Some inversion formulas for sums of quotients

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In this note we establish some formulas for certain sums of quotients of a positive integer n, which are closely related to an identity established by Préville-Ratelle in Problem M40 of the April 2003 issue of this magazine [1]. We also establish some elementary facts that are not well-known about quotients and remainders. Our main result is the following theorem.

Theorem 1 Let n and k be any positive integers with $k \leq n$. Then

$$\sum_{d=1}^{k} \lfloor n/d \rfloor - \sum_{d=\lfloor n/k \rfloor+1}^{n} \lfloor n/d \rfloor = k\lfloor n/k \rfloor. \qquad (F_k)$$

The first sum is clearly the sum of the quotients of n from $\lfloor n/1 \rfloor$ through $\lfloor n/k \rfloor$. We show below that the second sum is the sum of the quotients of n that are equal to one of $\{1, \ldots, k-1\}$. The "inversion" aspect of the formula is that the quotient sums are being taken in opposite directions.

As an illustration, here are the quotients for $n = 15$:

divisor d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quotient $\lfloor n/d \rfloor$	15	7	5	3	3	2	2	1	1	1	1	1	1	1	1

Theorem 1 is trivial for $k = 1$. When $k = 2$, for example, the formula says that $15+7-(1+1+1+1+1+1+1+1) = 2\lfloor 15/2 \rfloor$. For $k = 4$ the formula gives $15+7+5+3-(3+3+2+2+1+1+1+1+1+1+1+1) = 4\lfloor 15/4 \rfloor$.

In order to prove Theorem 1 we need the following result.

Lemma 1 Let d, $i \leq n$ be positive integers. Then

$$\lfloor n/i \rfloor = d \quad \text{if and only if} \quad \lfloor n/(d+1) \rfloor < i \leq \lfloor n/d \rfloor.$$

Proof: We first suppose $\lfloor n/i \rfloor = d$. By definition of the floor function, d is the unique integer such that $d \leq n/i < d+1$. Inverting the inequality yields $n/(d+1) < i \leq n/d$. Certainly, $\lfloor n/(d+1) \rfloor \leq n/(d+1)$. On the other hand, since i is an integer, we get $i \leq \lfloor n/d \rfloor$ from $i \leq n/d$. Hence, $\lfloor n/(d+1) \rfloor < i \leq \lfloor n/d \rfloor$. As these steps are reversible the proof is complete.

Proof of Theorem 1: From Lemma 1 we get at once

$$\sum_{d=\lfloor n/(k+1) \rfloor+1}^{\lfloor n/k \rfloor} \lfloor n/d \rfloor = k(\lfloor n/k \rfloor - \lfloor n/(k+1) \rfloor).$$

Letting $Q(n,k) = \lfloor n/k \rfloor - \lfloor n/(k+1) \rfloor$, we have

$$\sum_{d=\lfloor n/k \rfloor+1}^{n} \lfloor n/d \rfloor = \sum_{j=1}^{k-1} \sum_{d=\lfloor n/(j+1) \rfloor+1}^{\lfloor n/j \rfloor} \lfloor n/d \rfloor = \sum_{j=1}^{k-1} jQ(n,j). \qquad (1)$$

On the other hand, it is easy to verify that for $d = 2, \ldots, k$ we have

$$(d-1)Q(n,d-1) + d\lfloor n/d \rfloor - (d-1)\lfloor n/(d-1) \rfloor = \lfloor n/d \rfloor.$$

Part of the left-hand side of this equality telescopes if we sum over d, starting at 2. Therefore, using (1), we obtain

$$\sum_{d=2}^{k} \lfloor n/d \rfloor = \sum_{d=1}^{k-1} dQ(n,d) + k\lfloor n/k \rfloor - n = \sum_{d=\lfloor n/k \rfloor+1}^{n} \lfloor n/d \rfloor + k\lfloor n/k \rfloor - n.$$

Adding n to each side of this expression completes the proof.

Remark. Expression (1) and Lemma 1 show that the second sum in the statement of Theorem 1 is, in fact, the sum of the quotients of n that are equal to one of $\{1, \ldots, k-1\}$.

