UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 11- Solutions
Fall 2014

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Problem 11.1

Let $Y = X - \mu$, then

$$E[e^{\lambda Y}] \leq e^{\frac{\lambda^2 \sigma^2}{2}}.$$

It is easy to see that

$$\left|\frac{e^{\lambda y} - 1}{\lambda}\right| = |\sum_{k \geq 1} \frac{\lambda^{k-1} y^k}{k!}| \leq \sum_{k \geq 1} \frac{|\lambda|^{k-1} |y|^k}{(k-1)!} = |y| e^{|\lambda||y|} \leq e^{(|\lambda|+1)|y|} \leq (e^{(|\lambda|+1)y} + e^{-(|\lambda|+1)y}).$$

In addition,

$$(e^{(|\lambda|+1)Y} + e^{-(|\lambda|+1)Y}) \leq 2e^{\frac{(|\lambda|+1)^2 \sigma^2}{2}} < \infty,$$

then it follows from Dominated convergence theorem that

$$E(Y) = \lim_{\lambda \to 0} E\left(\frac{e^{\lambda Y} - 1}{\lambda}\right).$$

Therefore,

$$E(Y) = \lim_{\lambda \to 0^+} E\left(\frac{e^{\lambda Y} - 1}{\lambda}\right) \leq \lim_{\lambda \to 0^+} \frac{e^{\frac{\lambda^2 \sigma^2}{2}} - 1}{\lambda} = 0,$$

and

$$E(Y) = \lim_{\lambda \to 0^-} E\left(\frac{e^{\lambda Y} - 1}{\lambda}\right) \geq \lim_{\lambda \to 0^-} \frac{e^{\frac{\lambda^2 \sigma^2}{2}} - 1}{\lambda} = 0,$$

Thus, $E(Y) = 0$ and hence $E(X) = \mu$.

(b) Similar to (a), we have

$$|\frac{e^{\lambda y} - 1 - \lambda y}{\lambda^2}| = |\sum_{k \geq 2} \frac{\lambda^{k-2} |y|^k}{k!}| \leq y^2 e^{|\lambda| y} \leq e^{(|\lambda|+2)|y|} \leq (e^{(|\lambda|+2)y} + e^{-(|\lambda|+2)y}),$$

and hence

$$E(Y^2) = 2\lim_{\lambda\to 0} E\left(\frac{e^{\lambda Y} - 1 - \lambda Y}{\lambda^2}\right).$$

Since $E(Y) = 0$, it holds that

$$E(Y^2) = 2\lim_{\lambda\to 0} E\left(\frac{e^{\lambda Y} - 1 - \lambda Y}{\lambda^2}\right) = 2\lim_{\lambda\to 0} E\left(\frac{e^{\lambda Y} - 1}{\lambda^2}\right) \le 2\lim_{\lambda\to 0}\frac{e^{\frac{\lambda^2\sigma^2}{2}} - 1}{\lambda^2} = \sigma^2.$$

Therefore, $Var(X) \le \sigma^2$.

(c) It is not true when $E(X-\mu)^3 \neq 0$. Now we prove it. Suppose $\sigma^2 = Var(X)$. Similar to (a) and (b), it holds that

$$E(Y^3) = 6\lim_{\lambda\to 0} E\left(\frac{e^{\lambda Y} - 1 - \lambda Y - \frac{\lambda^2}{2}Y^2}{\lambda^3}\right).$$

Then

$$E(Y^3) = 6\lim_{\lambda\to 0^+} E\left(\frac{e^{\lambda Y} - 1 - \frac{\lambda^2}{2}Y^2}{\lambda^3}\right) \le 6\lim_{\lambda\to 0^+} E\left(\frac{e^{\frac{\lambda^2\sigma^2}{2}} - 1 - \frac{\lambda^2\sigma^2}{2}}{\lambda^3}\right) = 0,$$

and

$$E(Y^3) = 6\lim_{\lambda\to 0^-} E\left(\frac{e^{\lambda Y} - 1 - \frac{\lambda^2}{2}Y^2}{\lambda^3}\right) \ge 6\lim_{\lambda\to 0^-} E\left(\frac{e^{\frac{\lambda^2\sigma^2}{2}} - 1 - \frac{\lambda^2\sigma^2}{2}}{\lambda^3}\right) = 0,$$

Thus, $E(X-\mu)^3 = EY^3 = 0$.

