UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 5
Fall 2014

Issued: Friday, October 10 **Due:** Thursday, October 16 (beginning of class)

Problem 5.1
If $X_1, \ldots, X_n$ are i.i.d. from $N(\theta, \theta)$, then two natural estimators of θ are the sample mean $\bar{X}$ and the sample variance S^2. Determine the asymptotic relative efficiency of S^2 with respect to $\bar{X}$.

Problem 5.2
Let $X_1, \ldots, X_n$ be i.i.d. from an exponential distribution with unit failure rate.

1. Suppose we are interested in the limiting distribution for $X_{(2)}$, the second order statistic. Naturally, $X_{(2)} \xrightarrow{p} 0$ as $n \to \infty$. For an interesting limit theory we should scale $X_{(2)}$ by an appropriate power of n, but the correct power is not $1/2$. Suppose $x > 0$. Find a value p so that $P(n^p X_{(2)} \leq x)$ converges to a value between 0 and 1. (If p is too small, the probability will tend to 1, and if p is too large the probability will tend to 0.)

2. Determine the limiting distribution for $X_{(n)} - \log n$.

Problem 5.3
Consider the loss function

$$L(\theta, a) \quad = \quad \begin{cases} k_1 \ |\theta - a| & \text{if } a \leq \theta \\ k_2 \ |\theta - a| & \text{if } a > \theta \end{cases}$$

where $k_1 > 0$ and $k_2 > 0$ are constants. In a Bayesian setting, suppose that the random variable $(\theta \mid X = x)$ has finite mean for each x. Show that under this loss function, Bayes estimators are p^{th} quantiles of the posterior distribution, where p is a suitable function of k_1 and k_2.

Problem 5.4
Given a fixed known integer $r > 1$, let X_{ij}, $j = 1, \ldots r$ and $i = 1, \ldots, n$ be i.i.d. samples from $N(\mu_i, \sigma^2)$. Find the MLE of $\theta = (\mu_1, \ldots, \mu_n, \sigma^2)$, and show that it is inconsistent for σ^2 as $n \to +\infty$.

Problem 5.5
Let $(X_1, \ldots, X_n)$ be an i.i.d. sample from the mixture distribution with density

$$f_\theta(x) \quad = \quad \theta f_1(x) + (1 - \theta) f_2(x),$$

where $f_i, i = 1, 2$ are two different known densities, and $\theta \in (0, 1)$ is unknown.

(a) Show that the conditions

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f_1(X_i)}{f_2(X_i)} > 1 \qquad \text{and} \qquad \frac{1}{n}\sum_{i=1}^{n}\frac{f_2(X_i)}{f_1(X_i)} > 1$$

are necessary and sufficient for the score equation (setting the derivative of the log likelihood to zero) to have a unique solution. Show that if there is a solution, then it is the MLE.

(b) Derive the MLE of θ when the score equation has no solution.