STAT215A - HW3

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September 25, 2014

Problem 1. UMVU

Proof. **(a)** We have:

$$\begin{aligned}
f_\theta(x) =& \prod_{i=1}^n \frac{\phi(x_i)}{\Phi(\theta)} \mathbb{I}[x_i < \theta] \\
=& \prod_{i=1}^n \frac{\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x_i^2\}}{\Phi(\theta)} \mathbb{I}[x_i < \theta] \\
=& \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right) \frac{1}{\Phi^n(\theta)} \prod_{i=1}^n \mathbb{I}[x_i < \theta] \\
=& \frac{1}{(2\pi)^{n/2}} \exp(-\frac{1}{2}\sum_{i=1}^n x_i^2) \frac{\mathbb{I}[\max_{i\in\{1,\ldots,n\}}\{x_i\} < \theta]}{\Phi^n(\theta)}
\end{aligned}$$

By the factorization theorem, $\max\{X_i\}$ is a sufficient statistic. We will prove that it is a complete statistics. In fact, we have for $t < \theta$:

$$\begin{aligned}
\mathbb{P}[T < t] =& \mathbb{P}[X_i < t, \forall 1 \le i \le n] \\
=& \mathbb{P}[X_i < t]^n \\
=& \left(\int_{-\infty}^t \frac{\phi(x)}{\Phi(\theta)} dx\right)^n \\
=& \frac{(\Phi(t))^n}{(\Phi(\theta))^n}
\end{aligned}$$

For $t \ge \theta, \mathbb{P}[T < t] = 1$. Thus the density of T is: $p_T(t) = n\frac{(\Phi(t))^{n-1}}{(\Phi(\theta))^n}\phi(t)\mathbb{I}[(-\infty, \theta)]$. So for any function $h(T)$, we have:

$$\begin{aligned}
\mathbb{E}h(T) =& c, \forall\theta \\
\Leftrightarrow \int_{-\infty}^\theta h(t)p_T(t)dt =& c, \forall\theta \\
\Leftrightarrow n\int_{-\infty}^\theta h(t)\Phi^{n-1}(t)\phi(t)dt =& c\Phi^n(\theta), \forall\theta \\
\Rightarrow \frac{\partial}{\partial\theta} n\int_{-\infty}^\theta h(t)\Phi^{n-1}(t)\phi(t)dt =& \frac{\partial}{\partial\theta} c\Phi^n(\theta), \forall\theta \\
\Rightarrow nh(\theta)\Phi^{n-1}(\theta)\phi(\theta) =& cn\Phi^{n-1}(\theta)\phi(\theta) \\
\Rightarrow h(\theta) =& c
\end{aligned}$$

So T is complete. Now we will try to find an unbiased estimator of $g(\theta)$ in the form of $h(T)$. We need:

$$\begin{aligned}
\mathbb{E}h(T) =& g(\theta), \forall\theta \\
\Leftrightarrow \int_{-\infty}^{\theta} h(t)p_T(t)dt =& g(\theta) \\
\Leftrightarrow \int_{-\infty}^{\theta} h(t)n\frac{\Phi^{n-1}(t)}{\Phi^n(\theta)}\phi(t)dt =& g(\theta) \\
\Leftrightarrow n\int_{-\infty}^{\theta} h(t)\Phi^{n-1}(t)\phi(t)dt =& g(\theta)\Phi^n(\theta) \\
(1) \Rightarrow nh(\theta)\Phi^{n-1}(\theta)\phi(\theta) =& g'(\theta)\Phi^n(\theta) + g(\theta)n\Phi^{n-1}(\theta)\phi(\theta) \\
\Leftrightarrow nh(\theta)\phi(\theta) =& g'(\theta)\Phi(\theta) + ng(\theta)\phi(\theta) \\
\Leftrightarrow h(\theta) =& \frac{g'(\theta)\Phi(\theta)}{n\phi(\theta)} + g(\theta)
\end{aligned}$$

