1. Fiber Connectivity

Step1:

Prove that Fiber Connectivity is in NP.

Certificate: Given a subset of roads T in C.

Certifier: Verify whether this subset of roads connecting all cities in C and every city has at most k incident roads in T in polynomial time $O(n)$.

For each city, first check if it is connected to any other cities. If no, this certificate is not a solution. If yes, check if this city is connected to at most 2 roads. If yes, go to the next city. If no, this certification is not a solution.

Each check operation takes $O(1)$ time so the time complexity is $O(n)$. Hence, a given certificate can be verified in polynomial time.

Step2:

Choose Hamiltonian Path as known NP-Completeness problem.

Hamiltonian Path: Given an undirected graph G = (V, E), does there exist a simple path Γ that contains every node in V?

Step3:

We prove that the fiber connectivity problem is NP complete by a polynomial-time reduction from the Hamiltonian Cycle problem.

◎ Show that an instance I_1 of Hamiltonian Cycle can be transformed into an instance I_2 of Fiber Connectivity.

For Ham Path, we have a graph of nodes and edges and a subset of edges which can visit each node exactly once. Consider each node as a city and each edge as a road. We get a set of roads that connect all cities and each city is connected to at most two roads (in fiber connectivity, k=2). This transformation didn't take any extra computational time. Hence, we can do this transformation in polynomial time.

◎ Show that if I_1(YES) then I_2(YES).

Assume we have a subset of edges in graph G which can visit each node exactly once. These edges represent the set of roads chosen in the Fiber Connection problem. Suppose this set is not a solution to the Fiber Connection problem (k = 2), which means there exists a city that is connected by more than 2 roads or is not connected to any other cities. Transform this case into Ham Path problem, which means there is a node that is not connected to any other nodes or is connected to more than 2 nodes. This means the given set of edges is not a solution to a Ham Path problem and this contradicts our assumption at first. Hence, if in a graph G, there exists a Hamiltonian Path, then there must exists a subset of roads that can connect all cities and each city is connected to at most 2 roads.

◎ Show that if I_2(YES) then I_1(YES).

Assume we have a set of roads such that all cities are connected and every city is connected to at most two roads. These roads equate to a set of edges in the Ham Path problem. Suppose this set of edges is not the solution to Ham Path. This means we have a node such that it is not connected to any other nodes or is connected to more than 2 nodes(be visited more than once). In the Fiber Connectivity problem, this means that there is a city which is connected to more than 2 roads or is not connected to any other cities. This contradicts the assumption that we started with a solution for Fiber Connectivity. Hence, if we had a subset of roads to solve Fiber Connectivity problem, we know how to solve the Hamiltonian Path problem.

2. Pygmalion

Step 1:

Prove that Pygmalion is in NP

Certificate: Given a graph G = (V, E), each town is a vertex and each road between two towns is an edge and a set of k bunker vertices.

Certify: Verity if each town has a bunker or is connected directly to a bunker town.
For each town, first check if it has a bunker. If yes, go to the next town. If no, check if one of the neighboring towns has a bunker. If yes, go to the next town. If no, this certification is not a solution.
Each town can have at most *n-1* neighboring towns so the time complexity is $O(n^2)$. Hence, a given certificate can be verified in polynomial time.

Step 2:

Choose Vertex Cover as known NP-Completeness problem.

Vertex Cover: Given a graph G = (V, E), is there a set of at most *k* vertices that cover each edge?

Step 3:

We prove that the Pygmalion problem is NP complete by a polynomial-time reduction from the Vertex Cover problem.

◎ Show that an instance I_1 of Vertex Cover can be transformed into an instance I_2 of Pygmalion.
For vertex cover, we have a graph of nodes and edges. Consider each node as a town and each edge as a road, then the *k* vertices represent for *k* bunker towns. This transformation didn't take any extra computational time. Hence, we can do this transformation in polynomial time.

◎ Show that if I_1(YES) then I_2(YES).
Assume we have a set of *k* vertices that for each edge (*u*, *v*), either *u* or *v* lies in the set. These *k* vertices represent *k* bunker towns in the Pygmalion problem. Suppose this set is not a solution to the Pygmalion problem, which means there exists a town which does not have a bunker and is not connected to any town with a bunker. Transform this case into VC problem, which means there is a node that is not in the set of *k* vertices and does not belong to any edge covering by the *k* vertices. This means the given set of k vertices is not a solution to a Vertex Cover problem and this contradicts our assumption at first. Hence, if we have a solution for Vertex Cover, we have a solution for Pygmalion.

◎ Show that if I_2(YES) then I_1(YES).
Assume we have a set of *k* towns with bunkers such that every town either has its own bunker, or is connected by a direct road to a town that has a bunker. These *k* towns equate to *k* vertices in the Vertex Cover problem. Suppose these *k* vertices are not the solution. This means we have a node such that it is not in the set of *k* vertices and does not belong to any edge covered by the *k* vertices. In the Pygmalion problem, this means that there is a town which has no bunker and is not connected

to any town with a bunker. This contradicts the assumption that we started with a solution for Pygmalion. Hence, if we have a solution for Pygmalion, we have a solution for Vertex Cover.