Let $X \sim B(p,1)$ be a Bernoulli variable where $0 < p < 1/2$, then

$$E(X-\mu)^3 = p(1-p)(1-2p) > 0.$$

On the other hand, since X is bounded, it is sub-Gaussian. Thus, $\sigma^2 > Var(X)$ otherwise when λ is small,

$$Ee^{\lambda Y} = 1 + \sigma^2\lambda^2/2 + E(Y^3)\lambda^3/6 + o(\lambda^3) > e^{\lambda^2\sigma^2/2}.$$

Problem 11.2

(a) Without loss of generality, we assume $\sigma^2 = 1$, otherwise we replace X_i by X_i/σ. Then for any $t > 0$,

$$P(|X_i| \ge t) = \sqrt{\frac{2}{\pi}}\int_t^\infty e^{-\frac{x^2}{2}}dx \le \sqrt{\frac{2}{\pi}}t^{-1}e^{-\frac{t^2}{2}}$$

and hence

$$P(Z \geq t) \leq n\sqrt{\frac{2}{\pi}}t^{-1}e^{-\frac{t^2}{2}}.$$

Then,

$$\begin{aligned}
E(Z) = \int_0^\infty P(Z \geq t)dt &= \int_0^c P(Z \geq t)dt + \int_c^\infty P(Z \geq t)dt \\
&\leq c + n\sqrt{\frac{2}{\pi}}\int_c^\infty t^{-1}e^{-\frac{t^2}{2}}dt \\
&\leq c + c^{-2}n\sqrt{\frac{2}{\pi}}\int_c^\infty te^{-\frac{t^2}{2}}dt \\
&\leq c + \sqrt{\frac{2}{\pi}}c^{-2}ne^{-\frac{c^2}{2}}
\end{aligned}$$

Let $c = \sqrt{2\log n}$, then

$$E(Z) \leq \sqrt{2\log n} + \sqrt{\frac{2}{\pi}}(2\log n)^{-1}$$

When $n \geq 2$, $2\log n \geq \log 4 > 1$, thus,

$$E(Z) \leq \sqrt{2\log n} + \frac{4}{\sqrt{2\log n}}$$

For general σ^2, since $Z \stackrel{d}{=} \sigma \cdot (Z/\sigma)$, it is easy to see that

$$E(Z) \leq \sqrt{2\sigma^2 \log n} + \frac{4\sigma}{\sqrt{2\log n}}$$

(b) Assume $\sigma^2 = 1$. Let $c = 1 - e^{-1}$. Notice that if

$$E\#\{i : |X_i| \geq c\sqrt{2\log n}\} \geq 1, \tag{1}$$

then

$$E(Z) \geq c\sqrt{2\log n}E\#\{i : |X_i| \geq c\sqrt{2\log n}\} \geq c\sqrt{2\log n}$$

It is easy to see that (1) is equivalent to

$$nP(|X_1| \geq c\sqrt{2\log n}) \geq 1.$$

Recall that

$$P(|X_1| \geq t) = \sqrt{\frac{2}{\pi}}\int_t^\infty e^{-\frac{x^2}{2}}dx \geq \sqrt{\frac{2}{\pi}}\int_t^\infty \frac{(x^2+1)^2-2}{(x^2+1)^2}e^{-\frac{x^2}{2}}dx = \sqrt{\frac{2}{\pi}}\frac{z}{z^2+1}e^{-\frac{z^2}{2}}$$

Thus it is left to show that

$$\sqrt{\frac{2}{\pi}}n\frac{c\sqrt{2\log n}}{2c^2\log n + 1}e^{-c^2\log n} \geq 1$$

