UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 8- Solutions
Fall 2014

Issued: Oct 31
Due: Thursday, Nov 6

Problem 8.1

(a) For $\theta_i \sim Beta(\alpha, \beta)$ and $X_i|\theta_i \sim B(m, \theta_i)$, the joint distribution satisfies:

$$P(\theta_i, X_i) \propto p(\theta_i)p(X_i|\theta_i) \propto \theta_i^{\alpha+X_i-1}(1-\theta_i)^{\beta+m-X_i-1}.$$

Therefore $\theta_i|X_i \propto Beta(\alpha+X_i, \beta+m-X_i)$. The bayes estimator under quadratic loss is $E(\theta_i|X_i) = (\alpha+X_i)/(\alpha+\beta+m)$.

(b) By tower property: $EX_i = EE[X_i|\theta_i] = m\frac{\alpha}{\alpha+\beta}$ and

$$EX_i^2 = EE[X_i^2|\theta_i] = E[m\theta_i(1-\theta_i)+m^2\theta_i^2] = m\frac{\alpha}{\alpha+\beta} + m(m-1)\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

Natural estimator for EX_i and EX_i^2 are $\overline{X}$ and $\overline{X_i^2}$. Then solve the following equations:

$$\overline{X} = m\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}, \ \overline{X^2} = m\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} + m(m-1)\frac{\hat{\alpha}(\hat{\alpha}+1)}{(\hat{\alpha}+\hat{\beta})(\hat{\alpha}+\hat{\beta}+1)}$$

which gives

$$\hat{\alpha} = \frac{\overline{X}(m\overline{X}-\overline{X^2})}{m\overline{X^2}-m\overline{X}-(m-1)(\overline{X})^2}, \quad \hat{\beta} = \frac{(m\overline{X}-\overline{X^2})(m-\overline{X^2})}{m\overline{X^2}-m\overline{X}-(m-1)(\overline{X})^2}.$$

(c) Plug in the $\hat{\alpha}$ and $\hat{\beta}$ yields the empirical bayes estimator:

$$\hat{\Theta}_i = \frac{\hat{\alpha}+X_i}{\hat{\alpha}+\hat{\beta}+m}.$$

Problem 8.2

a) Let $T(X) = \max_{1\le i\le n} X_i$ and $\phi(X) \in \{0,1\}$ be any test such that:

H_0 is rejected if $\phi(X) = 1$ and accepted otherwise;

$\mathbb{E}_\theta\left(\phi(X)\right) \geq \alpha$ for all $\theta \leq \theta_0$;

We want to prove:

$$\mathbb{E}_{\theta_1}\left(\phi(X)\right) \leq \mathbb{E}_{\theta_1}\left(\delta(X)\right), \qquad \forall \theta_1 > \theta_0$$

To prove that, first notice that for any θ:

$$\begin{aligned}
\mathbb{E}_\theta\left(\phi(X)\right) &= \mathbb{E}_\theta\left(\phi(X)\mathbb{I}(T(X) \leq \theta_0)\right) + \mathbb{E}_\theta\left(\phi(X)\mathbb{I}(T(X) > \theta_0)\right) \\
\mathbb{E}_\theta\left(\delta(X)\right) &= \mathbb{E}_\theta\left(\delta(X)\mathbb{I}(T(X) \leq \theta_0)\right) + \mathbb{E}_\theta\left(\delta(X)\mathbb{I}(T(X) > \theta_0)\right)
\end{aligned}$$

Since $\phi(X) \leq 1$ and $T(X) > \theta_0$ implies $\delta(X) = 1$, we have:

$$\mathbb{E}_{\theta_1}\left(\delta(X)\mathbb{I}(T(X) > \theta_0)\right) \geq \mathbb{E}_{\theta_1}\left(\phi(X)\mathbb{I}(T(X) > \theta_0)\right)$$

It is then enough to prove that:

$$\mathbb{E}_{\theta_1}\left[\left(\delta(X) - \phi(X)\right)\mathbb{I}(T(X) \leq \theta_0)\right] \geq 0, \qquad \text{for all } \theta_1 > \theta_0$$

