The Homotopy Method for ℓ_1 Norm Minimisation

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1 Overview

The algorithm aims to solve the problem

$$\min ||\underline{x}||_1 \text{ subject to } \underline{\underline{A}}\underline{x} = \underline{y} \tag{1}$$

by finding the minimisers $\{x_0, x_1, \cdots\}$ of the functional

$$J_\lambda(x) = \frac{1}{2}\left|\left|\underline{\underline{A}}\underline{x} - \underline{y}\right|\right|_2^2 + \lambda\,||\underline{x}||_1\ . \tag{2}$$

for decreasing λ. The algorithm does this by tracing out the *homotopy path*, approaching the limit

$$\lim_{\lambda \to 0} J_\lambda(x) = \frac{1}{2}\left|\left|\underline{\underline{A}}\underline{x} - \underline{y}\right|\right|_2^2\ . \tag{3}$$

2 Variables

We have the following variables,

- $m \times n$ sensing matrix, $\underline{\underline{A}}$
- signal vector of m elements, $\underline{y}$
- solution vector of n elements, $\underline{x}$
- element index $\ell = \{0, 1, \cdots, n\}$ for solution vector $\underline{\underline{x}}$
- support Γ_j, i.e., the indices of the non-zero entries in x_j; the subscript Γ_j on a matrix refers to the columns in that matrix corresponding to the support indices
- path segment index j
- residual vector $\underline{c}^{(j)} = \underline{\underline{A}}^T\left(\underline{y} - \underline{\underline{A}}\underline{x}^{(j-1)}\right)$
- the infinity-norm of the residual vector $\lambda_{j-1} = \left|\left|\underline{c}^{(j)}\right|\right|_\infty$ (corresponding to λ in Eq. (1))
- direction vector $\underline{d}^{(j)}$, giving the direction of the j^{th} path segment
- magnitude γ_j of the j^{th} path segment
- minimizers $\left\{\underline{x}^{(0)}, \underline{x}^{(1)}, \cdots\right\}$, where $\underline{x}^{(j+1)} = \underline{x}^{(j)} + \gamma_{j+1}\underline{d}^{(j+1)}$

We evaluate these variables to yield the homotopy path towards the solution (for which $\lambda \to 0$),

$$\underline{x} = \underline{x}^{(0)} + \gamma_1\underline{d}^{(1)} + \gamma_2\underline{d}^{(2)} + \cdots \tag{4}$$

3 Algorithm

Step 1
In the first step, we

- initialise the guess solution

$$\underline{x}^{(0)} = \underline{0} \tag{5}$$

- initialise the first residual vector

$$\underline{c}^{(1)} = \underline{\underline{A}}^T \left(\underline{y} - \underline{\underline{A}}\underline{x}^{(0)} \right) = \underline{\underline{A}}^T \underline{y} \tag{6}$$

- find the first index to be added to the support

$$l_1 = \underset{\ell=1\cdots N}{\operatorname{argmax}} \left| \left\{ c_l^{(1)} \right\} \right| \tag{7}$$

$$\Gamma_1 \leftarrow l_1 \tag{8}$$

- evaluate the first direction vector $\underline{d}^{(1)}$, from solving

$$\underline{\underline{A_{\Gamma_1}}}^T \underline{\underline{A_{\Gamma_1}}} \underline{d}^{(1)} = \operatorname{sign}\left(\underline{c}_{\Gamma_1}^{(1)} \right); \text{ while } \forall l \notin \Gamma_i, \text{ set } d_l = 0 \tag{9}$$

now since $|\Gamma_1| = |\{l_1\}| = 1$,

$$\underline{\underline{A_{\Gamma_1}}}^T \underline{\underline{A_{\Gamma_0}}} = ||a_{l_1}||_2^2 \Rightarrow d_\ell^{(1)} = \begin{cases} \frac{\operatorname{sign}\left(\underline{c}_{l_1}^{(1)}\right)}{||a_{l_1}||_2^2} & \text{for } \ell = \ell_1 \\ 0 & \text{for } \ell \neq \ell_1 \end{cases} \tag{10}$$

