ST210A - HW7

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Problem 1. Method of moments estimation

Proof. According to CLT:

$$\sqrt{n}\left(\mu(\hat{\theta}) - \mu(\theta)\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2(\theta)\right)$$

Note that μ is strictly increasing, thus μ^{-1} exists, and $\frac{d}{dx}\mu^{-1} = \frac{1}{\mu'(\mu^{-1}(x))}$ exists since the denominator is always positive (as μ is strictly increasing so μ' is positive). Now applying the Delta method,

$$\begin{aligned}
\sqrt{n}\left(\mu^{-1}\mu(\hat{\theta}) - \mu^{-1}\mu(\theta)\right) &\xrightarrow{d} \mathcal{N}\left(0, \left[\mu^{-1}(\mu(\theta))'\right]^2 \sigma^2(\theta)\right) \\
\Leftrightarrow \sqrt{n}\left(\hat{\theta} - \theta\right) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2(\theta)}{\mu'(\mu^{-1}(\mu(\theta)))^2}\right) \\
\Leftrightarrow \sqrt{n}\left(\hat{\theta} - \theta\right) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2(\theta)}{(\mu'(\theta))^2}\right)
\end{aligned}$$

□

Problem 2. Empirical Process - Distribution of Root of Equation

Proof. (a) Let $W_n = t^{X_n} - X_n^2$ be random function, for $t \in [0,1]$ compact. We have:

$$\begin{aligned}
\mathbb{E}W_n =& \mathbb{E}\left[t^X - X^2\right] \\
=& \int_0^1 t^x dx - \int_0^1 x^2 dx \\
=& \frac{t-1}{\ln t} - \frac{1}{3} \\
\|W_n\|_\infty =& \sup_{t\in[0,1]} t^{X_n} - X_n^2 = 1 - X_n^2 \\
\mathbb{E}\|W_n\|_\infty =& 1 - \mathbb{E}X_n^2 = \frac{2}{3} < \infty
\end{aligned}$$

So the conditions of Uniform Weak Law of Random Function are met, thus we have:

$$\|\bar{W}_n - \frac{t-1}{\ln t} + \frac{1}{3}\|_\infty \xrightarrow{\mathbb{P}} 0$$

Now take $G_n = \bar{W}_n, g(t) = \frac{t-1}{\ln t} - \frac{1}{3}$, then $g(t)$ has a unique solution in $[0,1], c \approx 0.06$. The conditions of Theorem 9.4 in Keener are met, thus we have $T_n \xrightarrow{\mathbb{P}} c$, as $n \to \infty$ for c is the solution of $g(t)$ as mentioned above.

(b) We follow the procedure as in the proof for Theorem 9.14 in Keener.

Applying the Tayor expansion of W_n we have:

$$\bar{W}_n(T_n) = \bar{W}_n(c) + \bar{W}_n'(\tilde{T}_n)\,(T_n - c) \tag{1}$$

for $\tilde{T}_n$ is between T_n and c. Since T_n is the solution to $\bar{W}_n$, the LHS of (1) is zero. Thus:

$$\sqrt{n}\,(T_n - c) = \frac{\sqrt{n}\bar{W}_n(c)}{-\bar{W}_n'(\tilde{T}_n)}$$

For the numerator, we have $W_i(c)$ are i.i.d with mean zero. We now calculate its variance, and note that c sastify $g(c) = 0 \Leftrightarrow \ln c = 3(c-1)$, we have:

$$\begin{aligned}
\mathrm{Var} W_i(c) =& \mathbb{E}\left[W_i^2(c)\right] \\
=& \mathbb{E}\left[\left(c^X - X^2\right)^2\right] \\
=& \mathbb{E}\left[c^{2X}\right] - 2\mathbb{E}\left[c^X X^2\right] + \mathbb{E}X^4 \\
=& \frac{c^2-1}{2\ln c} - \frac{4 - 4c + 4c\ln c - 2c\ln^2 c}{\ln^3 c} + \frac{1}{5} \\
=& \frac{45c^3 - 171c^2 + 147c + 59}{270(c-1)^2} := \sigma^2 \\
\approx& 0.2812
\end{aligned}$$

By the CLT, $\sqrt{n}\bar{W}_n(c) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$

