Probability on Coin Toss Space

2.1

(i) $\because \sum_{\omega\in\Omega} P(\omega) = \sum_{\omega\in\mathrm{A}} P(\omega) + \sum_{\omega\in\mathrm{A^c}} P(\omega) = P(A) + P(A^c) = 1$
$\therefore P(A^c) = 1 - P(A)$

(ii) $\because P(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} P(A_n) - \sum_{j\neq k}^{N} P\big(A_k \cap A_j\big) \leq \sum_{n=1}^{N} P(A_n)$

$$\textit{The equality holds when } \sum_{j\neq k}^{N} P\big(A_k \cap A_j\big) = 0$$

2.2

(i) $S_3 = \begin{cases} 32, P = \frac{1}{8}; \\ 8, P = \frac{3}{8}; \\ 2, P = \frac{3}{8}; \\ \frac{1}{2}, P = \frac{1}{8} \end{cases}$ (ii) $\tilde{E}(S_1) = 5, \tilde{E}(S_2) = \frac{25}{4}, \tilde{E}(S_3) = \frac{125}{16}, \bar{\delta} = \frac{1}{4}$

(iii) $S_3 = \begin{cases} 32, P = \frac{8}{27}; \\ 8, P = \frac{4}{9}; \\ 2, P = \frac{2}{9}; \\ \frac{1}{2}, P = \frac{1}{27} \end{cases}$ $\tilde{E}(S_1) = 6, \tilde{E}(S_2) = 9, \tilde{E}(S_3) = \frac{27}{2}, \bar{\delta} = \frac{1}{2}$

2.3 $\forall n \in \{1, 2, 3, \dots\}, E_n\big(\phi(M_{n+1})\big) \geq \phi(E_n(M_{n+1})) = \phi(M_n)$
$\therefore \phi(M_n)\ is\ a\ sub - martingale$

2.4

(i) $\because \forall n \in \{1,2,3,\ldots\}, E_n(M_{n+1}) = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) + M_n = M_n$

$\therefore \{M\}_n$ *is a martingale*

(ii) $\forall n \in \{1,2,3,\ldots\}, E_n(S_{n+1}) = e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} E_n(e^{\sigma X_{n+1}})$

$$= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} (\frac{1}{2}(e^{\sigma}+e^{-\sigma})) = e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n} = S_n$$

2.5

(i) $I_n = -\frac{1}{2}\sum_{j=0}^{n-1} X_{j+1}^2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(M_{j+1}^2 - M_j^2\right)$

$$= \frac{1}{2}(M_0^2 + M_n^2) - \frac{n}{2} = \frac{1}{2}M_n^2 - \frac{n}{2}$$

(ii) $E_n\big(f(I_{n+1})\big) = E_n\big(f(I_n + M_n X_{n+1})\big)$

$$= \frac{1}{2}f(I_n + M_n) + \frac{1}{2}f(I_n - M_n)$$

$$= \frac{1}{2}f\left(I_n + \sqrt{2I_n + n}\right) + \frac{1}{2}f\left(I_n - \sqrt{2I_n + n}\right) = g(I_n)$$

2.6 $E_n(I_{n+1}) = I_n + \Delta_n(E_n(M_{n+1}) - M_n) = I_n$

2.7 *Denote* V_n *is the price of look back option at time* n.

By design $\frac{V_n}{(1+r)^n}$ *is a martingale but* $\frac{V_{n+1}}{(1+r)^{n+1}}$ *depends on*

$every\ step\ from\ time\ 0\ to\ time\ n. So\ it\ is\ not\ Markov.$

2.8

(i) $\forall n \le N, E_n(M_N) = E_n(M'_N) = M_n = M'_n$

(ii) $\tilde{E}_n\left(\frac{V_{n+1}}{(1+r)^{n+1}}\right) = \frac{\frac{1}{(1+r)}(\tilde{p}uV_n+\tilde{q}dV_n)}{(1+r)^n} = \frac{V_n}{(1+r)^n}$

(iii) $\tilde{E}_n\left(\frac{V'_{n+1}}{(1+r)^{n+1}}\right) = \tilde{E}_n\left(\tilde{E}_{n+1}\left(\frac{V_N}{(1+r)^N}\right)\right) = \frac{\tilde{E}_n\left(\frac{V_N}{(1+r)^{N-n}}\right)}{(1+r)^n} = \frac{V'_n}{(1+r)^n}$

(iv) $\because \frac{V'_N}{(1+r)^N} = \frac{\tilde{E}_N(V_N)}{(1+r)^N} = \frac{V_N}{(1+r)^N} \therefore \forall n: \frac{V'_n}{(1+r)^n} = \frac{V_n}{(1+r)^n}\ and\ so\ V_n = V'_n$

