UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 4
Fall 2014

Issued: Thursday, September 25 **Due:** Thursday, October 2 (beginning of class)

Problem 4.1
Let X have a geometric distribution with success probability θ, so $P_\theta(X = x) = \theta(1-\theta)^x, x = 0, 1, \ldots$. What is the smallest possible variance for an unbiased estimate of θ? Compare this variance with the lower bound in Theorem 4.9 in Keener.

Problem 4.2
Let $\Theta = (0, +\infty)$ and $\mathcal{A} = [0, \infty)$, and suppose that $X \sim \text{Poi}(\theta)$. Consider the loss function $L(\theta, a) = (\theta - a)^2/\theta$.

(a) Find Bayes estimators with respect to the family of Gamma(a, b) priors.

(b) Show that the estimator $\delta(X) = X$ can be obtained by a suitable limit from (a).

Problem 4.3
Let $(X_1, \ldots, X_n)$ be an i.i.d. sample from the uniform distribution on $(0, \theta)$, where $\theta > 0$ is unknown. Suppose that the prior distribution of θ is log-normal with parameters (μ_0, σ_0^2) where $\mu_0 \in \mathbb{R}$ and $\sigma_0^2 > 0$ are known constants.

(a) Find the posterior density of $\log \theta$.

(b) Suppose that we are interested in estimating θ under the loss function

$$L(\delta, \theta) \quad = \quad \begin{cases} 0 & \text{if } \delta = \theta \\ 1 & \text{otherwise.} \end{cases}$$

Find the Bayes estimator of θ under this loss function. (*Hint*: Part (a) is related.)

Problem 4.4
Consider the Bayesian model in which $\vec{\Theta}$ has distribution Λ, and conditioned on $\vec{\Theta} = \theta$, the random variable X has distribution $\mathbb{P}_\theta$. Suppose that we are interested in estimating $g(\theta)$ under quadratic loss. Prove that no unbiased estimator $\delta(X)$ of $g(\theta)$ can be a Bayesian estimator unless the Bayesian risk $r(\Lambda, \delta) = 0$. This shows that Bayes estimators and unbiased estimators agree only in pathological cases.

Problem 4.5
Suppose that X has density in exponential family form

$$p(x; \theta) \quad \propto \quad \exp\left(\sum_{i=1}^{d} \theta_i T_i(x) - A(\theta)\right) \; h(x).$$

where each T_i is differentiable.

(a) Prove the following lemma. If X has support on the real line and g is any differentiable function such that $\mathbb{E}|g'(X)| < +\infty$, then

$$\mathbb{E}\left\{\left[\frac{h'(X)}{h(X)} + \sum_{i=1}^{d} \theta_i T_i'(X)\right] g(X)\right\} \quad = \quad -\mathbb{E}[g'(X)].$$

(b) Specializing to the $N(\mu, \sigma^2)$ case, use part (a) to show that $\mathrm{cov}\{g(X), X\} = \sigma^2 \mathbb{E}[g'(X)]$.

(c) Use part (b) to compute the third and fourth moments of the $N(\mu, \sigma^2)$ distribution.