Problem 20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(**a**) $A = \{x \times y : y = 0\}$

(**b**) $B = \{x \times y : x > 0 \text{ and } y \neq 0\}$

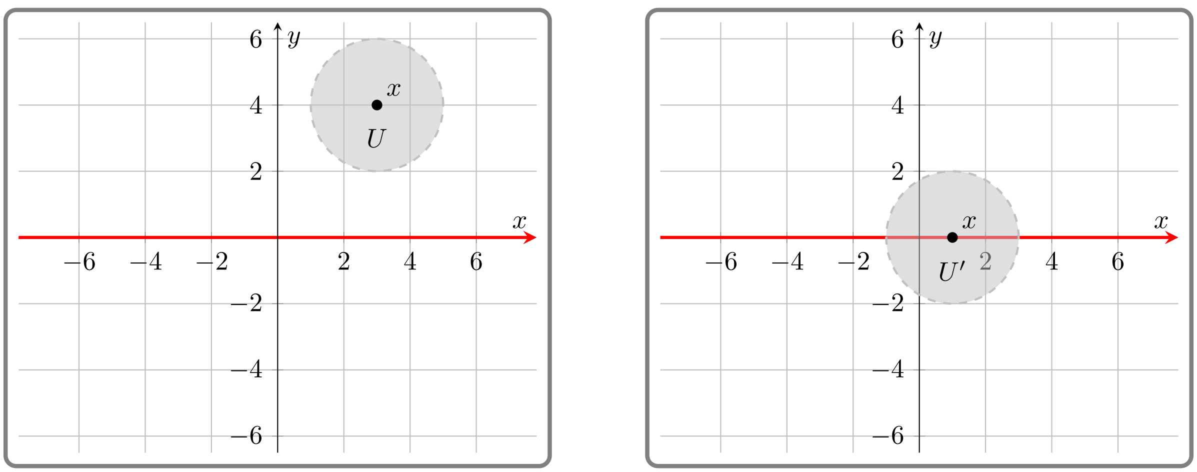
(**c**) $C = A \cup B$

(**d**) $D = \{x \times y : x \in \mathbb{Q}\}$

(**e**) $E = \{x \times y : 0 < x^2 - y^2 \leq 1\}$

(**f**) $A = \{x \times y : x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

Proof. **Part A:** Consider A and see that in R^2, A is the x-axis. Now consider any point $x \in A^c$ and see that we can always find an open neighborhood, $x \in U$, such that $U \subseteq A^c$. Hence x is an interior point and $A^c = \text{Int}(A^c)$. So A^c is open and A must be closed. Now consider some point $y \in A$ and see that any open neighborhood, $y \in U'$, must also contain points in A^c ($A^c \cap U' \neq \emptyset$). So y is **not** and interior point of A, and since y is an arbitrary point we have that $\text{Int}(A) = \emptyset$.



Left Fig: red is A.
Right Fig: red is A.

Question 19 gives us the following equality,

$$\overline{A} = \text{Int}(A) \cup \partial A.$$

Notice that this is a disjoint union and $A = \overline{A}$ since A is closed. So we have that,

$$A = \overline{A} = \text{Int}(A) \cup \partial A = \emptyset \cup \partial A = \partial A.$$

Part B: Notice that B in R^2 is the first and the fourth quadrant **not** including either axis. Now consider some point $x \in B$ and see that you can always find an open neighborhood, $x \in U$, such that $U \subseteq B$. Thus