2.9

(i) $\because \tilde{P}(HH|H) = \frac{1+r_1(H)-1}{1.5-1} = \frac{1}{2}, \tilde{P}(TT|T) = \frac{4-(1+r_1(T))}{4-1} = \frac{5}{6},$

$$and \because \tilde{P}(H) = \frac{1+r_0-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{2},$$

$$\therefore \tilde{P}(HH) = \tilde{P}(HT) = \frac{1}{4}, \tilde{P}(HH) = \tilde{P}(HT) = \frac{1}{4},$$

$$\tilde{P}(TH) = \frac{1}{12}, \tilde{P}(TT) = \frac{5}{12}$$

(ii) $V_1(H) = \frac{(5\tilde{P}(HH|H)+\tilde{P}(HT|H))}{1+r_1(H)} = \frac{12}{5}, V_1(T) = \frac{\tilde{P}(TH|T)}{1+r_1(T)} = \frac{1}{9}$

$$V_0 = \frac{\tilde{P}(H)V_1(H) + \tilde{P}(T)V_1(T)}{1+r_0} = \frac{226}{225}$$

(iii) $(1+r_0)V_0 + \Delta_0(S_1 - (1+r_0)S_0) = V_1, yields\ \Delta_0 = \frac{103}{270}$

(iv) $\left(1+r_1(H)\right)V_1(H) + \Delta_1\left(S_2 - \left(1+r_1(H)\right)S_1(H)\right) = V_2(H),$

$which\ yields\ \Delta_0 = 1$

2.10

(i) $\tilde{E}_n\left(\frac{X_{n+1}}{(1+r)^{n+1}}\right) = \frac{X_n}{(1+r)^n} + \frac{\Delta_n S_n\left(E_n(Y)-(1+r)\right)}{(1+r)^{n+1}},$

$If\ \frac{X_n}{(1+r)^n}\ is\ a\ martingale\ then\ E_n(Y) - (1+r) = 0$

$\therefore risk - neutral\ measure\ makes\ it\ a\ martingale$

(ii) $To\ prevent\ arbitrage\ V_n = X_n\ for\ \forall n, then\ risk -$
$neutral\ pricing\ formula\ still\ applies.$

(iii) $\tilde{E}_n\left(\frac{S_{n+1}}{(1+r)^{n+1}}\right) = \frac{\tilde{E}_n(1-A_{n+1})S_n}{(1+r)^n} \therefore\ \frac{S_n}{(1+r)^n}\ is\ not\ a\ martingale$

$If\ A\ is\ constant, \tilde{E}_n\left(\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}\right) = \frac{S_n}{(1-a)^n(1+r)^n},$

$which\ means\ \frac{S_n}{(1-a)^n(1+r)^n}\ is\ a\ martingale$

2.11

(i) $(K-S_N)^+ + S_N - K = \begin{cases} S_N - K, if\ S_N > K \\ 0, if\ S_N \le K \end{cases} = (S_n - K)^+ = C_N$

(ii) $E_n\left(\frac{C_N}{(1+r)^N}\right) = \frac{C_n}{(1+r)^n} = E\left(\frac{F_N+P_N}{(1+r)^N}\right) = \frac{F_n}{(1+r)^n} + \frac{P_n}{(1+r)^n} \therefore C_n = P_n + F_n$

(iii) $F_0 = E\left(\frac{F_N}{(1+r)^N}\right) = S_0 - \frac{K}{(1+r)^N}$

(iv) $At\ N, S_N + (1+r)^N(F_0 - S_0) = S_N - K = F_N$

(v) $C_0 = P_0 + F_0 = P_0$

(vi) $No.\ C_n = P_n + S_n - \frac{K}{(1+r)^{N-n}} \therefore only\ if\ S_n = \frac{K}{(1+r)^{N-n}}, C_n = P_n$

2.12 $At\ m, value\ V_m = \max(P_m, C_m) = P_m + (S_m - \frac{K}{(1+r)^{N-m}})^+$

$(S_m - \frac{K}{(1+r)^{N-m}})^+$ *is the value of a call option with strike*

$\frac{K}{(1+r)^{N-m}}$ *with maturity* m.

2.13

(i) $E_n(g(Y_{n+1}, S_{n+1})) = pg(Y_n + uS_n, uS_n) + qg(Y_n + dS_n, dS_n)$

$let\ f(Y_n, S_n) = pg(Y_n + uS_n, uS_n) + qg(Y_n + dS_n, dS_n),$
$\therefore$ *the process is Markov*.

(ii) $v_N(s, y) = V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right) = f(\frac{y}{N+1})$

$$v_n(s, y) = \frac{\tilde{p}v_{n+1}(us, y + us) + \tilde{q}v_{n+1}(ds, y + ds)}{(1+r)}$$

2.14

(i) $E_n\big(g(S_{n+1}, Y_{n+1})\big)$

$$= \begin{cases} \tilde{p}g(uS_n, 0) + \tilde{q}g(dS_n, 0), if\ 0 \leq n \leq M \\ \tilde{p}g(uS_n, Y_n + uS_n) + \tilde{q}g(dS_n, Y_n + dS_n), if\ M+1 \leq n \leq N \end{cases}$$

$\therefore It\ is\ Markov$

(ii) $v_n(S_n, Y_n) = \tilde{p}v_n(uS_n, Y_n + uS_n) + \tilde{q}v_n(dS_n, Y_n + dS_n)$

$v_n(S_n) = \tilde{p}g(uS_n) + \tilde{q}g(dS_n)$

$$V_n = \begin{cases} v_n(S_n), if\ 0 \leq n \leq M \\ v_n(S_n, Y_n), if\ M+1 \leq n \leq N \end{cases}$$