On the other hand, the only step in the above derivation of h that is not equivalent is the part of taking partial derivative. Since in general $f(x) = g(x) \Rightarrow F(x) = G(x) + c$, we need to double check:

$$\begin{aligned}
& n\int_{-\infty}^{\theta} \left(\frac{g'(t)\Phi(t)}{n\phi(t)} + g(t)\right)\Phi^{n-1}(t)\phi(t)dt \\
=& \int_{-\infty}^{\theta} \Phi^n(t)g'(t)dt + \int_{-\infty}^{\theta} ng(t)\Phi^{n-1}(t)\phi(t)dt \\
=& \int_{-\infty}^{\theta} \Phi^n(t)dg(t) + \int_{-\infty}^{\theta} g(t)d\Phi^{n-1}(t) \\
=& \Phi^n(t)g(t)\Big|_{-\infty}^{\theta} \\
=& \Phi^n(\theta)g(\theta) - \lim_{t\to-\infty}\Phi^n(t)g(t) \\
=& \Phi^n(\theta)g(\theta) - \lim_{t\to-\infty}\frac{1}{(2\pi)^{n/2}}\exp(-\frac{n}{2}t^2)g(t) \\
=& \Phi^n(\theta)g(\theta) - \lim_{t\to-\infty}\frac{1}{(2\pi)^{n/2}}\frac{g(t)}{\exp(\frac{n}{2}t^2)}
\end{aligned}$$

With the assumption that $g(t)$ is dominated by $\exp(\frac{n}{2}t^2)$ as t approaches $-\infty$, we have the $\Leftarrow$ direction in (1) as well. Thus $h(T)$ is an unbiased estimator of $g(\theta)$. Since $h(T) = \frac{g'(T)\Phi(T)}{n\phi(T)} + g(T)$ is a function of T, it is UMVU.

(b) It is obvious that $g(\theta) = \theta^2$ is differentiable and goes to infinity as $\theta \to \pm\infty$ much slower than $\exp(\theta^2)$. So we can apply (a) and the estimate is $\frac{2T\Phi(T)}{n\phi(T)} + T^2$ for $T = \max\{-2.3, -1.2, 0\} = 0$, thus the estimate is 0. ☐

Problem 2. Fisher Information

Proof. (a) The Fisher information:

$$
\begin{aligned}
I(\theta) =&\mathbb{E}_\theta \left(\frac{\partial \log p_\theta(X)}{\partial \theta} \right)^2 \\
=&\mathbb{E}_\theta \left(\frac{\partial \left(-\log \theta + \log f\left(x/\theta\right)\right)}{\partial \theta} \right)^2 \\
=&\mathbb{E}_\theta \left(-\frac{1}{\theta} - \frac{f'(x/\theta)}{f(x/\theta)} \frac{x}{\theta^2} \right)^2 \\
=&\mathbb{E}_\theta \left[\frac{1}{\theta^2} \left(1 + \frac{x f'(x/\theta)}{\theta f(x/\theta)} \right)^2 \right] \\
=&\frac{1}{\theta^2} \int \left(1 + \frac{x f'(x/\theta)}{\theta f(x/\theta)} \right)^2 \frac{1}{\theta} f\left(\frac{x}{\theta}\right) dx
\end{aligned}
$$

Change variable $y = x/\theta$, then $dy = dx/\theta$. Then:

$$
I(\theta) =\frac{1}{\theta^2} \int \left(1 + \frac{y f'(y)}{f(y)} \right)^2 f(y) dy
$$

(b) $\xi = \log \theta \Rightarrow \theta = \exp \xi \Rightarrow h(\xi) = \exp \xi$. The Fisher information now is:

$$
\begin{aligned}
\tilde{I}(\xi) =& \left[h'(\xi)\right]^2 \mathbb{E}_\theta \left(\frac{\partial \log p_\theta(X)}{\partial \theta} \right)^2 \\
=& \exp^2(\xi) \frac{1}{\theta^2} \int \left(1 + \frac{y f'(y)}{f(y)} \right)^2 f(y) dy \\
=&\theta^2 \frac{1}{\theta^2} \int \left(1 + \frac{y f'(y)}{f(y)} \right)^2 f(y) dy \\
=& \int \left(1 + \frac{y f'(y)}{f(y)} \right)^2 f(y) dy
\end{aligned}
$$