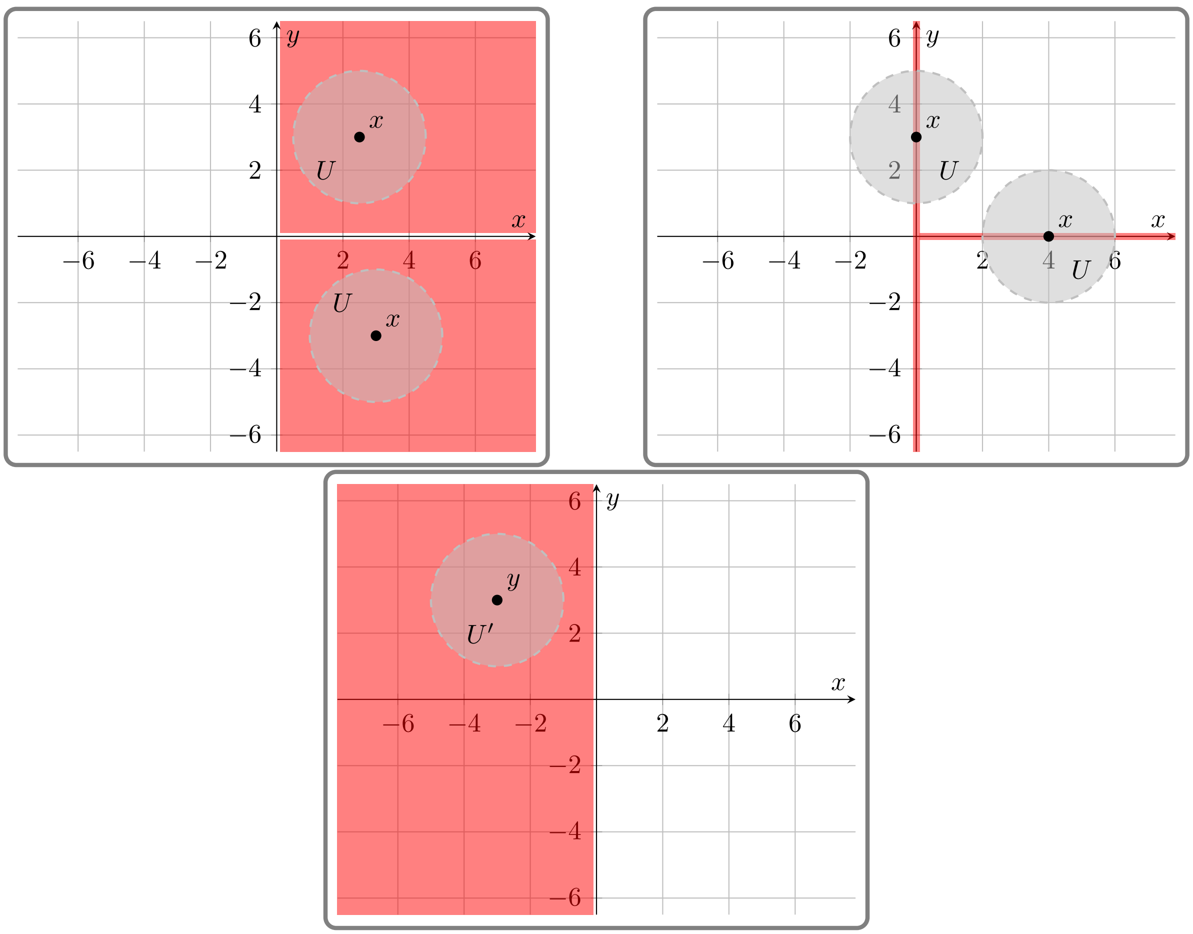
x is an interior point of B. Since x is arbitrary, $\text{Int}(B) = B$. Now consider the following sets,

$$S_1 = \{(x, y) : x > 0 \text{ and } y = 0\},$$

$$S_2 = \{(x, y) : x = 0\}.$$

Notice that for **every** open neighborhood around every point in either S_1 or S_2 will contain points in B, thus points in S_1 and S_2 must be limit points of B. Therefore $B \cup S_1 \cup S_2 \subseteq \overline{B}$. Now see that for any point $y \in (B \cup S_1 \cup S_2)^c$, we can find an open neighborhood, $y \in U'$, such that $U' \subseteq (B \cup S_1 \cup S_2)^c$. So $B \cup S_1 \cup S_2$ is closed, hence $B \cup S_1 \cup S_2 = \overline{B}$. So

$$\begin{aligned} \overline{B} = B \cup S_1 \cup S_2 = \text{Int}(B) \cup \partial B = B \cup \partial B \\ \implies S_1 \cup S_2 = \partial B. \end{aligned}$$



Top-Left Fig: red is B.

Top-Right Fig: red is $S_1 \cup S_2$.

Bottom-Center Fig: red is $(B \cup S_1 \cup S_2)^c$.

Part C: See that,

$$\overline{C} = \overline{A \cup B} = \overline{A} \cup \overline{B} = A \cup \overline{B}.$$

