The Iterative Reweighted Least Squares Method for ℓ_1 Norm Minimisation

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1 Overview

The algorithm aims to solve the problem

$$\min ||\underline{x}||_1 \text{ subject to } \underline{\underline{A}}\underline{x} = \underline{y} \tag{1}$$

by finding the minimisers $\{x_0, x_1, \cdots\}$ of the functional

$$J_\lambda(x) = \frac{1}{2}\left|\left|\underline{\underline{A}}\underline{x} - \underline{y}\right|\right|_2^2 + \lambda\,||\underline{x}||_1 \ . \tag{2}$$

for decreasing λ. The algorithm does this by tracing out the *homotopy path*, approaching the limit

$$\lim_{\lambda\to 0} J_\lambda(x) = \frac{1}{2}\left|\left|\underline{\underline{A}}\underline{x} - \underline{y}\right|\right|_2^2 \ . \tag{3}$$

2 Variables

We have the following variables,

- $m \times n$ sensing matrix, $\underline{\underline{A}}$
- signal vector of m elements, $\underline{y}$
- solution vector of n elements, $\underline{x}$
- weights vector of n elements, $\underline{w}$
- the error, ϵ
- iteration index j

3 Algorithm

The iterative reweighted least square algorithms requires initialisation of

- the value of $\epsilon_0 = 0$
- the weights vector to $\underline{w} = \underline{1}$ (i.e., a vector with all elements equal to one)

In each iterative step j, the iterative reweighted least square algorithms updates the solution approximation, $\underline{x}^{(j)}$, and the weights, $\underline{w}^{(j)}$, as follows.

- the approximate solution is updated

$$\underline{x}^{(j+1)} = \underline{\underline{D}}^n_j \underline{\underline{A}}^T \left(\underline{\underline{A}} \underline{\underline{D}}^{-1} \underline{\underline{A}}^T \right) \underline{y} \tag{4}$$

  where the matrix $\underline{\underline{D}}$ is a diagonal matrix containing the weights,

$$D_{ii} = w_i^{(j)} \tag{5}$$

- the value of epsilon is updated

$$\epsilon_{j+1} = \min \left(\epsilon_n, \frac{r \left(x^{(j+1)} \right)_{K+1}}{N} \right) \tag{6}$$

  where the vector $\underline{r}\left(\underline{x}\right)$ is defined to contain the non-increasing arrangement of the absolute values of the elements in $\underline{x}$; $r\left(\underline{x}\right)_i$ is thus the i^{th} largest element of $\underline{x}$; a vector $\underline{x}$ is k-sparse if and only if $r\left(\underline{x}\right)_{k+1} = 0$

- the weights are updated

$$w_j^{(j+1)} = \frac{1}{\sqrt{\left(x_j^{(j+1)} \right)^2 + \epsilon_{j+1}^2}} \tag{7}$$

The loop breaks either

- when reaching the maximum number of iterations, N_{iter}, or
- when the infinity-norm of the residual vector (i.e., the lambda parameter in Eq. (1)) becomes smaller than the pre-set tolerance,

whichever happens first.