UC Berkeley

Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 7- Solutions

Fall 2014

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Problem 7.1

Since μ is strictly monotonic and continuously differentiable, so its inverse function exits. By delta method and CLT, we have

$$\sqrt{n}(\hat{\theta}_n - \theta) \to \mathcal{N}(0, (\frac{\sigma}{\mu'(\theta)})^2)$$

where we use the fact that $(\mu^{-1})' = 1/\mu'(\theta)$.

Problem 7.2

(a) Let $W_i = t^{X_i} - X_i^2$ and $G_n = \frac{1}{n}\sum_{i=1}^n t^{X_i} - X_i^2$. Since $|W_i(t)| \leq |t^{X_i}| + |X_i^2| \leq 2$, $E\|W_i(t)\|_\infty < \infty$ and then mean function is

$$EG_n := G(t) = \begin{cases} \frac{1}{3} & t = 0 \\ \frac{t-1}{\log t} - \frac{1}{3} & t \in (0,1) \\ \frac{2}{3} & t = 1. \end{cases}$$

By Theorem 9.2 in Keener, we have $\|G_n - G\|_\infty \xrightarrow{p} 0$. Since $G(t) = 0$ has a unique solution c in $(0,1)$ which satisfies $3c - \log c - 3 = 0$ then By Theorem 9.3(3), $T_n \xrightarrow{p} c$.

(b) Writing the taylor expansion of G_n about c, gives us:

$$G_n(T_n) = G_n(c) + G_n'(t_n^*)(T_n - c) \tag{1}$$

where t_n^* lies between c and T_n. Note that $G_n(t_n) = 0$ then:

$$\sqrt{n}(T_n - c) = -\frac{\sqrt{n}G_n(c)}{G_n'(t_n^*)}.$$

Considering the denominator $G_n'(t) = \frac{1}{n}\sum_{i=1}^n X_i t^{X_i - 1}$, write is as:

$$G_n'(t) = \frac{1}{n}\sum_{i=1}^n X_i t^{X_i-1}\mathbf{1}_{t\in[c/2,3c/2]} + X_i t^{X_i-1}\mathbf{1}_{t\notin[c/2,3c/2]} := Y_n(t) + Z_n(t).$$

Because $\|Y_n(t)\|_\infty < \infty$, Theorem 9.2 implies that the supreme norm:

$$\|Y_n(t) - EX_it^{X_i-1}\mathbf{1}_{t\in[c/2,3c/2]}\|_\infty \xrightarrow{p} 0.$$

By the use of Lemma 9.15 and Theorem 9.4,

$$G_n'(t_n^*) \xrightarrow{p} EX_ic^{X_i-1} = \frac{c\log c - c + 1}{c(\log c)^2} := C_1$$

For the numerator, by CLT $\sqrt{n}G_n(c)$ goes to $\mathcal{N}(0, var(c^{X_i} - X_i^2))$ where

$$var(c^{X_i} - X_i^2) = \frac{c^2-1}{2\log c} + \frac{-4c - 2c[\log c - 2]\log c + 4}{\log^3 c} + \frac{1}{5} := C_2.$$

Combine the above two arguments in (1), we have:

$$\sqrt{n}(T_n - c) \xrightarrow{d} \mathcal{N}(0, C_2/C_1^2). \tag{2}$$

Problem 7.3

(a) We know that $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0,1)$ from the central limit theorem. We then apply the delta method to the function $h(u) = 1/u$, which for $u \neq 0$ has derivative $h'(u) = -1/u^2$ to conclude that

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) \quad \xrightarrow{d} \quad N\left(0, \frac{1}{\theta^4}\right)$$

as claimed.

(b) Here we write (formally) that

$$\begin{aligned}\sqrt{\frac{2\pi}{n}}\mathbb{E}(1/\bar{X}_n) &= \int_{y\neq 0}\frac{1}{y}\exp\left(-\frac{n}{2}(y-\theta)^2\right)dy \\ &= \int_0^\infty \frac{1}{y}\exp\left(-\frac{n}{2}(y-\theta)^2\right)dy + \int_{-\infty}^0 \frac{1}{y}\exp\left(-\frac{n}{2}(y-\theta)^2\right)dy\end{aligned}$$

For $\theta > 0$, there exist ε such that $0 < \varepsilon_1 < \theta$. Then,

$$\int_0^\infty \frac{1}{y}\exp\left(-\frac{n}{2}(y-\theta)^2\right)dy \quad > \quad \exp\left(-\frac{n}{2}\theta^2\right)\int_0^{\varepsilon_1}\frac{1}{y}dy = \infty$$

for $\theta < 0$, there exist ε such that $\theta < \varepsilon_2 < 0$. Then,

$$\int_{-\infty}^0 \frac{1}{y}\exp\left(-\frac{n}{2}(y-\theta)^2\right)dy \quad < \quad \exp\left(-\frac{n}{2}\theta^2\right)\int_{\varepsilon_2}^0\frac{1}{y}dy = -\infty$$

Thus, $\mathbb{E}(1/\bar{X}_n)$ fails to exist for all n. But this does not contradict the result from (a) because convergence in distribution does not imply convergence in expectation. Actually, when $\theta \neq 0$, $\frac{1}{\bar{X}_n} \xrightarrow{p} \frac{1}{\theta}$, because $\bar{X}_n \xrightarrow{p} \theta$ and $h(\theta) = 1/\theta$ is continuous at $\theta \neq 0$.