This does not depend on θ.
(c) Cauchy distribution has density $\frac{1}{\pi(x^2+1)}$. Thus the scale family has density $\frac{1}{\theta} \frac{1}{\pi(\frac{x^2}{\theta^2}+1)} = \frac{\theta}{\pi(x^2+\theta^2)}$.

$$
\begin{aligned}
I(\theta) =& -\mathbb{E}_\theta \frac{\partial^2 \log p_\theta(x)}{\partial \theta^2} \\
=& -\mathbb{E}_\theta \frac{\partial^2 (\log\theta - \log\pi - \log(x^2+\theta^2))}{\partial\theta^2} \\
=& -\mathbb{E}_\theta \left[-\frac{1}{\theta^2} + \frac{\partial \frac{2\theta}{(x^2+\theta^2)}}{\partial\theta} \right] \\
=& -\mathbb{E}_\theta \left[-\frac{1}{\theta^2} + \frac{2(x^2+\theta^2) - 2\theta 2\theta}{(x^2+\theta^2)^2} \right] \\
=& \frac{1}{\theta^2} + \mathbb{E}_\theta \left[\frac{2(\theta^2 - x^2)}{(x^2+\theta^2)^2} \right] \\
=& \frac{1}{\theta^2} + \int_{-\infty}^{\infty} \frac{2(x^2-\theta^2)}{(x^2+\theta^2)^2} \frac{\theta}{\pi(x^2+\theta^2)} dx \\
=& \frac{1}{\theta^2} + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \frac{\frac{x^2}{\theta^2} - 1}{\pi \left(\frac{x^2}{\theta^2} + 1\right)^3} d\frac{x}{\theta} \\
=& \frac{1}{\theta^2} + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \frac{y^2-1}{\pi(y^2+1)^3} dy \\
=& \frac{1}{\theta^2} + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \left[\frac{1}{\pi(y^2+1)^2} - \frac{2}{\pi(y^2+1)^3} \right] dy
\end{aligned}
$$

We have:

$$
\begin{aligned}
\int \frac{1}{(y^2+1)} dy &= \tan^{-1}(y) + c \\
\int \frac{1}{(y^2+1)^2} dy &= \frac{1}{2} \left(\frac{y}{(y^2+1)} + \tan^{-1}(y) \right) + c \\
\int \frac{1}{(y^2+1)^3 dy} &= \frac{1}{8} \left(\frac{3y^3+5y}{(y^2+1)^2} + \tan^{-1}(y) \right) + c
\end{aligned}
$$

So:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left[\frac{1}{\pi(y^2+1)^2} - \frac{2}{\pi(y^2+1)^3} \right] dy \\
=& -\left. \frac{y^3 + 3y + (y^2+1)^2 \tan^{-1}(y)}{4\pi(y^2+1)^2} \right|_{-\infty}^{\infty} \\
=& -\frac{1}{4\pi} \left(\lim_{y\to\infty} \tan^{-1}(y) - \lim_{y\to-\infty} \tan^{-1}(y) \right) \\
=& -\frac{1}{4\pi} \left(\frac{\pi}{2} - -\frac{\pi}{2} \right) = -\frac{1}{4}
\end{aligned}
$$

Thus

$$
I(\theta) = \frac{1}{\theta^2} - \frac{2}{\theta^2} \frac{1}{4} = \frac{1}{2\theta^2}
$$

