Algebraic Topology HW 2

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Problem 1. Prove the Barratt-Whitehead lemma:

$$
\begin{array}{ccccccccccccc}
\dots & \to & A_n & \to^{f_n} & B_n & \to^{g_n} & C_n & \to^{h_n} & A_{n-1} & \to & \dots \\
\downarrow^{\cong} & & \downarrow^{\alpha_n} & & \downarrow^{\beta_n} & & \downarrow^{\gamma_n \cong} & & \downarrow^{\alpha_{n-1}} & & \\
\dots & \to & A'_n & \to^{f'_n} & B'_n & \to^{g'_n} & C'_n & \to & A'_{n-1} & \to & \dots \\
\hline
\end{array}
$$

If the above ladder of abelian groups and homomoirphisms has exact rows and commutative squares (and each γ_n is an isomorphism for all n), then the sequence:

$$
\begin{array}{ccccccccccc}
\dots & \to & A_n & \to & A'_n \oplus B_n & \to & B'_n & \to & A_{n-1} & \to & \dots \\
& & a & \mapsto & (\alpha_n(a), -f_n(a)) & & b & \mapsto & h_n\gamma_n^{-1}g'_n(b) & & \\
& & & & (a,b) & \mapsto & f'_n(a) + \beta_n(b) & & & &
\end{array}
$$

is exact.

Proof. Let our maps be as follows:

$$\delta : A_n \to A'_n \oplus B_n$$

$$\kappa : A'_n \oplus B_n \to B'_n$$

$$\epsilon : B'_n \to A_{n-1}$$

Now let $(\alpha_n(a), -f_n(a)) \in \operatorname{Im}\delta$. We need to show that $(\alpha_n(a), -f_n(a)) \in \ker\kappa$ (i.e. $f'_n(\alpha_n(a)) + \beta_n(-f_n(a)) = 0$). Since squares commute we have that $f'_n(\alpha_n(a)) = \beta_n(f_n(a))$. Thus:

$$f'_n\alpha_n(a) - \beta_n f_n(a) = \beta_n f_n(a) - \beta_n f_n(a) = \beta_n f_n(a-a) = \beta_n f_n(0) = 0$$

So $\operatorname{Im}\delta \subseteq \ker\kappa$. Now let $(f'_n(a) + \beta_n(b)) \in \operatorname{Im}\kappa$. We need to show that $h_n\gamma_n^{-1}g'_n(f'_n(a) + \beta_n(b)) = 0$. See that:

$$
\begin{aligned}
h_n\gamma_n^{-1}g'_n(f'_n(a) + \beta_n(b)) &= h_n\gamma_n^{-1}g'_n(f'_n(a)) + h_n\gamma_n^{-1}g'_n(\beta_n(b)) && \text{Homomorphisms} \\
&= h_n\gamma_n^{-1}(0) + h_n\gamma_n^{-1}g'_n(\beta_n(b)) && f'_n(a) \in \ker g'_n \text{ by exactness} \\
&= 0 + h_n\gamma_n^{-1}g'_n(\beta_n(b)) && \\
&= 0 + h_n g_n(b) && \gamma^{-1}g'_n\beta_n(b) = g_n(b) \text{ by commutativity of squares} \\
&= 0 + 0 = 0 &&
\end{aligned}
$$

So $\operatorname{Im}\kappa \subseteq \ker\epsilon$. Let $h_n\gamma_n^{-1}g'_n(b) \in \operatorname{Im}\epsilon$, we must show that $(\alpha_{n-1}(h_n\gamma_n^{-1}g'_n(b)), -f_{n-1}(h_n\gamma_n^{-1}g'_n(b))) = (0,0)$. See that:

$$
\begin{aligned}
f_{n-1}(h_n\gamma_n^{-1}g'_n(b)) &= 0 && \text{Since } h_n\gamma_n^{-1}g'_n(b) \in \operatorname{Im} h_n \text{ so } h_n\gamma_n^{-1}g'_n(b) \in \ker f_{n-1}. \\
\alpha_{n-1}(h_n\gamma_n^{-1}g'_n(b)) &= h'_n g'_n(b) && \text{By commutativity of squares.} \\
&= 0 && \text{Since } g'_n(b) \in \operatorname{Im} g'_n \text{ and thus } g'_n(b) \in \ker h'_n.
\end{aligned}
$$

So $\operatorname{Im}\epsilon \subseteq \ker \delta_{n-1}$. Now it suffices to show reverse Inclusions

Exactness at κ_n: Let $(a, -b) \in \ker \kappa$, we now claim that $(a, -b) \in \operatorname{Im}\delta$. See that:

$$f'_n(a) - \beta_n(b) = 0 \implies f'_n(a) = \beta_n(b).$$

So we have:

$$0 = g'_n(f'_n(a)) = g'_n(\beta_n(b)) = \gamma_n(g_n(b))$$

Since γ is an isomorphism, we have that:

$$g_n(b) = 0$$

So $b \in \operatorname{Im} f_n$. Next since:

$$f'_n(a) = \beta_n(b) = \beta_n(f_n(a'))$$

by commutativity of squares there exists some $a'' \in A_n$ such that:

$$f'_n(a) = f'_n(\alpha_n(a'')) \implies a = \alpha_n(a'').$$

So $a \in \operatorname{Im}\alpha_n$.

