Algebraic Topology HW 5

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Problem 1. Let $c : [0,1] \to X$ be a path and $r : [0,1] \to [0,1]$ be such that $r(0) = 0$ and $r(1) = 1$. Then c_1 is path homotopic to $c_1 \circ r$.

Proof. To prove that these two are path homotopic it suffices to give the homotopy maps. Define $H : [0,1] \times [0,1] \to X$ by,

$$H(s,t) = c(r(t))(1-s) + c(t)s$$

To confirm that this path homotopy is correct we ought to evaluate it at the end points for each parameter. See that

$$\begin{aligned}
H(0,t) &= c(r(t)) \cdot (1) + c(t) \cdot 0 = c(r(t)) \\
H(1,t) &= c(r(t)) \cdot 0 + c(t) \cdot 1 = c(t) \\
H(s,0) &= c(0) \cdot (1-s) + c(0) \cdot s = c(0) - c(0)s + c(0)s = c(0) \\
H(s,1) &= c(1) \cdot (1-s) + c(1)s = c(1) - c(1)s + c(1)s = c(1).
\end{aligned}$$

Which is exactly what we would expect if H was a valid homotopy. □

Problem 2. Let $c : [0,1] \to X$ be a path with $c(0) = x_0$. Let $r : [0,1] \to [0,1]$ be such that $r(0) = 0$ and $r(1) = 0$. Then $c \circ r$ is homotopic to c_{x_0} (the constant path at x_0).

Proof. Define $H : [0,1] \times [0,1] \to X$ by,

$$H(s,t) = c(r(t))(1-s) + c_{x_0}(t)s.$$

Check the endpoints,

$$\begin{aligned}
H(0,t) &= c(r(t)) \cdot (1) + c_{x_0}(t) \cdot 0 = c(r(t)) \\
H(1,t) &= c(r(t)) \cdot 0 + c_{x_0}(t) \cdot 1 = c_{x_0}(t) \\
H(s,0) &= c(0) \cdot (1-s) + c_{x_0}(0) \cdot s = c(0) - c(0)s + c_{x_0}(0)s = x_0 - x_0 s + x_0 s = x_0 \\
H(s,1) &= c(0) \cdot (1-s) + c_{x_0}(1)s = c(0) - c(0)s + c_{x_0}(1)s = x_0 - x_0 s + x_0 s = x_0.
\end{aligned}$$

This is a valid homotopy. □

Problem 3. Prove that $c * \bar{c} \cong c_{x_0} \cong \bar{c} * c$ if c is a loop at x_0.

Proof. To prove this we will prove that if c is a loop at x_0, $c * \bar{c} \cong c^* \circ r \cong \bar{c} * c$ (where $c^* \circ r$ is as seen in problem (2)). Define $H_1 : [0,1] \times [0,1] \to X$ by,

$$H_1(s,t) = c(\bar{c}(t))(1-s) + c_{x_0}(t)s.$$

Check the endpoints,

$$\begin{aligned}
H_1(0,t) &= c(\bar{c}(t)) \cdot (1) + c_{x_0}(t) \cdot 0 = c(\bar{c}(t)) \\
H_1(1,t) &= c(\bar{c}(t)) \cdot 0 + c_{x_0}(t) \cdot 1 = c_{x_0}(t) \\
H_1(s,0) &= c(\bar{c}(0)) \cdot (1-s) + c_{x_0}(0) \cdot s = c(0) - c(0)s + c_{x_0}(0)s = x_0 - x_0 s + x_0 s = x_0 \\
H_1(s,1) &= c(\bar{c}(1)) \cdot (1-s) + c_{x_0}(1)s = c(1) - c(1)s + c_{x_0}(1)s = x_0 - x_0 s + x_0 s = x_0.
\end{aligned}$$

This is a valid homotopy. For the other side define $H_2 : [0,1] \times [0,1] \to X$ by,

$$H_2(s,t) = \bar{c}(c(t))(1-s) + c_{x_0}(t)s.$$

Check the endpoints,

$$\begin{aligned}
H(0,t) &= \bar{c}(c(t)) \cdot (1) + c_{x_0}(t) \cdot 0 = \bar{c}(c(t)) \\
H(1,t) &= \bar{c}(c(t)) \cdot 0 + c_{x_0}(t) \cdot 1 = c_{x_0}(t) \\
H(s,0) &= \bar{c}(c(0)) \cdot (1-s) + c_{x_0}(0) \cdot s = \bar{c}(0) - \bar{c}(0)s + c_{x_0}(0)s = x_0 - x_0 s + x_0 s = x_0 \\
H(s,1) &= \bar{c}(c(1)) \cdot (1-s) + c_{x_0}(1)s = \bar{c}(1) - \bar{c}(1)s + c_{x_0}(1)s = x_0 - x_0 s + x_0 s = x_0.
\end{aligned}$$

This is a valid homotopy. □

Problem 4. Prove:

$$(c_1 * c_2) * c_3 \cong c_1 * c_2 * c_3 \cong c_1 * (c_2 * c_3).$$

Where,

$$(c' * c'')(t) = \begin{cases} c'(2t) & 0 \leq t \leq \frac{1}{2} \\ c''(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$(c_1 * c_2 * c_3)(t) = \begin{cases} c_1(3t) & 0 \leq t \leq \frac{1}{3} \\ c_2(3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ c_3(3t-2) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

[Note that c_1, c_2, c_3 are arbitrary paths with $c_1(1) = c_2(0)$ and $c_2(1) = c_3(0)$].

