Math 491 HW 2

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Problem 3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \{x : \frac{1}{2} < |x| < 1\},$$

$$B = \{x : \frac{1}{2} < |x| \leq 1\},$$

$$C = \{x : \frac{1}{2} \leq |x| < 1\},$$

$$D = \{x : \frac{1}{2} \leq |x| \leq 1\},$$

$$E = \{x : 0 < |x| < 1 \quad \text{and} \quad \frac{1}{x} \notin \mathbb{Z}_+\},$$

Proof. We will consider the standard topology on $\mathbb{R}$ and the natural subspace topology on Y. Regarding A, we can clearly say that $A = (\frac{1}{2}, 1) \cup (-1, -\frac{1}{2})$. $(\frac{1}{2}, 1)$ and $(-1, -\frac{1}{2})$ are clearly open intervals in $\mathbb{R}$, so $(\frac{1}{2}, 1), (-1, -\frac{1}{2}) \in \mathbb{R}_{std}$. Since a topology must be closed under arbitrary unions, we can also say that $(\frac{1}{2}, 1) \cup (-1, -\frac{1}{2}) \in \mathbb{R}_{std}$. Now see that

$$((\frac{1}{2}, 1) \cup (-1, -\frac{1}{2})) \cap Y = (\frac{1}{2}, 1) \cup (-1, -\frac{1}{2}).$$

Thus $(\frac{1}{2}, 1) \cup (-1, -\frac{1}{2})$ is in the standard subspace topology on Y.

Regarding B, see that

$$B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1].$$

Notice that $(-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$ is in the standard topology on $\mathbb{R}$ and

$$((-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)) \cap [-1, 1] = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1].$$

Thus, $B \in \mathcal{T}_Y$. However, B is clearly not open in $\mathbb{R}$. To prove it; notice that if it were open in $\mathbb{R}$ then there must be a union of open intervals in $\mathbb{R}$, $\bigcup(a, b)$ where $a, b \in \mathbb{R}$, such that for *at least one* of the open intervals, we have $1 \in (a, b)$ **and** for any real number x such that $x > 1$, $x \notin (a, b)$. So $a < 1 < b$, but notice that by the density of $\mathbb{R}$ we can say that there exists some $y \in \mathbb{R}$ such that $a < y < 1 < b$. Thus, our set has failed to be open in $\mathbb{R}$.

Regarding C, see that

$$C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1).$$

C is open in neither Y or $\mathbb{R}$ since no matter how we create an open interval around $\frac{1}{2}$ (so (x, y) with $x < \frac{1}{2} < y$); see that (x, y) **must** contain points that land in the interval $(\frac{1}{2}, 0)$.

D is open in neither Y or $\mathbb{R}$ for the same reasons as before.

Regarding E. Note that this set is equivalent to

$$(-1, 0) \cup \left(\bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \right).$$

Clearly, this set is open in $\mathbb{R}$ since we are representing it as a union of open intervals. Now notice that

$$(-1, 0) \cup \left(\bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \right) \subseteq Y$$

hence,

$$(-1, 0) \cup \left(\bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \right) \cap Y = (-1, 0) \cup \left(\bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \right).$$

Thus E is also open in Y. $\square$

Problem 4. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof. Let U be open in $X \times Y$. So $U = \bigcup V_X \times V_Y$ such that each V_X is open in X and each V_Y is open in Y.

If π_1 is the canonical projection map then notice that

$$\pi_1(U) = \pi_1 \left(\bigcup V_X \times V_Y \right) = \bigcup \pi_1(V_X \times V_Y) = \bigcup V_X.$$

Since each V_X is open in X, $\bigcup V_X$ is also open in X. Hence $\pi_1(U)$ is is open in X and π_1 is an open map.

If π_2 is the canonical projection map then $\pi_2(U) = \bigcup V_Y$ (for the same reasons as in the previous argument) and since each V_Y is open in Y, $\bigcup V_Y$ is also open in Y. Hence $\pi_2(U)$ is open in Y and π_2 must be an open map. $\square$

Problem 6. Show that the countable collection

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d \text{ and } a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$.