□

Problem 3. Poisson Processes

Proof. (a) (I swap Y to X, and k through out this problem). We have:

$$\begin{aligned}
\mathbb{P}[X_0 = k_0] =& \frac{\theta^{k_0} e^{-\theta}}{k_0!} \\
\mathbb{P}[X_1 = k_1 \mid X_0 = k_0] =& \frac{\mathbb{P}[X_1 = k_1, X_0 = k_0]}{\mathbb{P}[X_0 = k_0]} \\
\Rightarrow \mathbb{P}[X_1 = k_1, X_0 = k_0] =& \mathbb{P}[X_1 = k_1 \mid X_0 = k_0]\mathbb{P}[X_0 = k_0] \\
=& \frac{(\theta k_0)^{k_1} e^{-\theta k_0}}{k_1!} \frac{\theta^{k_0} e^{-\theta}}{k_0!} \\
\Rightarrow& \mathbb{P}[X_2 = k_2, X_1 = k_1, X_0 = k_0] \\
=& \mathbb{P}[X_2 = k_2 \mid X_1 = k_1, X_0 = k_0]\mathbb{P}[X_1 = k_1, X_0 = k_0] \\
=& \frac{(\theta k_1)^{k_2} e^{-\theta k_1}}{k_2!} \frac{(\theta k_0)^{k_1} e^{-\theta k_0}}{k_1!} \frac{\theta^{k_0} e^{-\theta}}{k_0!}
\end{aligned}$$

By an inductive reasoning, we have:

$$\begin{aligned}
&\mathbb{P}[X_n = k_n, X_{n-1} = k_{n-1}, ..., X_0 = k_0] \\
=& \frac{\theta^{k_0} \prod_{i=1}^{n} (\theta k_{i-1})^{k_i}}{\prod_{i=0}^{n} k_i!} \exp\left(-\theta\left(\sum_{i=0}^{n-1} k_i + 1\right)\right)
\end{aligned}$$

The log-likelihood is then:

$$L(\theta) = -\theta(\sum_{i=0}^{n-1} k_i + 1) + \log\theta \sum_{i=0}^{n} k_i + f(k_0, k_1, ..., k_n)$$

To find the critical point:

$$\begin{aligned}
L'(\theta) =& -\sum_{i=0}^{n-1} -1 + \frac{1}{\theta}\sum_{i=0}^{n} k_i = 0 \\
\Leftrightarrow \theta =& \frac{k_0 + k_1 + ... + k_n}{k_0 + k_1 + ... + k_{n-1} + 1}
\end{aligned}$$

Since $L''(\theta) = -\frac{1}{\theta^2}\sum_{i=0}^{n} k_i < 0$ thus L is concave up on $(0, \infty)$. Thus the critical point is the max. MLE is given by $(\sum_{i=0}^{n} k_i) \,/\, \left(1 + \sum_{i=0}^{n-1} k_i\right)$

(b) The information is:

$$
\begin{aligned}
I(\theta) =& -\mathbb{E}\frac{\partial^2 \log p_\theta(X)}{\partial \theta^2} \\
=&\mathbb{E}\left[\frac{1}{\theta^2}\sum_{i=0}^n X_i\right] \\
=&\frac{1}{\theta^2}\sum_{i=0}^n \mathbb{E}X_i \\
\mathbb{E}X_0 =&\theta \\
\mathbb{E}X_1 =&\mathbb{E}\mathbb{E}[X_1 \mid X_0] \\
=&\mathbb{E}\theta X_0 = \theta\mathbb{E}X_0 = \theta^2 \\
\mathbb{E}X_2 =&\mathbb{E}\mathbb{E}[X_2 \mid X_1] \\
=&\mathbb{E}\theta X_1 = \theta\mathbb{E}X_1 = \theta^3 \\
\mathbb{E}X_i =&\theta^{i+1}, i = 0,1,2,...,n \\
\Rightarrow I(\theta) =&\frac{1}{\theta^2}(\theta + \theta^2 + ... + \theta^{n+1}) \\
=&\frac{1-\theta^{n+1}}{\theta(1-\theta)}, \theta \neq 1
\end{aligned}
$$

Intuitively, if $\theta < 1$, the later X_i will have smaller and smaller information on θ, and this will approach 0. When $\theta > 1$, the later the X_i in the sequence, the more information it contains about θ. This amount of information increases exponentially. □

