Math 439 Homework 5

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Problem 1. Let a_n denote the number of ternary strings of length n that contain an even number of 0s, b_n the number of ternary strings of length n that contain an even number of 0s and even number of 1s, and c_n the number of ternary strings that contain at least one 0 and at least one 1.

(**a**) Find an exponential generating function for a_n, and use it to find a closed form formula for a_n.

(**b**) Find an exponential generating function for b_n, and use it to find a closed form formula for b_n.

(**c**) Find an exponential generating function for c_n, and use it to find a closed form formula for c_n.

Answer 1. The answers are given below.

(**a**) We know that the generating function for distributing n different balls to r different boxes is

$$f(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots)^r.$$

Now imagine each ball (each are different) is labelled such that the ball reflects a digit in our ternary string. We know that we need to distribute all n labelled balls and that our boxes will be three labelled boxes, one labelled as 0 (for the spots that will have a 0 digit), one labelled as 1, and the last box being labelled 2. Hence, we just solve the previous problem except only the first box needs to be even. So our generating function for a_n is

$$g(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)^2 = \frac{1}{2}(e^x + e^{-x})(e^{2x}) = \frac{1}{2}(e^{3x} + e^x).$$

Hence, our formula for c_n (coefficients not part c) is

$$\frac{1}{2}(\frac{3^n}{n!} + \frac{1}{n!}).$$

Since $b_n = c_n \cdot n!$ we get,

$$\frac{1}{2}(3^n + 1).$$

(**b**) This problem is similar to the last except we require that boxes 0 and 1 have an even number of balls, so we get

$$f(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots)^2(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots) = (\frac{1}{2}(e^x + e^{-x}))^2(e^x).$$

By the binomial theorem,

$$f(x) = \frac{1}{4}(e^{2x} + e^{-2x} + 2)(e^x) = \frac{1}{4}(e^{3x} + e^{-x} + 2e^x).$$

So our formula for c_n (coefficients not part c) is

$$\frac{1}{4}(\frac{3^n}{n!} + \frac{(-1)^n}{n!} + 2\frac{1}{n!}).$$

Since $b_n = c_n \cdot n!$ we get,

$$\frac{1}{4}(3^n + (-1)^n + 2).$$

(**c**) For part c we now remove the first terms of our generating functions for the 0 and 1 boxes since we do not allow 0 balls to go into those boxes. So our generating function is

$$f(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)^2 (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots) = (e^x - 1)^2 (e^x) = e^{3x} - 2e^{2x} + e^x.$$

So our formula for the coefficients is

$$Coeff_n = \frac{3^n}{n!} + (-2)\frac{2^n}{n!} + \frac{1}{n!}$$

so

$$b_n = 3^n + (-1)(2^{n+1}) + 1.$$

Problem 2. Answer the following.

(**a**) Let a_n denote the number of ways to place n different people into five different rooms with at least one person in each room. Find an exponential generating function for a_n. Use it to find the number of ways to place 20 people into five different rooms with at least one person in each room.

(**b**) Let b_n denote the number of ways to distribute n different gifts to four different children with the first child getting at least two toys. Find an exponential generating function for b_n. Use it to find the number of ways to distribute eight different gifts subject to the above constraints.

Answer 2. The answers are offerred below.

(**a**) Similar to the last problem our generating function is

$$f(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)^5 = (e^x - 1)^5 = e^{5x} - 5e^{4x} + 10e^{3x} - 10e^{2x} + 5e^x - 1.$$

So our formula for c_n is

$$c_n = \frac{5^n}{n!} + (-5)\frac{4^n}{n!} + (10)\frac{3^n}{n!} + (-10)\frac{2^n}{n!} + (5)\frac{1}{n!}.$$

Hence,

$$a_n = 5^n + (-5)4^n + (10)3^n + (-10)2^n + 5.$$

In particular,

$$a_{20} = 5^{20} + (-5)4^{20} + (10)3^{20} + (-10)2^{20} + 5.$$

(**b**) We now restrict just the first box such that it must have at least two items. So our generating function is

$$f(x) = (\frac{x^2}{2!} + \frac{x^3}{3!} + \dots)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)^3 = (e^x - 1 - x)(e^x)^3 = e^{4x} - e^x - xe^x.$$

