Advanced Statistics - Tutorial 16/05/2021

Part 1: Histograms

- Let X random variable with values in $[0,\infty)$ and continuous density $f_X(x)$
- $0 = a_0 < a_1 < a_2 < \dots$ partition of $[0,\infty)$ into bins

$\rightarrow$ Histogram has rectangles with

width $a_i - a_{i-1}$

height $\dfrac{\int_{a_{i-1}}^{a_i} f_X(x)\,dx}{a_i - a_{i-1}}$

Rescaling: let $c > 0$ and let

- $Y := cX$ rescaled random variable
- $b_i := c a_i$ rescaled partition

$\rightarrow$ Density of Y

$$f_Y(y) = c\, f_X\left(\frac{y}{c}\right)$$

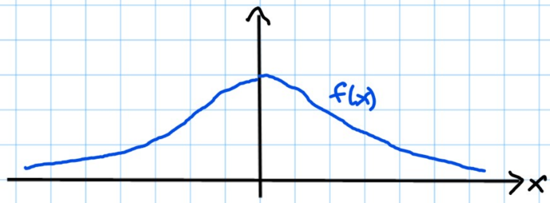
$\rightarrow$ Histogram has rectangles with

width $b_i - b_{i-1} = c(a_i - a_{i-1})$

height $\dfrac{\int_{b_{i-1}}^{b_i} f_Y(y)\,dy}{b_i - b_{i-1}} = \dfrac{1}{c}\,\dfrac{\int_{a_{i-1}}^{a_i} f_X(x)\,dx}{a_i - a_{i-1}}$

Part 2: Distributions where mean/variance do not exist

Example: Standard Cauchy distribution



Density $f(x) = \dfrac{1}{\pi(1+x^2)}$

$$\int_0^\infty \frac{x}{\pi(1+x^2)}\,dx = \infty \, ; \quad \int_{-\infty}^0 \frac{x}{\pi(1+x^2)}\,dx = -\infty$$

$$\Rightarrow E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} \text{ is not defined}$$

Part 3: Bayesian statistics

Example: Toss a coin n times and estimate probability ϑ of "head"

Likelihood: Binomial(n, ϑ) distribution

$$\Rightarrow f(k \mid \vartheta) = \binom{n}{k} \vartheta^k (1-\vartheta)^{n-k} \propto \vartheta^k (1-\vartheta)^{n-k} \quad \text{for } k \in \{0, 1, \dots, n\}$$

Prior: depends on our beliefs, e.g. Beta (a, b) distribution

$$\Rightarrow \quad f(\vartheta) = \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}_{= \frac{1}{B(a,b)} \text{ Beta function}} \vartheta^{a-1}(1-\vartheta)^{b-1} \propto \vartheta^{a-1}(1-\vartheta)^{b-1} \qquad \text{for } \vartheta \in (0,1)$$

Posterior:

$$\Rightarrow \quad f(\vartheta \mid k) \propto f(k \mid \vartheta)\, f(\vartheta)$$

$$\propto \vartheta^{k}(1-\vartheta)^{n-k} \cdot \vartheta^{a-1}(1-\vartheta)^{b-1}$$

$$= \vartheta^{k+a-1}(1-\vartheta)^{n-k+b-1}$$

Beta $(k+a,\ n-k+b)$ distribution

$\Rightarrow$ Prior and posterior distribution are from the same class of distributions ("conjugate prior").