Advanced Statistics - Tutorial 22/07/2021

Transformation Theorem

Let X be a random variable with values in $\mathbb{R}^n$ and pdf f_X.

Let $A, B \subset \mathbb{R}^n$ open/closed subsets and $g: A \to B$ such that:

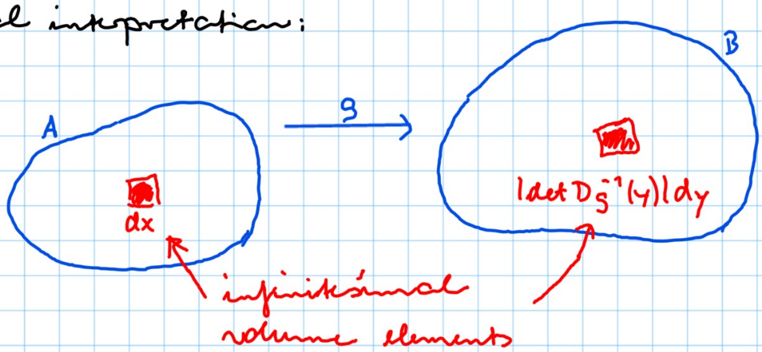
- $P(X \in A) = 1$, i.e. $\int_A f_X(x)\,dx = 1$
- $g: A \to B$ is bijective
- $g^{-1}: B \to A$ is continuously differentiable.

Then $g(X)$ has pdf

$$f_{g(X)}(y) = f_X(g^{-1}(y))\,|\det \underbrace{Dg^{-1}(y)}_{\text{Jacobi matrix}}|$$

$$Dg^{-1}(y) = \begin{pmatrix} \frac{\partial (g^{-1})_1}{\partial y_1}(y) & \cdots & \frac{\partial (g^{-1})_1}{\partial y_n}(y) \\ \vdots & & \vdots \\ \frac{\partial (g^{-1})_n}{\partial y_1}(y) & \cdots & \frac{\partial (g^{-1})_n}{\partial y_n}(y) \end{pmatrix}$$

Analytical interpretation:



Example 1:

$X \sim Exp(\lambda)$, i.e. $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$

$\to$ Find distribution of X^2

$g: (0,\infty) \to (0,\infty)$, $g(x) = x^2$ bijective

$g^{-1}: (0,\infty) \to (0,\infty)$, $g^{-1}(y) = \sqrt{y}$

$$\frac{dg^{-1}}{dy}(y) = \frac{1}{2\sqrt{y}}$$

$$\Rightarrow f_{X^2}(y) = \lambda e^{-\lambda\sqrt{y}} \cdot \left|\frac{1}{2\sqrt{y}}\right| = \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}}$$

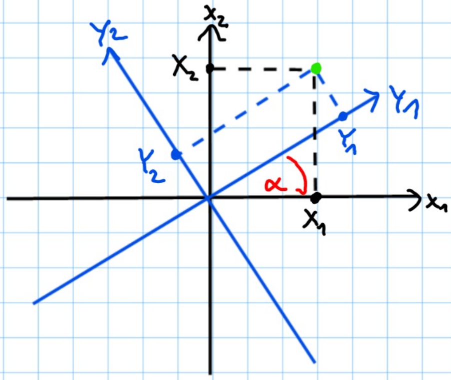
$$-\lambda\sqrt{y} = -\left(\frac{y}{1/\lambda^2}\right)^{1/2}$$

Weibull distribution with

- shape parameter $\frac{1}{2}$
- scale parameter $\frac{1}{\lambda^2}$

Example 2:

$$\left.\begin{array}{l} X_1 \sim \mathcal{N}(0,1) \\ X_2 \sim \mathcal{N}(0,1) \end{array}\right\}$$ independent, i.e.



$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right)$$

x_1-x_2-coordinates $\xrightarrow{\text{rotation by } \alpha}$ Y_1-Y_2-coordinates

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Inverse function:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Y_1\cos\alpha - Y_2\sin\alpha \\ Y_1\sin\alpha + Y_2\cos\alpha \end{pmatrix}$$

$$\det \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left((y_1\cos\alpha - y_2\sin\alpha)^2 + (y_1\sin\alpha + y_2\cos\alpha)^2\right)\right)\cdot|1|$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right),$$

i.e. $Y_1, Y_2 \sim \mathcal{N}(0,1)$ independent