Transformation of a bivariate Γ-distribution

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1 Statement

Let X and Y be independent random variables $\sim \Gamma(2,5)$; we call f_X and f_Y their density function. Compute the density function of the variable $T = X+Y$.

2 Straight of the theory

X and Y are random variables, that is they are $X : \mathcal{S} \longrightarrow \mathbb{R}^+$ and $Y : \mathcal{S} \longrightarrow \mathbb{R}^+$. They are Gamma variables of parameters $\alpha = 2$ and $\beta = 5$ (see [Ano21]), thus, they are distributed in such a way that given a set of real-numbers E:

$$P(Y \in E) = P(X \in E) = \int_{t\in E} \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} dt = \int_{t\in E} \frac{5^2}{1} t e^{-5t} dt = \int_{t\in E} 25 t e^{-5t} dt$$

Let us call $f_X(t) = f_Y(t) = 25te^{-5t}$.

3 Transformations?

Given a multivariate continuous random variable $X : \mathcal{S} \to D$ (having values in any set) and a mapping $g : D \to E$ a mapping, the **transformed random variable** $g \circ X$, often written as $g(X)$ is defined by $g(X)(s) = g(X(s))$.

We assume $E, D \subset \mathbb{R}^n$. Then this is the same as saying: given a series of random variables $X_i : \mathcal{S} \longrightarrow \mathbb{R}$ where $0 \le i \le n$ and $(X_1, \ldots, X_n) \in D$ and a series of functions $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ where $(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)) \in E$ when $(x_1, \ldots, x_n) \in E$.

Suppose that D and E are subsets of $\mathbb{R}^n$ and that g is a derivable tranformation (which means: $g_i(x_1, ..., x_n)$ is derivable, i.e. $\frac{\partial g_i}{\partial x_j}$ exists for each $0 \le i, j \le n$).

The transformation theorem, proved in [HMC20, 2.7], says that:

- $g(X)$ It is a continuous random variable.
- if $D, E \subset \mathbb{R}^n$ and X is a continuous random variable with pdf $f_X : D \to \mathbb{R}$ then the pdf of $g(X)$, noted $f_{g(X)}$, is equal to the following, $\forall \mathbf{a} \in E$:

$$f_{g(x)}(\mathbf{a}) = f_X(g^{-1}(\mathbf{a})) \cdot |J_{g^{-1}}(\mathbf{a})|$$

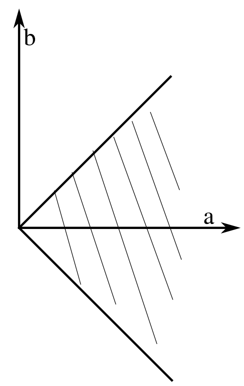
It is important to note that D and E are rarely equal to complete $\mathbb{R}^n$. E.g. It could be a subset of a plane where the lowest x depends on y.

4 Calculation with a transformation

Introduce the transformation

$$\begin{aligned} g: \quad \mathbb{R}^+ \times \mathbb{R}^+ &\longrightarrow \quad O \\ (x, y) \qquad &\mapsto \quad (a, b) = g(x, y) = (x + y, x - y) \end{aligned}$$

This transformation is a bijection if O is:



$$O = \{(a, b) \in \mathbb{R}^2 \mid a + b \geq 0 \text{ and } a - b \geq 0\}$$

Its inverse is:

$$g^{-1} : O \longrightarrow \mathbb{R}^+ \times \mathbb{R}^+$$
$$(a, b) \mapsto (x, y) = g^{-1}(a, b) = \left(\frac{1}{2}(a + b), \frac{1}{2}(a - b)\right)$$

because $g^{-1}(g(x, y)) = g^{-1}(x + y, x - y) = \left(\frac{1}{2}(x + y + x - y), \frac{1}{2}(x + y - x + y)\right) = \left(\frac{1}{2}2x, \frac{1}{2}2y\right) = (x, y)$

We apply the multivariate transformation theorem ([HMC20, 2.7]) and thus compute the Jacobian:

$$J_{g^{-1}}(a, b) = \frac{\partial g_1^{-1}}{\partial a} \cdot \frac{\partial g_2^{-1}}{\partial b} - \frac{\partial g_2^{-1}}{\partial a} \cdot \frac{\partial g_1^{-1}}{\partial b} = \frac{1}{2} \cdot -\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}$$

The joint density of (X, Y) is $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ as the variables are independent.

The theorem thus states that the joint density of $g(X, Y)$ is:

$$\begin{aligned} f_{A,B}(a, b) &= f_{X,Y}\left(g_1^{-1}(a, b), g_2^{-1}(a, b)\right) \cdot |J_{g^{-1}}| \\ &= |-\frac{1}{2}| \cdot f_X(\frac{1}{2}(a + b)) \cdot f_Y(\frac{1}{2}(a - b)) \end{aligned}$$

$$= \frac{1}{2} \cdot 25(\frac{1}{2}(a + b))e^{-\frac{5}{2}(a+b)} \cdot 25(\frac{1}{2}(a - b))e^{-\frac{5}{2}(a-b)} =$$

then we can compute the density of the variable A which is $X + Y$ as a marginal distribution:

$$f_A(a) = \int_{b \in \mathbb{R}^+} f_{A,B}(a, b)db = \int_{-a}^{a} \frac{1}{2} \cdot 25(\frac{1}{2}(a+b))e^{-\frac{5}{2}(a+b)} \cdot 25(\frac{1}{2}(a-b))e^{-\frac{5}{2}(a-b)}$$

4.1 Result

The random variable $T = X + Y$ is the same as the random variable $A = g_1(X, Y)$. Its density is given by:

$$\frac{625}{6} a^3 e^{-5a}$$

as computed on WolframAlpha [inc20] with:

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integral between -a to a of  (1/2)*25*(1/2)*(a+b)*e^(-2.5*(a+b))
* 25/2*(a-b)*e^(-2.5*(a-b))db
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References

[Ano21] Anonymous. Gamma distribution. web-page, February 2021. See `https://en.wikipedia.org/wiki/Gamma_distribution`.

[HMC20] Robert Hogg, Joseph McKean, and Allen Craig. *Introduction to Mathematical Statistics*. Pearson Education, eigth edition (global) edition, 2020.

[inc20] Wolfram Alpha inc. Wolfram alpha, 2020. See `https://wolframalpha.com`.