Applications to Genetics

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July 31, 2024

A farmer has a large population of plants with the varying three genotypes of AA, Aa, aa with initial percentage distributions of a_0, b_0, and c_0. This initial generation is fertilized with genotype AA, creating the first generation, which is fertilized with genotype Aa, creating the second generation, which is fertilized with genotype AA, and this alternating fertilization pattern repeats. We want to derive an expression for the distribution of genotypes for any number of generations.
For $n = 0, 1, 2, \ldots$ let us set:

$$
\begin{aligned}
a_n &= \text{fraction of plants in the n-th generation with AA genotype} \\
b_n &= \text{fraction of plants in the n-th generation with Aa genotype} \\
c_n &= \text{fraction of plants in the n-th generation with aa genotype}
\end{aligned}
$$

and for any generation, n, we say that:

$$
x^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}.
$$

We know a_0, b_0, and c_0 to represent the initial distribution, and we know that $a_n + b_n + c_n = 1$.
For the first generation, $x^{(1)}$, from AA fertilization of the initial distribution, we have:

$$
x^{(1)} = \begin{bmatrix} a_0 + \frac{1}{2}b_0 \\ c_o + \frac{1}{2}b_0 \\ 0 \end{bmatrix}.
$$

For any even generation, $x^{(n)}$, such that $n \geq 2$ and n is even, from Aa fertilization of an odd generation, we have:

$$
x^{(n)} = \begin{bmatrix} \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1} \\ \frac{1}{2}a_{n-1} + \frac{1}{2}b_{n-1} + \frac{1}{2}c_{n-1} \\ \frac{1}{4}b_{n-1} + \frac{1}{2}c_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1} \\ \frac{1}{2} \\ \frac{1}{4}b_{n-1} + \frac{1}{2}c_{n-1} \end{bmatrix}.
$$

For any odd generation, $x^{(n+1)}$, such that $n \geq 2$ and $n+1$ is odd, from AA fertilization of an even generation, we have:

$$x^{(n+1)} = \begin{bmatrix} a_n + \frac{1}{2}b_n \\ c_n + \frac{1}{2}b_n \\ 0 \end{bmatrix}$$

Note that these equations come from table 12.1 in the assigned text. These sets of equations for even and odd generations can be written into matrix formation. For even generations:

$$x^{(n)} = Nx^{(n-1)}.$$

For odd generations:

$$x^{(n+1)} = Mx^{(n)}$$

where

$$x^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} \quad N = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \quad M = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, for even generations,

$$x^{(n)} = Nx^{(n-1)} = NMx^{(n-2)} = NMNx^{(n-3)} = (NM)^2x^{(n-4)} = \cdots = (NM)^{\frac{n}{2}}x^{(0)}.$$

We compute NM:

$$(NM) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}.$$

To compute $(NM)^{\frac{n}{2}}$ we can diagonalize (NM) into PDP^{-1}. By computing eigenvalues, we obtain $\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$, and $\lambda_3 = 0$ with corresponding eigenvectors:

$$e_1 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \quad e_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Then we construct our matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 5 & -1 & 1 \\ 6 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad P^{-1} = \frac{1}{12}\begin{bmatrix} 1 & 1 & 1 \\ -4 & 2 & 8 \\ 3 & -3 & 3 \end{bmatrix}.$$

It follows that we can compute,

$$
\begin{aligned}
(NM)^{\frac{n}{2}} &= PD^{\frac{n}{2}}P^{-1} \\
&= \begin{bmatrix} 5 & -1 & 1 \\ 6 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{12} \begin{bmatrix} 1 & 1 & 1 \\ -4 & 2 & 8 \\ 3 & -3 & 3 \end{bmatrix} \\
&= \frac{1}{12} \begin{bmatrix} \frac{4}{2^n}+5 & 5-\frac{2}{2^n} & 5-\frac{8}{2^n} \\ 6 & 6 & 6 \\ 1-\frac{4}{2^n} & \frac{2}{2^n}+1 & \frac{8}{2^n}+1 \end{bmatrix}.
\end{aligned}
$$

