Algebraic Topology HW 1

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Problem 1. Prove that Hamming distance is a metric on strings. (For the sake of this assignment, you can assume that the strings in question are finite binary strings: 011, 101110, 0111011, etc.)

Proof. To prove that something is a metric we must satisfy the following four qualities:

(a) The distance from a string to itself must be 0 (i.e. $d(x, x) = 0$).
First, consider the binary string $\alpha = x_1x_2x_3...x_n$, where $x_i \in \{0, 1\}$ and $n \in \mathbb{N}$. To find the hamming distance of α to α, we will use induction. Let d_i be the distance between the binary string $x_1x_2 \dots x_i$ to itself.
Base Step: When $i = 1$ See that $x_1 = x_1$, thus $d_1 = 0$.
Inductive Step: Suppose for some $i \geq 2$ that $x_1x_2 \dots x_i = x_1x_2 \dots x_i$ and thus $d_i = 0$. Now consider the binary string $x_1x_2 \dots x_ix_{i+1}$. The distance of this string to itself is by definition $d_i + d(x_{i+1}, x_{i+1})$. Since $x_{i+1} = x_{i+1}$ we know that $d(x_{i+1}, x_{i+1}) = 0$ and thus $d_i + d(x_{i+1}, x_{i+1}) = 0 + 0 = 0$. By the induction hypothesis $d(\alpha, \alpha) = 0$.

(b) The distance between two distinct strings is positive (i.e. if $x \neq y$, then $d(x, y) > 0$).
Let α and β be two distinct binary strings. Since $\alpha \neq \beta$ then there must exist some positive number of bits in α and β that differ. Hamming distance is a count of the number of bits that differ. A count of positive objects is, by the definition of count, positive. Thus $d(\alpha, \beta) > 0$.

(c) The distance from x to y is the same as the distance from y to x (i.e. $d(x, y) = d(y, x)$).
Let x and y be two binary strings. Let $d(x, y) = d_x$ and $d(y, x) = d_y$ and suppose for contradiction that $d_x \neq d_y$. This implies that there exist bits, $x_n \in x$ and $y_n \in y$, such that one of two things is true:

(a) $x_n = y_n$ but $y_n \neq x_n$.

(b) $y_n = x_n$ but $x_n \neq y_n$.

Both of these are contradictions, thus $d_x = d_y$.

(d) The triangle inequality holds (i.e. $d(x, z) \leq d(x, y) + d(y, z)$).
Let x, y, x be binary strings. To show that the triangle inequality holds, it suffices to show that if a bit x_n disagrees with z_n, then one or both of two things happens:

(a) $x_n \neq y_n$.

(b) $y_n \neq z_n$.

By showing this we prove that whenever the quantity $d(x, z)$ increases, so too must $d(x, y) + d(y, z)$ increase by an equivalent or greater amount. First, suppose that $x_n \neq z_n$ (So $d(x, z)$ is increased by 1 at this bit). There are now two cases:

- Case 1: $x_n \neq y_n$. In this case, we have that the quantity $d(x, y)$ is increased by 1 and thus $d(x, y) + d(y, z)$ is increased by at least 1.

- Case 2: $x_n = y_n$. In this case we have $x_n = y_n \neq z_n$, and thus the quantity $d(y,z)$ is increased by 1. So $d(x,y) + d(y,z)$ is increased by at least 1.

Therefore, $d(x,z) \leq d(x,y) + d(y,z)$.

□

Problem 2. Prove that the standard Euclidean metric on $\mathbb{R}^n$ is actually a metric. [Edit (2/14):You can use the Cauchy-Schwarz inequality:

$$|u \cdot v| \leq ||u||\,||v||$$

for all vectors u, v in $\mathbb{R}^n$. (Here $u \cdot v$ is the dot product of u and v, and $||u||$ is the usual norm of u: the square root of $u \cdot u$.) It helps to reduce the problem to proving: $||u|| + ||v|| \geq ||u + v||$, for all vectors in $\mathbb{R}^n$.]