Problem 4. Error Bound

Proof. (a) Let $q(X,\theta) = \exp(n\theta - \sum X_i)\mathbb{I}[\min X_i \geq \theta]$ be the distribution of vector X. By CS inequality, we have:

$$
\begin{aligned}
\mathrm{Var}_\theta(\delta)\mathrm{Var}_\theta(\psi) \geq& \mathrm{Cov}^2_\theta(\delta,\psi) \\
\Rightarrow \mathrm{Var}_\theta(\delta) \geq& \frac{\mathrm{Cov}^2_\theta(\delta,\psi)}{\mathrm{Var}_\theta(\psi)}.
\end{aligned}
$$

Now note that if $\theta' > \theta$ then $q_\theta(X) = 0 \Rightarrow q_{\theta'}(X) = 0$.(This is true when $\theta' > \theta$ because if $q_\theta(X) = 0 \Rightarrow \min X_i < \theta \Rightarrow \min X_i < \theta' \Rightarrow q_{\theta'}(X) = 0$). Under regularity condition, with the appropriate choice of $\psi(X) = L(x) - 1, L(X) = q_{\theta'}(x)/q_\theta(x), q_\theta(x) > 0, L(X) = 1$, otherwise for $\theta' > \theta$, we have: $\mathrm{Cov}_\theta(\delta,\psi) = g(\theta') - g(\theta)$. Plugging this back to the inequality above, we have:

$$
\begin{aligned}
\mathrm{Var}_\theta(\delta) \geq& \frac{(g(\theta') - g(\theta))^2}{\mathbb{E}\left(\frac{q(X,\theta')}{q(X,\theta)} - 1\right)^2}, \forall \theta' > \theta \\
\Rightarrow \mathrm{Var}_\theta(\delta) \geq& \sup_{\theta' > \theta} \frac{(g(\theta') - g(\theta))^2}{\mathbb{E}\left(\frac{q(X,\theta')}{q(X,\theta)} - 1\right)^2}
\end{aligned}
$$

(b) We will calculate:

$$
\begin{aligned}
A =& \mathbb{E}_\theta\left[\left(\frac{q(X,\theta')}{q(X,\theta)}-1\right)^2\right] \\
=& \mathbb{E}_\theta\left[\left(\frac{q(X,\theta')}{q(X,\theta)}-1\right)^2 \middle| \min X_i < \theta'\right]\mathbb{P}[\min X_i < \theta'] + \\
&+ \mathbb{E}_\theta\left[\left(\frac{q(X,\theta')}{q(X,\theta)}-1\right)^2 \middle| \min X_i \geq \theta'\right]\mathbb{P}[\min X_i \geq \theta'] \\
=& \mathbb{E}_\theta\left[\left(\frac{0}{q(X,\theta)}-1\right)^2\right](1-\mathbb{P}[\min X_i \geq \theta']) + \\
&+ \mathbb{E}_\theta\left[\left(\frac{q(X,\theta')}{q(X,\theta)}-1\right)^2 \middle| \min X_i \geq \theta'\right]\mathbb{P}[\min X_i \geq \theta']
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbb{P}[\min X_i \geq \theta'] =& \mathbb{P}[X_i \geq \theta', \forall X_i] \\
=& \mathbb{P}^n[X_1 \geq \theta'] \\
=& \exp^n(-\theta' + \theta) \\
=& \exp(-n(\theta'-\theta))
\end{aligned}
$$

Thus the above expectation is:

$$
\begin{aligned}
A =& 1-\exp(-n(\theta'-\theta)) + \mathbb{E}_\theta\left[\left(\frac{q(X,\theta')}{q(X,\theta)}-1\right)^2 \middle| \min X_i \geq \theta'\right]\exp(-n(\theta'-\theta)) \\
=& 1-\exp(-n(\theta'-\theta)) + \mathbb{E}_\theta\left[\left(\exp\left\{n\theta' - \sum X_i - n\theta + \sum X_i\right\}-1\right)^2\right]\exp(-n(\theta'-\theta)) \\
=& 1-\exp(-n(\theta'-\theta)) + \left(\exp\left\{n\theta'-n\theta\right\}-1\right)^2\exp(-n(\theta'-\theta)) \\
=& 1-\frac{1}{B}+(B-1)^2\frac{1}{B}, B=\exp(n(\theta'-\theta)) \\
=& 1-\frac{1}{B}+B-2+\frac{1}{B} = B-1 = \exp(n(\theta'-\theta))-1
\end{aligned}
$$

