UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 4- Solutions
Fall 2014

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Due: Thursday, Oct 2

Problem 4.1

Geometric distribution belongs to a full rank exponential family, thus X is both sufficient and complete statistics for θ. Let $T = \mathbf{1}(X=0)$, then $E[T] = P(X=0) = \theta$ which indicates that T is an unbiased estimator. Since $E[T|X] = T$ we know T is a UMVU. Moreover we have for T,

$$Var(T) = Var(\mathbf{1}(X=0)) = \theta(1-\theta). \tag{1}$$

Calculate the fisher information matrix $I(\theta) = \frac{1}{\theta^2(1-\theta)}$. Therefore, CR lower bound gives us:

$$Var(\delta(X)) \geq \theta^2(1-\theta). \tag{2}$$

Compared (1) and (2), we have $Var(T)$ is strictly larger than the CR lower bound.

Problem 4.2

Bayes estimator of θ under loss function $L(\theta,a) = (\theta-a)^2/\theta$ is $\frac{1}{\mathbb{E}(\frac{1}{\theta}|X)}$. Basically you could write $E[L(\theta,a)|X]$ out and take derivative w.r.t a and setting the derivative to 0.

Useful Fact

If $Z \sim$ Gamma(α,β), $\mathbb{E}\left(\frac{1}{Z}\right) = \int_0^\infty \frac{z^{\alpha-2}e^{-\beta z}}{\Gamma(\alpha)\beta^{-\alpha}}dz = \frac{\beta}{\alpha-1}$ except $\alpha = 1$.

If $\alpha = 1$, $\mathbb{E}\left(\frac{1}{Z}\right) = \int_0^\infty \frac{1}{z}e^{-z}dz > e^{-1}\int_0^1 \frac{1}{z}dz = \infty$

(a) The joint distribution of (X,θ) is proportional to $e^{-\theta}\theta^x\theta^{a-1}e^{-b\theta} = \theta^{x+a-1}e^{-(b+1)\theta}$

Thus, the posterior distribution of θ is given by Gamma$(x+a, b+1)$. The Bayes estimator $\delta_{a,b}(X) = \dfrac{X+a-1}{b+1}$

(b) If $a \to 1$ and $b \to 0$, then $\delta_{a,b}(X) \to \delta(X) = X$.

Note : $\delta(X) = X$ is a Bayes with respect to the improper prior that is uniform on $(0,\infty)$. The posterior distribution of θ is given by Gamma$(x+1,1)$ since the joint distribution of (X,θ) is proportional to $e^{-\theta}\theta^x$.

Thus, $\delta(X) = \dfrac{1}{\mathbb{E}\left(\frac{1}{\theta}|X\right)} = X$ is Bayes. Observe that when $a \to 1, b \to 0$, Gamma(a, b) is almost same as uniform distribution on $(0, \infty)$ which is improper prior.

Problem 4.3

(a) The joint distribution of $((X_1, \ldots, X_n), \log\theta)$ is proportional to

$$\exp\left(-\frac{(\log\theta - \mu_0)^2}{2\sigma_0^2}\right)\exp(-n\log(\theta))\mathbf{1}(\theta \geq \mathbf{X_{(n)}})$$

Thus, the posterior distribution of $\log\theta$ is given by $N(\mu_0 - n\sigma_0^2, \sigma_0^2)$ given $\log\theta \geq \log(X_{(n)})$.

(b) $\delta^{Bayes}(X) = \arg\min_a \mathbb{E}(L(a,\theta)|X) = \arg\min_a \mathbb{P}(\theta \neq a|X) = \arg\max_a \mathbb{P}(\theta = a|X)$. But note that $\arg\max_a \mathbb{P}(a = \theta|X) \neq \arg\max_b \mathbb{P}(\log\theta = \log b|X)$ because of the extra factor when you do a change of variable. Find the posterior distribution of θ :

$$p(\theta|X) \propto \exp\left(-\frac{(\log\theta - (\mu_0 - (n+1)\sigma_0^2))^2}{2\sigma_0^2}\right)\mathbf{1}(\theta \geq \mathbf{X_{(n)}}).$$

If $\log(X_{(n)}) \geq \mu_0 - (n+1)\sigma_0^2$, then $\arg\max_a \mathbb{P}(\theta = a|X) = X_{(n)}$.

If $\log(X_{(n)}) < \mu_0 - (n+1)\sigma_0^2$, then $\arg\max_a \mathbb{P}(a = \theta|X) = e^{\mu_0 - (n+1)\sigma_0^2}$.

