Algebraic Topology HW 4

Caleb Alexander

Problem 3. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Let α denote the map from A to B, and β denote the map from B to C. Prove:

(**a**) there is a map $s : B \to A$ such that $s\alpha = idA$ iff there is a map $t : C \to B$ such that $\beta t = idC$.

(**b**) Prove that, in either case, B is isomorphic to $A \times C$, by arguing that B is the internal direct product of the groups $\alpha(A)$ and $t(C)$.

Proof. Suppose that there is a map $s : B \to A$ such that $s\alpha = id_A$. First note that α is injective and β is surjective by exactness. Surjectivity gives us that β has a right inverse, $t : C \to B$, such that $\beta t = id_C$ (which also implies that t is injective). Now it suffices to show that t is a homomorphism. Consider that

$$t(c_1) + t(c_2) = b_1 + b_2 = b_3.$$

Notice that $b_3 \in \operatorname{Im}\beta$ and thus there must exist some c_3 such that $t(c_3) = b_3$. But β is a homomorphism so $\beta(b_3) = \beta(b_1 + b_2) = \beta(b_1) + \beta(b_2)$ thus $c_3 = \beta(t(c_3)) = \beta(t(c_1) + t(c_2)) = \beta(t(c_1)) + \beta(t(c_2)) = c_1 + c_2$. Hence,

$$t(c_1) + t(c_2) = b_1 + b_2 = b_3 = t(c_3) = t(c_1 + c_2).$$

So we have that $s \implies t$. Now suppose that $t : C \to B$ exists with $\beta t = id_C$. Similar to before see that α has a left inverse, $s : A \to B$ such that $s\alpha = id_A$ and s is surjective. Consider some $b_1, b_2 \in \operatorname{Im}\alpha$ (so there exists $a_1, a_2 \in A$ such that $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$) and also see that $b_1 + b_2 = b_3 \in \operatorname{Im}\alpha$ since $b_3 = b_1 + b_2 = \alpha(a_1) + \alpha(a_2) = \alpha(a_1 + a_2) = \alpha(a_3)$ (α is a homomorphism) for some $a_3 \in A$. Furthermore since $\alpha(a_1 + a_2) = \alpha(a_1) + \alpha(a_2) = b_3 = \alpha(a_3)$, by injectivity, $a_3 = a_1 + a_2$. Now see that:

$$s(b_1 + b_2) = s(b_3) = a_3 = a_1 + a_2 = s(b_1) + s(b_2).$$

Now we claim that the existence of this splitting implies that $B \cong A \oplus C$. Thus it suffices to show that there exists an isomorphism between the two. Define θ as follows:

$$\theta : B \to A \oplus C$$

$$\theta(b) = (s(b), \beta(b)).$$

First we tackle injectivity. Let $b \in B$ such that $\theta(b) = (0, 0)$. If $\beta(b) = 0$ then $b \in \ker\beta$, thus $b \in \operatorname{Im}\alpha$. Hence, there exists some $a \in A$ such that $\alpha(a) = b$, but $0 = s(b) = s(\alpha(a)) = a$. So $b = \alpha(a) = 0$. Therefore, $\ker\theta$ is trivial and θ is injective.

Now for surjectivity, let $(a, c) \in A \oplus C$. Since α is injective, $\alpha(a) = b$ for some $b' \in B$. Thus $s(b') = a$. Furthermore, $t(c) = b''$ for some $b'' \in B$. Notice that for b', $\beta(b') = 0$ since $b' \in \operatorname{Im}\alpha$ and similarly $s(b'') = 0$. Finally, see that

$$\begin{aligned}\theta(b' + b'') &= (s(b' + b''), \beta(b' + b'')) = (s(b') + s(b''), \beta(b') + \beta(b'')) \\ &= (a + 0, 0 + c) = (a, c) = (a, 0) + (0, c) = \theta(b') + \theta(b'').\end{aligned}$$

We have found an element $b' + b'' \in B$ such that $\theta(b' + b'') = (a, c)$. This suffices to show surjectivity. Furthermore we have proven that θ is a homomorphism.

$\square$

Problem 4. Let A be a subset of X, where X is a topological space. We say that $r : X \to A$ is a retraction if the composition $A \to X \to A$ (the second map is r) is equal to the identity. If such an r exists, we also say that A is a retract of X. (Don't confuse a retraction with a strong deformation retraction. Retractions need not be homotopy equivalences, for instance.) Prove: if $r : X \to A$ is a retraction, then the long exact sequence in homology splits into (infinitely many) short exact sequences:

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0,$$

all of which are split. (You can get started by arguing that the map from $H_n(A)$ is injective.) In particular, show that S^n is never a retract of B^{n+1}.

Proof. The maps $r \circ i = id_A$ induce homomorphisms on homology such that $r^* \circ i^* = id_{H_n(A)}$. Note that $r^* : H_n(X) \to H_n(A)$ so $i^* : H_n(A) \to H_n(X)$. Clearly for the composition to be the identity, i^* is injective. Now consider the following short exact sequence of chain groups:

$$0 \to C_n(A) \to^i C_n(X) \to^j C_n(X, A) \to 0,$$

which induces the following long exact sequence on homology:

$$\ldots \to H_{n+1}(X, A) \to^{\partial} H_n(A) \to^{i^*} H_n(X) \to^{j^*} H_n(X, A) \to^{\partial} H_{n-1}(A) \to \ldots.$$

Since i^* is still injective, $\{0\} = \ker i^* = \operatorname{Im} \partial_n$, therefore $\ker \partial = H_n(X, A)$. By exactness $\ker \partial = \operatorname{Im} j^* = H_n(X, A)$ and so j^*, whatever map it may be, is certainly surjective. Given this, we claim that exactness of the following sequence is proven:

$$0 \to H_n(A) \to^{i^*} H_n(X) \to^{j^*} H_n(X, A) \to 0.$$

Now suppose for contradiction that there does exist $r : B^{n+1} \to S^n$ which gives us the following short exact sequence:

$$0 \to H_n(S^n) \to^{i^*} H_n(B^{n+1}) \to^{j^*} H_n(B^{n+1}, S^n) \to 0.$$

Consider reduced homology and note that the reduced homology of B^{n+1} is always 0 and the reduced homology of S^n is 0 in every dimension except for $n - 1$ in which it is $\mathbb{Z}$. So $i^*_{n-1} : \mathbb{Z} \to 0$, but i^*_{n-1} is injective, which clearly is not possible. Hence a contradiction. $\square$