Plugging this result of A back to our inequality obtained from (a), we have:

$$
\mathrm{Var}_\theta(\delta) \geq \sup_{\theta' \geq \theta} \frac{(\theta'-\theta)^2}{\exp(n(\theta'-\theta))-1}
$$

Let $y=\theta'-\theta$, limit to $y>0$, then the derivative of $h(y)=\frac{y^2}{\exp(ny)-1}$ is equal to zero iff:

$$
\begin{aligned}
& 2y(\exp(ny)-1) - y^2 n\exp(ny) = 0 \\
\Leftrightarrow & \frac{2}{ny} - \frac{\exp(ny)}{\exp(ny)-1} = 0 \\
\Leftrightarrow & 2\exp(ny)-2 = ny\exp(ny) \\
\Leftrightarrow & (2-ny)\exp(ny) = 2
\end{aligned}
$$

Let $C>0$ be the solution to $(2-x)e^x=2$, then C is unique (˜1.59) and constant with respect to n. Note that $C=na^*$ for a^* from the problem ($a^*=C/n$ is dependent on n). Since $\lim_{y\to 0^+} h(y) = \lim_{y\to\infty} h(y) = 0$, and $h(y) > 0, \forall y > 0$. Thus there is a maximum for $h(y)$. Thus the maximum is attained at the unique critical point of $a^* = C/n$. At this point a^*, we have:

$$h(y^*) = \frac{(\frac{C}{n})^2}{\exp C - 1} = \frac{1}{n^2} \frac{C^2}{\exp C - 1} = \frac{1}{n^2} \frac{C^2}{\frac{2}{2-C} - 1} = \frac{1}{n^2} C(2 - C).$$

We should stop here since this is a $\mathcal{O}(1/n^2)$ bound, better than the $\mathcal{O}(1/n^3)$ bound a^*/n^2 that the problem asks for. However to get the bound the problem ask for, we need to prove:

$$\begin{aligned} C(2 - C) \geq & a^* = \frac{C}{n} \\ \Leftrightarrow 2 - C \geq & \frac{1}{n} \\ \Leftrightarrow n \geq & \frac{1}{2 - C} \sim 2.46 \end{aligned}$$

So the problem's inequality is only true for $n \geq 3$.

(c) The lower bound for error $\mathcal{O}(1/n^2)$ that we gets from part (b) is better than smaller than the lower bound for error $\mathcal{O}(1/n)$ that one normally gets. That means we can have lower error for this problem. We make a guess this is the case because for this problem, the distribution is one one side of θ, thus we get additional precision when predicting θ because we know for sure θ must be smaller than $\min X_i$.

(d) From part (b), we have $\mathbb{P}[\min X_i \geq t] = \exp(-n(t - \theta)) \Rightarrow \mathbb{P}[\min X_i \leq t] = 1 - \exp(-n(t - \theta))$. Thus the density for $\min X_i$ is $f(t) = n \exp(-n(t - \theta))$ for $t \geq \theta$, and zero otherwise.

$$\begin{aligned} \mathbb{E} \min X_i = & \int_\theta^\infty tn \exp(-n(t - \theta)) dt \\ = & \exp(n\theta) \int_\theta^\infty tn \exp(-nt) \frac{1}{n} dnt \\ = & \exp(n\theta) \frac{1}{n} \int_{n\theta}^\infty y \exp(-y) dy \\ = & \frac{1}{n} \exp(n\theta) \left(-\frac{y + 1}{\exp x} \right) \Bigg|_{n\theta}^\infty \\ = & \frac{n\theta + 1}{n} = \theta + \frac{1}{n} \end{aligned}$$

