Math 491 Exam 1 Take-Home Question

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Problem 10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Proof. For this problem, we will examine the bases of each of these topologies and then in the final section the topologies will be compared.

0.1 The Standard Topology on I^2

First consider the standard topology on I, which will be the subspace topology generated by the standard topology on $\mathbb{R}$. By theorem 16.1, basic elements for this topology will look like

$$(x, x') \cap [0, 1] \quad \text{where} \quad x, x' \in \mathbb{R},$$

so,

$$\mathcal{C} = \{(x, x') \cap [0, 1] : x, x' \in \mathbb{R} \text{ and } x < x'\}$$

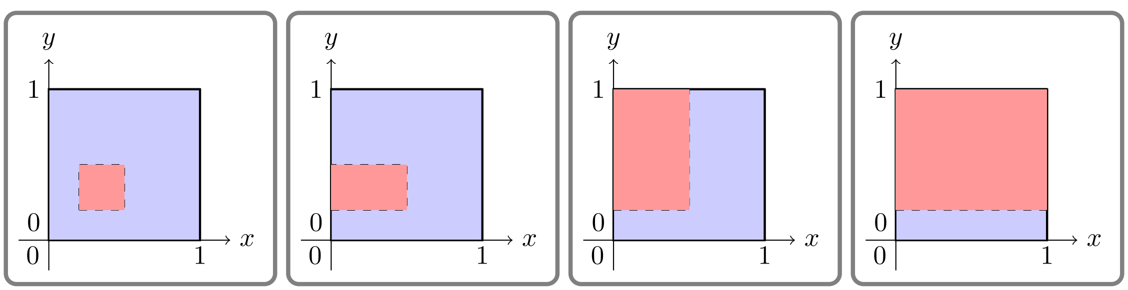
is a basis for I. By theorem 15.1,

$$\mathcal{B}_1 = \{((x_1, x_2) \cap [0, 1]) \times ((y_1, y_2) \cap [0, 1]) : x_1, x_2, y_1, y_2 \in \mathbb{R} \text{ and } x_1 < x_2 \wedge y_1 < y_2\},$$

is a basis for the natural product topology on I^2. Hence, it is clear that basic elements for the product topology on I^2 take the form,

$$((x_1, x_2) \times (y_1, y_2)) \cap [0, 1] \quad \text{where} \quad x_1, x_2, y_1, y_2 \in \mathbb{R} \text{ and } x_1 < x_2 \wedge y_1 < y_2.$$

Refer to the figures below for a visual representation of these basic elements.



Light blue is in I^2. Dark red is in our basic element for I^2.

0.2 The Dictionary-Order Topology on I^2

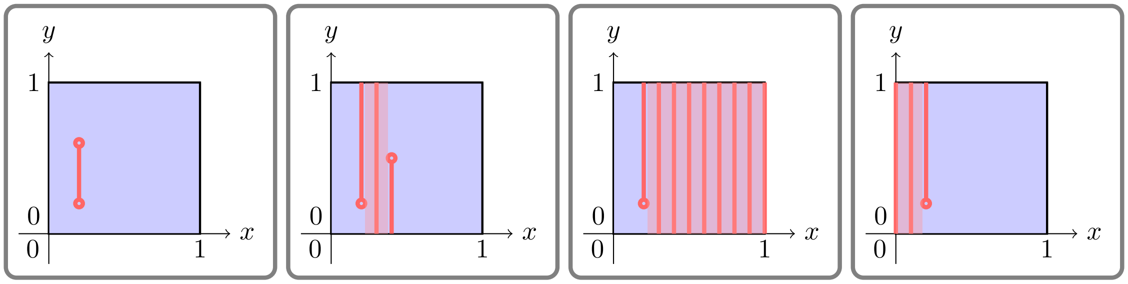
For any order topology we know that basic elements take one of the following forms,

(a) (a, b) where $a, b \in I^2$ (recall that a and b are tuples of elements in I).

(**b**) $[a, b)$ where a is the smallest element of I^2.

(**c**) $(a, b]$ where b is the largest element of I^2.

Note: it should be clear that for the dictionary order on I^2 the smallest element is $(0, 0)$ and the largest element is $(1, 1)$.
In practice, basic elements in I^2 will look like, one of the examples below.



For all figs, dark red is included in the basic element inside of I^2 and light red inside of I^2 denotes that the space between lines is included in our basic element.

0.3 The Dictionary-Order Subspace Topology on I^2

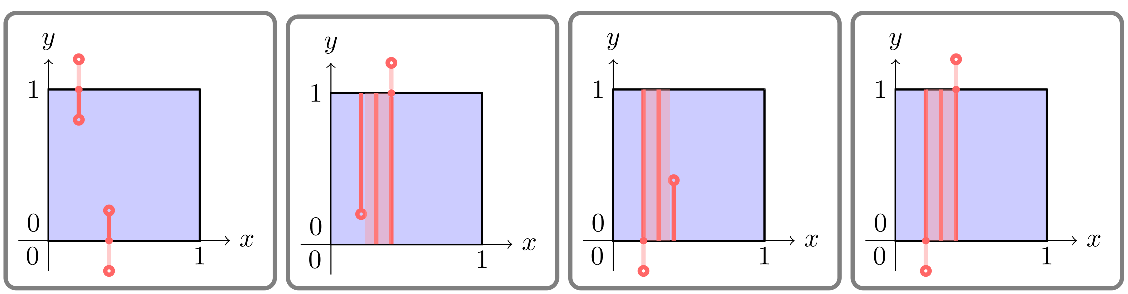
Recall that for $a, b, c, d \in \mathbb{R}$ the set,

$$\{(a \times b, c \times d) : a < c\} \cup \{(a \times b, c \times d) : a = c \text{ and } b < d\}$$

is a basis for the dictionary order topology on $\mathbb{R}^2$. Thus by theorem 16.1, the set

$$(\{(a \times b, c \times d) : a < c\} \cap [0, 1]) \cup (\{(a \times b, c \times d) : a = c \text{ and } b < d\} \cap [0, 1])$$

is a basis for our desired subspace topology on I^2. Notice that if we choose a basic element with endpoints entirely in I^2 we can recreate any of the basic elements from the previous topology. Additionally, we can construct new basic elements if we let at least one of our endpoints be outside of I^2. See below.