Exactness at ϵ_n: Let $b' \in \ker \epsilon$. So $h_n\gamma_n^{-1}g'_n(b') = 0$. We claim that $b' = f'_n(a) + \beta_n(b)$ where $b \in B_n$ and $a \in A'_n$. Since $\gamma_n^{-1}g'_n(b') \in \ker h_n$ we know that $\gamma_n^{-1}g'_n(b') \in \operatorname{Im} g_n$ so there exists some $b \in B_n$ such that $g_n(b) = \gamma_n^{-1}g'_n(b')$. Since γ is an isomorphism we know also that $\gamma_n g_n(b) = g'_n(b')$, but by commutativity of squares, this means that $g'_n(b') = g'_n\beta_n(b)$. See that $\beta_n(b) - b' \in \ker g'_n$ since:

$$g'_n(\beta_n(b) - b') = g'_n(\beta_n(b)) - g'_n(b') = g_n\gamma_n(b) - g'_n(b') = \gamma_n(\gamma_n^{-1}(g'_n(b')) - g'_n(b') = 0.$$

So $\beta_n(b) - b' \in \operatorname{Im} f'_n$ and thus there exists $a \in A'_n$ such that $f'_n(a) = \beta_n(b) - b'$. Hence:

$$\kappa(-a, b) = -f'_n(a) + \beta_n(b) = -(\beta_n(b) - b') + \beta_n(b) = b'.$$

So $b' = f'_n(a) + \beta_n(b)$ where $b \in B_n$ and $a \in A'_n$.

Exactness at δ_{n-1}: Let $a \in \ker \delta_{n-1}$, we claim that $a \in \operatorname{Im}\epsilon$ (i.e. there exists $b \in B'_n$ such that $h_n\gamma_n^{-1}g'_n(b) = a$). Notice that $a \in \ker f_{n-1}$ and thus $a \in \operatorname{Im} h_n$ hence $a = h_n(c)$ where $c \in C_n$. Since γ is an isomorphism there also exists $c' \in C'_n$ such that $h_n(\gamma_n^{-1}(c') = a)$. By commutativity of squares $0 = \alpha_{n-1}(a) = \alpha_{n-1}(h_n(\gamma_n^{-1}(c'))) = h'_n(c')$. So $c' \in \ker h'_n$ and therefore $c' \in \operatorname{Im} g'_n$. So there should exists $b \in B'$ such that we have:

$$g'_n(b) = c' \implies h_n(\gamma_n^{-1}(g'_n(b))) = a.$$

Hence, $a \in \operatorname{Im}\delta_{n-1}$.

$\square$

Problem 2. Prove the "Five-Lemma": If

$$\begin{array}{ccccccccc}
A & \to^{\alpha} & B & \to^{\beta} & C & \to^{\gamma} & D & \to^{\delta} & E \\
\downarrow^{f_1} & & \downarrow^{f_2} & & \downarrow^{f_3} & & \downarrow^{f_4} & & \downarrow^{f_5} \\
A' & \to^{\alpha'} & B' & \to^{\beta'} & C' & \to^{\gamma'} & D' & \to^{\delta'} & E' \\
\hline
\end{array}$$

is a diagram of abelian groups and homomorphisms such that rows are exact, all squares commute, and f_1, f_2, f_4, f_5 are all isomorphisms, then f_3 is also an isomorphism.

[(Easier than 1). Note that exactness of the top row does not imply injectivity of α (actually, it directly implies nothing at all, except that $\operatorname{Im}\alpha = \ker\beta$). Similarly δ is not necessarily surjective.]

Proof. To show that f_3 is an isomorphism it suffices to show that it is a bijection (Note that we are given it being a homomorphism). First let $c \in \ker f_3$ so $f_3(c) = 0$, but since homomorphisms preserve 0 we have that $\gamma' f_3(c) = 0$ and by commutativity of squares we also have that $\gamma' f_3(c) = f_4\gamma(c)$. but since f_4 is an isomorphism, $f_4\gamma(c) = 0 \implies \gamma(c) = 0$, thus $c \in \ker(\gamma)$. Therefore there exists $b \in B$ such that $\beta(b) = c$. Furthermore $\beta'(f_2(b)) = 0$ by commutativity of squares. Since $f_2(b) \in \ker\beta'$ there exists an $a' \in A'$ such that $\alpha'(a') = f_2(b)$. Because f_1 is an isomorphism, $\alpha(f_1^{-1}(a')) = b$, thus $b \in \operatorname{Im}\alpha$ so $b \in \ker\beta$ and we have that $0 = \beta(b) = c$. Thus $\ker f_3$ is trivial and f_3 is injective.

Next, let $c' \in C'$. We have two cases:

(**a**) $c' \in \ker\gamma'$. In which case there exists $b' \in B'$ such that $\beta'(b') = c'$. Since f_2 is an isomorphism and squares commute, we also have that there exists a $b \in B$ such that $\beta'(f_2(b)) = f_3(\beta(b)) = c'$. Since $\beta(b) \in C$ we have that f_3 is surjective.

(**b**) There exists $d \in D'$ such that $\gamma'(c') = d'$. So $d' \ker\delta'$ which implies that $f_4^{-1}(d') \in \ker\delta$ so there exists $c \in C$ such that $f_4(\gamma(c)) = d'$. By commutativity of squares:

$$f_4(\gamma(c)) = \gamma'(f_3(c)) = d' = f_3(c').$$

Note that $\gamma \circ f_4$ is injective since f_4 is, and since $\gamma \circ f_4 = f_3 \circ \gamma'$ we have that $f_3\gamma'$ is also injective. Thus $f_3(c) = c'$ which shows surjectivity.

In either case, f_3 is surjective. We have shown a bijection and thus f_3 is an isomorphism.

$\square$