State Prices

3.1

(i) $\because \tilde{P}(\omega) > 0\ for\ \forall\omega\ \therefore\ \tilde{P}\left(\frac{1}{Z}\right) > 0$

(ii) $\tilde{E}\left(\frac{1}{Z}\right) = \sum \frac{\tilde{P}(\omega)}{Z(\omega)} = \sum P(\omega) = 1$

(iii) $E(Y) = \sum Y(\omega)P(\omega) = \sum Y(\omega)\frac{\tilde{P}(\omega)}{Z(\omega)} = \tilde{E}(\frac{Y(\omega)}{Z(\omega)})$

3.2

(i) $P(\Omega) = \sum_{\omega\in\Omega}\tilde{P}(\omega) = \sum_{\omega\in\Omega} Z(\omega)P(\omega) = E(Z) = 1$

(ii) $\tilde{E}(Y) = \sum Y(\omega)\tilde{P}(\omega) = \sum Y(\omega)Z(\omega)P(\omega) = E(ZY)$

(iii) $\tilde{P}(A) = \sum_{\omega\in A} Z(\omega)P(\omega) = \sum_{\omega\in A} Z(\omega) \times 0 = 0$

(iv) $P(A) = \sum_{\omega\in A}\frac{\tilde{P}(\omega)}{Z(\omega)} = \sum_{\omega\in A}\frac{0}{Z(\omega)} = 0$

(v) $If\ P(A^c) = \tilde{P}(A^c) = 0\ then\ P(A) = \tilde{P}(A) = 1$

(vi) $P(A) = p, P(B) = 1 - p, P(A^c \cup B^c) = 0;$

$$Z(A) = \frac{1}{p}, Z(B) = 0, Z(A^c \cup B^c) = 0$$

3.3 $M_3(HHH) = 32, M_3(HHT) = M_3(HTH) = M_3(THH) = 8,$

$M_3(HTT) = M_3(THT) = M_3(TTH) = 2, M_3(TTT) = \frac{1}{2},$

$M_2(HH) = 24, M_2(HT) = M_2(TH) = 6, M_2(TT) = \frac{3}{2}$

$M_1(H) = 18, M_1(T) = \frac{9}{2}, M_0 = \frac{27}{2}$

$M_n = E_n(M_m)\ for\ \forall n < m\ as\ expected, M_n\ is\ a\ martingale$

3.4

(i) $\zeta_3(HHH) = \frac{27}{125}, \zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) = \frac{54}{125}$

$\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) = \frac{108}{125}, \zeta_3(TTT) = \frac{216}{125}$

(ii) $v_0 = \frac{152}{125}$

(iii)$\zeta_2(HT) = \zeta_2(TH) = \frac{18}{25}$

(iv) $V_2(HT) = \frac{1}{\zeta_2(HT)} E_2\big(\zeta_3(HT)V_3(HT)\big) = 1$

$V_2(TH) = \frac{1}{\zeta_2(HT)} E_2\big(\zeta_3(TH)V_3(TH)\big) = \frac{1}{5}$

3.5

(i) $Z(HH) = \frac{9}{16}, Z(HT) = \frac{9}{8}, Z(TH) = \frac{3}{8}, Z(TT) = \frac{15}{4}$

(ii) $Z_1(H) = \frac{3}{4}, Z_1(T) = \frac{3}{2}, Z_0 = E(Z_1) = 1$

(iii) $\zeta_2(HH) = \frac{9}{25}, \zeta_2(HT) = \frac{18}{25}, \zeta_2(TH) = \frac{1}{5}, \zeta_2(TT) = 2$

$V_0 = \frac{226}{225}$