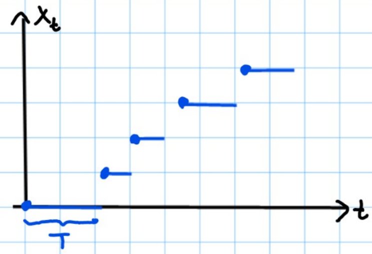
Advanced Statistics - Tutorial 15/07/2021

Part 1: Exponential Distribution

① Motivation

Let $t > 0$ (time), $\lambda > 0$ (rate) and let $X_t \sim \text{Poisson}(\lambda t)$ (number of events in $[0, t]$). Let T be the waiting time for the first event. Then:



$$
\left.
\begin{aligned}
& P(T > t) = P(X_t = 0) = e^{-\lambda t} \\
\Rightarrow \text{CDF: } & F_T(t) = 1 - e^{-\lambda t} \\
\text{PDF: } & f_T(t) = \frac{d}{dt} F_T(t) = \lambda e^{-\lambda t}
\end{aligned}
\right\} T \sim \text{Exp}(\lambda)
$$

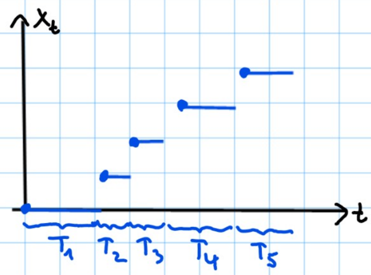
② Mean, variance

$$E[T] = \frac{1}{\lambda}$$

$$\text{Var}[T] = \frac{1}{\lambda^2}$$

Part 2: Gamma Distribution

① Motivation



Let $n \in \mathbb{N}$, $\lambda > 0$ and let $T_1, \dots, T_n$ be i.i.d. $\text{Exp}(\lambda)$-distributed.

Let $Y := \sum_{i=1}^{n} T_i$ (waiting time for n subsequent events).

Then:

$$f_Y(y) = \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}, \quad \text{i.e. } Y \sim \Gamma(\underset{\text{shape}}{n}, \underset{\text{rate}}{\lambda})$$

Alternative convention: $\Gamma(\underset{\text{shape}}{n}, \underset{\text{scale}}{\tfrac{1}{\lambda}})$

In other words:

$$\underbrace{\text{Exp}(\lambda) * \dots * \text{Exp}(\lambda)}_{\substack{\text{convolution of} \\ n \ \text{Exp}(\lambda)\text{-distributions}}} = \Gamma(n, \lambda)$$

② Generalization

Density of Gamma distribution $\Gamma(k, \lambda)$ for $k > 0$, $\lambda > 0$:

$$f(y) = \frac{\lambda^k}{\Gamma(k)} y^{k-1} e^{-\lambda y}$$

$\Gamma(k)$ is the suitable normalization:

$$\Gamma(k) = \int_0^\infty y^{k-1} e^{-y} \, dy$$

For $n \in \mathbb{N}$: $\Gamma(n) = (n-1)!$

Generalized convolution property:

$$\Gamma(k_1, \lambda) * \Gamma(k_2, \lambda) = \Gamma(k_1 + k_2, \lambda)$$

③ Mean, variance

Let $Y \sim \Gamma(k, \lambda)$.

$E[Y] = \frac{k}{\lambda}$

$Var[Y] = \frac{k}{\lambda^2}$