UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 6- Solutions

Fall 2014

Issued: Friday, Oct 10
Due: Thursday, Oct 16

Problem 6.1

By CLT, the mean:

$$\sqrt{n}(\bar{X} - \theta) \xrightarrow{d} \mathcal{N}(0, \theta). \tag{1}$$

On the other hand, for sample variance $S^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2$, we prove it as follows that $\frac{n-1}{\theta}S^2$ can be written as a sum of $n-1$ square of independent normal r.v. Surely there exists an orthogonal matrix $A = (a_{ij})$ with first row equals to $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$. Let $Y_i = \sum_{k=1}^n a_{ik}X_k,\ (i = 1, ..., n)$. By orthogonality, $Y_1, ..., Y_n$ are independent and $Y_1 \sim \mathcal{N}(\sqrt{n}\mu, \theta),\ Y_i \sim \mathcal{N}(0, \theta)$. Note we have: $\bar{X} = \frac{1}{\sqrt{n}}Y_1 \sim \mathcal{N}(\mu, \theta)$ and $\sum_{i=1}^n X_i^2 = \sum_{i=1}^n Y_i^2$. Therefore:

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n(\bar{X})^2 = \sum_{i=2}^n Y_i^2$$

which suggests $\frac{n-1}{\theta}S^2$ is sum of $n-1$ independent r.v with bounded second moment(satisfies χ^2_{n-1}). Since $Var(Y_i^2) = 2\theta^2$:

$$\sqrt{n-1}(S^2 - \theta) \xrightarrow{d} \mathcal{N}(0, 2\theta^2). \tag{2}$$

Combine (1) and (2) then the asymptotic ARE of S^2 with respect to $\bar{X}$ equals to $\frac{1}{2\theta}$.

Problem 6.2

(1) Write the density function of $X_{(2)}$: $\mathbb{P}(X_{(2)} = x) = n(n-1)(1 - e^{-x})e^{-x}e^{-(n-2)x}$. Therefore:

$$\begin{aligned}
\mathbb{P}(X_{(2)} \le y) &= \int_0^y n(n-1)[e^{-(n-1)x} - e^{-nx}]dx = n(1 - e^{-(n-1)y}) - (n-1)(1 - e^{-ny}) \\
&= 1 - ne^{-(n-1)y} + (n-1)e^{-ny}
\end{aligned}$$

and

$$\mathbb{P}(X_{(2)} \le xn^{-p}) = 1 - ne^{-(n-1)n^{-p}x} + (n-1)e^{-n^{1-p}x}.$$

For p=1, the above equals to $1 + e^{-x}(n - 1 - ne^{-x/n}) = 1 - e^{-x}(1 + x)$ by taylor expansion $e^{-x/n} = 1 - x/n + o(x/n)$.

(2) Similarly $\mathbb{P}(X_{(n)} \leq x) = (1 - e^{-x})^n$ therefore:

$$\mathbb{P}(X_{(n)} \leq x + \log n) = (1 - e^{-x - logn})^n = e^{-e^{-x}}.$$

Problem 6.3

Let a_p is p^{th} quantiles of the posterior distribution, where $p = \frac{k_1}{k_1+k_2}$. *i.e.*

$$\mathbb{P}(\theta \leq a_p|X) \geq \frac{k_1}{k_1 + k_2} \; and \; \mathbb{P}(\theta \geq a_p|X) \geq \frac{k_2}{k_1 + k_2}$$

Then for $a < a_p$,

$$\begin{aligned}
\mathbb{E}(L(\theta, a_p)|X) &= \mathbb{E}\left(k_1(\theta - a_p)I(a_p \leq \theta) + k_2(a_p - \theta)I(a_p > \theta)|X\right) \\
&= \mathbb{E}\left(k_1(\theta - a)I(a \leq \theta) + k_2(a - \theta)I(a > \theta)|X\right) \\
&\quad +(a - a_p)\left((k_1 + k_2)\mathbb{P}(\theta \geq a_p|X) - k_2\right) + (k_1 + k_2)\mathbb{E}\left((a - \theta)\mathbf{1}(a \leq \theta < a_p)\right) \\
&\leq \mathbb{E}\left(k_1(\theta - a)I(a \leq \theta) + k_2(a - \theta)I(a > \theta)|X\right)
\end{aligned}$$

For $a > a_p$,

$$\begin{aligned}
\mathbb{E}(L(\theta, a_p)|X) &= \mathbb{E}\left(k_1(\theta - a_p)I(a_p \leq \theta) + k_2(a_p - \theta)I(a_p > \theta)|X\right) \\
&= \mathbb{E}\left(k_1(\theta - a)I(a \leq \theta) + k_2(a - \theta)I(a > \theta)|X\right) \\
&\quad +(a - a_p)\left(k_1 - (k_1 + k_2)\mathbb{P}(\theta \leq a_p|X)\right) + (k_1 + k_2)\mathbb{E}\left((\theta - a)\mathbf{1}(a_p < \theta \leq a)\right) \\
&\leq \mathbb{E}\left(k_1(\theta - a)I(a \leq \theta) + k_2(a - \theta)I(a > \theta)|X\right)
\end{aligned}$$

Thus, Bayes estimators are p^{th} quantiles of the posterior distribution, where $p = \frac{k_1}{k_1+k_2}$.

