UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 5
Fall 2014

Issued: Friday, October 3 **Due:** Thursday, October 9 (beginning of class)

Problem 5.1
Consider a Bayesian model in which the prior distribution for θ is uniform on $(0, 1)$ and, given θ, the observations $X_1, \ldots, X_n$ are i.i.d. Bernoulli with success probability θ. Find

$$P(X_{n+1} = 1 \mid X_1, \ldots, X_n).$$

Problem 5.2
Let $X_1, X_2, \ldots, X_n$ be sampled conditionally independently from $N(\mu, \sigma^2)$, where μ and σ^2 are considered random. In class we presented a conjugate prior for μ when σ^2 is treated as fixed and a conjugate prior for σ^2 when μ is treated as fixed. Show that the prior obtained by assuming μ and σ^2 are independent and taking the product of the priors presented in class is not conjugate for the joint parameter (μ, σ^2). Provide a conjugate prior. Discuss a practical data analysis situation in which the conjugate prior seems appropriate and a data analysis situation in which the non-conjugate product prior seems appropriate.

Problem 5.3
Find a transform of θ, $\eta = h(\theta)$, such that the Fisher information $I(\eta)$ is constant (and therefore the Jeffreys prior is constant) for:

- the binomial distribution, $\text{Bin}(n, \theta)$;
- the gamma distribution, $\text{Ga}(a, \theta)$, with $a = 1, 2, 3$; and
- the Maxwell distribution, $\text{Max}(\theta) : p(x|\theta) \propto \theta^{3/2} x^2 e^{-\theta x/2}, x \geq 0, \theta > 0$.

Problem 5.4
In a linear regression model the n-vector of responses y has distribution $(y \mid \beta) \sim N(X\beta, I_n)$, with mean response vector $\mu = E(y \mid \beta) = X\beta$ and identity variance matrix, where X is the $n \times p$ design matrix of rank p and β is the p-vector of regression coefficients. Suppose that the prior for β is $\beta \sim N(0, g^{-1}(X'X)^{-1})$ for some number $g > 0$.

1. What is the posterior distribution of $(\beta \mid y)$?
2. Show that posterior mean $E(\beta \mid y)$ can be expressed as a function of $\hat{\beta}$, the usual MLE of β.
3. What is the posterior mean $E(\mu \mid y)$?

4. What is the posterior variance matrix of μ?

5. Consider the special case of an orthogonal design, so that $X'X = I_p$. Denote by μ_i the ith element of μ. Under the posterior $p(\mu \,|\, y)$ are μ_j and μ_k independent for $j \neq k$?

Problem 5.5

Consider a Bayesian model in which given θ the observations $X_1, \ldots, X_n$ are i.i.d. Bernoulli with success probability θ.

1. Let $(\pi(1), \ldots, \pi(n))$ be a permutation of $(1, \ldots, n)$. Show that

$$(X_{\pi(1)}, \ldots, X_{\pi(n)}) \text{ and } (X_1, \ldots, X_n)$$

   have the same distribution. When this holds the variables are said to be *exchangeable*.

2. Show that $\mathrm{Cov}(X_i, X_j) \geq 0$. When will this covariance be zero?