- evaluate the next point along the homotopy path

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \gamma_1 \underline{d}^{(1)} \tag{11}$$

where the homotopy path parameter γ_1 is

$$\gamma_1 = \min_{\ell} \left\{ \gamma_1^+, \gamma_1^- \right\} \tag{12}$$

where we consider the *positive* terms of the following sets

$$\gamma_1^+ = \min_{\ell \notin \Gamma_1} \left\{ \frac{||\underline{c}^{(1)}||_\infty - c_\ell^{(1)}}{1 - \left(\underline{\underline{A}}^T \underline{\underline{A}} \underline{d}^{(1)} \right)\Big|_\ell}, \frac{||\underline{c}^{(1)}||_\infty + c_\ell^{(1)}}{1 + \left(\underline{\underline{A}}^T \underline{\underline{A}} \underline{d}^{(1)} \right)\Big|_\ell} \right\} \tag{13}$$

$$\gamma_1^- = \min_{\ell \in \Gamma_1} \left\{ -\frac{x_l^{(0)}}{d_l^{(1)}} \right\} \tag{14}$$

- find the next index to add or remove to the support

$$\ell_2 = \underset{\ell}{\operatorname{argmin}} \left\{ \gamma_1^+, \gamma_1^- \right\} \tag{15}$$

Step $j = 2, 3, \cdots, N_{iter}$
In the following steps, indexed by j, we follow a similar procedure.

- adjust the support

$$\text{if } \ell_j \in \Gamma_{j-1} \quad \text{then} \quad \Gamma_j = \Gamma_{j-1} \setminus \ell_j \tag{16}$$
$$\text{if } \ell_j \notin \Gamma_{j-1} \quad \text{then} \quad \Gamma_j = \Gamma_{j-1} \cup \ell_j \tag{17}$$

- update the residual vector

$$\underline{c}^{(j)} = \underline{\underline{A}}^T \left(\underline{y} - \underline{\underline{A}}\underline{x}^{(j-1)} \right) \tag{18}$$

- find the direction vector $\underline{d}^{(j)}$ from solving

$$\underline{\underline{A_{\Gamma_j}}}^T \underline{\underline{A_{\Gamma_j}}} \underline{d}^{(j)}_{\Gamma_j} = \text{sign}\left(\underline{c}^{(j)}_{\Gamma_j} \right); \text{ while } \forall \ell \notin \Gamma_i, \text{ set } d_\ell^{(j)} = 0 \tag{19}$$

- evaluate homotopy path parameter λ_{j-1}

$$\lambda_{j-1} = \left|\left| \underline{c}^{(j)} \right|\right|_\infty \tag{20}$$

- update the solution approximation $\underline{x}$

$$\underline{x}^{(j)} = \underline{x}^{(j-1)} + \gamma_j \underline{d}^{(j)} \tag{21}$$

  where the homotopy path parameter γ_j is

$$\gamma_j = \min_\ell \left\{ \gamma_j^+, \gamma_j^- \right\} \tag{22}$$

  where we consider the *positive* terms of the following sets

$$\gamma_j^+ = \min_{\ell \notin \Gamma_1} \left\{ \frac{||\underline{c}^{(j)}||_\infty - c_\ell^{(j)}}{1 - \left(\underline{\underline{A}}^T \underline{\underline{A}}\underline{d}^{(j)} \right)\Big|_\ell}, \frac{||\underline{c}^{(j)}||_\infty + c_\ell^{(j)}}{1 + \left(\underline{\underline{A}}^T \underline{\underline{A}}\underline{d}^{(j)} \right)\Big|_\ell} \right\} \tag{23}$$

$$\gamma_j^- = \min_{\ell \notin \Gamma_j} \left\{ -\frac{\underline{x}_l^{(j-1)}}{\underline{d}_\ell^{(j)}} \right\} \tag{24}$$

- find the next index to add or remove to the support

$$\ell_{j+1} = \underset{\ell}{\text{argmin}} \left\{ \gamma_j^+, \gamma_j^- \right\} \tag{25}$$

The loop breaks either

- when reaching the maximum number of iterations, N_{iter}, or
- when the infinity-norm of the residual vector (i.e., the lambda parameter in Eq. (1)) becomes smaller than the pre-set tolerance,

whichever happens first.