This can be further simplified to

$$f(n,c) = a + (1-c^2)\log n + \log c + \frac{1}{2}\log\log n - \log(2c^2\log n + 1) \geq 0 \tag{2}$$

where $a = \log\sqrt{2/\pi} + \log 2/2 < 0.1208$ is a constant not relying on n and c. Let $g(x,c) = f(e^x, c)$, then for $x \geq \log 5 > 1.5$,

$$g(x,c) \geq a+(1-c^2)x+\log c+\frac{1}{2}\log x-\log(2c^2x+\frac{2}{3}x) = a+\log c-\log(2c^2+\frac{2}{3})-\frac{1}{2}\log x \triangleq h(x).$$

Note that $h(x)$ is increasing in $[1.5,\infty)$ since

$$h'(x) = 1-c^2-\frac{1}{2x} \geq 1-c^2-\frac{1}{3} > 0.$$

Thus,

$$f(n,c) = g(\log n, c) \geq h(\log n) \geq h(\log 5) > 0.008 > 0.$$

Therefore, the inequality holds for $n \geq 5$. For general σ^2, it is easy to see that

$$E(Z) \geq (1-e^{-1})\sqrt{2\sigma^2\log n}$$

(c) Assume $\sigma^2 = 1$. Recall (2) that for any $c < 1$,

$$\lim_{n\to\infty} f(n,c) = \lim_{n\to\infty} a + (1-c^2)\log n + \log c + \frac{1}{2}\log\log n - \log(2c^2\log n + 1) = \infty,$$

thus for sufficiently large n, $f(n,c) \geq 0$ and hence

$$\liminf_{n\to\infty} E(Z) \geq c\sqrt{2\log n}.$$

Since c is arbitrary, it holds that

$$\liminf_{n\to\infty} E(Z) \geq \sup_{c<1} c\sqrt{2\log n} = \sqrt{2\log n}.$$

On the other hand, it foloows from (a) that

$$\limsup_{n\to\infty} E(Z) \leq \sqrt{2\log n}.$$

Therefore,

$$\lim_{n\to\infty} \frac{E(Z)}{\sqrt{2\log n}} = 1.$$

For general σ^2, we have

$$\lim_{n\to\infty} \frac{E(Z)}{\sqrt{2\sigma^2\log n}} = 1.$$

Problem 11.3

(a) It is easy to see that

$$P(Z \geq t) \leq \min\{1, C\exp(-\frac{t^2}{2(\nu^2+Bt)})\}$$

Then,

$$E(Z) = \int_0^\infty P(Z \geq t)dt = \int_0^{\frac{\nu^2}{B}} P(Z \geq t)dt + \int_{\frac{\nu^2}{B}}^\infty P(Z \geq t)dt \triangleq I_1 + I_2$$

First, we consider I_1.

$$\begin{aligned}
&I_1 \\
\leq &\int_0^{\frac{\nu^2}{B}} \min\{1, C\exp(-\frac{t^2}{2(\nu^2+Bt)})\}dt \leq \int_0^{\frac{\nu^2}{B}} \min\{1, C\exp(-\frac{t^2}{4\nu^2})\}dt \\
\leq &\int_0^\infty \min\{1, C\exp(-\frac{t^2}{4\nu^2})\}dt \\
= &\int_0^{2\nu(\sqrt{\log C}+1)} 1dt + \int_{2\nu(\sqrt{\log C}+1)}^\infty C\exp(-\frac{t^2}{4\nu^2})dt \\
\leq &2\nu\sqrt{\log C} + 2\nu + \frac{2\nu C}{\sqrt{2\log C}+1}\exp(-(\sqrt{\log C}+1)^2) \\
= &2\nu(\sqrt{\log C} + 1 + \frac{e^{-1-2\sqrt{\log C}}}{1+\sqrt{2\log C}}) \\
\leq &2\nu(\sqrt{\log C} + 1 + e^{-1}) \leq 2\nu(\sqrt{\log C} + \sqrt{\pi})
\end{aligned}$$

Next, we consider I_2.

$$\begin{aligned}
I_2 &\leq \int_{\frac{\nu^2}{B}}^\infty \min\{1, C\exp(-\frac{t^2}{2(\nu^2+Bt)})\}dt \leq \int_{\frac{\nu^2}{B}}^\infty \min\{1, C\exp(-\frac{t}{4B})\}dt \\
&\leq \int_0^\infty \min\{1, C\exp(-\frac{t^2}{4B})\}dt \leq \int_0^{4B\log C} 1dt + \int_{4B\log C}^\infty C\exp(-\frac{t}{4B})dt \\
&= 4B(\log C + 1)
\end{aligned}$$

