# General Probability Theory

1.1

(i) $\because A \subset B \therefore A \cap B = A$

$\therefore P(B) = P(A \cap B) + P(B \cap A^c) = P(A) + P(B \cap A^c)$

$\therefore P(B) \geq P(A)$

(ii)$\because A \subset A_n \; for \; \forall n \therefore A \cap A_n = A$

$\therefore 0 = \lim_{n \to \infty} P(A_n) \geq \lim_{n \to \infty} P(A \cap A_n) = P(A)$

$\therefore P(A) = 0$

1.2

(i)$An\ element\ of\ that\ does\ not\ appear\ in\ this\ list\ is$

$the\ sequence\ whose\ first\ 2\ components\ are\ heads\ if\ \omega_1, \omega_2$

$are\ tails\ and\ are\ tails\ if\ \omega_1, \omega_2\ are\ heads\ \ldots \ldots$

(ii)$Assume\ n\ is\ even, then$:

$Define\ A_n = \{\omega = \omega_1 \omega_2 \omega_3 \ldots \omega_n; \omega_1 = \omega_2, \omega_3 = \omega_4, \ldots \omega_{n-1} = \omega_n\}$

$\therefore P(A_n) = \left(1 - 2 \times P(1 - P)\right)^n \therefore \lim_{n \to \infty} P(A_n) = 0\ if\ 0 < P < 1$

$\because A_n \subset A\ for\ \forall\ n \therefore P(A) = P(\lim_{n \to \infty} A_n) = \lim_{n \to \infty} P(A_n) = 0$

1.3

**1.1.3**: $\because \Omega \backslash \emptyset = \Omega \therefore P(\emptyset) = 1 - P(\Omega) = 0$

**1.1.4**: $Assume\ A\ and\ B\ are\ disjoint,$

$If\ both\ of\ them\ are\ finite:$

$A \cup B\ is\ still\ finite \therefore P(A) + P(B) = P(A \cup B) = 0$

$If\ at\ least\ A\ is\ finite:$

$P(A \cup B) \geq P(A) = \infty \therefore P(A) + P(B) = P(A \cup B) = \infty$

**1.1.5:** $If\ all\ of\ them\ are\ finite:$

$$Their\ finite\ unite\ is\ still\ finite \therefore\ P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} A_n = 0$$

$$Else: P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} A_n = \infty = P(some\ infinite\ A_n)$$

**1.1.2:** $Their\ infinite\ unit\ is\ the\ full\ collection:$

$$\therefore P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1\ while\ \sum_{\mathrm{n}=1}^{\infty} A_n = \infty$$

1.4

(i) $Construct\ the\ uniform\ distribution\ by: X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n}, denote$ $the\ CDF\ of\ normal\ distribution\ as\ N(x), then\ the\ normal\ distributed$ $valriable\ Z = N^{-1}(X).$

(ii) $\because Z_1 \leq Z_2 \leq Z_3 \leq \cdots \leq Z_n \leq \cdots Z\ and\ N^{-1}\ is\ monotone$

$$\therefore \lim_{n\to\infty} Z_n = \lim_{n\to\infty} N^{-1}(X_n) = \mathrm{N}^{-1}(\lim_{n\to\infty} X_n) = N^{-1}(X) = Z$$

1.5 $\because \int_{\Omega} \int_0^{\infty} I_{[0,X(\omega))}(x) dx dP(\omega) = \int_0^{\infty} \int_{\Omega} I_{[0,X(\omega))}(x) dP(\omega) dx$

$$and \because \int_{\Omega}\int_{0}^{\infty} I_{[0,X(\omega))}(x)dxdP(\omega)$$

$$= \int_{\Omega}\int_{0}^{X(\omega)} dxdP(\omega) = \int_{\Omega} X(\omega)dP(\omega) = E(x)$$

$$and \because \int_{0}^{\infty}\int_{\Omega} I_{[0,X(\omega))}(x)dP(\omega)dx = \int_{0}^{\infty} P(x < X(\omega))dx$$

$$= \int_{0}^{\infty} 1 - F(x)dx$$

$$\therefore E(x) = \int_{0}^{\infty} 1 - F(x)dx$$

1.6

(i) $\because E(e^{ux}) = \int\ e^{ux}f(x)dx = \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}+ux}dx$

$$\therefore \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}+ux}dx$$

$$= e^{u\mu+\frac{1}{2}u^2\sigma^2} \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(x-(\mu+\sigma^2 u)\right)^2}{2\sigma^2}}dx = e^{u\mu+\frac{1}{2}u^2\sigma^2}$$