Proof. Let $u, v \in \mathbb{R}^n$. If $u = v$, then $||u - v|| = ||u - u|| = ||0|| = \sqrt{0} = 0$.
Now consider $||u - v||$ in general. See that:

$$||u - v|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Since $x^2 \geq 0$, we have that each term $(u_i - v_i)^2 \geq 0$ and thus the sum $(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2 \geq 0$.
Finally, we conclude that $\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \geq 0$.
Next consider $||u - v||$ and $||v - u||$:

$$\begin{aligned} ||u - v|| &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \\ &= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + \cdots + (v_n - u_n)^2} && \text{since } x^2 = (-x)^2 \\ &= ||v - u|| \end{aligned}$$

Thus $d(u,v) = d(v,u)$.
Now treat u and v as the difference between some other vectors $x, y, z \in \mathbb{R}^n$, with $u = x - y$ and $v = y - z$

(notice that $u+v=x-z$). Now we claim, $||u||+||v|| \geq ||u+v||$. To prove this, see that:

$$\begin{aligned}
&\left(\sqrt{u_1^2+\cdots+u_n^2}+\sqrt{v_1^2+\cdots+v_n^2}\right)^2 \\
&= (u_1^2+\cdots+u_n^2)+(v_1^2+\cdots+v_n^2)+2\left(\sqrt{(u_1^2+\cdots+u_n^2)(v_1^2+\cdots+v_n^2)}\right) \\
&= (u_1^2+v_1^2)+\cdots+(u_n^2+v_n^2)+2\left(\sqrt{(u_1^2+\cdots+u_n^2)(v_1^2+\cdots+v_n^2)}\right) \\
&= (u_1^2+v_1^2)+\cdots+(u_n^2+v_n^2)+2\left(\sqrt{u_1^2(v_1^2+\cdots+v_n^2)+\cdots+u_n^2(v_1^2+\cdots+v_n^2)}\right) \\
&= (u_1^2+v_1^2)+\cdots+(u_n^2+v_n^2)+2\left(\sqrt{(u_1^2v_1^2+\cdots+u_1^2v_n^2)+\cdots+(u_n^2v_1^2+\cdots+u_n^2v_n^2)}\right) \\
&\geq^* (u_1^2+v_1^2)+\cdots+(u_n^2+v_n^2)+2\left(\sqrt{u_1^2v_1^2+\cdots+u_n^2v_n^2}\right) \\
&\geq^{**} (u_1^2+v_1^2)+\cdots+(u_n^2+v_n^2)+2(u_1v_1+\cdots+u_nv_n) \\
&= (u_1^2+v_1^2+2u_1v_1)+\cdots+(u_n^2+v_n^2+2u_nv_n) \\
&= (u_1+v_1)^2+\cdots+(u_n+v_n)^2.
\end{aligned}$$

* We have removed terms of the form $x^2y^2 \geq 0$.

** $\sqrt{x^2} \geq x$ and thus $\sqrt{x^2y^2} = \sqrt{(xy)^2} \geq xy$.

We conclude that:

$$\sqrt{u_1^2+\cdots+u_n^2}+\sqrt{v_1^2+\cdots+v_n^2} \geq \sqrt{(u_1+v_1)^2+\cdots+(u_n+v_n)^2}.$$

□

Problem 3. Prove that the "prefix" metric d on infinite binary strings is, indeed, a metric. (Here $d(w_1, w_2) = e^{-n}$, where n is the length of the longest prefix that is common to both w_1 and w_2.) Argue that d satisfies the following strong form of the triangle inequality: for all w_1, w_2, w_3,

$$d(w_1, w_3) \leq \max\{d(w_1, w_2), d(w_2, w_3)\}.$$

[A metric that satisfies the above inequality is called an ultrametric.] [Edit(2/14): This has been changed so that the metric is on infinite binary strings.]

Proof. Let x, y be infinite binary strings. If $x = y$ then their longest shared prefix is the length of themselves, which in this case is infinite. Thus, the distance should be $\lim_{n\to\infty} e^{-n} = 0$.
Now to prove that the distance between x and y is always positive note a few things:

(a) The length of the longest common prefix of x and y cannot be less than 0. Given any two infinite binary strings they are either:

- Different at every bit, in which case the size of the longest common prefix is 0.
- Not different at every bit, in which case there is at least one shared bit and the length of the longest common prefix is at least 1.

(**b**) That $e^{-n} \geq 0$ since $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n \geq 0$

Thus the distance between x and y must be positive.
Now see that if the distance between x and y is their longest shared prefix, then the distance between y and x is the same longest shared prefix. Thus the length of that shared prefix, let's say n, is equivalent and so too will $e^{-n} = e^{-n}$.

Let z be an infinite binary string and n_{xz}, n_{xy}, n_{yz} be the length of the common shared prefixes for their respective binary strings. We have three cases:

(**a**) $n_{xz} > n_{xy}$. In this case $e^{n_{xz}} < e^{n_{xy}}$ and we are finished.

(**b**) $n_{xz} < n_{xy}$. Because we are working with prefixes we know that the sequence shared by x and z is a subsequence of the one shared by x and y. Thus y contains a prefix such that $n_{xz} \geq n_{yz}$. Therefore $e^{n_{xz}} \leq e^{n_{yz}}$.