Thus $\delta(X) = \min X_i - \frac{1}{n}$ is an unbiased estimator of θ. Now we calculate the variance:

$$
\begin{aligned}
\mathrm{Var}(\min X_i - \frac{1}{n}) =& \mathrm{Var}(\min X_i) \\
=& \mathbb{E}\left[(\min X_i)^2\right] - (\mathbb{E}\min X_i)^2 \\
\mathbb{E}\left[(\min X_i)^2\right] =& \int_\theta^\infty t^2 n \exp(-n(t-\theta))dt \\
=& \exp(n\theta) \int_\theta^\infty t^2 n^2 \frac{1}{n} \exp(-nt) \frac{1}{n} dnt \\
=& \frac{1}{n^2} \exp(n\theta) \int_{n\theta}^\infty y^2 \exp(-y) dy \\
=& - \frac{\exp(n\theta)}{n^2} \frac{x^2+2x+2}{\exp x}\Big|_{n\theta}^\infty \\
=& \frac{n^2\theta^2 + 2n\theta + 2}{n^2} \\
\Rightarrow \mathrm{Var}(\min X_i - \frac{1}{n}) =& \frac{n^2\theta^2 + 2n\theta + 2}{n^2} - \frac{\theta^2 n^2 + 2n\theta + 1}{n^2} \\
=& \frac{1}{n^2}
\end{aligned}
$$

□

Problem 5. Poisson Truncated

Proof. We have the density for truncated Poisson for $k \geq 1$ is:

$$
\begin{aligned}
\mathbb{P}\left[X_i = k\right] =& \frac{1}{1-\exp(-\lambda)} \frac{\lambda^k \exp(-\lambda)}{k!} \\
\Rightarrow p_X(x) = \mathbb{P}[X_i = k_i, \forall X_i] =& \frac{1}{(1-\exp(-\lambda))^n} \frac{\lambda^{\sum x_i} \exp(-n\lambda)}{\prod_{i=1}^n k_i!} \\
\Rightarrow \log p_X(x,\lambda) =& -n\log(1-\exp(-\lambda)) + \sum x_i \log\lambda - n\lambda + h(k_1, k_2, ..., k_n) \\
\Rightarrow \frac{\partial \log p_X(x,\lambda)}{\partial\lambda} =& -n + \frac{1}{\lambda}\sum x_i - n\frac{\exp(-\lambda)}{1-\exp(-\lambda)} \\
\Rightarrow \frac{\partial \log p_X(x,\lambda)}{\partial\lambda} =& \frac{1}{\lambda}\sum x_i - n\frac{1}{1-\exp(-\lambda)} \\
\Rightarrow \frac{\partial^2 \log p_X(x,\lambda)}{\partial\lambda^2} =& -\frac{1}{\lambda^2}\sum x_i + n\frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2} \\
\Rightarrow I(\theta) =& \mathbb{E}\left[\frac{1}{\lambda^2}\sum x_i - n\frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}\right]
\end{aligned}
$$

Now for $Y \sim Poi(\lambda)$ (not truncated), then:

$$
\begin{aligned}
\lambda = \mathbb{E}Y =& \mathbb{E}\left[Y \mid Y=0\right]\mathbb{P}[Y=0] + \mathbb{E}[Y \mid Y>0]\mathbb{P}[Y>0] \\
\Rightarrow \lambda =& \mathbb{E}[Y \mid Y>0](1-\exp(-\lambda)) \\
\Rightarrow \mathbb{E}[Y \mid Y>0] =& \frac{\lambda}{1-\exp(-\lambda)} \\
\Rightarrow \mathbb{E}X_i =& \frac{\lambda}{1-\exp(-\lambda)} \\
\Rightarrow I(\theta) =& \frac{1}{\lambda^2} n \frac{\lambda}{1-\exp(-\lambda)} - n\frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2} \\
=& \frac{n}{(1-\exp(-\lambda))^2\lambda}\left(1-\exp(-\lambda)-\lambda\exp(-\lambda)\right)
\end{aligned}
$$

Thus the information bound for any unbiased estimator of λ is:

$$
\mathrm{Var}(\delta) \geq \frac{(g'(\lambda))^2}{I(\theta)} = \frac{\lambda(1-\exp(-\lambda))^2}{n(1-\exp(-\lambda)-\lambda\exp(-\lambda))}
$$

$\square$

P/S. This is not even a homework. This is like a chapter in a book. Good one.