Proof. Let r be the map as described in problem 1. Now consider $(c_1 * c_2 * c_3) \circ r$ which, by problem 1, we know is homotopic to $c_1 * c_2 * c_3$. It now suffices to show that $(c_1 * c_2) * c_3$ and $c_1 * (c_2 * c_3)$ are also homotopic to $(c_1 * c_2 * c_3) \circ r$.

Consider the following homotopy. Define $H_1 : [0,1] \times [0,1] \to X$ by,

$$H_1(s,t) = (c_1 * c_2 * c_3)(r(t))(1-s) + ((c_1 * c_2) * c_3)(t)s$$

To confirm that this path homotopy is correct we ought to evaluate it at the end points for each parameter. See that

$$\begin{aligned}
H_1(0,t) &= (c_1 * c_2 * c_3)(r(t)) \cdot (1) + ((c_1 * c_2) * c_3)(t) \cdot 0 = (c_1 * c_2 * c_3)(r(t)) \\
H_1(1,t) &= (c_1 * c_2 * c_3)(r(t)) \cdot 0 + ((c_1 * c_2) * c_3)(t) \cdot 1 = ((c_1 * c_2) * c_3)(t) \\
H_1(s,0) &= (c_1 * c_2 * c_3)(0) \cdot (1-s) + ((c_1 * c_2) * c_3)(0) \cdot s = c_1(0) - c_1(0)s + c_1(0)s = c_1(0) \\
H_1(s,1) &= (c_1 * c_2 * c_3)(1) \cdot (1-s) + ((c_1 * c_2) * c_3)(1)s = c_3(1) - c_3(1)s + c_3(1)s = c_3(1).
\end{aligned}$$

Which is exactly what we would expect if H_1 was a valid homotopy. Now define $H_2 : [0,1] \times [0,1] \to X$ by,

$$H_2(s,t) = (c_1 * c_2 * c_3)(r(t))(1-s) + (c_1 * (c_2 * c_3))(t)s$$

To confirm that this path homotopy is correct we ought to evaluate it at the end points for each parameter. See that

$$\begin{aligned}
H_2(0,t) &= (c_1 * c_2 * c_3)(r(t)) \cdot (1) + (c_1 * (c_2 * c_3))(t) \cdot 0 = (c_1 * c_2 * c_3)(r(t)) \\
H_2(1,t) &= (c_1 * c_2 * c_3)(r(t)) \cdot 0 + (c_1 * (c_2 * c_3))(t) \cdot 1 = (c_1 * (c_2 * c_3))(t) \\
H_2(s,0) &= (c_1 * c_2 * c_3)(0) \cdot (1-s) + (c_1 * (c_2 * c_3))(0) \cdot s = c_1(0) - c_1(0)s + c_1(0)s = c_1(0) \\
H_2(s,1) &= (c_1 * c_2 * c_3)(1) \cdot (1-s) + (c_1 * (c_2 * c_3))(1)s = c_3(1) - c_3(1)s + c_3(1)s = c_3(1).
\end{aligned}$$

Which is exactly what we would expect if H_2 was a valid homotopy. Hence, they are all homotopic. $\square$

Problem 5. Prove that $c_{x_0} * c \cong c \cong c * c_{x_0}$. [$c$ is a loop at x_0].

Proof. Let r be the map as described in problem 1. Now consider $c \circ r$ which, by problem 1, we know is homotopic to c. It now suffices to show that $c_{x_0} * c$ and $c * c_{x_0}$ are also homotopic to $c \circ r$.

Consider the following homotopy. Define $H_1 : [0,1] \times [0,1] \to X$ by,

$$H_1(s,t) = c(r(t))(1-s) + (c_{x_0} * c)(t)s$$

To confirm that this path homotopy is correct we ought to evaluate it at the end points for each parameter. See that

$$\begin{aligned}
H_1(0,t) &= c(r(t)) \cdot (1) + (c_{x_0} * c)(t) \cdot 0 = c(r(t)) \\
H_1(1,t) &= c(r(t)) \cdot 0 + (c_{x_0} * c)(t) \cdot 1 = (c_{x_0} * c)(t) \\
H_1(s,0) &= c(0) \cdot (1-s) + (c_{x_0} * c)(0) \cdot s = c(0) - c(0)s + c_{x_0}(0)s = x_0 - x_0 s + x_0 s = x_0 \\
H_1(s,1) &= c(1) \cdot (1-s) + (c_{x_0} * c)(1)s = c(1) - c(1)s + c(1)s = c(1) = x_0.
\end{aligned}$$

Which is exactly what we would expect if H_1 was a valid homotopy. Now define $H_2 : [0,1] \times [0,1] \to X$ by,

$$H_2(s,t) = c(r(t))(1-s) + (c * c_{x_0})(t)s$$

To confirm that this path homotopy is correct we ought to evaluate it at the end points for each parameter. See that

$$\begin{aligned}
H_2(0,t) &= c(r(t)) \cdot (1) + (c * c_{x_0})(t) \cdot 0 = c(r(t)) \\
H_2(1,t) &= c(r(t)) \cdot 0 + (c * c_{x_0})(t) \cdot 1 = (c * c_{x_0})(t) \\
H_2(s,0) &= c(0) \cdot (1-s) + (c * c_{x_0})(0) \cdot s = c(0) - c(0)s + c(0)s = c(0) = x_0 \\
H_2(s,1) &= c(1) \cdot (1-s) + (c * c_{x_0})(1)s = c(1) - c(1)s + c_{x_0}(1)s = x_0 - x_0 s + x_0 s = x_0.
\end{aligned}$$

Which is exactly what we would expect if H_2 was a valid homotopy. Hence, they are all homotopic. $\square$