Thus, $\delta^{Bayes}(X) = \max\left(X_{(n)}, e^{\mu_0 - (n+1)\sigma_0^2}\right)$.

Problem 4.4

If we have an estimator that both unbiased and bayes w.r.t quadratic loss, by definition, from unbiasedness, we have that $E(\delta(X)|\theta) = g(\theta)$ and by bayes we have $\delta(X) = E\left(g(\theta)|X\right)$. Consider the Bayes risk:

$$E\left[(g(\theta) - \delta(X))^2\right] = E\left[g(\theta)^2 - 2g(\theta)\delta(X) + \delta(X)^2\right].$$

Calculate the bayes risk by first condition on θ, then take the expectation over X, we have:

$$E\left[(g(\theta) - \delta(X))^2\right] = E\left[g(\theta)^2 - 2g(\theta)E\left(\delta(X)|\theta\right) + E\left(\delta(X)^2|\theta\right)\right]$$

by unbiasedness:

$$E\left[(g(\theta) - \delta(X))^2\right] = -E\left[g(\theta)^2\right] + E\left(\delta(X)^2\right). \tag{3}$$

If, on the other hand, we condition on X first and take expectation over posterior distribution of θ, we have:

$$E\left[(g(\theta)-\delta(X))^2\right] = E\left[E\left(g(\theta)^2|X\right) - 2E\left(g(\theta)|X\right)\delta(X) + \delta(X)^2\right]$$

by δ is an bayes estimator:

$$E\left[(g(\theta)-\delta(X))^2\right] = E\left[g(\theta)^2\right] - E\left(\delta(X)^2\right). \tag{4}$$

From these two expressions(3),(4)

$$E\left[(g(\theta)-\delta(X))^2\right] = -E\left[(g(\theta)-\delta(X))^2\right]$$

and hence the bayes risk must be zero.

Problem 4.5

(a) To avoid dealing with $\lim_{x\to\infty} p(x,\theta)g(x)$ we break the integral into two parts:

$$E\left[g'(X)\right] = \int_{-\infty}^{0} g'(x)p(x,\theta)dx + \int_{0}^{\infty} g'(x)p(x,\theta)dx \tag{5}$$

and deal with each part using integration by part. We have:

$$\begin{aligned}
\int_{-\infty}^{0} g'(x)p(x,\theta)dx &= \int_{-\infty}^{0} g'(x)\int_{-\infty}^{x} p'(y,\theta)dydx \\
&= \int_{-\infty}^{0} g'(x)\int_{-\infty}^{x} \exp\left(\theta_i T_i(y) - A(\theta)\right)h(y)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)dydx \\
&= \int_{-\infty}^{0}\int_{y}^{0} g'(x)dx \exp\left(\theta_i T_i(y) - A(\theta)\right)h(y)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)dy \\
&= \int_{-\infty}^{0} [g(0)-g(y)] \exp\left(\theta_i T_i(y) - A(\theta)\right)h(y)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)dy.
\end{aligned}$$

The second last equality is due to Fubini's theorem.

Similarly we have:

$$\int_{0}^{\infty} g'(x)p(x,\theta)dx = \int_{0}^{\infty} [g(0)-g(y)] \exp\left(\theta_i T_i(y) - A(\theta)\right)h(y)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)dy.$$

Combine the above two parts we have:

$$\begin{aligned}
E\left[g'(X)\right] &= g(0)\int_{-\infty}^{\infty} p'(x,\theta)dx - E\left[g(X)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)\right] \\
&= -E\left[g(X)\left(\frac{h'(y)}{h(y)} + \sum_i \theta_i T_i'(y)\right)\right]
\end{aligned}$$

The last equality because $p(x,\theta)$ is a density so $p(x,\theta)\to 0$ when $|x|\to\infty$.

(b) For $X \sim \mathcal{N}(\mu, \sigma^2)$, $h(x) = \frac{1}{\sqrt{2\pi}}$, $T_1(x) = x^2$, $T_2(x) = x$, $\theta_1 = -1/2\sigma^2$, $\theta_2 = \mu/2\sigma^2$. Plug these into the equality yields what we want.

(c) Using part (b) and set $g(x) = x^2$, we get: $E[X]^3 - E[X]E\left[X^2\right] = \sigma^2 E[2X]$ which suggests: $E\left[X^3\right] = \mu(\mu^2 + 3\sigma^2)$. Further let $g(x) = x^3$ similarly we have $E\left[X^4\right] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$.