# Advanced Statistics - Tutorial 05/08/2021

## Part 1: Sum of two independent random variables / convolution

**Theorem:**

$X_1, X_2$ real-valued independent random variables with prob. densities $f_1(x)$, $f_2(x)$

$\Rightarrow X_1 + X_2$ has prob. density $g(x) = \int_{-\infty}^{\infty} f_1(y)\, f_2(y-x)\, dy$

$= \int_{-\infty}^{\infty} f_1(y-x)\, f_2(y)\, dy$

## Part 2: Bayesian statistics

① Classical vs. Bayesian statistics

Determine parameter $\vartheta/\theta$

based on observations $x = (x_1, \dots, x_n)$

### Classical statistics

- $x$ is a realisation of a random variable $X$
- $\vartheta$ is an unknown but deterministic parameter
- Mathematical model:

  $f_\vartheta(x)$ likelihood

- Point estimators:
  - Maximum likelihood of $f_\vartheta(x)$
  - ...

### Bayesian statistics

- $x$ is a realisation of a random variable $X$
- $\vartheta$ is a realisation of a random variable $\theta$
- Mathematical model:

$$f(\vartheta|x) \propto f(x|\vartheta)\, f(\vartheta)$$

$f(\vartheta|x)$: a posterior density of $\theta$; $f(x|\vartheta)$: likelihood; $f(\vartheta)$: a prior density of $\theta$

proportional with const

$\left(\int f(x|\vartheta)\, f(\vartheta)\, d\vartheta\right)^{-1}$ continuous case

$\left(\sum_\vartheta f(x|\vartheta)\, f(\vartheta)\right)^{-1}$ discrete case

- Point estimators:
  - Mode
  - Median
  - Mean

  } of $f(\vartheta|x)$
  - ...

• Interval estimations:

$\vartheta \in [a(x), b(x)]$ with confidence level $1-\alpha$

$\Leftrightarrow P_\vartheta\left(a(X) \leq \vartheta \leq b(X)\right) \geq 1-\alpha$ for all $\vartheta$

($a(X)$, $b(X)$: random variable)

• Interval estimations:

$\theta \in [a(x), b(x)]$ with probability $1-\alpha$

$\Leftrightarrow P\left(\theta \in [a(x), b(x)] \mid X = x\right) \geq 1-\alpha$

② Example 1: AIDS test

$f(\text{pos} \mid \text{AIDS}) = 0.999$ (accuracy of the test)

$f(\text{pos} \mid \neg\text{AIDS}) = 0.05$ (false positive rate)

$f(\text{AIDS}) = 0.00035$ (prior / prevalence of AIDS)

$$\Rightarrow f(\text{AIDS} \mid \text{pos}) = \frac{f(\text{pos} \mid \text{AIDS})\, f(\text{AIDS})}{f(\text{pos} \mid \text{AIDS})\, f(\text{AIDS}) + f(\text{pos} \mid \neg\text{AIDS})\, f(\neg\text{AIDS})} \approx 0.065$$

③ Example 2: Toss a coin $n$ times and estimate probability $\vartheta$ of "head"

likelihood: Binomial $(n, \vartheta)$ distribution

$$\Rightarrow f(k \mid \vartheta) = \binom{n}{k} \vartheta^k (1-\vartheta)^{n-k} \quad \text{for } k \in \{0, 1, \dots, n\}$$

Prior: depends on our beliefs, e.g. Beta $(a, b)$ distribution

$$\Rightarrow f(\vartheta) = \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}}_{= \frac{1}{B(a,b)}} \vartheta^{a-1} (1-\vartheta)^{b-1} \propto \vartheta^{a-1} (1-\vartheta)^{b-1}$$

Beta function

Posterior:

$$\begin{aligned} \Rightarrow f(\vartheta \mid k) &\propto f(k \mid \vartheta)\, f(\vartheta) \\ &\propto \vartheta^k (1-\vartheta)^{n-k} \cdot \vartheta^{a-1} (1-\vartheta)^{b-1} \\ &= \vartheta^{k+a-1} (1-\vartheta)^{n-k+b-1} \end{aligned}$$

Beta $(k+a, n-k+b)$ distribution

⇒ Prior and posterior distribution are from the same class of distributions ("conjugate prior")