Advanced Statistics - Tutorial 12/08/2021

Statistics on variances of normal distributions

Section 1: One set of i.i.d. variables

Let $n \in \mathbb{N}$ and let $X_1, \dots, X_n$ be independent, $\mathcal{N}(\mu, \sigma^2)$-distributed under P_{μ,σ^2}, where $\mu \in \mathbb{R}$, $\sigma^2 > 0$.

Goal: Find tests with significance level α / confidence level $1-\alpha$ for the following hypotheses:

① $H_0: \sigma \le \sigma_0$; $H_1: \sigma > \sigma_0$

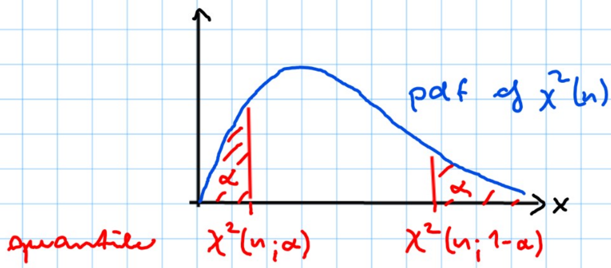
② $H_0: \sigma \ge \sigma_0$; $H_1: \sigma < \sigma_0$

Idea: Compare empirical variance to σ_0^2

Case 1: Known mean μ, i.e. $\mu = \mu_0$ → rather theoretical case

- Empirical variance is $V = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2$
- Test statistic is $\frac{V}{\sigma_0^2}$
- $\frac{n}{\sigma^2} V$ has $\chi^2(n)$ distribution under P_{μ_0,σ^2}

  "degrees of freedom"



- Regions of rejection for hypothesis tests:

  ① $\left\{ \frac{n}{\sigma_0^2} V > \chi^2(n; 1-\alpha) \right\}$

  ② $\left\{ \frac{n}{\sigma_0^2} V < \chi^2(n; \alpha) \right\}$

Case 2: Unknown mean μ, i.e. $\mu \in \mathbb{R}$ → typical case in applications

- Empirical mean is $M = \frac{1}{n} \sum_{i=1}^{n} X_i$
- Empirical variance is $V^* = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - M)^2$
- Test statistic is $\frac{V^*}{\sigma_0^2}$

- $\frac{n-1}{\sigma^2} V^*$ has $\chi^2(n-1)$ distribution under P_{μ,σ^2}

- Regions of rejection for hypothesis tests:

  ① $\left\{ \frac{n-1}{\sigma_0^2} V^* > \chi^2(n-1;\, 1-\alpha) \right\}$

  ② $\left\{ \frac{n-1}{\sigma_0^2} V^* < \chi^2(n-1;\, \alpha) \right\}$

Section 2: Two sets of i.i.d. variables

Let $m, n \in \mathbb{N}$, let $X_{1,1}, \dots, X_{1,m};\ X_{2,1}, \dots, X_{2,n}$ be independent, and let

$X_{1,1}, \dots, X_{1,m}$ be $\mathcal{N}(\mu_1, \sigma_1^2)$-distributed

$X_{2,1}, \dots, X_{2,n}$ be $\mathcal{N}(\mu_2, \sigma_2^2)$-distributed

under $P_{\mu_1,\sigma_1^2;\,\mu_2,\sigma_2^2}$, where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1^2, \sigma_2^2 > 0$.

Goal: Find a test for the following hypothesis:

③ $H_0: \sigma_1 = \sigma_2;\quad H_1: \sigma_1 > \sigma_2$

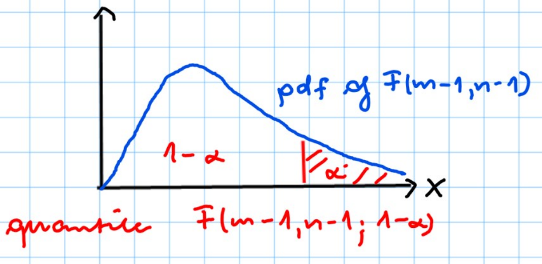
Idea: Compare empirical variances of both samples

Assume means μ_1, μ_2 to be unknown.

- Let V_1^*, V_2^* empirical variances of $X_{1,1}, \dots, X_{1,m}$ and $X_{2,1}, \dots, X_{2,n}$

- Test statistic is $\frac{V_1^*}{V_2^*}$

- $\frac{\sigma_2^2}{\sigma_1^2} \cdot \frac{V_1^*}{V_2^*}$ has $F(m-1, n-1)$ distribution under $P_{\mu_1,\sigma_1^2;\,\mu_2,\sigma_2^2}$



- Region of rejection for hypothesis test:

  ③ $\left\{ \frac{\sigma_2^2}{\sigma_1^2} \cdot \frac{V_1^*}{V_2^*} > F(m-1, n-1;\, 1-\alpha) \right\}$