The Binomial No-Arbitrage Pricing Model

1.1 $If\ X_0 = 0, X_1 = \Delta_0(S_1 - (1+r)S_0)$

$\therefore X_1 \geq 0\ if\ \frac{S_1}{S_0} \geq (1+r)\ and\ X_1 < 0\ if\ \frac{S_1}{S_0} < (1+r)$

$\therefore u > (1+r) > d\ and\ this\ precludes\ arbitrage$

1.2 $X_1\ is\ either\ 3\Delta_0 + 1.5\Gamma_0\ for\ H\ or - (3\Delta_0 + 1.5\Gamma_0)\ for\ T$

$\because P(T) > 0\ and\ P(H) > 0$

$\therefore If\ P(X_1 > 0) > 0\ then\ P(X_1 < 0) > 0$

1.3 $Equivalent\ to\ underlying\ stock\ unless\ its\ price\ goes\ negative$

$\therefore S_0 = V_0$

1.4 $\because X_{n+1}(T) = (1+r)X_n + \Delta_n(S_{n+1}(T) - (1+r)S_n)$

$$= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}(dS_n(T) - (1+r)S_n)$$

$$= (\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)) - \tilde{p}\big(V_{n+1}(H) - V_{n+1}(T)\big)$$

$$= V_{n+1}(T)$$

1.5 $\mathrm{X}_2(\mathrm{HH}) = (1+\mathrm{r})\mathrm{X}_1(\mathrm{H}) + \Delta_1(H)\big(S_2(HH) - (1+r)S_1(H)\big)$

$$= 1.25 \times 2.24 + \frac{3.2 - 2.4}{16 - 4} \times (16 - 1.25 \times 8)$$

$$= 3.2$$

$$X_2(HT) = (1+r)X_1(H) + \Delta_1(H)\big(S_2(HT) - (1+r)S_1(H)\big)$$

$$= 1.25 \times 2.24 + \frac{3.2 - 2.4}{16 - 4} \times (4 - 1.25 \times 8)$$

$$= 2.4$$

$$X_3(HTH) = (1+r)X_2(HT) + \Delta_2(HT)\big(S_3(HTH) - (1+r)S_2(HT)\big)$$

$$= 1.25 \times 2.4 + \frac{-6}{8 - 2} \times (8 - 1.25 \times 4)$$

$$= 0$$

$$X_3(HTT) = (1+r)X_2(HT) + \Delta_2(HT)\big(S_3(HTT) - (1+r)S_2(HT)\big)$$

$$= 1.25 \times 2.4 + \frac{-6}{8 - 2} \times (2 - 1.25 \times 4)$$

$$= 6$$

1.6 $\mathit{To\ earn}\ 25\%\ \mathit{interest}, X_0\ \mathit{and}\ \Delta_0\ \mathit{must\ satisfy}$:

$(1+r)X_0 + \Delta_0\big(S_1 - S_0(1+r)\big) + (S_1 - K)^+ = 1.25V_0,$
$\because (S_1 - K)^+ = 3\ \mathit{when}\ S_1 = uS_0\ \mathit{and}\ (S_1 - K)^+ = 0\ \mathit{when}\ S_1 = dS_0,$
$\therefore X_0 = 0\ \mathit{and}\ \Delta_0 = -0.5.$

This means we should short sell 0.5 *share of stock*.
When $S_1 = 2S_0$, *we earn* 3 *from the option but loss* 1.5 *from short selling the stock*; *when* $S_1 = \frac{1}{2}S_0$, *we earn nothing from the option but gain* 1.5 *from short selling stock*. *Both yield* 1.5 *return*.

