STAT 210 - Homework 9

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Problem 1. $X \sim \mathcal{N}_p(\theta, I)$. $H_0 : \theta \in \Omega_0, H_1 : \theta \notin \Omega_0$.

Proof. (a) We have the likelihood ratio statistic λ is defined as:

$$\begin{aligned}
\lambda =& \frac{\sup_{\Omega_1} L(\theta)}{\sup_{\Omega_0} L(\theta)} \\
=& \frac{\sup_{\theta' \in \Omega_1} \exp\left\{-\frac{1}{2}\sum (X_i - \theta'_i)^2\right\}}{\sup_{\theta \in \Omega_0} \exp\left\{-\frac{1}{2}\sum (X_i - \theta_i)^2\right\}} \\
=& \exp\left\{-\frac{1}{2}\inf_{\theta' \in \Omega_1} \sum (X_i - \theta'_i)^2 + \frac{1}{2}\inf_{\theta \in \Omega_0} \sum (X_i - \theta_i)^2\right\}
\end{aligned}$$

Thus the likelihood ratio test:

$$\begin{aligned}
\lambda >& C \\
\Leftrightarrow 2\log\lambda >& 2\log C \\
\Leftrightarrow \inf_{\theta \in \Omega_0} \|X - \theta\| >& 2\log C + \inf_{\theta' \in \Omega_1} \|X - \theta'\|
\end{aligned}$$

So the likelihood ratio test is equivalent to the distance D between X and Ω_0
(b) The significant level:

$$\begin{aligned}
\mathbb{E}\phi(X) &= \sup_{\theta \in \Omega_0} \mathbb{P}_\theta\left[D > c\right] && (1) \\
&= \sup_{\theta \in \Omega_0} \mathbb{P}_\theta\left[\inf_{\theta \in \Omega_0}\left\{(X_1 - \theta_1)^2 + (X_2 - \theta_2)^2\right\} > c\right] && (2)
\end{aligned}$$

First we notice the the probability in (2) is maximized when $\theta = (0, 0)$.

Now for $c > 0$, the event $\inf_{\theta \in \Omega_0}\left\{(X_1 - \theta_1)^2 + (X_2 - \theta_2)^2\right\} > c$ can be partitioned into three mutually exclusive events:

$$\begin{cases}
X_1 > \sqrt{c}, X_2 \leq 0 \\
X_2 > \sqrt{c}, X_1 \leq 0 \\
X_1^2 + X_2^2 > c, X_1 > 0, X_2 > 0
\end{cases}$$

Together with the fact that X_1 and X_2 are independent (conditioning on θ), we have the probability of the above event is the sum of these three events (for $\mathbb{P}_0$ denote the bivariate standard normal):

$$
\begin{aligned}
(2) =& \mathbb{P}_0\left[X_1 > \sqrt{c}\right]\mathbb{P}_0\left[X_2 \le 0\right] + \mathbb{P}_0\left[X_2 > \sqrt{c}\right]\mathbb{P}_0\left[X_1 \le 0\right] + \\
&\left.+ \mathbb{P}_0\left[X_1^2 + X_2^2 > c, X_1 > 0, X_2 > 0\right]\right\} \\
=& 1 - \Phi(\sqrt{c}) + \frac{1}{4}\left(1 - F_2(c)\right)
\end{aligned}
$$

For Φ is CDF of standard normal, F_2 is CDF of standard $\mathcal{X}_2^2$ (degree of freedom 2). The above number is the significant level. □

Problem 2. $\mathcal{N}(\mu, \sigma^2)$

Proof. (a) We have $\bar{Y}_n \sim \mathcal{N}\left(\mu, \sigma^2/n\right)$, thus:

$$
\begin{aligned}
\mathbb{P}_{\mu_0}\left[\bar{Y} \ge t_\alpha\right] =& \mathbb{P}_{\mu_0}\left[\frac{\bar{Y} - \mu_0}{\sigma/\sqrt{n}} \ge \frac{t_\alpha - \mu_0}{\sigma/\sqrt{n}}\right] && (3) \\
=& 1 - \Phi\left(\frac{t_\alpha - \mu_0}{\sigma/\sqrt{n}}\right) && (4)
\end{aligned}
$$

We want (4) to be equal to α, $\alpha = 1 - \Phi\left(\frac{\sqrt{n}(t_\alpha - \mu_0)}{\sigma}\right)$. Thus α is the solution to this function (expressed in term of inverse of Gaussian CDF).

(b) We have the power of the test:

$$
\begin{aligned}
\beta(\mu_1) =& \mathbb{P}_{\mu_1}\left[\bar{Y}_n \ge t_\alpha\right] \\
=& 1 - \Phi\left(\frac{t_\alpha - \mu_1}{\sigma/\sqrt{n}}\right) \\
=& \Phi\left(\frac{\sqrt{n}(\mu_1 - t_\alpha)}{\sigma}\right) \\
=& \Phi\left(z_\alpha + \delta_n\right)
\end{aligned}
$$

As $\mu_1 - \mu_0$ increases, we have δ_n increases, thus we have higher power (since CDF is an increasing function). This make sense as if the true mean is very large in comparision with μ_0, the probability of us successfully rejecting H_0 is higher.