So our formula for c_n is

$$c_n = \frac{4^n}{n!} - \frac{1}{n!} - \frac{1}{(n-1)!}.$$

Hence,

$$b_n = 4^n - 1 - n.$$

In particular,

$$b_8 = 4^8 - 9.$$

Problem 3. Let c_n denote the number of ways to choose any number of flags from n different flags and place them onto three distinct flag poles (where the order of the flags on each pole matters). Prove that

$$f(x) = (\frac{1}{1-x})^3 \cdot e^x$$

is an exponential generating function for c_n. (hint: using the expansions of $(\frac{1}{1-x})^3$ and e^x and the rule for series multiplication you can find the coefficient d_n of x^n in $f(x)$. You then just need to verify that $c_n = d_n \cdot n!$.)

Answer 3. We have that,

$$f(x) = (1-x)^{-3}(1 + x + \frac{x^2}{2!} + \dots) = (\sum_{k=0}^{\infty} \binom{k+2}{2} x^k)(1 + x + \frac{x^2}{2!} + \dots).$$

So,

$$d_n = \frac{\binom{2}{2}}{n!} + \frac{\binom{3}{2}}{(n-1)!} + \dots + \frac{\binom{(n-1)+2}{2}}{1!} + \frac{\binom{n+2}{2}}{0!}.$$

Hence,

$$c_n = 1 + \lfloor n \rfloor_1 \binom{3}{2} + \lfloor n \rfloor_2 \binom{4}{2} + \dots + \lfloor n \rfloor_{n-1} \binom{(n-1)+2}{2} + n! \binom{n+2}{2}.$$

Problem 4. (**a**) Let $n \geq 3$. Determine the number of permutations of $\{1, \dots, n\}$ that contain exactly 3 cycles and with elements $1, 2, 3$ all in different cycles.

(**b**) Let $n \geq 3$. Determine the number of permutations of $\{1, \dots, n\}$ in which $1, 2, 3$ are in the same cycle.

(**c**) Let $n \geq 2$. Find a simple formula for $c(n, 2)$ and provide some justification for it.

Answer 4. The answers are offerred below.

(**a**) If we disregard the constraint of $1, 2, 3$ all being in different cycles we know that there are $c(n, 3)$ ways of permuting $[n]$ such that it has exactly 3 cycles. Since we want to assure that $1, 2, 3$ are all in different cycles, we can just refrain from adding them until the end, so we start with $c(n-3, 3)$ ways of permuting our remaining elements into exactly three cycles, now we just choose which cycle to insert $1, 2, 3$. So we get,

$$c(n-3, 3) \cdot 3!.$$

(**b**) The answer is the same as the previous, except at the end we just add all three numbers to the same cycle, so we get

$$c(n-3, 3) \cdot 3.$$

(**c**) We know that $(x + n - 1)_n$ is a generating function for $c(n, k)$ where the coefficient for x^k is equivalent to $c(n, k)$. Hence we need to find the coefficient of x^2 from

$$x(x+1)(x+2) \cdots (x+n-1).$$

Notice that

$$x(x+1)(x+2) \cdots (x+n-1) = (x^2 + x)(x+2) \cdots (x+n-1).$$

Problem 5. Answer the following problems.

(**a**) Let a_n denote the number of ways to select a subset of non-consecutive numbers from $[n]$. Find a recurrence relation for a_n together with initial conditions and use that to find a_8.

(**b**) Let b_n denote the number of ternary strings of length n that do not contain 110. Find a recurrence for b_n.

(**c**) Let c_n denote the number of different permutations of $[5n]$ where each cycle has length 5. Find a recurrence for c_n.

Answer 5. The answers are offerred below.

(**a**) We call a subset as "good" iff none of the members are consecutive. First notice that any "good" subset of a_{n-1} is also a good subset of a_n. In fact, the only subsets that are not already considered by a_{n-1} are those containing the number n itself. So consider those subsets which contain n which also must be good. We know that they also cannot contain $n-1$ and that they must be a good subset themselves. So now we must just count the number of good subsets which do not contain $n-1$, notice that this is equivalent to the number of good subsets of $[n-2]$, hence

$$a_n = a_{n-1} + a_{n-2}.$$

So (assuming that $a_0 = a_1 = 1$),

$$a_8 = a_7 + a_6 = (a_6 + a_5) + (a_5 + a_4) = ((a_5 + a_4) + (a_4 + a_3)) + ((a_4 + a_3) + (a_3 + a_2))$$

$$\ldots = (8+5) + (5+3) + (5+3) + (3+2) = 13 + 8 + 8 + 5 = 34$$

(**b**) We call a string "good" iff it does not contain 110. We will break into cases. Consider any ternary string of length n.

