Advanced Statistics - Tutorial 19/08/2021

Transformation Theorem

Let X be a random variable with values in $\mathbb{R}^n$ and pdf f_X.

Let $A, B \subset \mathbb{R}^n$ open / closed subsets and $g: A \to B$ such that:

- $P(X \in A) = 1$, i.e. $\int_A f_X(x)\, dx = 1$.
- $g: A \to B$ is bijective
- $g^{-1}: B \to A$ is continuously differentiable

Then $Y := g(X)$ has pdf

$$f_Y(y) = f_X(g^{-1}(y)) \, |\det \underbrace{Dg^{-1}(y)}_{\text{Jacobi matrix}}|$$

$$Dg^{-1}(y) = \begin{pmatrix} \frac{\partial (g^{-1})_1}{\partial y_1}(y) & \cdots & \frac{\partial (g^{-1})_1}{\partial y_n}(y) \\ \vdots & & \vdots \\ \frac{\partial (g^{-1})_n}{\partial y_1}(y) & \cdots & \frac{\partial (g^{-1})_n}{\partial y_n}(y) \end{pmatrix}$$

Example 1: Generation of random variables from uniform distribution (Application: random number generator)

- $U \sim \mathcal{U}([0,1])$ uniform distribution on $[0,1]$
- $f: \mathbb{R} \to \mathbb{R}$ some continuous pdf
- $F: \mathbb{R} \to [0,1]$, $F(x) = \int_{-\infty}^{x} f(y)\, dy$ corresponding cdf with inverse F^{-1}

$\Rightarrow Y := F^{-1}(U)$ has density $f(y)$

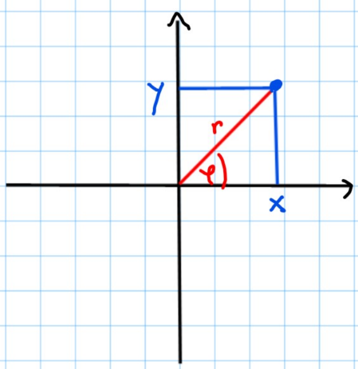
Proof:

$\frac{dF}{dy}(y) = f(y)$, $\left|\frac{dF}{dy}(y)\right| = f(y)$ since pdf's are ≥ 0

$$f_Y(y) = \underset{\substack{\uparrow \\ f_U(u) = 1,\ u \in [0,1]}}{f_U(F(y))} \cdot \left|\frac{dF}{dy}(y)\right| = f(y)$$

Example 2:

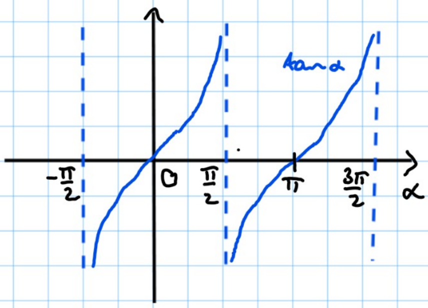
$$\left.\begin{array}{l} X \sim \mathcal{N}(0,1) \\ Y \sim \mathcal{N}(0,1) \end{array}\right\} \text{independent, i.e.}$$

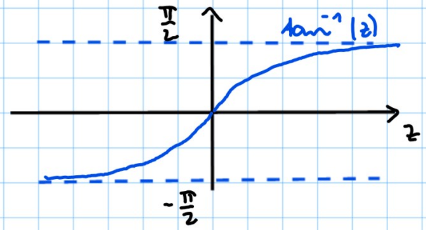


$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2 + y^2)\right)$$

change to polar coordinates:

$$\begin{pmatrix} R \\ \Phi \end{pmatrix} = g \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \begin{pmatrix} r \\ \varphi \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arg(x,y) \end{pmatrix}$$





$$\text{where } \arg(x,y) = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } x>0,\ y\geq 0 \\ \tan^{-1}\frac{y}{x}+\pi & \text{if } x<0 \\ \tan^{-1}\frac{y}{x}+2\pi & \text{if } x>0,\ y<0 \\ \frac{\pi}{2} & \text{if } x=0,\ y>0 \\ \frac{3\pi}{2} & \text{if } x=0,\ y<0 \end{cases}$$

$$g^{-1}\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$Dg^{-1}\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\det Dg^{-1}\begin{pmatrix} r \\ \varphi \end{pmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$\Rightarrow f_{R,\Phi}(r,\varphi) = \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right) r$$

$$\Rightarrow f_R(r) = \exp\left(-\frac{r^2}{2}\right)\cdot r, \quad f_\Phi(\varphi) = \frac{1}{2\pi}, \quad f_{R,\Phi}(r,\varphi) = f_R(r)\, f_\Phi(\varphi)$$

$$\Rightarrow \left.\begin{array}{l} R \sim \text{Weibull distribution} \\ \Phi \sim \mathcal{U}([0,2\pi]) \end{array}\right\} \text{independent}$$