Proof. We proved that the set of open intervals with rational endpoints

$$\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$$

is a basis for $\mathbb{R}$. Thus,

$$\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\} \times \{(c, d) : c, d \in \mathbb{Q} \text{ and } c < d\}$$

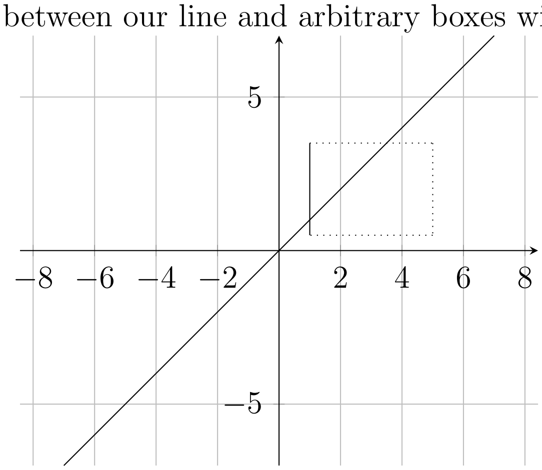
ought to be a basis for $\mathbb{R}^2$. See that

$$\begin{aligned}\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\} \times \{(c, d) : c, d \in \mathbb{Q} \text{ and } c < d\} \\ = \{(a, b) \times (c, d) : a < b \text{ and } c < d \text{ and } a, b, c, d \in \mathbb{Q}\}\end{aligned}$$

□

Problem 8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$. In each case, it is a familiar topology.

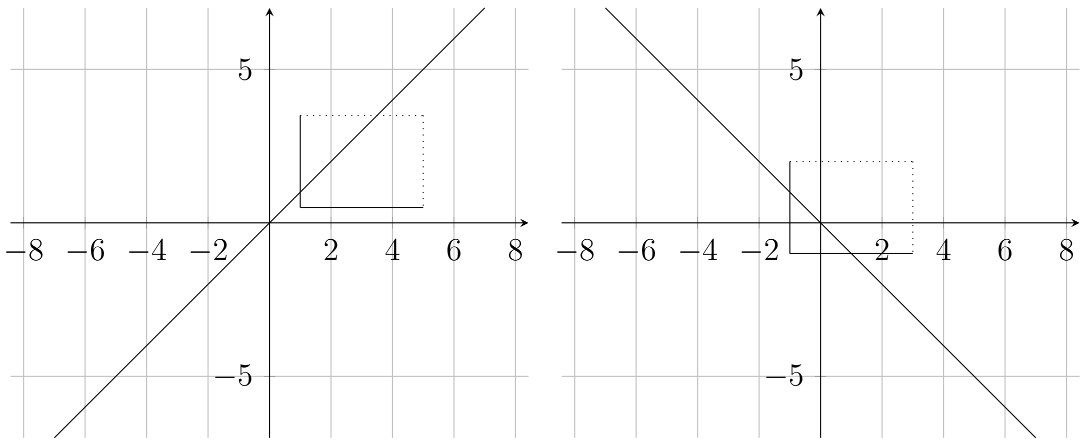
Proof. Regarding $\mathbb{R}_l \times \mathbb{R}$, see that in practice our basic elements will look like the intersections between our line and arbitrary boxes with closed left side.



See that regardless of line, we can always find some box such that the intersection looks like a line segment with the left endpoint included and the right not included. Because L is homeomorphic to $\mathbb{R}$, each of these line segments can be said to be half open intervals in $\mathbb{R}$ of the form

$[a, b)$ where $a, b \in \mathbb{R}$. Notice that we can safely disregard the intervals that include neither endpoint (i.e. open intervals of the form (a, b)) because these intervals can be written as an arbitrary union of half-open intervals. The topology formed by these intersections will be $\mathbb{R}_l$.

Regarding $\mathbb{R}_l \times \mathbb{R}_l$, we have two cases based on whether or not L has positive or negative slope.



Notice that if L has positive slope; then we have exactly the same scenario as with $\mathbb{R}_l \times \mathbb{R}$. However, if L has a negative slope; then it is always possible to pick a basic element of $\mathbb{R}_l \times \mathbb{R}_l$ such that the intersection with L results in a closed interval of the form $[a, b]$ where $a, b \in \mathbb{R}$. However, since a topology must be closed under finite intersection, $[a, a] = \{a\}$ must be in our topology. If every singleton is a member of our topology then the topology must be the discrete topology. $\square$