As δ increases, we have δ_n decreases as $\mu_1 - \mu_0$ is positive. So the power decreaese. This also makes sense as when there is more variance, it should be harder to tell whether the true parameter is μ_0 or not.

(c) When μ is μ_0, we have Z_i is Bernouli with probability 1/2, expectation 1/2, and variance 1/4. Thus $\bar{Z}_n$ by CLT is approximately $\mathcal{N}\left(\frac{1}{2}, \frac{1}{4n}\right)$.

The significant level for rejecting at s is:

$$
\begin{aligned}
\mathbb{P}_{\mu_0}\left[\bar{Z}_n \ge s\right] =& \mathbb{P}_{\mu_0}\left[\frac{\bar{Z}_n - \frac{1}{2}}{\frac{1}{2\sqrt{n}}} \ge \frac{s - \frac{1}{2}}{\frac{1}{2\sqrt{n}}}\right] \\
\approx& 1 - \Phi\left(2\sqrt{n}(s - 1/2)\right) \\
=& 1 - \Phi\left(2\sqrt{n}s - \sqrt{n}\right) \\
=& \Phi(\sqrt{n} - 2\sqrt{n}s)
\end{aligned}
$$

So if we want:

$$\begin{aligned}\alpha =& \Phi(\sqrt{n} - 2\sqrt{n}s) \\ \Leftrightarrow z_\alpha =& \sqrt{n} - 2\sqrt{n}s \\ \Leftrightarrow s =& \frac{\sqrt{n} - z_\alpha}{2\sqrt{n}}\end{aligned}$$

We have to pick $s_\alpha = \frac{z_\alpha + 1}{2\sqrt{n}}$

Now when μ is μ_1, Z_i is Bernoulli with success probability $\mathbb{P}_{\mu_1}[Y_i \geq \mu_0] = \mathbb{P}_{\mu_1}\left[\frac{Y_i - \mu_1}{\sigma} \geq \frac{\mu_0 - \mu_1}{\sigma}\right] = 1 - \Phi\left(-\frac{\delta_n}{\sqrt{n}}\right) = \Phi(\frac{\delta_n}{\sqrt{n}})$. Thus $\bar{Z}_n$ by CLT is approximately $\mathcal{N}\left(\Phi\left(\frac{\delta_n}{\sqrt{n}}\right), \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right] \Big/ n\right)$

The power function for level-α test is:

$$\begin{aligned}B_Z(\mu_1) =& \mathbb{P}_{\mu_1}\left[\bar{Z}_n \geq s_\alpha\right] \\ =& \mathbb{P}_{\mu_1}\left[\frac{\bar{Z}_n - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)}{\sqrt{\Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right] \Big/ n}} \geq \frac{s_\alpha - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)}{\sqrt{\Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right] \Big/ n}}\right] \\ \approx& \Phi\left\{\frac{\Phi\left(\frac{\delta_n}{\sqrt{n}}\right) - \frac{\sqrt{n} - z_\alpha}{2\sqrt{n}}}{\sqrt{\Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right] \Big/ n}}\right\} \\ =& \Phi\left\{\frac{n\Phi\left(\frac{\delta_n}{\sqrt{n}}\right) - n/2 + \sqrt{n}z_\alpha/2}{\sqrt{n\Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right]}}\right\}\end{aligned}$$

(d) Using the first order Taylor approximation $\Phi(\delta_n/\sqrt{n}) \approx \frac{1}{2} + \left(\delta_n/\sqrt{2\pi n}\right)$, we have:

$$\begin{aligned}1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right) =& \frac{1}{2} - \frac{\delta_n}{\sqrt{2\pi n}} \\ \Rightarrow \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\left[1 - \Phi\left(\frac{\delta_n}{\sqrt{n}}\right)\right] =& \frac{1}{4} - \frac{\delta_n^2}{2\pi n} \\ =& \frac{1}{4}\left(1 - \frac{2\delta_n^2}{\pi n}\right) \\ \approx& \frac{1}{4} \\ n\Phi\left(\frac{\delta_n}{\sqrt{n}}\right) - n/2 + \sqrt{n}z_\alpha/2 =& \frac{n}{2} + \frac{\sqrt{n}\delta_n}{\sqrt{2\pi}} - \frac{n}{2} + \frac{\sqrt{n}z_\alpha}{2} \\ =& \frac{\sqrt{n}}{2}\left(z_\alpha + \sqrt{\frac{2}{\pi}}\delta_n\right) \\ \Rightarrow B_Z(\mu_1) \approx& \Phi\left(z_\alpha + \sqrt{\frac{2}{\pi}}\delta_n\right)\end{aligned}$$

The amount of power loss is approximately:

$$\Phi(z_\alpha+\delta_n)-\Phi(z_\alpha+\sqrt{\frac{2}{\pi}}\delta_n)$$

If we use Taylor series to approximate the Gaussian cdf, we see that this power is less than the previous power before thresholding an approximate amount about $\frac{1}{\sqrt{2\pi}}\delta_n\left(1-\sqrt{\frac{2}{\pi}}\right)$. □

Problem 3. Nonparametric Hypothesis Test

Proof. (a) Under the null hypothesis, $\mathbb{P}[Y_i>\mu_0]=1/2$. Thus S is binomial with n trial, success probability 1/2. This can be useful because in hypothesis testing the significant level is the probability of S belong to some interval conditioning on the null hypothesis is true.

