Math 439 Homework 1

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Problem 1. There are 10 different toys and 4 children: Amy, Bobby, Caroline, and Daniel.

(**a**) How many different ways are there to distribute the toys so that each child gets exactly one toy?

(**b**) How many different ways are there to distribute the toys so that all 10 toys are given away and that the two girls combined get 6 of the toys and the boys get 4 of the toys? (hint: first decide which 6 toys are to be given to the girls.)

Answer 1. Part a. There are $\lfloor 10 \rfloor_4 = 10 \times 9 \times 8 \times 7 = 5040$ different ways to distribute the toys. We know this because we can think about this situation as a string. Each space represents a child being assigned a toy and the number under the space represents the # of choices for a toy that can be given to the child. Note that once a toy has been given to a child, it can no longer be given to a different child and therefore must be subtracted from our pool of total choices.

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10	9	8	7

Part b.

$$\binom{10}{6} * 2^{10}$$

This is because we calculate all the ways to choose 6 spots out of 10 (for the 6 toys that will be given to the girls). Essentially we have chosen all of the ways to distribute 6 toys to the girls and 4 toys to the boys. From there we must multiply by 2^{10} to account for all of the different ways to give out the toys within the boy group and girl group. Note that one girl or one boy is allowed to receive all of the toys in their group and thus we do not subtract from our choice pool, thus 2^{10}.

Problem 2. There are five different mathematicians, six different physicists, and four different chemists. There are five different awards to be handed out to them. Each award is given to one and only one person and a person could receive multiple awards. How many different ways are there to hand out the awards

(**a**) altogether?

(**b**) if the mathematicians receive exactly three awards?

(**c**) if at least one award is given to a chemist?

Answer 2. Below the answers are given.

(**a**) 15^5. We have 5 awards and thus 5 spots to fill. A person can receive multiple awards and thus every time we choose who receives an award they are not removed from the pool.

(**b**) $\binom{5}{3} * 5^3 * 10^2$. We first get all of the ways to choose 3 spots out of 5, since there are 5 awards and the mathematicians must take 3 of them. Then we multiply by the # of ways to give those spots to the 5 mathematicians, 5^3. Finally, we multiply by the number of ways to give out the last two spots 2^10, since the mathematicians cannot receive any more and thus our pool of choices is now 10.

(**c**) $\binom{5}{1} * 4 * 15^4$. We get the number of ways to choose 1 spot out of 5 awards, then multiply by four to account for the number of chemists who could receive the award. Finally, multiply by the number of ways to assign 4 spots with 15 choices.

Problem 3. How many ternary (i.e. (0, 1, 2)-strings) of length 10 are there

(**a**) that contain at least one 1?

(**b**) that contain at most two 1s?

(**c**) that contain exactly five 1s?

(**d**) that contain twice as many 1s as 0's. (The possibility where there are no 1's and 0's is allowed.)

Answer 3. Below the answers are given.

(**a**) $\binom{10}{1} * 3^9$. You find the # of ways there are to choose 1 spot of a 10-length string. Then multiply by all of the different ways to fill the other 9 spots.

(**b**) $(\binom{10}{0} * 2^10) + (\binom{10}{1} * 2^9) + (\binom{10}{2} * 2^8)$. Each term in the summation is the ways to choose the # of 1's in the string (under 2) multiplied by the ways to arrange the leftover spots.

(**c**) $\binom{10}{5} * 2^5$. You get the number of ways to choose 5 spots and multiply the ways to assign 0 or 2 to the leftover spots.

(**d**) $(\binom{10}{6} * \binom{4}{3}) + (\binom{10}{4} * \binom{6}{2}) + (\binom{10}{2} * \binom{8}{1}) + 1$. Since there are only 10 spots to choose from there are only 4 scenarios in which the # of ones can be twice as many as the # of 0's (6 1's and 3 0's, 4 1's and 2 0's, 2 1's and 1 0, and the case where every spot is 2). So at each step, we take the # of ways to choose our 1's and multiply by the ways to choose 0 for the remaining choices. At the end, we add 1 to account for the 1 case where every spot is 2.

Problem 4. How many ways are there to draw 10 cards from a standard 52-card deck that contains

(**a**) all different numbers.

(**b**) two pairs and two three of a kind.

(**c**) either no king or no ace. (So we miss king or ace or both.)

Answer 4. Below the answers are given.

(**a**) $\binom{13}{10} * 4^{10}$. We choose 10 different numbers of 13 and then each number can be assigned 1 of 4 suits.

(**b**) $(\binom{13}{2} * \binom{4}{2} * \binom{4}{2}) * (\binom{11}{2} * \binom{4}{3} * \binom{4}{3})$ We choose our 2 pairs and then choose the suits of the pairs, then choose our 2 triples and choose the suits of those.