Consider the following sets,

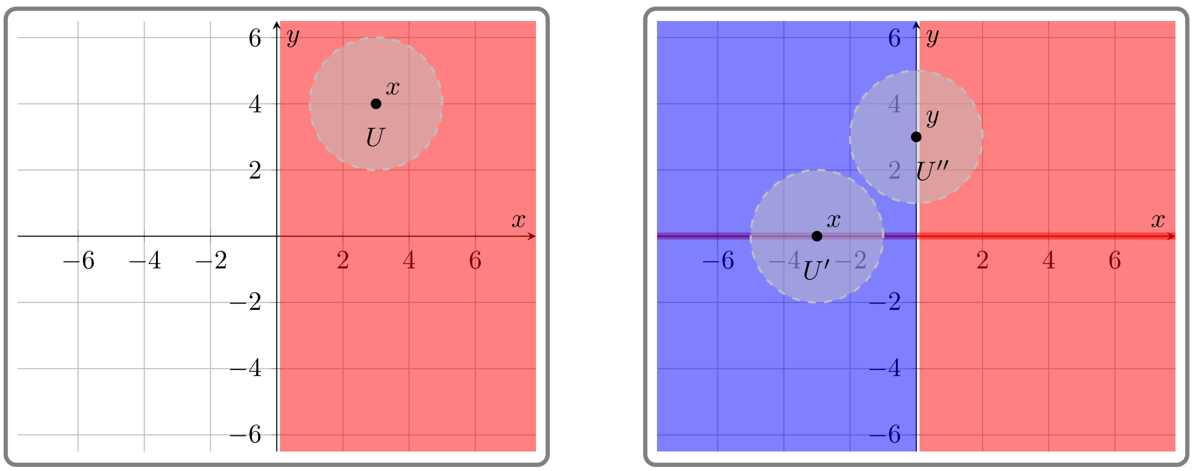
$$S_1 = \{(x, y) : x > 0\} \quad S_2 = \{(x, y) : x \leq 0\}.$$

and notice that

$$S_1 \cup S_2 = \mathbb{R}^2 \quad \text{and} \quad S_1 \cap S_2 = \emptyset.$$

See that for every point $x \in S_1 \subset C$, we can find an open neighborhood, $x \in U$, such that $U \subseteq S_1$. Furthermore, for every point $y \in S_2$, every open neighborhood, $y \in U'$, contains points in C^c. Hence $S_1 = \text{Int}(C)$. So,

$$\begin{aligned} \overline{C} &= A \cup \overline{B} = S_1 \cup \partial C \\ &\implies \partial C = A \cup \overline{B} - S_1 = \{(x, y) : x < 0 \text{ and } y = 0\} \cup \{(x, y) : x = 0\}. \end{aligned}$$



Left Fig: red is S_1.
Right Fig: red is C and blue is S_2.

Part D: Consider any point $(x, y) \in D$ and look at some open neighborhood, $(x, y) \in U$. Notice that the projection of U onto the x-axis is some open interval in $\mathbb{R}$, $(x - \epsilon, x + \epsilon)$ where $\epsilon > 0$. Since $x, x - \epsilon \in \mathbb{R}$ and $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$ we have that there exists some $r, i \in (x - \epsilon, x)$ such that $r \in \mathbb{Q}$ and $i \in \mathbb{R} - \mathbb{Q}$ (with $x \neq x - \epsilon \neq r \neq i$). Hence $(r, y) \in U$ and $(r, y) \in D$ but $(i, y) \in U$ and $(i, y) \notin D$. Notice that in particular, this tells us two things about the arbitrary point (x, y).

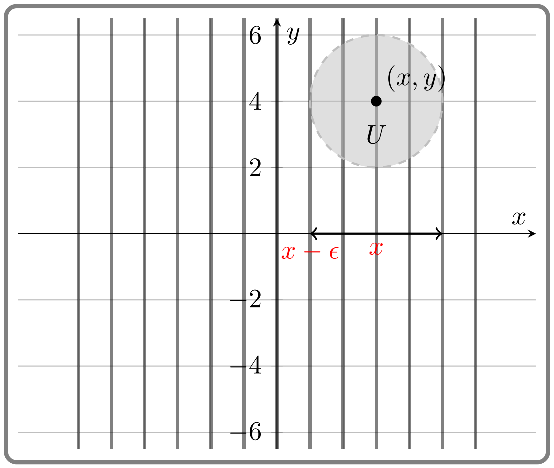
(**a**) (x, y) cannot be an interior point, since every open neighborhood around it contains points in D^c.

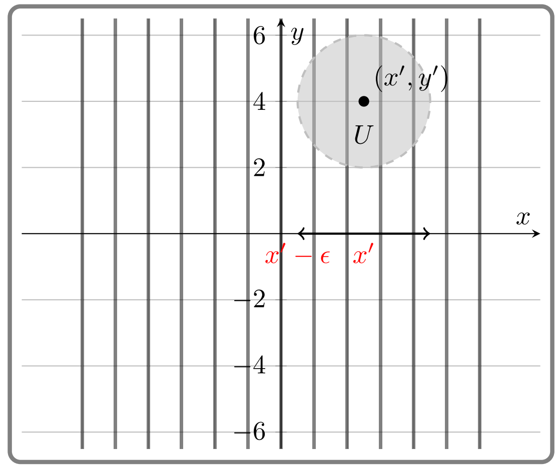
(**b**) (x, y) is a point on the boundary of D, since every open neighborhood around it contains points in D and D^c.