For the denominator, since $\left|\tilde{T}_n - c\right| \le |T_n - c|$, and $T_n \xrightarrow{\mathbb{P}} c$, thus $\tilde{T}_n \xrightarrow{\mathbb{P}} c$. By Theorem 9.2 in Keener, $\|\bar{W}_n' - \mu\|_\infty \xrightarrow{\mathbb{P}} 0$, for:

$$\begin{aligned}
\mu(t) =& \mathbb{E}W'(t) \\
=& \mathbb{E}\left[(X-1)t^X\right] \\
=& \int_0^1 x t^x\,dx - \int_0^1 t^x\,dx \\
=& \frac{-t + t\ln t + 1}{\ln^2 t} - \frac{t-1}{\ln t} \\
=& \frac{-t + \ln t + 1}{\ln^2 t} \\
\mu(c) =& -0.2363
\end{aligned}$$

So we have $\|\bar{W}_n' - \mu\|_\infty \xrightarrow{\mathbb{P}} 0$ with μ a non-random function, and $\tilde{T}_n \xrightarrow{\mathbb{P}} c$, thus by Theorem 9.4 in Keener part 1, we have:

$$\bar{W}_n'(\tilde{T}_n) \xrightarrow{\mathbb{P}} \mu(c)$$

By the Slutky theorem (8.13), we have:

$$\begin{aligned}
\sqrt{n}\,(T_n - c) \xrightarrow{d}& \mathcal{N}\left(0, \frac{\sigma^2}{\mu^2(c)}\right) \\
\xrightarrow{d}& \mathcal{N}\left(0, \frac{3}{40}\left(45c^3 - 171c^2 + 147c + 59\right)\right) \\
\approx& \mathcal{N}\left(0, 5.03649\right)
\end{aligned}$$

□

Problem 3. Normal location $\mathcal{N}(\theta, 1)$

Proof. (a) Let $\Phi(x, \mu, \sigma)$ be the CDF of normal mean μ, variance σ^2. Let the CDF of $1/\bar{X}_n$ be F_n, and PDF be f_n, then we see that f_n is continuous everywhere except 0. We have $\bar{X}_n \sim \mathcal{N}\left(\theta, \frac{1}{n}\right)$.

$$\begin{aligned}
F_n(b) =& \mathbb{P}\left[\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) < b\right] \\
=& \mathbb{P}\left[\frac{1}{\bar{X}_n} < \frac{b}{\sqrt{n}} + \frac{1}{\theta}\right] \\
=& \mathbb{P}\left[\frac{1}{\bar{X}_n} < \frac{b\theta + \sqrt{n}}{\theta\sqrt{n}}\right] \\
=& \mathbb{P}\left[\frac{1}{\bar{X}_n} < \frac{1}{a}\right], \text{ for a=}\frac{\theta\sqrt{n}}{b\theta + \sqrt{n}} \\
=& \begin{cases} \mathbb{P}\left[\bar{X}_n > a \vee \bar{X} < 0\right] & \text{if a} > 0 \\ \mathbb{P}\left[0 > \bar{X}_n > a\right] & \text{if a} < 0 \end{cases} \\
=& \begin{cases} 1 - \Phi\left(a, \theta, \frac{1}{n}\right) + \Phi\left(0, \theta, \frac{1}{n}\right) & \text{if a} > 0 \\ \Phi\left(0, \theta, \frac{1}{n}\right) - \Phi\left(a, \theta, \frac{1}{n}\right) & \text{if a} < 0 \end{cases}
\end{aligned}$$

For both cases $a > 0$ and $a < 0$, we share the same form of density, which is:

$$\begin{aligned}
f_n(b) =& \frac{\partial F_n(b)}{\partial b} \\
=& \frac{\partial F_n(b)}{\partial a}(a)\frac{\partial a}{\partial b}(b) \\
=& -\frac{\partial \Phi\left(a, \theta, \frac{1}{n}\right)}{\partial a}(a) \times \left(-\frac{\sqrt{n}\theta^2}{(b\theta + \sqrt{n})^2}\right) \\
=& \frac{\sqrt{n}}{\sqrt{2\pi}} \exp\left\{-\frac{n}{2}\left(\frac{\theta\sqrt{n}}{b\theta + \sqrt{n}} - \theta\right)^2\right\}\left(\frac{\sqrt{n}\theta^2}{(b\theta + \sqrt{n})^2}\right) \\
=& \frac{n\theta^2}{\sqrt{2\pi}(b\theta + \sqrt{n})^2} \exp\left\{-\frac{n\theta^2}{2}\left(\frac{-b\theta}{b\theta + \sqrt{n}}\right)^2\right\} \\
=& \frac{\theta^2 n}{\sqrt{2\pi}} \frac{n}{(b\theta + \sqrt{n})^2} \exp\left\{-\frac{\theta^4}{2} b^2 \frac{n}{(b\theta + \sqrt{n})^2}\right\} \\
\Rightarrow \lim_{n\to\infty} f_n(b) =& \frac{\theta^2}{\sqrt{2\pi}} \exp\left\{-\frac{\theta^4}{2} b^2\right\}, \forall b \neq 0
\end{aligned}$$

