right] \\
=& -\left(\frac{1}{\beta^2} - \frac{2}{\beta^3}\beta\right) \\
=& \frac{1}{\beta^2}
\end{aligned}$$

By the asymptotic theory of MLE, we have:

$$\sqrt{n}(\beta - \hat{\beta}_n) \to^d \mathcal{N}\left(0, \frac{1}{I(\beta)}\right) = \mathcal{N}\left(0, \beta^2\right)$$

(c) We have:

$$
\begin{aligned}
G(X) =& \log p(X; \hat{\beta}_n) - \log p(X; 1) \\
=& - n \log \hat{\beta}_n - \sum_{i=1}^{n} \log t_i - \frac{1}{\hat{\beta}_n} \sum \frac{x_i}{t_i} + \sum_{i=1}^{n} \log t_i + \sum \frac{x_i}{t_i} \\
=& - n \log \hat{\beta}_n - n + n\hat{\beta}_n \\
=& n(\hat{\beta}_n - \log \hat{\beta}_n - 1)
\end{aligned}
$$

since $\hat{\beta}_n = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{t_i}$.

(d) Let $\hat{\alpha}_n = \hat{\beta}_n - 1$, under the null, we have: $\sqrt{n}(\hat{\alpha}_n) \to^d \mathcal{N}(0, 1)$, thus $n\hat{\alpha}_n^2 \to^d \mathcal{X}_1^2$.

Now using Taylor series for $\log(x + 1)$, we have:

$$
\begin{aligned}
2G(X) =& 2n \left(\hat{\beta}_n - \log \hat{\beta}_n - 1 \right) \\
=& 2n \left(\hat{\alpha}_n - \log \left(\hat{\alpha} + 1 \right) \right) \\
=& 2n \left(\hat{\alpha}_n - \hat{\alpha}_n + \frac{\left(\hat{\alpha}_n \right)^2}{2} - \frac{\left(\hat{\alpha}_n^* \right)^3}{3} \right) \\
=& n \left(\hat{\alpha}_n \right)^2 - \frac{2n}{3} \left(\hat{\alpha}_n^* \right)^3
\end{aligned}
$$

For $\hat{\alpha}_n^*$ is between 0 and $\hat{\alpha}_n$. Now we have $n \left(\hat{\alpha}_n \right)^2 \Rightarrow \mathcal{X}_0^1$, and $n \left| \hat{\alpha}_n^* \right|^3 \leq n \left| \hat{\alpha}_n \right|^3$ converges to 0 in probability since. So by the Slutsky theorem, we have $2G(X) \to^d \mathcal{X}_0^1$. □

Problem 4. Bayesian Testing

Proof. (a) For each $X = x$, given that the true model is p_0, the chance of wrongly accepting p_1 is $\varphi(x)$. Given that the true model is p_1, the chance of wrongly accepting p_0 is $1 - \varphi(x)$. Thus taking the expectation, we have:

$$
\begin{aligned}
R(\varphi) =& \mathbb{E} \left[\mathbb{I}_{\{\Theta=0\}} \varphi(X) + \mathbb{I}_{\{\Theta=1\}} (1 - \varphi(X)) \right] \\
=& \mathbb{E} \left[\varphi(X)^{1-\Theta} (1 - \varphi(X))^{\Theta} \right]
\end{aligned}
$$

(b) Using tower property, we have:

$$
\begin{aligned}
R(\varphi) =& \mathbb{E} \left[\mathbb{E} \left[\mathbb{I}_{\{\Theta=0\}} \varphi(X) + \mathbb{I}_{\{\Theta=1\}} (1 - \varphi(X)) \mid \Theta \right] \right] \\
=& (1 - p) \mathbb{E}_0(\varphi) + p \mathbb{E}_1(\varphi) \\
=& \int (1 - p) \varphi(x) p_0(x) dx + \int p \varphi(x) p_1(x) dx
\end{aligned}
$$

(c) For those x such that $(1 - p)p_0(x) < pp_1(x)$, $R(\varphi)$ is minimized when $\varphi(x) = 0$.

Similarly for those x such that $(1 - p)p_0(x) > pp_1(x)$, $R(\varphi)$ is minimized when $\varphi(x) = 1$. So the test function φ^* minimizing $R(\varphi)$ is:

$$
\varphi^*(x) = \begin{cases} 1, & \text{if } p_0(x) > \frac{p}{1-p} p_1(x) \\ 0, & \text{if } p_1(x) < \frac{p}{1-p} p_1(x) \end{cases}
$$

The critical value $k = \frac{p}{1-p}$. □