We have:

$$\begin{aligned}
\mathbb{E}_{\theta_1}\left[\left(\delta(X) - \phi(X)\right)\mathbb{I}(T(X) \leq \theta_0)\right] &= \frac{\int \left(\delta(X) - \phi(X)\right)}{\theta_1^n}\mathbb{I}(x_i \leq \theta_0)\mathbb{I}(x_i \leq \theta_1)dx \\
&= \frac{1}{\theta_1^n}\underbrace{\left[\int \left(\delta(X) - \phi(X)\right)\mathbb{I}(x_i \leq \theta_0)dx\right]}_{h(\delta,\phi,\theta_0)} \\
&= \frac{h(\delta, \phi, \theta_0)}{\theta_1^n}
\end{aligned}$$

and so, it is enough to prove that $h(\delta, \phi, \theta_0) \geq 0$. To prove that, notice:

$$\begin{aligned}
\frac{h(\delta, \phi, \theta_0)}{\theta_0^n} &= \mathbb{E}_{\theta_0}\left[\left(\delta(X) - \phi(X)\right)\mathbb{I}(T(X) \leq \theta_0)\right] \\
&= \mathbb{E}_{\theta_0}\left[\left(\delta(X) - \phi(X)\right)\right] \\
&\geq 0
\end{aligned}$$

where:

the second equality is due to the fact that $\mathbb{P}_{\theta_0}(X_i \geq \theta_0) = 0$;

the inequality follows from the fact that $\mathbb{E}_{\theta_0}\phi(X) \leq \mathbb{E}_{\theta_0}\delta(X) = \alpha$.

b) First, we show that $\mathbb{E}_{\theta_0}(\delta(X)) = \alpha$. To do that notice that:

$$\mathbb{P}_\theta\left(X_{(n)} \leq x\right) = \begin{cases} 0, & \text{for } x \in (-\infty, 0); \\ \left(\frac{x}{\theta}\right)^n, & \text{for } x \in [0, 1]; \\ 1, & \text{for } x \in (1, \infty); \end{cases}$$

so:

$$\begin{aligned} \mathbb{E}_{\theta_0}(\delta(X)) &= \mathbb{P}_{\theta_0}\left(T(X) \leq \theta_0 \alpha^{\frac{1}{n}}\right) + \underbrace{\mathbb{P}_{\theta_0}(T(X) > \theta_0)}_{=0} \\ &= \left(\frac{\theta_0 \alpha^{\frac{1}{n}}}{\theta_0}\right)^n = \alpha \end{aligned}$$

Define $\phi(X) \in \{0, 1\}$ to be any function satisfying:

H_0 is rejected if $\phi(X) = 1$ and accepted otherwise;

$\mathbb{E}_{\theta_0}(\phi(X)) \geq \alpha$;

We want to prove:

$$\mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\right] \geq 0, \qquad \forall \theta_1 \neq \theta_0$$

As before, we partition the problem according to:

$$\begin{aligned} \mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\right] &= \mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\, \mathbb{I}(\theta_0 \alpha^{\frac{1}{n}} \leq T(X) \leq \theta_0)\right] \\ &\quad + \mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\, \mathbb{I}\left(T(X) \in (-\infty, \theta_0 \alpha^{\frac{1}{n}}) \cup (\theta_0, \infty)\right)\right] \end{aligned}$$

Based on similar arguments as those used in item a, we have:

$$\mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\, \mathbb{I}\left(T(X) \in (-\infty, \theta_0 \alpha^{\frac{1}{n}}) \cup (\theta_0, \infty)\right)\right] \geq 0$$

so, it is enough to prove:

$$\mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\, \mathbb{I}(\theta_0 \alpha^{\frac{1}{n}} \leq T(X) \leq \theta_0)\right], \qquad \text{for } \theta_1 \geq \theta_0 \alpha^{\frac{1}{n}}$$

For $\theta_1 \geq \theta_0 \alpha^{\frac{1}{n}}$ and h as defined in item a:

$$\mathbb{E}_{\theta_1}\left[(\delta(X) - \phi(X))\, \mathbb{I}(\theta_0 \alpha^{\frac{1}{n}} \leq T(X) \leq \theta_0)\right] = \frac{h(\delta, \phi, \theta_0)}{\theta^n}$$

Again, it is enough to prove that $h(\delta, \phi, \theta_0) \geq 0$. To prove that, notice that:

$$\mathbb{E}_{\theta_0}\left[(\delta(X) - \phi(X))\, \mathbb{I}(\theta_0 \alpha^{\frac{1}{n}} \leq T(X) \leq \theta_0)\right] = \frac{h(\delta, \phi, \theta_0)}{\theta_0^n} = \alpha - \mathbb{E}_{\theta_0}(\phi(X)) \geq 0$$

Problem 8.3

(a) Let $Z_i = \frac{X_i}{t_i}$. It follows that $Z_i, i = 1, \ldots, n$ are i.i.d. exponentially distributed random variables with $\mathbb{E}_\beta(Z_i) = \beta$. We know that the sufficient statistic is given by $T(Z) = \sum_{i=1}^n Z_i$, so using moment matching leads us to:

$$n\hat{\beta} = \mathbb{E}_{\hat{\beta}} T(Z) = \sum_{i=1}^n Z_i$$

and hence $\hat{\beta} = \frac{1}{n}\sum_{i=1}^n \frac{X_i}{t_i}$.