This limit is the density for normal mean 0 variance $\frac{1}{\theta^2}$. Thus the probability density of $\sqrt{n}\left(1/\bar{X}_n - 1/\theta\right)$ converges to $\phi(x, 0, \frac{1}{\theta^2})$. By the Scheffe Lemma, the CDF also converges to the corresponding CDF $\Phi(x, 0, \frac{1}{\theta^2})$. So $\delta = \frac{1}{\bar{X}_n} \xrightarrow{\mathbb{P}} \mathcal{N}\left(0, \frac{1}{\theta^2}\right)$.

(b) WLOG assuming that $\theta > 0$. For $n \in \mathbb{N}$ arbitrary, we have:

$$\mathbb{E}\left[\frac{1}{\bar{X}_n}\right] = \int_{-\infty}^{0} \frac{1}{x}\phi(x, \theta, \frac{1}{n})dx + \int_{0}^{\theta} \frac{1}{x}\phi(x, \theta, \frac{1}{n})dx + \int_{\theta}^{\infty} \frac{1}{x}\phi(x, \theta, \frac{1}{n})dx$$

Now we have $\phi(x, \theta, \frac{1}{n})$ is a a function symmetric at $x = \theta$, decreasing with respect to x when $0 < x < \theta$. Thus:

$$\int_0^\theta \frac{1}{x}\phi(x,\theta,\frac{1}{n})dx > \int_0^\theta \frac{1}{x}\phi(0,\theta,\frac{1}{n})dx$$
$$= \int_0^\theta \frac{c}{x}dx \text{ for c} > 0$$
$$= \infty$$

Thus $\forall n, \mathbb{E}\left[\frac{1}{\bar{X}_n}\right]$ also diverges. This does not contradict the result in part (a). We only have convergence in distribution implying convergence in expectation of bounded continuous function. Since function $1/x$ is not continuous at 0, and not bounded, it needs not sastify that condition. □

Problem 4. Poisson Asymptotic Confidence Interval

Proof. (a) The variance of Poisson is also θ. By CLT,

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} \mathcal{N}(0,\theta)$$

By the Delta method for any g differentiable at θ,

$$\sqrt{n}\left(g(\hat{\theta})-g(\theta)\right) \xrightarrow{d} \mathcal{N}\left(0,(g'(\theta))^2\theta\right)$$

We want $(g'(\theta))^2\theta = 1 \Leftrightarrow g'(\theta) = \frac{1}{\sqrt{\theta}} \Rightarrow g(\theta) = 2\sqrt{\theta}$.
(b) $Z_n = 2\sqrt{n}\left(\sqrt{\bar{X}_n}-\sqrt{\theta}\right)$.

$$\mathbb{P}\left[|Z_n| < z_{\alpha/2}\right] \to 1-\alpha$$
$$\mathbb{P}\left[-z_{\alpha/2} < 2\sqrt{n}\left(\sqrt{\bar{X}_n}-\sqrt{\theta}\right) < z_{\alpha/2}\right] \to 1-\alpha$$
$$\mathbb{P}\left[z_{\alpha/2} > 2\sqrt{n}\left(\sqrt{\theta}-\sqrt{\bar{X}_n}\right) > -z_{\alpha/2}\right] \to 1-\alpha$$
$$\mathbb{P}\left[\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} > \sqrt{\theta} > \frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right] \to 1-\alpha$$