(ii) $\because E\big(\phi(\mathrm{X})\big) = e^{u\mu+\frac{1}{2}u^2\sigma^2}, \phi\big(E(X)\big) = e^{u\mu}\ and\ e^{\frac{1}{2}u^2\sigma^2} \geq 1$

$\therefore E\big(\phi(\mathrm{X})\big) \geq \phi\big(E(X)\big)$

1.7

(i) $f(x) = 0$

(ii) $\because f_n(x)\ is\ the\ PDF\ of\ N(0,n)$

$\therefore \int_{-\infty}^{\infty} f_n(x)dx = 1\, and \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 1$

(iii) $With\ larger\ n, the\ PDF\ of\ normal\ distribution\ will\ be\ more\ fat - tailed. If\ m < n, we\ should\ expect\ that\ f_m(x)\ is\ smaller\ than\ f_n(x)\ when\ x\ is\ far\ from\ 0, but\ larger\ than\ f_n(x) when\ x\ is\ close\ to\ 0.\ f_m(x) \le f_n(x)\ everywhere\ for\ m < n\ is\ wrong.$

$Therefore\ the\ result\ does\ not\ violate\ the\ theorem\ because\ the\ given\ PDF\ function\ violates\ the\ theorem's\ assumption\ first.$

1.8

(i) $\because \theta \le t \therefore |Y_n| = |\frac{e^{tX} - e^{s_nX}}{t - s_n}| = Xe^{\theta X} \le Xe^{tX}$

$\because E(Xe^{tX}) < \infty\ for\ t \in R \therefore \lim_{n\to\infty} E(Y_n) = E(\lim_{n\to\infty} Y_n)$

$As\ s_n \to t, \theta \to t, \therefore \lim_{n\to\infty} E(Y_n) = E(Xe^{tX})$

(ii) $\because E(|X|e^{tX}) < \infty$

$$\therefore |Y_n^+| = |\frac{e^{tX} - e^{s_nX}}{t - s_n}| = X^+e^{\theta X^+} \le X^+e^{tX^+}, E\left(X^+e^{tX^+}\right) < \infty;$$

$$and\ |Y_n^-| = |\frac{e^{tX} - e^{s_nX}}{t - s_n}| = X^-e^{\theta X^-} \le X^-e^{tX^-}, E(X^-e^{tX^-}) < \infty$$

$$\therefore \lim_{n\to\infty} E(Y_n^+) = E(\lim_{n\to\infty} Y_n^+), \lim_{n\to\infty} E(Y_n^-) = E(\lim_{n\to\infty} Y_n^-)$$

$$\therefore \lim_{n\to\infty} E(Y_n) = E(\lim_{n\to\infty} Y_n^+) - E(\lim_{n\to\infty} Y_n^-)$$
$$= \int X^+e^{tX^+}dP(\omega) - \int X^-e^{tX^-}dP(\omega) = E(Xe^{tX})$$

1.9 $\because X\ is\ independent\ from\ A \therefore X(\Omega)\ is\ independent\ from\ \omega = A$

$$\int_A g\big(X(\omega)\big)dP(\omega) = \int_\Omega I_A\big(X(\omega)\big)g\big(X(\omega)\big)dP(\omega) = P(A) \times E(g(X))$$

1.10

(i) $Z \geq 0, and\ E(Z) = 0 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1$

(ii) $For\ both\ 2\ cases\ Z\ is\ constant,$

$\therefore \tilde{P}(\omega) = Z(\omega)\int_A dP(\omega) = Z(\omega)P(\omega) \therefore if\ P(\omega) = 0, \tilde{P}(\omega) = 0$