Next we establish some facts about quotients that are a direct consequence of Lemma 1.

Corollary 1 Let d, $i \leq n$ be positive integers.

(a) $d \leq \lfloor n/\lfloor n/d \rfloor \rfloor$.

(b) $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor = \lfloor n/d \rfloor$.

(c) $\lfloor n/i \rfloor = \lfloor n/d \rfloor$ if and only if $\lfloor n/(\lfloor n/d \rfloor + 1) \rfloor < i \leq \lfloor n/\lfloor n/d \rfloor \rfloor$.

(d) $\lfloor n/\lfloor n/i \rfloor \rfloor = d$ if and only if $\lfloor n/i \rfloor = \lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$.

Proof: We use freely the fact that $\lfloor n/d \rfloor \geq \lfloor n/(d+1) \rfloor$ for any $d \in \mathbb{N}$.

(a) Let $k = \lfloor n/\lfloor n/d \rfloor \rfloor$. From Lemma 1 it follows that $\lfloor n/(k+1) \rfloor < \lfloor n/d \rfloor$. Thus $k+1 > d$, that is, $k \geq d$.

(b) We use (a). Replacing d by $\lfloor n/d \rfloor$, we get $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \geq \lfloor n/d \rfloor$. On the other hand we have $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \leq \lfloor n/d \rfloor$ because $\lfloor n/d \rfloor$ is a decreasing function of d.

(c) This follows immediately from Lemma 1 by replacing d with $\lfloor n/d \rfloor$.

(d) Case $i = d$ follows from Lemma 1 by replacing i with $\lfloor n/d \rfloor$. Now we prove the general case. Suppose $\lfloor n/\lfloor n/i \rfloor \rfloor = d$. Thus, $\lfloor n/\lfloor n/\lfloor n/i \rfloor \rfloor \rfloor = \lfloor n/d \rfloor$. From (b) we have $\lfloor n/i \rfloor = \lfloor n/d \rfloor$, so $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, from the case $i = d$, we get $\lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Now suppose $\lfloor n/i \rfloor = \lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Thus, $\lfloor n/\lfloor n/i \rfloor \rfloor = \lfloor n/\lfloor n/d \rfloor \rfloor$. Using again the case $i = d$, we obtain $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, $\lfloor n/\lfloor n/i \rfloor \rfloor = d$.

Remark. Using (a) and (b) the following facts can also be easily proven:

(1) $d \leq \lfloor n/i \rfloor$ if and only if $i \leq \lfloor n/d \rfloor$.

(2) $i \leq d$ implies $\lfloor n/\lfloor n/i \rfloor \rfloor \leq \lfloor n/\lfloor n/d \rfloor \rfloor$.

(3) $\lfloor n/\lfloor n/i \rfloor \rfloor = \lfloor n/\lfloor n/d \rfloor \rfloor$ if and only if $\lfloor n/i \rfloor = \lfloor n/d \rfloor$.

We can also reformulate Corollary 1 in terms of $n \bmod d = n - d\lfloor n/d \rfloor$, the remainder on division of n by d. For example, reformulation of (a) and the case $i = d$ of (d) yields the following result.

Corollary 2 Let $d \leq n$ be a positive integer.

(a) $n \bmod \lfloor n/d \rfloor \leq n \bmod d$.

(b) $n \bmod \lfloor n/d \rfloor < n \bmod d$ if and only if $\lfloor n/d \rfloor = \lfloor n/(d+1) \rfloor$.

Furthermore, from Theorem 1 and Lemma 1 we get some unusual expressions for $n \bmod k$, and hence, a criterion for divisibility of n by k.

Corollary 3 Let n and k be any positive integers with $k \leq n$.

(a) $n \bmod k = \sum\limits_{d=k+1}^{n} \lfloor n/d \rfloor - \sum\limits_{d=2}^{\lfloor n/k \rfloor} \lfloor n/d \rfloor = \sum\limits_{d=\lfloor n/k \rfloor+1}^{n} \lfloor n/d \rfloor - \sum\limits_{d=2}^{k} \lfloor n/d \rfloor$.