(c) $\binom{52}{10}$ - $(\binom{4}{1} * \binom{4}{1} * \binom{50}{8})$ We get the total number of ways to choose 10 cards of 52 and then subtract the ways which have both a king and an ace.

Problem 5. **(a)** How many different ways are there to deal a standard 52-card deck to 13 different players such that each player gets 4 cards? (order doesn't matter).

(b) If 13 different players are each dealt 4 cards from a standard 52-card deck. What is the probability that each player gets one card of each suit? (assuming order doesn't matter)

Answer 5. **(a)** $\binom{52}{4} * \binom{48}{4} * \binom{44}{4} * \binom{40}{4} * \binom{36}{4} * \binom{32}{4} * \binom{28}{4} * \binom{24}{4} * \binom{20}{4} * \binom{16}{4} * \binom{12}{4} * \binom{8}{4}$ At each stage we choose 4 cards of our remaining cards.

(b)

Problem 6. **(a)** There is a group of eleven children. Five of them can only play tennis. Four of them can only play badminton. Two of them Joe and Jen, can play both tennis and badminton. How many ways are there to pick a 2 children to play tennis and to pick 2 children to play badminton. (hint: consider different cases.)

(b) There are 6 men and 10 women. How many ways are there to form a 5-person committee with more women than men?

(c) How many 10-letter strings with no repetition of letters are there that contain 3 vowels and 7 consonants and so that the 3 vowels appear in their natural alphabetical order?

Answer 6. **(a)** Assuming that a player can be chosen for both games. $1+(\binom{2}{1} * \binom{5}{1} * \binom{4}{1})+(\binom{5}{2} * \binom{4}{2})$ The first scenario is for if both Jen and Joe are picked then the scenario in which one of them is picked and we must choose a remaining player for both tennis and badminton. The final scenario is if neither of them is picked and we need 2 players from each of the other groups.

(b) $(\binom{10}{5}) + (\binom{10}{4} * \binom{6}{1}) + (\binom{10}{3} * \binom{6}{2})$. In the first scenario, the committee is all women so we just choose 5 women of 10. In the second scenario the committee is 4 women, so choose 4 of 10, and 1 man, so choose 1 of 6. The final scenario is to choose 3 women of 10 and 2 men of 6. There are no other scenarios because if the number of women is less than 3 then there must be more men on the committee.

(c) $(\binom{5}{3})*(\binom{20}{7} * \binom{10}{3} * \lfloor 7 \rfloor_7)$. We choose 3 vowels of 5, 7 consonants of 20, and then choose 3 spots in our 10-letter string for the vowels such that we can ensure they appear in alphabetical order. Finally, we have all the ways to distribute out 7 consonants in the string.

Problem 7. **(a)** Let n, m, k be positive integers where $n \geq m \geq k$. Give a combinatorial proof of the identity:

$$\binom{n}{m} \cdot \binom{m}{k} = \binom{n}{k} \cdot \binom{n-k}{m-k}$$

(b) Let n, k be positive integers. Prove that

$$\binom{n}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \cdots + \binom{n-1}{k-1}$$

(hint: Consider counting number of binary strings of length n that contain exactly k many 1's in two different ways.) (Comment: a combinatorial proof of a combinatorial identity is a proof you design a counting problem so that the two sides are answers to the same problem, but derived in two different ways.)

Answer 7. (**a**) For this problem imagine that this is a problem in which you are choosing numbers in a ternary string. You want m-k to be the number of 1's in the string, and you wank k 2's. The rest should be 0's. On the LHS you choose m spots, $\binom{n}{m}$, which will be non-zero. Then you choose k spots among the m spots to be 2's. The rest m-k will be 1. For the RHS, another way you could do this is to choose the 2's first, so choose k spots from n to be 2, $\binom{n}{k}$. Then with the remaining spots, $n-k$, choose $m-k$ spots to be 1, $\binom{n-k}{m-k}$. The rest are 0.

(**b**) For this problem assume we are choosing 1's in an n-length binary string. For the LHS, $\binom{n}{k}$, is akin to choosing exactly k-spaces to be 1 and the rest will be 0. For the RHS, imagine choosing the last place that a 1 will appear. We have k 1's to distribute so the first place for a last 1 would be if all 1's are in the first k spots. From there you can distribute k-1 to k-1 spots, $\binom{k-1}{k-1}$. If the last 1 is at the k+1th spot, then you can distribute k-1 to k spots, $\binom{k}{k-1}$. You will keep summing these quantities until the last 1 is at the last spot, the nth spot. Thus for the last quantity you can distribute k-1 into n-1 spots, or $\binom{n-1}{k-1}$.