

Advanced Statistics - Tutorial 12/08/2021

Statistics on variances of normal distributions

Section 1: One set of i.i.d. variables

Let $n \in \mathbb{N}$ and let X_1, \dots, X_n be independent, $N(\mu, \sigma^2)$ -distributed under P_{μ, σ^2} , where $\mu \in \mathbb{R}$, $\sigma^2 > 0$.

Goal: Find tests with significance level α / confidence level $1 - \alpha$ for the following hypotheses:

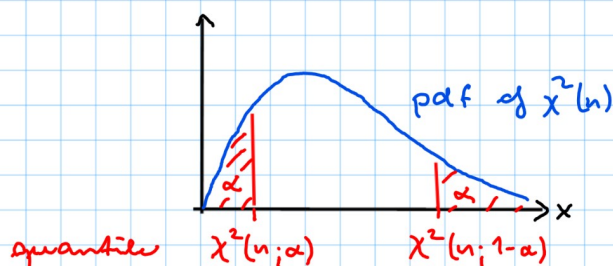
① $H_0: \sigma \leq \sigma_0; H_1: \sigma > \sigma_0$

② $H_0: \sigma \geq \sigma_0; H_1: \sigma < \sigma_0$

Idea: Compare empirical variance to σ_0^2

Case 1: Known mean μ , i.e. $\mu = \mu_0$ \rightarrow rather theoretical case

- Empirical variance is $V = \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2$
- Test statistic is $\frac{V}{\sigma_0^2}$
- $\frac{n}{\sigma_0^2} V$ has $\chi^2(n)$ distribution under P_{μ_0, σ_0^2}
↑
"degrees of freedom"



• Regions of rejection for hypothesis tests:

① $\left\{ \frac{V}{\sigma_0^2} > \frac{1}{n} \chi^2(n; 1-\alpha) \right\}$

② $\left\{ \frac{V}{\sigma_0^2} < \frac{1}{n} \chi^2(n; \alpha) \right\}$

Case 2: Unknown mean μ , i.e. $\mu \in \mathbb{R}$ \rightarrow typical case in applications

- Empirical mean is $M = \frac{1}{n} \sum_{i=1}^n X_i$
- Empirical variance is $V^* = \frac{1}{n-1} \sum_{i=1}^n (X_i - M)^2$
- Test statistic is $\frac{V^*}{\sigma_0^2}$

- $\frac{n-1}{\sigma^2} V^*$ has $\chi^2(n-1)$ distribution under P_{μ, σ^2}

- Region of rejection for hypothesis test:

$$\textcircled{1} \left\{ \frac{V}{\sigma^2} > (n-1) \chi^2(n-1; 1-\alpha) \right\}$$

$$\textcircled{2} \left\{ \frac{V}{\sigma^2} < (n-1) \chi^2(n-1; \alpha) \right\}$$

Section 2: Two sets of i.i.d. variables

let $m, n \in \mathbb{N}$, let $X_{1,1}, \dots, X_{1,m}; X_{2,1}, \dots, X_{2,n}$ be independent, and let

$X_{1,1}, \dots, X_{1,m}$ be $\mathcal{N}(\mu_1, \sigma_1^2)$ -distributed

$X_{2,1}, \dots, X_{2,n}$ be $\mathcal{N}(\mu_2, \sigma_2^2)$ -distributed

under $P_{\mu_1, \sigma_1^2; \mu_2, \sigma_2^2}$, where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1^2, \sigma_2^2 > 0$.

Goal: Find a test for the following hypothesis:

$$\textcircled{3} H_0: \sigma_1 = \sigma_2; \quad H_1: \sigma_1 > \sigma_2$$

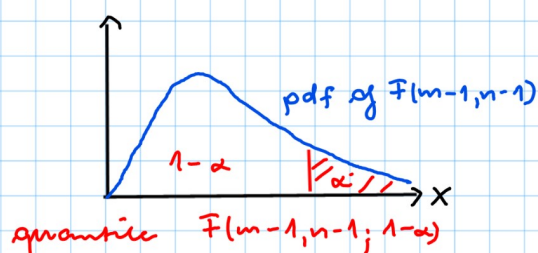
Idea: Compare empirical variances of both samples

Assume means μ_1, μ_2 to be unknown.

- Let V_1^*, V_2^* empirical variances of $X_{1,1}, \dots, X_{1,m}$ and $X_{2,1}, \dots, X_{2,n}$

- Test statistic is $\frac{V_1^*}{V_2^*}$

- $\frac{\sigma_1^2}{\sigma_2^2} \cdot \frac{V_1^*}{V_2^*}$ has $F(m-1, n-1)$ distribution under $P_{\mu_1, \sigma_1^2; \mu_2, \sigma_2^2}$



- Region of rejection for hypothesis test:

$$\textcircled{3} \left\{ \frac{V_1^*}{V_2^*} > F(m-1, n-1; 1-\alpha) \right\}$$