(b) $n \bmod k = (F(k) + F(\lfloor n/k \rfloor))/2$, where

$$F(k) = \sum_{d=k+1}^{n} \lfloor n/d \rfloor - \sum_{d=2}^{k} \lfloor n/d \rfloor.$$

Moreover, $n \bmod k = F(k)$ if and only if $\lfloor n/(k+1) \rfloor < k = \lfloor n/k \rfloor$.

(c) $k|n$ if and only if $n/k = \sum_{d=\lfloor n/k \rfloor+1}^{n} \lfloor n/d \rfloor - \sum_{d=2}^{k-1} \lfloor n/d \rfloor$.

Proof: (a) We get the second identity by partitioning the sum $\sum_{d=2}^{n} \lfloor n/d \rfloor$ in two obvious ways. To prove the first identity, we add and subtract $\sum_{d=1}^{\lfloor n/k \rfloor} \lfloor n/d \rfloor$ to the left-hand side of (F_k) to obtain the following equivalent formula:

$$\sum_{d=1}^{\lfloor n/k \rfloor} \lfloor n/d \rfloor - \sum_{d=k+1}^{n} \lfloor n/d \rfloor = k \lfloor n/k \rfloor.$$

Thus, replacing $k\lfloor n/k \rfloor$ by $n - n \bmod k$, canceling n, and multiplying both sides of the equation by -1, we complete the proof of (a).

(b) The formula for $n \bmod k$ holds because the right member is the arithmetic mean of the second and third member of (a). From this we have $n \bmod k = F(k)$ if and only if $F(k) = F(\lfloor n/k \rfloor)$. Partitioning the sum $\sum_{d=2}^{n} \lfloor n/d \rfloor$ as was done in (a), we get $\sum_{d=2}^{k} \lfloor n/d \rfloor = \sum_{d=2}^{\lfloor n/k \rfloor} \lfloor n/d \rfloor$. Since each quotient is positive, we have $\lfloor n/k \rfloor = k$. These steps are reversible, so we have proved that $n \bmod k = F(k)$ if and only if $\lfloor n/k \rfloor = k$. Then, from Lemma 1, after we replace i and d by k, the proof of (b) is complete.

(c) This follows at once from (a) using the rightmost expression for $n \bmod k$.

Next we establish a generalization of Theorem 1 that clearly shows the process of inversion of the involved sums.

Corollary 4 Let n, j, and k be positive integers with $j \le k \le n$. Then

$$\sum_{d=j+1}^{k} \lfloor n/d \rfloor - \sum_{d=\lfloor n/k \rfloor+1}^{\lfloor n/j \rfloor} \lfloor n/d \rfloor = k\lfloor n/k \rfloor - j\lfloor n/j \rfloor. \qquad (F_{j,k})$$

Proof: This follows at once from Theorem 1 by subtracting (F_j) from (F_k).

Remark. Theorem 1 and Corollary 4 are logically equivalent because (F_k) follows from $(F_{1,k})$.

Consequently, we have generalized Préville-Ratelle's identity, as this is precisely the next result in the case $j|n$ and $k|n$.

Corollary 5 Let n, j, and k be any positive integers with $j \le k \le n$. Then

$$\sum_{d=j+1}^{k} \lfloor n/d \rfloor = \sum_{d=\lfloor n/k \rfloor+1}^{\lfloor n/j \rfloor} \lfloor n/d \rfloor \quad \text{if and only if} \quad n \bmod j = n \bmod k.$$

Proof: This follows at once from Corollary 4 after replacing $k\lfloor n/k \rfloor - j\lfloor n/j \rfloor$ by $n \bmod j - n \bmod k$ in $(F_{j,k})$.

Concluding remark. Préville-Ratelle's solution to Problem M40 gives a nice graphical interpretation of $(F_{j,k})$ for the case when j and k are divisors of n. Can the reader generalize that graphical approach to prove $(F_{j,k})$ for arbitrary j and k with $j \leq k$?

References

[1] Louis-François Préville-Ratelle; Problem M40, *Crux Mathematicorum* 29:3 (2003), 140-141.

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