STAT210A - Homework 11

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December 13, 2014

Problem 1. Sub-Gaussian bounds and means/variances. Given:

$$\mathbb{E}\left[\exp\left\{\lambda(X-\mu)\right\}\right] \le \exp\left\{\frac{\lambda^2\sigma^2}{2}\right\} \tag{1}$$

Proof. (a) We have:

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\lambda(X-\mu)\right\}\right] =&\mathbb{E}\left[\sum_{i=0}^{\infty}\frac{(\lambda(X-\mu))^i}{i!}\right] = \sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\mathbb{E}(X-\mu)^i && (2)\\
\Rightarrow 1+\lambda\mathbb{E}(X-\mu)+\frac{\lambda^2}{2}\mathbb{E}(X-\mu)^2 \le&\mathbb{E}\exp\left\{\lambda(X-\mu)\right\},\text{ as } \lambda\to 0 && (3)\\
\le&\exp\left\{\frac{\lambda^2\sigma^2}{2}\right\} && (4)\\
=&1+\frac{\lambda^2\sigma^2}{2}+o(\lambda^4) && (5)\\
\lambda\mathbb{E}(X-\mu)+\frac{\lambda^2}{2}\mathbb{E}(X-\mu)^2 \le&\frac{\lambda^2\sigma^2}{2}+o(\lambda^4),\text{ as } \lambda\to 0 && (6)
\end{aligned}$$

Dividing both sides of (6) by $\lambda > 0$, and letting $\lambda \to 0^+$, we have $\mathbb{E}\left[X-\mu\right] \le 0$.
Dividing both sides of (6) by $\lambda < 0$, and letting $\lambda \to 0^-$, we have $\mathbb{E}\left[X-\mu\right] \ge 0$.
Thus $\mathbb{E}X = \mu$
(b) Dividing both sides of (6) by $\lambda \ne 0$, and letting $\lambda \to 0$, we have $\text{Var}X \le \sigma^2$
(c) It is not true. Assuming that $\text{Var}\left(X\right) = \sigma^2$, from (1) we have:

$$\frac{\lambda^2}{2}\text{Var}(X)+\frac{\lambda^3}{6}\mathbb{E}\left[X-\mu\right]^3+o(\lambda^4) \le \frac{\lambda^2}{2}\sigma^2+o(\lambda^4), \forall\lambda \tag{7}$$

So if we pick a sub-gaussian RV X that is skewed to the right, i.e. $\mathbb{E}\left[X-\mu\right]^3 > 0$, then (7) is not true as $\lambda \to 0$. We can construct for example X :

$$\begin{aligned}
\mathbb{P}\left[X=1\right] =&\mathbb{P}\left[X=3\right] = 0.25\\
\mathbb{P}\left[X=2\right] =&0.5
\end{aligned}$$

Then:

$$\begin{aligned}
\mathbb{E}X =&0\\
\mathbb{E}X^2 =&\frac{1}{4}\left(1+3^2-2\times 2^2\right) = \frac{1}{2}\\
\mathbb{E}X^3 =&\frac{1}{4}\left(1+3^3-2\times 2^3\right) = 3 > 0
\end{aligned}$$

To make sure we double check. Apparently X is sub-gaussian since it is bounded. The MGF is:

$$\mathbb{E}\left[\exp\left\{\lambda X\right\}\right] = \frac{1}{4}\left(\exp\lambda + \exp\left(3\lambda\right) - 2\exp\left(2\lambda\right)\right) \tag{8}$$

$$\exp\left\{\frac{\lambda^2}{2} \times \frac{1}{2}\right\} = \exp\left(\frac{\lambda^2}{4}\right) \tag{9}$$

We can check that (9) is not bigger than (8) for all λ; e.g. for $\lambda = 1$, (9) $<$ (8) . □

Problem 2. Gaussian Maxima

Proof. (a) First, to simplify notation, we note that we can just prove for the case of standard Gaussian, as we can rescale Z/σ in all the inequality. So we can assume that $\sigma = 1$.

