UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 8
Fall 2014

Issued: Friday, October 31 **Due:** Thursday, November 6 (beginning of class)

Problem 8.1
Consider estimating success probabilities $\theta_1, \dots, \theta_p$ for p independent binomial variables, $X_1, \dots, X_p$, each based on m trials, under compound squared error loss, $L(\theta, d) = \sum_{i=1}^{p}(\theta_i - d_i)^2$.

(a) Following a Bayesian approach, model the unknown parameters as random variables $\Theta_1, \dots, \Theta_p$ that are i.i.d. from a beta distribution with parameters α and β. Determine the Bayes estimators of $\Theta_1, \dots, \Theta_p$.

(b) In the Bayesian model, $X_1, \dots, X_p$ are i.i.d. Determine the first two moments for their common marginal distribution, EX_i and EX_i^2. Using these, suggest simple method of moments estimators for α and β.

(c) Give empirical Bayes estimators for θ_i combining the simple "empirical" estimates for α and β in (b) with the Bayes estimate for θ_i when α and β are known in (a).

Problem 8.2
Let $X_1, \dots, X_n$ be an i.i.d. sample from the uniform distribution on $[0, \theta]$.

(a) Consider the problem of testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Show that any test δ for which $\mathbb{E}_{\theta_0}[\delta(X)] = \alpha$, $\mathbb{E}_{\theta}[\delta(X)] \leq \alpha$ for all $\theta \leq \theta_0$ and $\delta(x) = 1$ when $x_{(n)} = \max\{x_1, \dots, x_n\} > \theta_0$ is UMP at level α.

(b) Now consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. Show that a unique UMP test exists, and is given by

$$\delta(x) = \begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0(\alpha)^{1/n} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 8.3
Suppose that $X_1, \dots, X_n$ are independent exponential random variables with $\mathbb{E}[X_i] = \beta t_i$ where $t_1, \dots, t_n$ are known constants, and $\beta > 0$ is an unknown parameter.

(a) Show that the MLE of β is given by $\widehat{\beta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i/t_i$.

(b) Prove that $\sqrt{n}\left(\beta - \widehat{\beta}_n\right) \xrightarrow{d} N(0, \beta^2)$.

(c) Suppose that we want to test $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$. In order to do so, we consider the statistic

$$G(X_1, \ldots, X_n) \quad = \quad \log p(X; \widehat{\beta}_n) - \log p(X; 1),$$

where $p(X; \beta) = \prod_{i=1}^{n} p(X_i; \beta, t_i)$ is the likelihood. Show that $G(X) = n\left(\widehat{\beta}_n - \log \widehat{\beta}_n - 1\right)$.

(d) Show that when H_0 is true, we have $2G(X) \xrightarrow{d} \chi_1^2$.

Problem 8.4

Consider simple versus simple testing from a Bayesian perspective. Let Θ have a Bernoulli distribution with $P(\Theta = 1) = p$ and $P(\Theta = 0) = 1 - p$. Given $\Theta = 0$, X will have density p_0 and given $\Theta = 1$, X will have density p_1.

(a) Show that the chance of accepting the wrong hypothesis in the Bayesian model using a test function φ is

$$R(\varphi) = E[I(\Theta = 0)\varphi(X) + I(\Theta = 1)(1 - \varphi(X))].$$

(b) Use the tower property to relate $R(\varphi)$ to $E_0(\varphi) = E(\varphi(X)\gamma\Theta = 0)$ and $E_1(\varphi) = E(\varphi(X)\gamma\Theta = 1)$.

(b) Find the test function φ^* minimizing $R(\varphi)$. Show that φ^* is a likelihood ratio test, identifying the critical value k.