For all figs, dark red is included in the basic element inside of I^2, light red outside of I^2 is included in the basic element of $\mathbb{R}^2$, light red inside of I^2 denotes that the space between lines is included in our basic element.

0.4 Comparisons

Claim 1: The subspace dictionary order topology on I^2 is strictly finer than the standard topology on I^2. See that for any basic element of the dictionary order topology on I^2, $b = (a \times b, c \times d)$, we can take a basic element of the dictionary order on $\mathbb{R}^2$, $b' = (a \times b, c \times d)$, and we have that,

$$b' \cap I^2 = b.$$

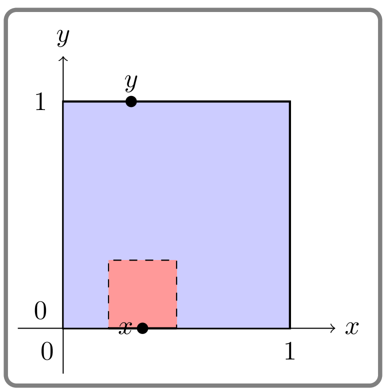
[Note: the pictures are quite helpful in seeing this.]

However, consider the case in which b' has an endpoint below the y-axis. So $b' = (a \times b, c \times d)$ with $b \leq 0$ (and $a \neq 0$), which gives that,

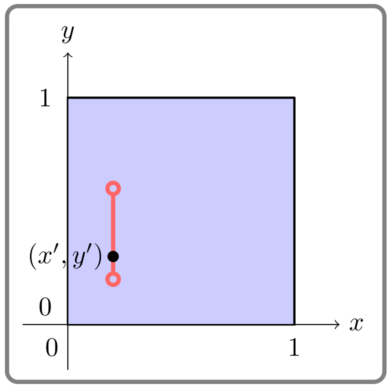
$$b' = (a \times b, c \times d) \cap I^2 = [a \times 0, c \times d).$$

This set is not open in the dictionary order topology on I^2 since there is clearly no way to write it as a union of basic elements in the dictionary order topology on I^2.

Claim 2: The dictionary order topology on I^2 not comparable to the standard topology on I^2. Consider a basic element such as below.



Let x be the origin and consider some union of basic elements in the subspace dictionary order topology on I^2. See that this union must have some basic element in it, b, such that $x \in b$. However, since the point x is neither maximal or minimal, we know that $b = (a \times b, c \times d)$ where $(a \times b) < x < (c \times d)$. So there is a point $y \in (a \times b, x)$ such that $y \in b$ and y is not in our basic element for the standard topology on I^2. For the other direction, consider some basic element for the dictionary order topology on I^2 that looks like,



If this were open in the standard topology on I^2 there ought to be a way to write this basic element as:

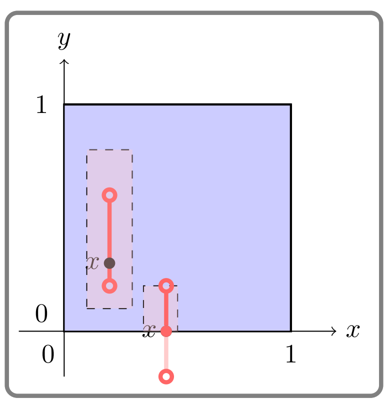
$$\bigcup (x_1, x_2) \times (y_1, y_2) : x_1 < x_2 \text{ and } y_1 < y_2.$$

So take some element (x', y') which is in both. Then notice that there must be some basic element of the standard topology on I^2, $(x_1, y_1) \times (x_2, y_2)$ such that $(x', y') \in (x_1, y_1) \times (x_2, y_2)$. Therefore $x_1 < x' < x_2$ and $y_1 < y' < y_2$. Which implies that there exists (z, l) with $x_1 < z < x'$ and $y_1 < l < y'$ such that,

$$(z, l) \in \bigcup (x_1, x_2) \times (y_1, y_2) : x_1 < x_2 \text{ and } y_1 < y_2.$$

but (z, l) is clearly not in our basic element of the dictionary order topology.

Claim 3: To prove that the subspace dictionary order topology is strictly finer than the standard topology on I^2, take some point $x \in I^2$ and then find some basic element for the standard topology on I^2, $x \in B$. Notice that if x is not on the top or bottom edge of B we can always draw some open vertical line (a basic element in the subspace topology, call it B') such that $x \in B' \subseteq B$.



If x is on the top or bottom edge of B notice that we can always draw a half-closed vertical line with x as the left endpoint which is a basic element of the subspace dictionary order topology on I^2. So the subspace dictionary order topology is finer, to prove that it is strictly finer just see that,

$$\mathcal{T}_{dict} \subset \mathcal{T}_{sub-dict} \wedge \mathcal{T}_{std} = \mathcal{T}_{sub-dict} \implies \mathcal{T}_{dict} \subset \mathcal{T}_{std}$$

but we already showed that the standard topology and dictionary order topology are not comparable.

$\square$