If $\frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} > 0$, we have: $\mathbb{P}\left[\left(\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right)^2 > \theta > \left(\frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right)^2\right] = \mathbb{P}\left[\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} > \sqrt{\theta} > \frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right]$.
If $\frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} \le 0$, we have $\left[\left(\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right)^2 > \theta\right] = \mathbb{P}\left[\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} > \sqrt{\theta} > \frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right]$.
So the $1-\alpha$ confidence interval is (δ_1,δ_2), for

$$\delta_1 = \begin{cases} 0 & \text{if } \frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n} \le 0 \\ \left(\frac{-z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right)^2 & \text{otherwise} \end{cases}$$
$$\delta_2 = \left(\frac{z_{\alpha/2}}{2\sqrt{n}}+\sqrt{\bar{X}_n}\right)^2$$

□

Problem 5. Asymptotic Confidence Interval. Normal

Proof. (a) According to Theorem 9.14 in Keener:

$$\sqrt{n}\left(\hat{\mu}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I(\mu)}\right)$$
$$\sqrt{nI(\mu)}\left(\hat{\mu}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

The likelihood function for a single X

$$\begin{aligned}
l(\mu) =& \log f_\mu(X) \\
=& -\frac{(X-\mu)^2}{2g^2(\mu)} - \log g(\mu) - \frac{1}{2}\log(2\pi) \\
\Rightarrow -l'(\mu) =& \frac{g'(\mu)}{g(\mu)} + \frac{-2(X-\mu)g^2(\mu) - (X-\mu)^2 2g'(\mu)g(\mu)}{2g^4(\mu)} \\
=& \frac{g'(\mu)}{g(\mu)} - \frac{X-\mu}{g^2(\mu)} - \frac{(X-\mu)^2 g'(\mu)}{g^3(\mu)} \\
\Rightarrow \mathbb{E}_\mu\left[\left(l'(\mu)\right)^2\right] =& \left(\frac{g'(\mu)}{g(\mu)}\right)^2 + \frac{1}{g^2(\mu)} + \frac{3(g'(\mu))^2}{g^2(\mu)} - 2\frac{(g'(\mu))^2}{g^2(\mu)} \\
=& 2\left(\frac{g'(\mu)}{g(\mu)}\right)^2 + \frac{1}{g^2(\mu)} = I(\mu)
\end{aligned}$$

Now we can use $I(\hat{\mu})$ instead of $I(\mu)$ to get the confident interval (according to Slutsky we still have convergence in distribution to normal). So we have:

$$\begin{aligned}
\mathbb{P}\left[-z_{\alpha/2} < \sqrt{nI(\hat{\mu}_n)}\left(\hat{\mu}_n - \mu\right) < z_{\alpha/2}\right] \to& 1-\alpha \\
\mathbb{P}\left[-z_{\alpha/2}/\sqrt{nI(\hat{\mu}_n)} < \mu - \hat{\mu}_n < z_{\alpha/2}/\sqrt{nI(\hat{\mu}_n)}\right] \to& 1-\alpha \\
\Rightarrow \mathbb{P}\left[\hat{\mu}_n - z_{\alpha/2}/\sqrt{nI(\hat{\mu}_n)} < \mu < \hat{\mu}_n + z_{\alpha/2}/\sqrt{nI(\hat{\mu}_n)}\right] \to& 1-\alpha
\end{aligned}$$

For

$$I(\hat{\mu}_n) = 2\left(\frac{g'(\hat{\mu}_n)}{g(\hat{\mu}_n)}\right)^2 + \frac{1}{g^2(\hat{\mu}_n)}$$

(b) Without the functional relation, we have CLT:

$$\sqrt{n}\left(\hat{\mu}'_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

Thus the confidence interval width is $2z_{\alpha/2}\sigma/\sqrt{n}$. Thus the ratio of the width of two confidence interval is:

$$\frac{Width(b)}{Width(a)} = \sigma\sqrt{I(\hat{\mu}_n)}$$

We have $\hat{\mu}_n \xrightarrow{\mathbb{P}} \mu$ (consistency of MLE), thus according to the continuous mapping theorem, $I(\hat{\mu}_n) \xrightarrow{\mathbb{P}} I(\mu)$. So the limit:

$$\frac{Width(b)}{Width(a)} \xrightarrow{\mathbb{P}} 2(g'(\mu))^2 + 1$$

Asymptotically, the confidence interval in part (b) is wider than that in part (a). This makes sense because we have more structure in part (a), as the mean and variance are related, thus we should be more confidence with our estimate. $\square$