Lead Poisoning

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1 Introduction

This report is investigating a case of lead poisoning for the patient Mr. Merton utilizing applications from first order differential equations to calculate lead concentration in the bones. In a letter sent on February 23$^{\text{rd}}$, it is stated that since August 30$^{\text{th}}$, Mr. Merton has been consuming lead from lead glazed ceramic mugs and a potential secondary source as well, as claimed by Mr. Phaze, who believes that there would not be enough lead transfer from the mugs to cause such a case of lead poisoning.
Let $B(t)$ and $S(t)$ be the blood lead content and bone lead content respectively. Hence $B'(t)$ and $S'(t)$ are the changes in blood and bone lead content. The following is known:

(**a**) 177 days have passed since our subject began to ingest lead (so $0 \leq t \leq 177$).

(**b**) On day 177 our subjects blood lead content was $0.033g$ (i.e. $B(177) = 0.033g$).

(**c**) $B(0) = 0$ and $S(0) = 0$. Mr. Merton was not absorbing lead before August 30th.

(**d**) Our subject has been ingesting $0.00795g$ of lead from the ceramic mugs plus potentially some extra amount of lead, L, from an unknown source, daily. And their body will retain 15% of that lead in the blood and tissues. (Intake of Lead $= 0.15(0.00795 + L)$).

(**e**) On any particular day, of all the lead in the subjects blood, 0.39% will leave his blood and enter his bones and 3.22% will leave his body entirely through natural processes. (Out take of lead $= 0.0322 \cdot B(t) + 0.00039 \cdot B(t) = 0.0361B(t)$).

Lastly, we are given that the change in bone level is the blood level ($S'(t) = 0.0039B(t)$) and the change in blood level is intake $-$ out take. Given all of this, we hope to find the bone lead content, $S(t)$.

2 Math

First notice that,

$$B'(t) = 0.15 \cdot (0.00795 + L) - 0.0361 \cdot B(t) \implies B'(t) + 0.0361 \cdot B(t) = 0.15(0.00795 + L).$$

Using the method of integrating factor, take $\mu(t) = e^{\int P(t)dt}$ where $P(t) = 0.0361$. Thus $\mu(t) = e^{0.0361t}$. Therefore,

$$\begin{aligned}
&\mu'(t) = e^{0.0361} \cdot 0.0361 = P(t) \cdot \mu(t) \\
&\implies \mu(t) \cdot B'(t) + \mu'(t)B(t) = \mu(t)0.15(0.00795 + L) \\
&\implies \frac{d}{dt}(e^{0.0361t} \cdot B(t)) = e^{0.0361} \cdot 0.15(0.00795 + L) && \text{by the product rule} \\
&\implies e^{0.0361t}B(t) = \frac{1}{0.0361} \cdot e^{0.0361t} \cdot 0.15(0.00795 + L) + C && \text{integrate both sides w.r.t. } t \\
&\implies B(t) = \frac{1}{0.0361} \cdot 0.15(0.00795 + L) + Ce^{-0.0361t} && \text{solve for } B(t)
\end{aligned}$$

Now we want to find C. We know that our bounds on t are 0 and 177 hence we get:

$$\begin{aligned}
B(0) = 0 &= \frac{1}{0.0361} \cdot 0.15(0.00795 + L) + Ce^{0} \\
\implies C &= -\frac{1}{0.0361} \cdot 0.15(0.00795 + L)
\end{aligned}$$

Substitute for C to find L,

$$\begin{aligned}
&B(177) = 0.033 = \frac{1}{0.0361} \cdot 0.15(0.00795 + L) + \left(-\frac{1}{0.0361} \cdot 0.15(0.00795 + L)\right) e^{-0.0361(177)} \\
&\implies 0.033 = \frac{0.15 \cdot 0.00795}{0.0361} - \frac{0.15 \cdot 0.00795}{0.0361} e^{-0.0361(177)} + L\left(\frac{0.15}{0.0361} - e^{-0.0361(177)} \frac{0.15}{0.0361}\right) \\
&\implies L = 0.000005355
\end{aligned}$$

This L value is approximate to zero, thus we can conclude it to be negligible. The ceramic mugs are the sole source of lead poisoning. This gives us that

$$C = -\frac{1}{0.0361} \cdot 0.15(0.00795).$$

We can conclude that,

$$\begin{aligned}
B(t) &= \frac{1}{0.0361} \cdot 0.15(0.00795) + \left(-\frac{1}{0.0361} \cdot 0.15(0.00795)\right) \cdot e^{-0.0361t} \\
&= \left(\frac{1}{0.0361} \cdot 0.15(0.00795)\right)(1 - e^{-0.0361}) \\
&= 0.03303(1 - e^{-0.0361}).
\end{aligned}$$

Now that we have found $B(t)$, we can use the relationship between $B(t)$ and $S'(t)$ to find $S(t)$:

$$\begin{aligned}
S'(t) &= 0.0039B(t) \\
&= 0.0039 \cdot 0.03303(1 - e^{-0.0361t}) \\
&= 0.0001288(1 - e^{-0.0361t}).
\end{aligned}$$

By integrating the above equation with respect to t, we can attain $S(t)$:

$$\begin{aligned} S(t) &= \int 0.0001288(1 - e^{-0.0361t})\ dt \\ &= 0.0001288 \int 1 - e^{-0.0361t}\ dt \\ &= 0.0001288 \left(t + \frac{1}{0.0361} \cdot e^{-0.0361t} \right) + C. \end{aligned}$$

Using $S(0) = 0$ to find C:

$$\begin{aligned} S(0) &= 0.0001288 \left(0 + \frac{1}{0.0361} \cdot 1 \right) + C \\ &= \frac{0.0001288}{0.0361} + C \\ &= 0.003568 + C = 0. \end{aligned}$$

Hence, $C = -0.003568$. Therefore,

$$S(t) = 0.0001288 \left(t + \frac{1}{0.0361} \cdot e^{-0.0361t} \right) - 0.003568.$$

We want to find the amount of lead in his bones on day 177, $S(177)$. So,

$$\begin{aligned} S(177) &= 0.0001288 \left(177 + \frac{1}{0.0361} \cdot e^{-0.0361 \cdot 177} \right) - 0.003568 \\ &= 0.0001288 \left(177 + \frac{1}{0.0361} \cdot e^{-6.3897} \right) - 0.003568 \\ &= 0.0001288 \left(177 + \frac{1}{0.0361} \cdot 0.00167 \right) - 0.003568 \\ &= 0.0001288(177 + 0.0465) - 0.003568 \\ &= 0.0001288(177.0465) - 0.003568 \\ &= 0.0228 - 0.003568 \\ &= 0.01924. \end{aligned}$$

Therefore, the level of lead in Mr. Merton's bones was $0.01924g$ on February 23^{rd}.

3 Conclusion

We can see from our analysis that Mr. Phaze's claim that the lead ingested from the usage of the mugs over 6 months was not enough to yield $0.033g$ of lead in his blood and tissues was incorrect. By our calculations, Mr. Merton ingested only an additional $0.000005g$ of lead daily, which is negligible, and approximately 0. Thus it is solely the ceramic mugs that are causing Mr. Merton's lead poisoning, and the creator of the ceramic mugs is the sole culprit. It can also be observed from the above analysis that Mr. Merton currently (on the 177^{th} day) has 0.01924g of lead in his bones. This is an extremely dangerous level of lead and will be difficult to remove.