# Advanced Statistics - Tutorial 02/05/2021

## Hypothesis tests

General setting:

- $\vartheta \in \Omega$ unknown parameter
- $X = (X_1, \dots, X_n)$ random sample with density $f_{X;\vartheta}(x)$ under $P_\vartheta$
- Split of $\Omega$: $\Omega = \Omega_0 \,\dot{\cup}\, \Omega_1$

$$\underbrace{H_0 : \vartheta \in \Omega_0}_{\text{null hypothesis}} \qquad \underbrace{H_1 : \vartheta \in \Omega_1}_{\text{alternative hypothesis}}$$

Goal: decide between $H_0$ and $H_1$, i.e. determine critical region $C$ s.t.

$$X \in C \Rightarrow \text{reject } H_0 \text{ (accept } H_1)$$

$$X \in C^c \Rightarrow \text{retain } H_0 \text{ (reject } H_1)$$

Potential decision errors:

| | $H_0$ is true | $H_1$ is true |
|---|---|---|
| Reject $H_0$ | Type I error | ✓ |
| Accept $H_0$ | ✓ | Type II error |

Constraints: $\sup\limits_{\vartheta \in \Omega_0} P_\vartheta(X \in C) \leq \alpha \qquad (*)$

$\alpha$: significance level

Minimize $\sup\limits_{\vartheta \in \Omega_1} P_\vartheta(X \in C^c)$ subject to $(*)$

$(\Leftrightarrow$ Maximize $\inf\limits_{\vartheta \in \Omega_1} \underbrace{P_\vartheta(X \in C)}_{\text{power of the test}} \dots )$

**Example**: Average lifetime of a device

Setting:

- $X_1, \dots, X_n$ i.i.d. with density $f_{X_i;\vartheta}(x_i) = \frac{1}{\vartheta} \exp(-\frac{1}{\vartheta} x_i)$ ($\text{Exp}(\frac{1}{\vartheta})$-distribution)
- $H_0 : \vartheta \geq \vartheta_0$; $\quad H_1 : \vartheta < \vartheta_0$

Joint density:

$$f_{X;\vartheta}(x) = \frac{1}{\vartheta^n} \exp\left(-\frac{1}{\vartheta} \sum_{i=1}^n x_i\right)$$

Monotonicity:

If $\vartheta < \vartheta'$, then $\frac{f_{X;\vartheta'}(x)}{f_{X;\vartheta}(x)} = \frac{\vartheta^n}{\vartheta'^n} \exp\left(\underbrace{-\left(\frac{1}{\vartheta'} - \frac{1}{\vartheta}\right)}_{>0} \sum_{i=1}^n x_i\right)$ is monotone increasing in $\sum_{i=1}^n x_i$

$\Rightarrow$ Define statistic $T := \sum_{i=1}^n X_i$ ; decision rule:

$$\left.\begin{aligned} T < c &\Rightarrow \text{reject } H_0 \\ T \geq c &\Rightarrow \text{retain } H_0 \end{aligned}\right\} \text{ where } P_{\vartheta_0}(T < c) = \alpha$$

$T$ is Gamma$(n, \frac{1}{\vartheta_0})$-distributed under $P_{\vartheta_0}$, i.e. $c$ is $\alpha$-quantile of Gamma$(n, \frac{1}{\vartheta_0})$.