So,

$$
\begin{aligned}
x^{(n)} &= (NM)^{\frac{n}{2}}x^0 \\
&= (PD^{\frac{n}{2}}P^{-1})x^0 \\
&= \frac{1}{12} \begin{bmatrix} \frac{4}{2^n}+5 & 5-\frac{2}{2^n} & 5-\frac{8}{2^n} \\ 6 & 6 & 6 \\ 1-\frac{4}{2^n} & \frac{2}{2^n}+1 & \frac{8}{2^n}+1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} \frac{1}{12}(a_0(\frac{4}{2^n}+5) + b_0(5-\frac{2}{2^n}) + c_0(5-\frac{8}{2^n})) \\ \frac{1}{2}a_0 + \frac{1}{2}b_0 + \frac{1}{2}c_0 \\ \frac{1}{12}(a_0(1-\frac{4}{2^n}) + b_0(\frac{2}{2^n}+1) + c_0(\frac{8}{2^n}+1)) \end{bmatrix}.
$$

For any even generation, when simplified, this gives us the following set of equations to determine the distribution of genotypes:

$$
\begin{aligned}
a_n &= \frac{5}{12} + \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
b_n &= \frac{1}{2} \\
c_n &= \frac{1}{12} - \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right).
\end{aligned}
$$

For any odd generation, we may solve $x^{(n+1)} = Mx^{(n)}$. We now know both m and $x^{(n)}$, so

$$
\begin{aligned}
x^{(n+1)} &= Mx^{(n)} \\
&= \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{0}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{12}(a_0(\frac{4}{2^n}+5)+b_0(5-\frac{2}{2^n})+c_0(5-\frac{8}{2^n})) \\ \frac{1}{2}a_0+\frac{1}{2}b_0+\frac{1}{2}c_0 \\ \frac{1}{12}(a_0(1-\frac{4}{2^n})+b_0(\frac{2}{2^n}+1)+c_0(\frac{8}{2^n}+1)) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{12}(a_0(\frac{4}{2^n}+5)+b_0(5-\frac{2}{2^n})+c_0(5-\frac{8}{2^n}))+\frac{1}{4}a_0+\frac{1}{4}b_0+\frac{1}{4}c_0 \\ \frac{1}{12}(a_0(1-\frac{4}{2^n})+b_0(\frac{2}{2^n}+1)+c_0(\frac{8}{2^n}+1))+\frac{1}{4}a_0+\frac{1}{4}b_0+\frac{1}{4}c_0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

When simplified, this gives us the following set of equations to to determine the distribution of genotypes for any odd generation beyond the first generation:

$$
\begin{aligned}
a_{n+1} &= \frac{2}{3} + \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
b_{n+1} &= \frac{1}{3} - \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
c_{n+1} &= 0.
\end{aligned}
$$

Organizing all of our findings, we have the following sets of equations:
Let $n \geq 2$ and be an even number,
The first generation is determined by:

$$
\begin{aligned}
a_1 &= a_0 + \frac{1}{2}b_0 \\
b_1 &= c_o + \frac{1}{2}b_0 \\
c_1 &= 0
\end{aligned}
$$

Any even generation is determined by:

$$
\begin{aligned}
a_n &= \frac{5}{12} + \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
b_n &= \frac{1}{2} \\
c_n &= \frac{1}{12} - \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right).
\end{aligned}
$$

Any odd generation except for the first is determined by:

$$
\begin{aligned}
a_{n+1} &= \frac{2}{3} + \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
b_{n+1} &= \frac{1}{3} - \frac{1}{2^n}\left(\frac{a_0}{3} - \frac{b_0}{6} - \frac{2c_0}{3}\right) \\
c_{n+1} &= 0.
\end{aligned}
$$