1.7 $\mathit{To\ earn}\ 25\%\ \mathit{interest}, X_0\ \mathit{and}\ \Delta_0\ \mathit{must\ satisfy}$:
$(1+r)X_0 + \Delta_0\big(S_1 - S_0(1+r)\big) + V_1 = 1.25V_0,$
$\because V_1 = 2.24\ \mathit{when}\ S_1 = uS_0\ \mathit{and}\ V_1 = 1.2\ \mathit{when}\ S_1 = dS_0,$

$\therefore X_0 = 0 \; and \; \Delta_0 = -\frac{13}{75}.$

$Then \; move \; to \; t = 1:$

$(1+r)X_1 + \Delta_0\big(S_2 - S_1(1+r)\big) + V_2 = 1.25^2 V_0$

$\dots \dots$

$Then \; move \; to \; t = 1: \dots \dots$

1.8

(i) $v_n(s,y) = \frac{2}{5}\left(v(2s, y+2s) + v\left(\frac{s}{2}, y+\frac{s}{2}\right)\right)$

(ii) $v_3(32,60) = 11, v_3(8,36) = 5, v_3(8,24) = 2,$

$v_3(8{,}18) = v_3(2{,}18) = \frac{1}{2}, v_3(2,12) = v_3(2{,}9) = v_3\left(\frac{1}{2}, \frac{15}{2}\right) = 0;$

$$v_2(16,28) = \frac{v_3(32,60) + v_3(8,36)}{2(1+r)} = \frac{32}{5},$$

$$v_2(4,16) = \frac{v_3(8,24) + v_3(2,18)}{2(1+r)} = 1,$$

$$v_2(4,10) = \frac{v_3(8,18) + v_3(2,12)}{2(1+r)} = \frac{1}{5},$$

$$v_2(1,7) = \frac{v_3(2,9) + v_3\left(\frac{1}{2}, \frac{15}{2}\right)}{2(1+r)} = 0;$$

$$v_1(8,12) = \frac{v_2(16,28) + v_2(4,16)}{2(1+r)} = \frac{74}{25},$$

$$v_1(2,6) = \frac{v_2(4,10) + v_2(1,7)}{2(1+r)} = \frac{2}{25};$$

$$v_0(4,4) = \frac{v_1(8,12) + v_1(2,6)}{2(1+r)} = \frac{152}{125} = 1.216$$

(iii)

$$From\begin{cases}(1+r)(v_n(s,y)-\delta_n(s,y)s)+u\delta_n(s,y)s=v_{n+1}(us,y+us)\\(1+r)(v_n(s,y)-\delta_n(s,y)s)+d\delta_n(s,y)s=v_{n+1}(ds,y+ds)\end{cases}$$

$$\delta_n(s,y)=\frac{v_{n+1}(2s,y+2s)-v_{n+1}\left(\frac{1}{2}s,y+\frac{1}{2}s\right)}{\frac{3}{2}s}$$

1.9

(i) $For\ every\ combination\ \omega_1\omega_2\ldots\omega_{N-1}$:

$$\begin{cases}(1+r_{N-1})V_{N-1}+\Delta_{N-1}(u_{N-1}S_{N-1}-(1+r_{N-1})S_{N-1})=V_N(H)\\(1+r_{N-1})V_{N-1}+\Delta_{N-1}(d_{N-1}S_{N-1}-(1+r_{N-1})S_{N-1})=V_N(T)\end{cases}$$

$Choose\ risk\ neutral\ probabilities\ and\ yield$:

$$V_{N-1}=\tilde{p}V_N(H)+\tilde{q}V_N(T), where\ \tilde{p}=\frac{(1+r_{N-1})-d_{N-1}}{u_{N-1}-d_{N-1}},\tilde{q}=1-\tilde{p}$$

$Then\ move\ to\ every\ combination\ \omega_1\omega_2\ldots\omega_{N-2}$: …

… …

$$V_0=\tilde{p}V_1(H)+\tilde{q}V_1(T), where\ \tilde{p}=\frac{(1+r_0)-d_0}{u_0-d_0}\ and\ \tilde{q}=1-\tilde{p}$$

(ii) From (i), $\Delta_n=\frac{V_{n+1}(H)-V_{n+1}(T)}{(u_{n+1}-d_{n+1})S_n}$

(iii) $\because u=1.1, d=0.9, 1+r=1\ \therefore \tilde{p}=\tilde{q}=\frac{1}{2}$

$V_0=9.375$