About Falcons

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1 Introduction

Falcons have their eyes located on either side of their head. As such, to keep their prey in sight without cocking their head, (as this decreases aerodynamic efficiency) falcons fly at a curved trajectory such that the tangent line of the falcon's path and the falcon's line of sight with the prey keep a constant angle of $40°$ for optimal view.

Assuming this constant angle we aim to find the falcon's trajectory. It should be noted that the falcon's movement can be assumed to be planar because we can assume that the falcon will take the shortest path to it's prey, who we will assume to be stationary. The shortest path between any two points is a line, hence the falcon is restricted to moving on the plane formed by the lines between it's prey and itself as θ (the azimuthal angle) changes.

2 Trajectory of Falcon

We let r be the radial distance to the prey (which we place at the origin), and θ the azimuthal angle. We then let $r(\theta)$ be the position of the falcon when the azimthal angle is θ. Consider two points, $(r(\theta), \theta)$ and $(r(\theta + \Delta\theta), \theta + \Delta\theta)$. Plotting the points gives us Figure 1.

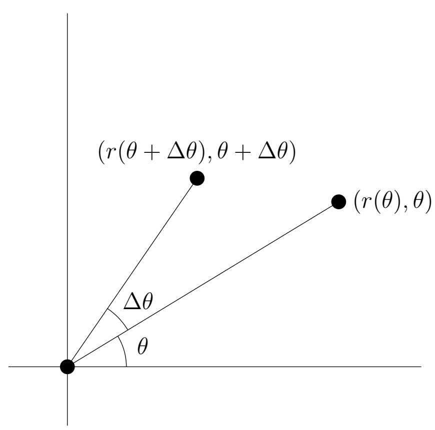

Figure 1: Points $(r(\theta), \theta)$ and $(r(\theta + \Delta\theta), \theta + \Delta\theta)$ graphed in polar coordinates.

As $\Delta\theta$ becomes arbitrarily small, our points better approximate the triangle formed by the radial distance lines and the tangent line (see Figure 2).

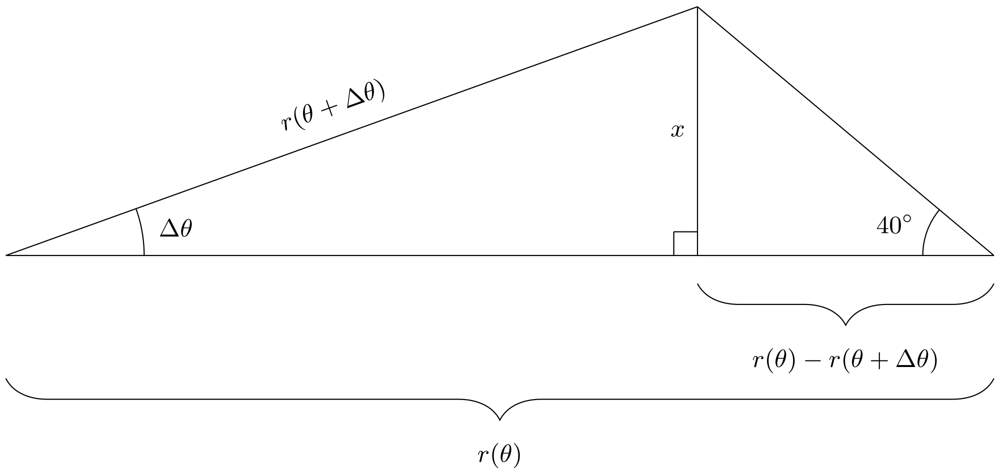

Figure 2: Triangle formed by the radial distance and tangent lines as described above.

Hence we have the following equalities:

$$r(\theta) - r(\theta + \Delta\theta) = x \cot 40^\circ \tag{1}$$

$$x = r(\theta + \Delta\theta) \sin \Delta\theta. \tag{2}$$

By solving (1) and (2) for x and setting these equalities equal to each other, these imply that

$$\begin{gathered} r(\theta) - r(\theta + \Delta\theta) = r(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ \implies \\ \frac{r(\theta) - r(\theta + \Delta\theta)}{\Delta\theta} = \frac{r(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ}{\Delta\theta}. \end{gathered}$$

See that,

$$\begin{aligned}
&\lim_{\Delta\theta \to 0} \frac{r(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ}{\Delta\theta} && \text{by L'Hôpital's rule} \\
&= \lim_{\Delta\theta \to 0} \frac{\frac{d}{d\Delta\theta} r(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ}{\frac{d}{d\Delta\theta} \Delta\theta} \\
&= \lim_{\Delta\theta \to 0} \frac{d}{d\Delta\theta} r(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ && \text{by the product rule} \\
&= \lim_{\Delta\theta \to 0} \cos \Delta\theta r(\theta + \Delta\theta) \cot 40^\circ + r'(\theta + \Delta\theta) \sin \Delta\theta \cot 40^\circ && \sin \Delta\theta \to 0 \\
&= \lim_{\Delta\theta \to 0} r(\theta + \Delta\theta) \cos \Delta\theta \cot 40^\circ \\
&= r(\theta) \cot 40^\circ.
\end{aligned}$$

By definition,

$$\begin{aligned}
r'(\theta) &= \lim_{\Delta\theta\to 0} \frac{r(\theta+\Delta\theta)-r(\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta\to 0} (-1)\frac{r(\theta)-r(\theta+\Delta\theta)}{\Delta\theta} \\
&= (-1)\lim_{\Delta\theta\to 0} \frac{r(\theta)-r(\theta+\Delta\theta)}{\Delta\theta} && \text{as shown above} \\
&= -r(\theta)\cot 40^\circ.
\end{aligned}$$

Now we solve the following ordinary differential equation,

$$\begin{aligned}
r'(\theta) &= -r(\theta)\cot 40^\circ \\
\implies r(\theta) &= c\, e^{-\cot{(40^\circ)}\theta}
\end{aligned}$$

This equation models the trajectory of the flight as a logarithmic spiral centered on the origin. Note that the constant c represents the initial radial distance to the origin.

3 Conclusion

Thus we can conclude that the trajectory is a spiral modeled by the equation $r(\theta) = c\, e^{-\cot{(40^\circ)}\theta}$ where c is the initial distance between the falcon and its prey. The falcon spirals around the target such that it always maintains a 40° angle between its line of sight and its tangential path of flight. Once the falcon reaches a close enough distance to its prey it will abandon this spiral path and dive at its prey.