Thus we can say that $\text{Int}(D) = \emptyset$ and $D \subseteq \partial D$. Now consider any point $(x', y') \in D^c$ and see that using the same logic as before we can conclude that (x', y') is a boundary point of D. So every point in D and every point in D^c is a boundary point of D. But $D \cup D^c = \mathbb{R}^2$. Thus $\partial D = \mathbb{R}^2$. It may also be of some interest to note that the following equality,

$$\overline{D} = \text{Int}(D) \cup \partial D$$

tells us that $\overline{D} = \mathbb{R}^2$.





Left Fig: Vertical gray is D.

Right Fig: Vertical gray is D.

Part E: First notice that since,

$$\begin{aligned} (y = x \vee y = -x) \wedge x^2 - y^2 = 1 \\ &\implies x^2 = y^2 \\ &\implies x^2 - y^2 = 0 \wedge x^2 - y^2 = 1. \end{aligned}$$

The curve $x^2 - y^2 = 1$ never intersects $y = x$ or $y = -x$. Consider the sets,

$$S_1 = \{x \times y : 0 < x^2 - y^2 < 1\} \quad S_2 = \{x \times y : x^2 - y^2 = 1\} \quad S_3 = \{x \times y : x = y \vee x = -y\}$$

and see that,

$$S_1 = E - S_2.$$

Now consider that for any point $x \in S^2$ and any open neighborhood, $x \in U$. The neighborhood U must contain points in E^c. However, for every point $y \in S_1$, we can find an open neighborhood $y \in U'$, such that $U' \subseteq E$. Hence, S_1 is precicesly the interior of E (i.e. $\text{Int}(E) = S_1$).

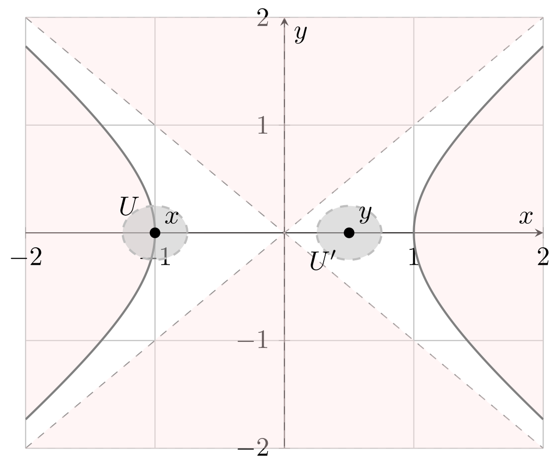
Now notice that every point S_3, is a limit point of E since every open neighborhood around one of these points will contain some points in E but not in S_3. Thus $E \cup S_3 \subseteq \overline{E}$. Now consider that every point in

$(E \cup S_2)^c$ is an interior point of $(E \cup S_2)^c$ and thus $(E \cup S_2)^c$ is closed. Therefore,

$$\overline{E} = E \cup S_3 = S_1 \cup \partial E,$$

which tells us that

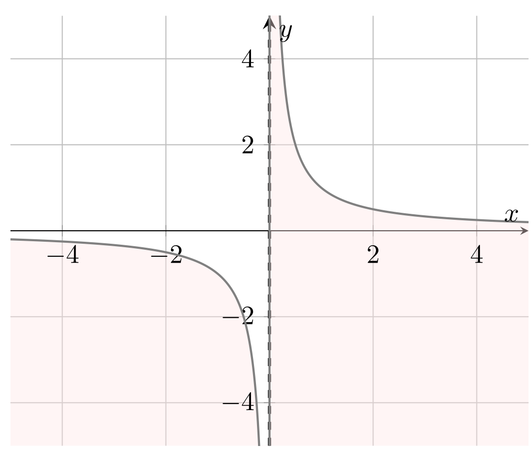
$$\partial E = S_3 \cup S_2.$$



red is everything outside of E.

Part F: Arguments previously made suffice to show that,

$$\partial F = \{x \times y : y = \frac{1}{x}\} \cup \{x \times y : x = 0\} \quad \text{Int}(F) = \{x \times y : x \neq 0 \text{ and } y < \frac{1}{x}\}.$$



red is everything inside of F.

$\square$