Math 439 Homework 6

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Problem 1. Solve the following recurrence relations using generating functions. (The last one requires the use of an exponential generating function.)

(a) $a_n = 3a_{n-1} + 4a_{n-2}, \quad a_0 = a_1 = 1.$

(b) $a_n = 2a_{n-1} - a_{n-2}, \quad a_0 = 1, a_1 = 5.$

(c) $a_n = na_{n-1} + 3^n, \quad a_0 = 1.$

Answer 1. The answers are offered below.

(a) We have,

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 3a_{n-1}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n.$$

Let $g(x)$ be the generating function for a_n. So,

$$g(x) - x - 1 = 3x \cdot (g(x) - 1) + 4x^2 \cdot g(x) \implies g(x)(1 - 3x - 4x^2) = 1 - 2x \implies g(x) = \frac{1 - 2x}{1 - 3x - 4x^2}.$$

With partial fraction decomposition we get that,

$$g(x) = \frac{3}{5} \cdot \frac{1}{1 - (-x)} + \frac{2}{5} \cdot \frac{1}{1 - 4x}.$$

So,

$$c_n = \frac{3}{5}(-1)^n + \frac{2}{5}4^n.$$

(b) We have,

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 3a_{n-1}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n.$$

Let $g(x)$ be the generating function for a_n. So,

$$g(x) - 5x - 1 = 2x \cdot (g(x) - 1) - x^2 \cdot g(x) \implies g(x) = \frac{3x + 1}{1 - 2x + x^2}$$

With partial fraction decomposition we get that,

$$g(x) = -3 \cdot (\frac{1}{1 - x}) + 4 \cdot (1 - x)^{-2}$$

So,

$$c_n = -3 + 4 \cdot (n + 1) = 4n + 1.$$

(c) We have,

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} na_{n-1}\frac{x^n}{n!} + \sum_{n=1}^{\infty} 3^n \frac{x^n}{n!} = x\sum_{n=1}^{\infty} a_{n-1}\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{(3x)^n}{n!}.$$

Let $g(x)$ be the generating function for a_n. So,

$$g(x) - 1 = xg(x) + e^{3x} \implies g(x) = \frac{e^{3x} + 1}{1 - x} = e^{3x} \cdot (\frac{1}{1 - x}) + \frac{1}{1 - x}$$

So,

$$c_n = \sum_{k=0}^{n} 3^k + n.$$

Problem 2. Find the general solutions to the following recurrence relations using characteristic equations.

(a) $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$.

(b) $2a_n = 7a_{n-1} - 3a_{n-2} + 2^n$.

Answer 2. The answers are offered below.

(a) Find the general solution first. Suppose $a_n = \lambda^n$, then

$$\lambda^n = 2\lambda^{n-1} + 8\lambda^{n-2} \implies \lambda^2 = 2\lambda + 8.$$

So,

$$\lambda^2 - 2\lambda - 8 = 0 \implies (\lambda - 4)(\lambda + 2) = 0.$$

Hence,

$$a_n = C_1 4^n + C_2(-2)^n.$$

For $f(n) = 81n^2$ we try a particular solution of the form $p(n) = B_2n^2 + B_1n + B_0$. So

$$B_2n^2 + B_1n + B_0 = 2[B_2(n-1)^2 + B_1(n-1) + B_0] + 8[B_2(n-2)^2 + B_1(n-2) + B_0] + 81n^2.$$

Solve this for B_0, B_1, B_2 to get that $B_2 = -9$, $B_1 = -36$, and $B_0 = -38$. So,

$$c_n = C_1 4^n + C_2(-2)^n - 9n^2 - 36n - 38$$

(b) Find the general solution first. Suppose $a_n = \lambda^n$, then

$$2\lambda^n = 7\lambda^{n-1} - 3\lambda^{n-2} \implies 2\lambda^2 = 7\lambda - 3.$$

So,

$$2\lambda^2 - 7\lambda + 3 = 0 \implies (2\lambda - 1)(\lambda - 3) = 0.$$

Hence,

$$a_n = C_1(\frac{1}{2})^n + C_2 3^n.$$

For $f(n) = 2^n$ we try a particular solution of the form $p(n) = A \cdot 2^n$. So

$$2(A \cdot 2^n) = 7(A \cdot 2^{n-1}) - 3(A \cdot 2^{n-2}) + 2^n.$$

Solve this for A to get that $A = -\frac{4}{3}$. So,

$$c_n = C_1(\frac{1}{2})^n + C_2 3^n - \frac{4}{3} \cdot 2^n$$

Problem 3. Do the following.