(b) From the properties of the exponential distribution and the definitions in item a above, we have $\mathbb{E}Z_i = \beta$ and $\mathrm{var} Z_i = \beta^2$. It follows from the central limit theorem that:

$$\sqrt{n}\left(\frac{\hat{\beta}_n - \beta}{\beta}\right) = \sqrt{n}\left(\frac{\frac{1}{n}\sum_{i=1}^n (Z_i - \beta)}{\beta}\right) \quad \xrightarrow{d} \quad \mathcal{N}(0,1)$$

which proves the result.

(c) In terms of Z_i, we can write the likelihood for the i-th observation as:

$$p(z, \beta) \quad = \quad \exp\left(-\frac{z}{\beta} - \log(\beta)\right)$$

Hence:

$$\begin{aligned}
\log \prod_{i=1}^n p(z_i, \hat{\beta}_n) &= -\frac{1}{\hat{\beta}_n}\sum_{i=1}^n z_i - n\log(\hat{\beta}_n) = -n - n\log(\hat{\beta}_n) \\
\log \prod_{i=1}^n p(z_i, 1) &= -\sum_{i=1}^n z_i - n\log(1) = -n\hat{\beta}_n
\end{aligned}$$

And finally:

$$\log p(Z; \hat{\beta}_n) - \log p(Z; 1) \quad = \quad n\hat{\beta}_n - n\log(\hat{\beta}_n) - n$$

(d) To prove this, let $g(x) = (x - \log x)$. We have:

$$\begin{aligned}
g'(\beta_0) &= 1 - \frac{1}{\beta_0} \\
g''(\beta_0) &= \frac{1}{\beta_0^2}
\end{aligned}$$

Under H_0, $g'(\beta_0) = g'(1) = 0$ and $g''(\beta_0) = g''(1) = 1$. Using the delta method with a second order expansion and perturbation $(\hat{\beta}_n - 1)$ yields:

$$\begin{aligned}
g(\hat{\beta}_n - g(1)) &= g'(1)\left(\hat{\beta}_n - 1\right) + \frac{1}{2}\left(\hat{\beta}_n - 1\right)^2 + o_p(1) \\
2n(\hat{\beta}_n - \log\hat{\beta}_n - 1) &= n\left(\hat{\beta}_n - 1\right)^2 + o_p(1).
\end{aligned}$$

By part c), $\sqrt{n}\left(\hat{\beta}_n - 1\right) \xrightarrow{d} \mathcal{N}(0, \beta^2)$ and continuous mapping theorem, we have $2G(X) \to \chi^2(1)$.

Problem 8.4

(a) By definition,

$$R(\varphi) = P(\Theta = 0, \varphi(X) = 1) + P(\Theta = 0, \varphi(X) = 0) = E(I(\Theta = 0)\varphi(X) + I(\Theta = 1)(1 - \varphi(X))).$$

(b) By tower property,

$$R(\varphi) = P(\Theta = 0)E(\varphi(X)|\Theta = 0) + P(\Theta = 1)E(1 - \varphi(X)|\Theta = 1) = (1-p)E_0\varphi(X) + p(1 - E_1\varphi(X))$$

(c) From part (b), we have

$$\begin{aligned} R(\varphi) &= p + (1-p)E_0\varphi(X) - pE_1\varphi(X) \\ &= p + \int [(1-p)p_0(x) - pp_1(x)]\varphi(x)dx \\ &= p + \int_{p_0/p_1 > p/1-p} |(1-p)p_0(x) - pp_1(x)|\varphi(x)dx - \int_{p_0/p_1 \le p/1-p} |(1-p)p_0(x) - pp_1(x)|\varphi(x)dx. \end{aligned}$$

To minimize $R(\varphi)$, it is easy to see from above that:

$$\varphi^*(X) = \begin{cases} 1 & \text{if } \frac{p_1}{p_0} \geq \frac{1-p}{p} \\ 0 & \text{if } \frac{p_1}{p_0} < \frac{1-p}{p} \end{cases}$$

which is a likelihood ratio test with critical value $(1-p)/p$.