Advanced Statistics - Tutorial 09/12/2021

Estimators - methods and desirable properties

Setting: • ϑ unknown parameter

• $X = (X_1, \dots, X_n)$ i.i.d. observations with density $f_{X;\vartheta}(x)$ under P_ϑ

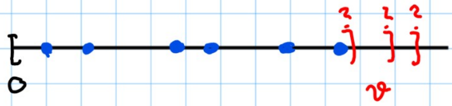
• $u_n(x)$ function of observations

⇝ Then $T_n := u_n(X)$ is an estimator (very general definition!)

Example:

Numbers $X_1, \dots, X_n$ are i.i.d. uniformly distributed on $[0, \vartheta]$.

What is a good estimator for ϑ?



Density of $X = (X_1, \dots, X_n)$:

$$f_{X;\vartheta}(x) = \begin{cases} \frac{1}{\vartheta^n} & \text{if } \forall i \in \{1, \dots, n\}: x_i \in [0, \vartheta] \\ 0 & \text{otherwise} \end{cases}$$

Classical ansatz: Maximum Likelihood Estimator (MLE)

→ choose $u_n(x)$ s.t.

$$f_{X;u_n(x)}(x) = \max_\vartheta f_{X;\vartheta}(x)$$

Example (ctd):

$T_n^{MLE} = \max_i X_i$ is MLE for ϑ

$\bar{T}_n = \frac{2}{n} \sum_i X_i$ is another estimator for ϑ

Desirable properties of estimators

- Consistency: $\forall \varepsilon > 0: \lim_{n \to \infty} P_\vartheta(|T_n - \vartheta| > \varepsilon) = 0$
- Unbiasedness: $E_\vartheta[T_n] = \vartheta$
- Effectiveness: $Var_\vartheta(T_n)$ as small as possible

} minimum variance unbiased estimator (MVUE)

- Sufficiency: $\dfrac{f_{X;\vartheta}(x)}{f_{T_n;\vartheta}(u_n(x))}$ does not depend on ϑ

Example (ctd):

Properties of T_n^{MLE}:

Let $c \in [0, \vartheta]$.

$\Rightarrow P_\vartheta(T_n^{MLE} \leq c) = P_\vartheta(\max_i X_i \leq c) = \prod_{i=1}^n P(X_i \leq c) = \left(\frac{c}{\vartheta}\right)^n$ (cdf of T_n^{MLE})

$$\Rightarrow \quad f_{T_n^{MLE};\vartheta}(t) = \frac{n}{\vartheta^n}\, t^{n-1}, \quad t \in [0,\vartheta] \qquad (\text{pdf of } T_n^{MLE})$$

Consistency: $P_\vartheta(|T_n^{MLE}-\vartheta| > \varepsilon) \underset{P_\vartheta(T_n^{MLE}>\vartheta)=0}{\overset{\uparrow}{=}} P_\vartheta(T_n^{MLE} \le \vartheta-\varepsilon) = \left(\frac{\vartheta-\varepsilon}{\vartheta}\right)^n \to 0$ for $n\to\infty$

Unbiasedness: $E_\vartheta[T_n^{MLE}] = \int_0^\vartheta \frac{n}{\vartheta^n}\, t^n \, dt = \frac{n}{n+1}\vartheta \neq \vartheta$

$\Rightarrow$ T_n^{MLE} is biased but $T_n^* := \frac{n+1}{n} T_n^{MLE}$ is unbiased.

Sufficiency: Let $x = (x_1,\dots,x_n) \in [0,\vartheta]^n$

$$\frac{f_{X;\vartheta}(x)}{f_{T_n^{MLE};\vartheta}(\max_i x_i)} = \frac{\frac{1}{\vartheta^n}}{\frac{n}{\vartheta^n}(\max_i x_i)^{n-1}} = \frac{1}{n\,(\max_i x_i)^{n-1}}$$

does not depend on ϑ

Remark:
- T_n^* is consistent, sufficient and a MVUE for ϑ.
- $\bar{T}_n$ is consistent and unbiased but not effective.