UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 11
Fall 2014

Issued: Weds, November 26 **Due:** Thurs, December 4 (beginning of class)

Problem 11.1
(Sub-Gaussian bounds and means/variances). Consider a random variable X such that

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2\sigma^2}{2}+\lambda\mu}$$

for all $\lambda \in \mathbb{R}$.

(a) Show that $\mathbb{E}[X] = \mu$.

(b) Show that $\mathrm{Var}(X) \leq \sigma^2$.

(b) Suppose that the smallest possible σ satisfying the inequality above is chosen. Is it then true that $\mathrm{Var}(X) = \sigma^2$? Prove or disprove.

Problem 11.2
(Gaussian maxima). Let $\{X_i\}_{k=1}^n$ be an i.i.d. sequence of $N(0, \sigma^2)$ random variables, and consider the random variable $Z = \max_{k=1,\ldots,n} |X_k|$.

(a) Prove that

$$\mathbb{E}[Z] \leq \sqrt{2\sigma^2 \log n} + \frac{4\sigma}{\sqrt{2\log n}} \quad \text{for all } n \geq 2.$$

(*Hint*): You may use the tail bound $P(U \geq \delta) \leq \sqrt{\frac{2}{\pi}}\frac{1}{\delta}e^{-\delta^2/2}$, valid for any standard Gaussian normal variate.

(b) Prove that

$$\mathbb{E}[Z] \geq (1 - 1/e)\sqrt{2\sigma^2 \log n} \quad \text{for all } n \geq 5.$$

(c) Prove that $\frac{\mathbb{E}[Z]}{\sqrt{2\sigma^2 \log n}} \to 1$ as $n \to \infty$.

Problem 11.3
(Bernstein and expectations). Consider a nonnegative random variable that satisfies a concentration inequality of the form

$$P(Z \geq t) \leq Ce^{-\frac{t^2}{2(\nu^2+B)}}$$

for positive constants (ν, b) and $C \geq 1$.

(a) Show that $\mathbb{E}[Z] \leq 2\nu(\sqrt{\pi} + \sqrt{\log C}) + 4B(1 + \log C)$.

(b) Let $\{X_k\}_{k=1}^n$ be an i.i.d. sequence of zero-mean variables satisfying the Bernstein condition (2.16). Use part (a) to show that

$$\mathbb{E}\left[|\frac{1}{n}\sum_{k=1}^{n} X_k|\right] \leq \frac{2\sigma}{\sqrt{n}}(\sqrt{\pi} + \sqrt{\log 2}) + \frac{4b}{n}(1 + \log 2).$$

Problem 11.4

(Sub-Gaussian random matrices). Consider the random matrix $Q = gB$, where $g \in \mathbb{R}$ is a zero-mean sub-Gaussian variable with parameter σ.

(a) Assume that g has a distribution symmetric around zero, and $B \in \mathcal{S}^{d\times d}$ is a deterministic matrix. Show that Q is sub-Gaussian with matrix parameter $V = c^2\sigma^2 B^2$, for some universal constant c.

(b) Now assume that $B \in \mathcal{S}^{d\times d}$ is random and independent of g, with the operator norm of B upper bounded by a constant b almost surely. Now show that Q is sub-Gaussian with matrix parameter $V = c^2 b^2 \sigma^2 I_{d\times d}$.