(**a**) Let $n \geq k$ be positive integers. Let a_n denote the number of permutations of $[n]$ such that each cycle has length at least k. Find a recurrence relation for a_n.

(**b**) Suppose $2n$ points $x_1, \ldots, x_{2n}$ are arranged in a circle and we pair them off into n pairs. A pairing is *non-crossing* if when we connect each pair with a line-segment the n line segments do not intersect. Let b_n denote the number of non-crossing pairs of $x_1, \ldots, x_{2n}$. Find a recurrence relation for b_n.

Answer 3. The answers are offered below.

(**a**) We have two cases.

  (a) If k does not divide n then we just take a_{n-1} and we decide where to put our last element ($n-1$ spots). So $(n-1)a_{n-1}$

  (b) If k does divide n then we can choose $k-1$ other elements to form a new cycle with or distribute into the existing cycles. So $\binom{(n-1)}{(k-1)} \cdot (k-1)! \cdot a_{n-k}$.

  So

$$a_n = (n-1)a_{n-1} + \binom{n-1}{k-1} \cdot (k-1)! \cdot a_{n-k}.$$

(**b**) Cases.

  (a) Case 1. Our two new points are in a pair themselves, in which case we just have b_{n-1}.

  (b) Case 2. Our new points choose two other points to pair themselves off with. We choose where our two other points to pair off with ($(2n-2)(2n-3)$ choices). So $(2n-2)(2n-3)b_{n-2}$.

  So,

$$b_n = b_{n-1} + (2n-2)(2n-3)b_{n-2}.$$

Problem 4. There are n students at a local middle school. Each student is to participate in exactly one of the two camps: Math and English. Each student in the Math camp must take an exam and all students will be ranked. Each student in the English camp must choose one of three topics to write an essay on. Using an exponential generating function, determine how many different outcomes there are.

Answer 4. The exponential generating function for this is,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \cdot k! \cdot 3^{n-k} \cdot (n-k)! \frac{x^n}{n!}.$$

By theorem 5.1 we know that the product of the two separate generating functions can give us a simple formula. We are working with a convolution of the sequences $\{n!\}$ and $\{3^n\}$. So,

$$a_n = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

and

$$b_n = \sum_{n=0}^{\infty} 3^n n! \frac{x^n}{n!} = \frac{1}{1-3x}.$$

Hence

$$c_n = \frac{1}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

The partial fraction decomposition gives us

$$c_n = \frac{-\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}.$$

Now we multiply by $n!$ for the nth coefficient and get,

$$(-1) \cdot n! \cdot \frac{1}{2} + n! \cdot \frac{3}{2} \cdot 3^n.$$

Problem 5. There are n senators planning to form two committees: education and urban planning. The education committee must have an even number of members (0 allowed) and the urban planning committee must have an odd number of members and designate a chair. Using an exponential generating function, determine the number of different ways the two committees can be formed.

Answer 5. The exponential generating functions are,

$$a_n = \frac{1}{2}(e^x + e^{-x})$$

and

$$b_n = \frac{1}{2}x(e^x - e^{-x}).$$

So

$$c_n = \frac{1}{4}x(e^{2x} - e^{-2x}).$$

So for the nth coeffeficient we have,

$$n! \cdot [\frac{1}{4} \cdot (\frac{2^{n-1}}{(n-1)!} - \frac{(-2)^{n-1}}{(n-1)!})] = \frac{n2^{n-1}}{4} - \frac{n(-2)^{n-1}}{4}.$$