Case 1: It begins with a 2 or 0. In this case, as long as the rest of the string is "good" we can be certain that the full string is good. Hence there are $2a_{n-1}$ n length good strings that start with a 2 or 0.

Case 2: It begins with a 1, in this case we have two requirements for the rest of the string. The first two digits cannot be 10 and the rest of the $n-3$ spots of the string must also be good. There are 8 ternary strings of length 2 and one of them (10) cannot be included, so we have $7a_{n-3}$.

Our full answer is,

$$a_n = 2a_{n-1} + 7a_{n-3}$$

(**c**) Consider a permutation of $[5n]$ to be "good" if every cycle has length 5. Consider a "good" permutation of $[5(n-1)]$ and notice that when we add in 5 more elements we can choose one of two options. We could form a cycle from our new 5 elements, or we could choose some number of other elements to swap with out new elements and then form a cycle out of the swapped elements. Hence our formula is,

$$c_n = c_{n-1} + \binom{5(n-1)}{1} c_{n-1} + \binom{5n-1}{2} c_{n-1} + \binom{5n-1}{3} c_{n-1} + \binom{5n-1}{4} c_{n-1} + \binom{5n-1}{5} c_{n-1}$$

Problem 6. In each problem find a closed form formula for a_n via generating functions.

(**a**) a_n is a sequence that is recursively defined by

$$a_n = 2a_{n-1} + 8a_{n-2}, \quad a_0 = a_1 = 1.$$

(**b**) a_n is a sequence defined by

$$a_n = \sum_{k=1}^{n} k \cdot 2^{n-k}.$$

Answer 6. The answers are offered below.

(**a**) Notice that

$$\sum_{k=2}^{n} a_n x^n = 2 \cdot \sum_{k=2}^{n} a_{n-1} x^n + 8 \cdot \sum_{k=2}^{n} a_{n-2} x^n,$$

if $g(x)$ is the generating function for a_n then we know that,

$$g(x) - 1 - x = 2x \cdot (g(x) - 1) + 8x^2 g(x) = 2xg(x) - 2x + 8x^2 g(x).$$

We now hope to solve for $g(x)$, with some re-arrangement we get

$$g(x) - 2xg(x) - 8x^2 g(x) = 1 - x \implies g(x)(1 - 2x - 8x^2) = 1 - x \implies g(x) = \frac{1-x}{1-2x-8x^2} = \frac{1-x}{(1-4x)(2x+1)}.$$

A partial fraction decomposition gives

$$g(x) = \frac{1}{2} \cdot \frac{1}{2x+1} - \frac{1}{2} \cdot \frac{1}{4x-1} = \frac{1}{2} \cdot \frac{1}{1+2x} + \frac{1}{2} \cdot \frac{1}{1-4x}.$$

Hence,

$$g(x) = \frac{1}{2} \cdot \sum_{k=0}^{\infty} (-2x)^k + \frac{1}{2} \cdot \sum_{k=0}^{\infty} (4x)^k.$$

Finally we have our closed form formula for a_n,

$$a_n = \frac{(-2)^n}{2} + \frac{4^n}{2}.$$

(**b**) Notice that this is a convolution between the sequences $\{n\}$ and $\{2^n\}$. We know that the generating function for $\{n\}$ is

$$f(x) = \frac{x}{(1-x)^2}.$$

Now we consider $\{2^n\}$, this can be represented as a geometric series with common ration $2x$ hence,

$$g(x) = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}.$$

If $h(x)$ is the generating function for a_n then,

$$h(x) = f(x) \cdot g(x) = \frac{x}{(1-x)^2(1-2x)}.$$

Use partial fraction decomposition to get,

$$h(x) = 2\frac{1}{1+2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} = 2 \cdot \sum_{k=0}^{\infty}(-2x)^k - \sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty} \binom{k+1}{1} x^k.$$

So,

$$a_n = 2 \cdot (-2)^n - 1 - \binom{n+1}{1} = (-1)^n \cdot 2^{n+1} - n - 2.$$