(b) We approximate the binomial distribution of S by normal: $\mathcal{N}\left(\frac{1}{2},\frac{1}{4n}\right)$ according to CLT. Then we have the level $\alpha(s)$ is:

$$\begin{aligned}\mathbb{E}_0\left[\delta_s(Y)\right]=&\mathbb{P}_0\left[S\geq s\right]\\ =&\mathbb{P}_0\left[\frac{S-\frac{1}{2}}{\frac{1}{2\sqrt{n}}}\geq\frac{s-1/2}{\frac{1}{2\sqrt{n}}}\right]\\ =&\Phi\left(2\sqrt{n}\left(s-1/2\right)\right)\\ =&\Phi\left(2\sqrt{n}s-\sqrt{n}\right)\end{aligned}$$

□

Problem 4. Bayes Risk and Hypothesis Testing

Proof. (a) We have $\delta(X)=\mathbb{I}_{\{X\in C\}}, l(\theta,\delta(X))=\left|\mathbb{I}_{\{\theta\in\Omega_1\}}-\delta(X)\right|$Under the prior of λ, we have the Bayes risk is:

$$\begin{aligned}r(\lambda,\delta)=&\mathbb{E}l(\theta,\delta(X))\\ =&\mathbb{E}\left[\mathbb{E}\left[l(\theta,\delta(X))\mid\theta\right]\right]\\ =&\mathbb{P}\left[\theta=\theta_0\right]\mathbb{E}\left[l(\theta,\delta(X))\mid\theta=\theta_0\right]+\mathbb{P}\left[\theta=\theta_1\right]\mathbb{E}\left[l(\theta,\delta(X))\mid\theta=\theta_1\right]\\ =&\lambda_0\mathbb{E}_0\delta(X)+(1-\lambda_0)(1-\mathbb{E}_1\delta(X))\end{aligned}$$

(b) We have:

$$\begin{aligned}r(\lambda,\delta)=&\lambda_0\int p_0(x)\delta(x)dx-(1-\lambda_0)\int p_1(x)(1-\delta(x))dx\\ =&1-\lambda_0+\int\left[\lambda_0p_0(x)-(1-\lambda_0)p_1(x)\right]\delta(x)dx\\ =&1-\lambda_0+\int_{p_1(x)/p_0(x)<\lambda_0/(1-\lambda_0)}\left|\lambda_0p_0(x)-(1-\lambda_0)p_1(x)\right|\delta(x)dx\\ &-\int_{p_1(x)/p_0(x)>\lambda_0/(1-\lambda_0)}\left|\lambda_0p_0(x)-(1-\lambda_0)p_1(x)\right|\delta(x)dx\end{aligned}$$

From here we see that the test that minimizes Bayes risk is a likelihood ratio test where:

$$\delta(x)=\begin{cases}0 & ,\frac{p_1(x)}{p_0(x)}<\frac{\lambda_0}{1-\lambda_0}\\ 1 & ,\frac{p_1(x)}{p_0(x)}>\frac{\lambda_0}{1-\lambda_0}\end{cases}$$

(c) The test:

$$\delta_n(x) = \begin{cases} 0 & , \frac{\prod p_1(x_i)}{\prod p_0(x_i)} < \frac{\lambda_0}{1-\lambda_0} \\ 1 & , \frac{\prod p_1(x_i)}{\prod p_0(x_i)} > \frac{\lambda_0}{1-\lambda_0} \end{cases}$$

For the derived Bayes test above, we have:

$$\begin{aligned} r(\lambda, \delta_n) =& \lambda_0 \mathbb{E}_0 \delta_n(X) + (1-\lambda_0)\left(1 - \mathbb{E}_1 \delta_n(X)\right) \\ =& \lambda_0 \mathbb{P}_0 \left[\frac{\prod p_1(x_i)}{\prod p_0(x_i)} > (1-\lambda_0)\right] + (1-\lambda_0)\left(1 - \mathbb{P}_1\left[\frac{\prod p_1(x_i)}{\prod p_0(x_i)} > \frac{\lambda_0}{1-\lambda_0}\right]\right) \\ =& \lambda_0 \int_{t^n}^{\infty} \end{aligned}$$

$\square$

Problem 5. Generalized Likelihood

Proof. (a) Let $\Omega_0 = \left\{\mu_x, \mu_y, \sigma_x, \sigma_y \mid \mu_x = \mu_y, \sigma_x^2 = \sigma_y^2\right\}$. We have:

$$\begin{aligned} \lambda =& \frac{\sup_{\theta \notin \Omega_0} L(\theta)}{L_{\theta \in W_0}(\theta)} \\ =& \sup_{\theta' \notin \Omega_0} \exp\left\{-\frac{1}{2}\right\} \end{aligned}$$

$\square$