Problem 7.4

(a) By CLT, we have

$$\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta).$$

Let $g(\theta) = 2\sqrt{\theta}$ and by delta method we could show:

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, g'(\theta)^2\theta) \sim \mathcal{N}(0, 1).$$

(b) Let Z_β to be the β-quantile of standard normal distribution, then

$$P(\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \in [Z_{\alpha/2}, Z_{1-\alpha/2}]) \to 1 - \alpha$$

this implies that:

$$\left[(\sqrt{\hat{\theta}_n} - \frac{Z_{1-\alpha/2}}{2\sqrt{n}})^2, (\sqrt{\hat{\theta}_n} - \frac{Z_{\alpha/2}}{2\sqrt{n}})^2\right]$$

is an $1 - \alpha$ asymptotic confidence interval.

Problem 7.5

(a) It follows the Theorem 9.14 the normality of MLE:

$$\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}(0, I^{-1}(\mu))$$

where

$$I(\mu) = E\left(\frac{\partial \log f(x,\mu)}{\partial \mu}\right)^2 = \left(E\frac{\partial \log f(x,\mu)}{\partial \mu}\right)^2 + var\left(\frac{\partial \log f(x,\mu)}{\partial \mu}\right).$$

Let $Z(\mu) = (X - \mu)/g(\mu)$ then $Z(\mu) \sim \mathcal{N}(0, 1)$ for any μ.

$$\begin{aligned}
\frac{\partial \log f(x,\mu)}{\partial \mu} &= \frac{\partial}{\partial \mu}\left(-\log g(\mu) - \frac{(x-\mu)^2}{2g^2(\mu)}\right) \\
&= -\frac{g'}{g} + \frac{X-\mu}{g^2} + \frac{(X-\mu)^2 g'}{g^3} \\
&= -\frac{g'}{g} + \frac{Z}{g} + \frac{Z^2 g'}{g}.
\end{aligned}$$

It is easier to check that $E\frac{\partial \log f(x,\mu)}{\partial \mu} = 0$ and

$$var\left(\frac{\partial \log f(x,\mu)}{\partial \mu}\right) = (\frac{g'}{g})^2 var(Z^2) + \frac{1}{g^2} var(Z) + 2\frac{g'}{g^2} cov(Z^2, Z) = \frac{2(g')^2+1}{g^2}.$$

Therefore

$$I(\mu) = \frac{2(g')^2+1}{g^2}.$$

By Slusky's theorem,

$$\frac{\sqrt{n(2(g'(\hat{\mu}_n))^2+1)}}{g(\hat{\mu}_n)}(\hat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}(0, I^{-1}(\mu))$$

so a $1-\alpha$ asymptotic confidence interval is

$$\left[\hat{\mu}_n - \frac{g(\hat{\mu}_n)}{\sqrt{n(2g'(\hat{\mu}_n)^2+1)}} Z_{\alpha/2},\ \hat{\mu}_n + \frac{g(\hat{\mu}_n)}{\sqrt{n(2g'(\hat{\mu}_n)^2+1)}} Z_{1-\alpha/2}\right]$$

(b) The length of asymptotic CI for part(a) equals to:

$$l_1(\hat{\mu}_n) = 2Z_{1-\alpha/2}\frac{g(\hat{\mu}_n)}{\sqrt{n(2g'(\hat{\mu}_n)^2+1)}}$$

while for t-test the CI length is

$$l_2(\hat{\mu}_n) = 2t_{n-1,1-\alpha/2}\sqrt{\frac{S_n^2}{n-1}}$$

here $S_n^2 = (n-1)^{-1}\sum(X_i - \bar{X})^2$. Therefore the ratio equals:

$$\frac{l_1(\hat{\mu}_n)}{l_2(\hat{\mu}_n)} = \frac{Z_{1-\alpha/2}\, g(\hat{\mu}_n)}{t_{n-1,1-\alpha/2} S_n \sqrt{2g'(\hat{\mu}_n)^2+1}}.$$

By the fact that $t_{n-1,1-\alpha/2} \to Z_{1-\alpha/2}$, $S_n^2 \xrightarrow{p} \sigma^2$ and $\hat{\mu}_n \xrightarrow{p} \mu$, we have the ratio:

$$\frac{l_1(\hat{\mu}_n)}{l_2(\hat{\mu}_n)} \xrightarrow{p} \frac{1}{\sqrt{2g'(\mu)^2+1}}.$$