Transformation of a bivariate Γ-distribution

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1 Statement

Let X and Y be independent random variables $\sim \Gamma(2,5)$; we call f_X and f_Y their density function. Compute the density function of the variable $T = X+Y$.

2 Straight of the theory

X and Y are random variables, that is they are $X : \mathcal{S} \longrightarrow \mathbb{R}^+$ and $Y : \mathcal{S} \longrightarrow \mathbb{R}^+$. They are Gamma variables of parameters $\alpha = 2$ and $\beta = 5$ (see [Ano21]), thus, they are distributed in such a way that given a set of real-numbers E:

$$P(Y \in E) = P(X \in E) = \int_{t \in E} \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} dt = \int_{t \in E} \frac{5^2}{1} t e^{-5t} dt = \int_{t \in E} 25 t e^{-5t} dt$$

Let us call $f_X(t) = f_Y(t) = 25te^{-5t}$.

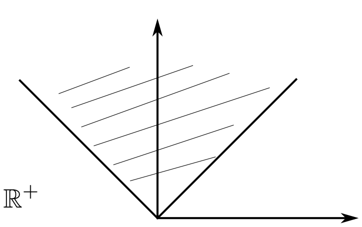
3 Calculation with a transformation

Introduce the transformation

$$g : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow O$$
$$(x, y) \mapsto (x+y, x-y)$$

This transformation is a bijection if O is defined to be

$$O = \{(a,b) \in \mathbb{R}^2 \mid a+b \geq 0 \text{ and } a-b \geq 0\}$$



Its inverse is:

$$g^{-1} : O \longrightarrow \mathbb{R}^+ \times \mathbb{R}^+$$
$$(a,b) \mapsto \left(\frac{1}{2}(a+b), \frac{1}{2}(a-b)\right)$$

because $g^{-1}(g(x,y)) = g^{-1}(x+y, x-y) = \left(\frac{1}{2}(x+y+x-y), \frac{1}{2}(x+y-x+y)\right) = \left(\frac{1}{2}2x, \frac{1}{2}2y\right) = (x,y)$

We apply the multivariate transformation theorem ([HMC20, 2.7]) and thus compute the Jacobian:

$$J_{g^{-1}} = \frac{\partial g_1^{-1}}{\partial a} \cdot \frac{\partial g_2^{-1}}{\partial b} - \frac{\partial g_2^{-1}}{\partial a} \cdot \frac{\partial g_1^{-1}}{\partial b} = \frac{1}{2} \cdot -\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}$$

The joint density of (X,Y) is $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ as the variables are independent.

The theorem thus states that the joint density of $g(X,Y)$ is:

$$f_{A,B}(a,b) = f_{X,Y}\left(g_1^{-1}(a,b), g_2^{-1}(a,b)\right) \cdot |J_{g^{-1}}|$$
$$= |-\frac{1}{2}| \cdot f_X(\frac{1}{2}(a+b)) \cdot f_Y(\frac{1}{2}(a-b))$$

$$= \frac{1}{2} \cdot 25(\frac{1}{2}(a+b))e^{-\frac{5}{2}(a+b)} \cdot 25(\frac{1}{2}(a-b))e^{-\frac{5}{2}(a-b)} =$$

then we can compute the density of the variable A which is $X+Y$ as a marginal distribution:

$$f_A(a) = \int_{b \in \mathbb{R}^+} f_{A,B}(a,b)db = \int_{-a}^{a} \frac{1}{2} \cdot 25(\frac{1}{2}(a+b))e^{-\frac{5}{2}(a+b)} \cdot 25(\frac{1}{2}(a-b))e^{-\frac{5}{2}(a-b)} =$$

3.1 Result

The random variable $T = X + Y$ is the same as the random variable $A = g_1(X,Y)$. Its density is given by:

$$\frac{625}{6}a^3e^{-5a}$$

as computed on WolframAlpha [inc20] with:

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integral between -a to a of  (1/2)*25*(1/2)*(a+b)*e^(-2.5*(a+b))
* 25/2*(a-b)*e^(-2.5*(a-b))db
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References

[Ano21] Anonymous. Gamma distribution. web-page, February 2021. See `https://en.wikipedia.org/wiki/Gamma_distribution`.

[HMC20] Robert Hogg, Joseph McKean, and Allen Craig. *Introduction to Mathematical Statistics*. Pearson Education, eigth edition (global) edition, 2020.

[inc20] Wolfram Alpha inc. Wolfram alpha, 2020. See `https://wolframalpha.com`.