For $\lambda > 0, \exp(\lambda x)$ is a convex function in x. Using Jensen inequality, we have:

$$\begin{aligned}
\exp\left(\lambda \mathbb{E} Z\right) \leq & \mathbb{E}\exp\left(\lambda Z\right) \quad \text{(Jensen)} && (10)\\
= & \mathbb{E} \max_{i=1,\ldots,n} \exp\left(\lambda \left|X_i\right|\right) && (11)\\
\leq & \sum_{i=1}^{n} \mathbb{E}\left[\exp\left(\lambda\left|X_i\right|\right)\right] && (12)\\
= & n\mathbb{E}\left[\exp\left(\lambda\left|X_i\right|\right)\right] && (13)\\
= & 2n\int_0^{\infty} \exp\left(\lambda x\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) && (14)\\
= & n\exp\left(\frac{\lambda^2}{2}\right)\left(\operatorname{erf}\left(\frac{\lambda}{\sqrt{2}}\right) + 1\right) && (15)
\end{aligned}$$

, for

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right) dt.$$

Taking log on both side of (15), and for $\lambda = \sqrt{2\log n}$, we have:

$$\begin{aligned}
\lambda \mathbb{E} Z \leq & \log n + \frac{\lambda^2}{2} + \log\left(\operatorname{erf}\left(\frac{\lambda}{\sqrt{2}}\right) + 1\right)\\
\Rightarrow \mathbb{E} Z \leq & \frac{\log n}{\lambda} + \frac{\lambda}{2} + \frac{1}{\lambda}\log\left(\operatorname{erf}\left(\frac{\lambda}{\sqrt{2}}\right) + 1\right)\\
\Rightarrow \mathbb{E} Z \leq & \sqrt{2\log n} + \frac{1}{\sqrt{2\log n}}\log\left(\operatorname{erf}\left(\sqrt{\log n}\right) + 1\right).
\end{aligned}$$

Now if we use the inequality for the error function:

$$\begin{aligned}
\operatorname{erf}(x) \leq & 1\\
\Rightarrow \mathbb{E} Z \leq & \sqrt{2\log n} + \frac{\log 2}{\sqrt{2\log n}}\\
\leq & \sqrt{2\log n} + \frac{4}{\sqrt{2\log n}}
\end{aligned}$$

Note that we obtain a slightly sharper bound as $\log 2 < 4$.

(b) We state a lemma without knowing how to prove. The lemma says for a constant c such that:

$$\mathbb{E}\left[\left\{i \mid |X_i| \geq c\right\}\right] \geq 1$$

, then

$$\begin{aligned}\mathbb{E}Z \geq& c\\ Z =& \max_{i=1,\ldots,n} X_i\end{aligned}$$

. In words, if we expect to have at least one $|X_i| \geq c$, then the maximum over all $|X_i|' s$ should have expectation at least c.

Using this lemma, and let $d > 0$, we check the condition:

$$\begin{aligned}\mathbb{E}\left[\left\{i \mid |X_i| \geq d\sqrt{2\log n}\right\}\right] =& n\mathbb{P}\left[|X_i| \geq d\sqrt{2\log n}\right]\\ =& n\left(1 - \operatorname{erf}\left(d\sqrt{\log n}\right)\right)\end{aligned}$$

We need:

$$\begin{aligned}n\left(1 - \operatorname{erf}\left(d\sqrt{\log n}\right)\right) \geq& 1\\ \Leftrightarrow \operatorname{erf}\left(d\sqrt{\log n}\right) \leq& 1 - \frac{1}{n}\end{aligned}$$

Using the inequality for erf function:

$$\operatorname{erf}(x) \leq \sqrt{1 - \exp\left(-4x^2/\pi\right)}$$

, it suffice if we have:

$$\begin{aligned}\sqrt{1 - \exp\left(-\frac{4d^2 \log n}{\pi}\right)} \leq& 1 - \frac{1}{n}\\ \Leftrightarrow 1 - \frac{1}{n^{4d^2/\pi}} \leq& 1 - \frac{2}{n} + \frac{1}{n^2}\\ \Leftrightarrow \frac{2}{n} \leq& \frac{1}{n^{4d^2/\pi}} + \frac{1}{n^2}\\ \Leftrightarrow 2n \leq& n^{2-4d^2/\pi} + 1\end{aligned}$$

Now for $d = 1 - \frac{1}{e}, \Rightarrow 2 - 4d^2/\pi \approx 1.49$, we have the last inequality is true for n big enough. In fact it is true for all $n \geq 3$.

(c) $\square$