Advanced Statistics – Tutorial 23/07/2021

Normal distribution

PDF of normal distribution with mean $\mu \in \mathbb{R}$, standard deviation $\sigma > 0$ (i.e. variance σ^2):

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Corresponding CDF:

$$\Phi_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} \phi_{\mu,\sigma^2}(x')\, dx'$$

Central Limit Theorem

Let $n \in \mathbb{N}$ and let $X_1, \dots, X_n$ be <u>independent, identically distributed</u> (i.i.d.) random variables with

$$\text{mean} \quad \mu = E[X_i]$$
$$\text{variance} \quad \sigma^2 = Var[X_i].$$

Consider $\sum_{i=1}^{n} X_i$.

① Normal approximation without standardization/scaling:

Note that $E\left[\sum_{i=1}^{n} X_i\right] = n\mu$

$$Var\left[\sum_{i=1}^{n} X_i\right] \underset{\substack{\uparrow \\ \text{independence}}}{=} n\sigma^2$$

$$\Rightarrow P\left(a < \sum_{i=1}^{n} X_i \leq b\right) \approx \Phi_{n\mu, n\sigma^2}(b) - \Phi_{n\mu, n\sigma^2}(a) \quad \text{for } n \text{ large}$$

② Normal approximation with standardization and scaling:

Let $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$.

Then $E[S_n] = 0$

$$Var[S_n] = \left(\frac{1}{\sqrt{n}}\right)^2 Var\left[\sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}\right] \underset{\substack{\uparrow \\ \text{independence}}}{=} \frac{1}{n} \sum_{i=1}^{n} \underbrace{Var\left[\frac{X_i-\mu}{\sigma}\right]}_{=1} = 1$$

$\rightsquigarrow P(c < S_n \leq d) \longrightarrow \Phi_{0,1}(d) - \Phi_{0,1}(c)$ for $n \to \infty$

more rigorous statement than 1