Therefore,

$$E(Z) = I_1 + I_2 \leq 2\nu(\sqrt{\pi} + \sqrt{\log C}) + 4B(1 + \log C)$$

(b)$X_k \sim$ (2.16) implies that X_k are sub-exponential variables with parameters $(\sqrt{2}\sigma, 2b)$ ($C = 2$), then $n^{-1}\sum_{k=1}^n X_k$ is also a sub-exponential variable with parameters $(\sqrt{\frac{2}{n}}\sigma, \frac{2b}{n})$, this entails that

$$P(|n^{-1}\sum_{k=1}^n X_k| \geq t) \leq 2\exp\left(-\frac{nt^2}{2(\sigma^2+bt)}\right).$$

Thus, it follows from (a) that

$$E(|n^{-1}\sum_{k=1}^{n} X_k|) \leq \frac{2\sigma}{\sqrt{n}}(\sqrt{\pi}+\sqrt{\log 2}) + \frac{4b}{n}(1+\log 2)$$

Problem 11.4

(a) First we prove for any positive integer k,

$$Eg^{2k} \leq 2^{k+1}k!\sigma^{2k}.$$

In fact, since g is sub-gaussian with parameter σ^2

$$\begin{aligned} Eg^{2k} &= \int_0^\infty 2kt^{2k-1}P(|g|\geq t)dt \leq 4k\int_0^\infty t^{2k-1}e^{-\frac{t^2}{2\sigma^2}}dt \\ &= \sigma^{2k}\cdot 4k\int_0^\infty t^{2k-1}e^{-\frac{t^2}{2}}dt = 2^{k+1}k!\sigma^{2k} \end{aligned}$$

Recall that g is symmetric, it holds that

$$\begin{aligned} Ee^{\lambda Q} &= \sum_{k\geq 0}\frac{\lambda^{2k}B^{2k}Eg^{2k}}{(2k)!} \preceq \sum_{k\geq 0}\frac{\lambda^{2k}B^{2k}\sigma^{2k}2^{k+1}k!}{(2k)!} \\ &= \sum_{k\geq 0}\lambda^{2k}B^{2k}\sigma^{2k}\frac{(2k-1)!!}{(2k)!}\frac{2^{k+1}k!}{(2k-1)!!} \\ &= \sum_{k\geq 0}\frac{\lambda^{2k}B^{2k}\sigma^{2k}}{2^k k!}\frac{2^{k+1}k!}{(2k-1)!!} \end{aligned}$$

where $n!! = n(n-2)(n-4)\cdots$. Note that for each k,

$$\frac{2^{k+1}k!}{(2k-1)!!} = 2\frac{2k}{2k-1}\frac{2k-2}{2k-3}\cdots\frac{2}{1} \leq 2^{k+1} \leq 2^{2k}$$

Let $c=4$ and $V = c^2\sigma^2B^2$, then

$$Ee^{\lambda Q} \preceq \sum_{k\geq 0}\frac{\lambda^{2k}B^{2k}\sigma^{2k}c^{2k}}{2^k k!} = e^{\frac{\lambda^2 V}{2}}$$

(b) Similar to (a), we have

$$Ee^{\lambda Q} = \sum_{k\geq 0}\frac{\lambda^{2k}E(B^{2k})E(g^{2k})}{(2k)!} \preceq \sum_{k\geq 0}\frac{\lambda^{2k}E(B^{2k})\sigma^{2k}c^{2k}}{2^k k!}.$$

Since $||B||_{op}\leq b$ $a.s.$, we have $B^2 \preceq b^2 I_{d\times d}$ and hence $B^{2k}\preceq b^{2k}I_{d\times d}$. Therefore,

$$Ee^{\lambda Q} \preceq \sum_{k\geq 0}\frac{\lambda^{2k}b^{2k}\sigma^{2k}c^{2k}I_{d\times d}}{2^k k!} = e^{\frac{\lambda^2}{2}V}$$

where $V = c^2b^2\sigma^2 I_{d\times d}$.