Advanced Statistics - Tutorial 14/10/2021

Estimators

Setting:
- ϑ unknown parameter
- $X = (X_1, \dots, X_n)$ i.i.d. observations with density $f_{X;\vartheta}(x)$ under P_ϑ
- $u_n(x)$ function of observations

$\rightsquigarrow$ Then $T_n := u_n(X)$ is an estimator (very general definition!)

Example: Yearly traffic collisions

Let $X_1, \dots, X_n$ be the number of yearly traffic collisions in n subsequent years.
Assume i.i.d. Poisson(ϑ) distribution.

$$\Rightarrow f_{X;\vartheta}(x) = \exp(-n\vartheta) \prod_{i=1}^{n} \frac{\vartheta^{x_i}}{x_i!}$$

Classical ansatz: Maximum Likelihood Estimator (MLE)

Choose $u_n(x)$ s.t.

$$f_{X;u_n(x)}(x) = \max_\vartheta f_{X;\vartheta}(x)$$

Example (ctd.):

Determine max of $\ln f_{X;\vartheta}(x)$ (easier than max of $f_{X;\vartheta}(x)$):

$$\ln f_{X;\vartheta}(x) = -n\vartheta + \ln(\vartheta)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!)$$

$$\Rightarrow \frac{d}{d\vartheta}\left(\ln f_{X;\vartheta}(x)\right) = -n + \frac{1}{\vartheta}\sum_{i=1}^{n} x_i \overset{!}{=} 0$$

$$\Rightarrow \vartheta = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \text{i.e.} \quad T_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Desirable properties of estimators:

- **Consistent:** $\forall \varepsilon > 0: \lim_{n\to\infty} P_\vartheta(|T_n - \vartheta| > \varepsilon) = 0$

  Ex.: ✓ because of weak law of large numbers
- **Unbiased:** $E_\vartheta[T_n] = \vartheta$

  Ex.: ✓ because $E_\vartheta[T_n] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E[X_i]}_{=\vartheta} = \vartheta$
- **Effective:** $\mathrm{Var}_\vartheta(T_n)$ as small as possible

  Ex.: ✓ follows from theory on "exponential families"

minimum variance unbiased estimator (MVUE)

- **Sufficient:** $\dfrac{f_{X;\vartheta}(x)}{f_{T_n;\vartheta}(u_n(x))}$ does not depend on ϑ

Ex.: ✓ Because of Neyman Theorem:

$$f_{X;\vartheta}(x) = \underbrace{\left(\exp(-n\vartheta)\,\vartheta^{n\cdot\frac{1}{n}\sum_i x_i}\right)}_{\text{function of } \frac{1}{n}\sum_i x_i \text{ and } \vartheta} \underbrace{\left(\prod_{i=1}^{n} \frac{1}{x_i!}\right)}_{\text{function of } x}$$