(**c**) $n_{xz} = n_{xy}$. In this case $e^{n_{xz}} = e^{n_{xy}}$ and we are finished.

Thus $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. □

Problem 4. Suppose that d is an ultrametric on X. Let $B_{e1}(x_1)$ and $B_{e2}(x_2)$ be metric balls in X. (Here $B_e(X) = \{y \in X : d(x,y) < e\}$)
Prove that, if $B_{e1}(x_1)$ and $B_{e2}(x_2)$ have a point in common, then one of the metric balls is contained in the other.

Proof. Let $x \in B_{e1}(x_1), B_{e2}(x_2)$, thus x is within e_1 distance of x_1 and e_2 distance of x_2. Since d is an ultrametric we know that $d(x_1,x_2) \leq \max\{d(x_1,x), d(x,x_2)\} \leq \max\{e_1,e_2\}$. Without loss of generality choose $e_1 \geq e_2$, so $d(x_1,x_2) \leq e_1$. Let $y \in B_{e2}(x_2)$, so y is within e_2 distance from x_2. Now see that:

$$d(y,x_1) \leq \max\{d(y,x_2), d(x_2,x_1)\} \leq \max\{e_2,e_1\} \leq e_1.$$

So $y \in B_{e1}(x_1)$ and thus $B_{e2}(x_2) \subseteq B_{e1}(x_1)$. □

Problem 5. Use the python program "ripshomology2.py" to generate a simplicial complex. (I get good results with 7 vertices, a maximum value of 2 for the metric, and a maximum diameter of 1. (Thus, you enter the numbers 7, 2, and 1, in that order, when prompted.)) (REVISED: I think that 6, 2, 1 (in that order) may be better than 7,2,1 (as originally indicated). You should use 6,2,1 instead.) Try to get a complex that has at least one 2-simplex, and non-trivial homology groups in more than one dimension. (This means that the vector of "Betti numbers" should have two non-zero entries.) It may take a few tries to get a suitable complex.

(**a**) Sketch the simplicial complex;

(**b**) Find matrices for all of the boundary operators;

(**c**) Directly compute the homology groups. (These will be with real coefficients.) You should do this with aid from a computer, such as an online matrix calculator. Find explicit bases for the homology groups, and compare your answers with the list of Betti numbers produced by "ripshomology2.py".

Proof. (**a**)
$$\begin{pmatrix} 0 & 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 2 & 0 & 2 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

Simplex: $['f',' e',' d',' df',' c',' cf',' b',' bf',' be',' bd',' bdf',' bc',' bcf',' a',' ad',' ac'] = \Delta$

Betti numbers: $[1, 1, 0]$

(**b**) $\partial_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \partial_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ $\quad \partial_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(**c**) $Z_2 = \ker \partial_2 = \text{null} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \{0\}$

$$Z_1 = \ker \partial_1 = \text{null} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$Z_0 = \ker \partial_0 = \text{null} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = C_1$$

$$B_2 = \operatorname{Im}\partial_3 = C_2 = \{0\} \quad B_1 = \operatorname{Im}\partial_2 = \operatorname{col}\begin{pmatrix}0 & 0\\1 & 1\\1 & 0\\0 & 1\\0 & 0\\1 & 1\end{pmatrix} = \left\{\begin{bmatrix}0\\1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}\right\}$$

$$B_0 = \operatorname{Im}\partial_1 = \operatorname{col}\begin{pmatrix}1&1&0&0&0&0&0&0\\0&0&1&1&1&1&0&0\\1&0&1&0&0&0&1&0\\0&1&0&1&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&1&1\end{pmatrix} = \left\{\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}\right\}$$

$H_2 = Z_2/B_2 = \{0\}/\{0\} = \{0\}$

$$H_1 = Z_1/B_1 = \left\{\begin{bmatrix}1\\-1\\-1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\\0\\0\\-1\\1\end{bmatrix}\right\}/\left\{\begin{bmatrix}0\\1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}\right\}$$

$= \{'ac' -' bc' +' bd' -' ad',' ac' -' ad' -' cf' +' df'\}/\{'bcf',' bdf'\} = \{'a',' b'\}$

$$H_0 = Z_0/B_0 = C_1/\left\{\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}\right\} \cong \mathbb{Z}$$ Everything is connected, so the homology group has one fully connected component and 0. The betti numbers make sense, dim 1 has 1 connected component, dim 2 has a single 1-dimensional hole, and

□