Problem 6.4

$$l(\theta) \propto -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij} - \mu_i)^2 - \frac{nr}{2}\log\sigma^2$$

Because it is normal distribution, $\frac{\partial l(\theta)}{\partial \theta} = 0 \;\Rightarrow\; \theta = \hat{\theta}^{MLE}$.

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial \mu_i} &= -\frac{1}{\sigma^2}\sum_{j=1}^{r}(\mu_i - x_{ij}) = 0 \\
\frac{\partial l(\theta)}{\partial \sigma^2} &= \frac{1}{2\sigma^2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij} - \mu_i)^2 - nr\right) = 0
\end{aligned}$$

Thus, $\hat{\mu_i}^{MLE} = x_{i\cdot}$, $\hat{\sigma^2}^{MLE} = \frac{1}{nr}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij} - x_{i\cdot})^2$, where $x_{i\cdot} = \frac{1}{r}\sum_{j=1}^{r}x_{ij}$.

Note that $\frac{\sum_{j=1}^{r}(x_{ij} - x_{i\cdot})^2}{\sigma^2} \sim$ *i.i.d.* $\chi^2(r-1)$ and $\mathbb{E}\left(\sum_{j=1}^{r}(x_{ij} - x_{i\cdot})^2\right) = (r-1)\sigma^2$. Thus, by Weak Law of Large Number(WLLN), $\hat{\sigma^2}^{MLE} \rightarrow \frac{r-1}{r}\sigma^2 \neq \sigma^2$ as $n \rightarrow \infty$.

Problem 6.5

(a) Let $l(\theta) = \log p(X_1, \ldots, X_n) = \dfrac{1}{n}\sum_{i=1}^{n} \log\left(\theta f_1(X_i) + (1-\theta) f_2(X_i)\right)$

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial \theta} &= \frac{1}{n}\sum_{i=1}^{n} \frac{f_1(X_i) - f_2(X_i)}{\theta f_1(X_i) + (1-\theta) f_2(X_i)} = g(\theta) \\
g'(\theta) &= -\frac{1}{n}\sum_{i=1}^{n} \frac{(f_1(X_i) - f_2(X_i))^2}{(\theta f_1(X_i) + (1-\theta) f_2(X_i))^2} \leq 0
\end{aligned}$$

If $f_1(X_i) = f_2(X_i)$ for all $i = 1, \ldots, n$, then $g(\theta) = 0$, $\forall\, \theta \in (0,1)$. Thus, solution is not unique. Thus, $g'(\theta) = 0 \;\Leftrightarrow \exists \theta_1 \neq \theta_2 \; s.t. \; g(\theta_1) = g(\theta_2) = 0$.

Therefore, to have a unique solution, $g'(\theta) < 0$ (Sufficient and necessary condition). Also, $g(1) < 0$ and $g(0) > 0$.

$$g(1) < 0,\; g(0) > 0 \;\Leftrightarrow\; \frac{1}{n}\sum_{i=1}^{n} \frac{f_1(X_i)}{f_2(X_i)} > 1 \; and \; \frac{1}{n}\sum_{i=1}^{n} \frac{f_2(X_i)}{f_1(X_i)} > 1$$

$g'(\theta) = \dfrac{\partial^2 l(\theta)}{\partial \theta^2}$. Thus, if there is a solution, trivially, it is MLE.

(b) If $\frac{1}{n}\sum_{i=1}^{n} \frac{f_1(X_i)}{f_2(X_i)} < 1$ *or* $\frac{1}{n}\sum_{i=1}^{n} \frac{f_2(X_i)}{f_1(X_i)} < 1$, there is no solution for the score function:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \frac{f_1(X_i)}{f_2(X_i)} < 1 \quad &\Leftrightarrow \quad g(\theta) < 0 \; \forall \theta \in (0,1) \Rightarrow \hat{\theta}^{MLE} = 0 \\
\frac{1}{n}\sum_{i=1}^{n} \frac{f_2(X_i)}{f_1(X_i)} < 1 \quad &\Leftrightarrow \quad g(\theta) > 0 \; \forall \theta \in (0,1) \Rightarrow \hat{\theta}^{MLE} = 1
\end{aligned}$$