(iii) $When\ A = \left[0, \frac{1}{2}\right), \tilde{P}(\omega) = 0\ for\ Z(A) = 0, but\ P(A) = \frac{1}{2}$

1.11 $\tilde{E}(e^{uY}) = E(Ze^{uY}) = e^{u\theta - \frac{1}{2}\theta^2}E\big(e^{-\theta x}\big) + e^{-\frac{1}{2}\theta^2}E\big(e^{(u-\theta)x}\big)$

$$= e^{u\theta} + e^{\frac{1}{2}u^2 - u\theta} = e^{\frac{1}{2}u^2}$$

1.12 $\because \hat{Z} \times Z = e^{\theta(Y-X)-\theta^2} and\ Y - X = \theta \therefore \hat{Z} \times Z = 1$

$$\therefore \hat{Z} = \frac{1}{Z}\ and\ \hat{P}(A) = \int_A \hat{Z}(\omega)d\tilde{P}(\omega) = \int_A Z(\omega)\hat{Z}(\omega)dP(\omega) = P(A)$$

1.13

(i) $By\ the\ Mean\ Value\ Theorem, \exists\omega\ that\ X(\omega) \in B\left(x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2}\right),$

$such\ that\ P\left(x+\frac{\epsilon}{2}\right)-P\left(x-\frac{\epsilon}{2}\right)=\epsilon\times P'\big(X(\omega)\big)=P\{X\in B(x,\epsilon)\}$

$\therefore \frac{P\{X\in B(x,\epsilon)\}}{\epsilon}=\frac{e^{\left(-\frac{X^2(\omega)}{2}\right)}}{\sqrt{2\pi}}$

$\because \bar{\omega}\ always\ in\ X\in B(x,\epsilon)\ and\ B\ is\ small$

$\therefore X(\omega)\approx X(\bar{\omega}), \frac{P\{X\in B(x,\epsilon)\}}{\epsilon}\approx\frac{e^{\left(-\frac{X^2(\bar{\omega})}{2}\right)}}{\sqrt{2\pi}}$

(ii) $Replace\ x\ with\ y, X\ with\ Y\ and\ P\ with\ \tilde{P}\ then\ yield\ the\ result.$

(iii) $Given\ X=Y-\theta\ \therefore x=X(\omega)=Y(\omega)-\theta=y-\theta$

$\therefore \{Y\in B(y,\epsilon)\}=\{(Y-\theta)\in B(x,\epsilon)\}=\{X\in B(x,\epsilon)\}$

(iv) $Divide\ the\ result\ from\ (iii)\ by\ the\ result\ of\ (ii)$:

$\frac{\tilde{P}\{Y\in B(y,\epsilon)\}}{P\{X\in B(x,\epsilon)\}}=e^{-\frac{Y^2(\omega)-X^2(\omega)}{2}}$

$substitute\ \{Y\in B(y,\epsilon)\}=\{X\in B(x,\epsilon)\}=A\ and\ Y=X+\theta$

$yield=\ \frac{\tilde{P}(A)}{P(A)}=e^{-\theta X(\bar{\omega})-\frac{1}{2}\theta^2}$

1.14

(i) $\tilde{P}(\Omega)=\int_\Omega Z\,dP=\int_\Omega \tilde{\lambda}e^{-\tilde{\lambda}X}dX=1$

(ii) $\tilde{P}(X\leq a)=\int_{-\infty}^{a}\tilde{\lambda}e^{-\tilde{\lambda}X}dX=1-e^{-\tilde{\lambda}a}$

1.15

(i) $E(Z) = \int_R Zf(X)dX = \int_R h\big(g(X)\big)g'(X)dX = \int_{-\infty}^{\infty} h(g)\, dg = 1$

(ii) $Given\ g(y)\ is\ monotone\ and\ \lim_{y \to \pm\infty} g(y) = \pm\infty$

$$\tilde{P}(Y \le y) = \tilde{P}\big(X \le g^{-1}(y)\big) = \int_{-\infty}^{g^{-1}(y)} h\big(g(X)\big)g'(X)dX$$

$$= \int_{-\infty}^{g^{-1}(y)} h\big(g(X)\big)dg(X) = \int_{-\infty}^{y} h(Y)dY$$