Notes from Warner's Modern Algebra

Aaron Bailey

December 22, 2022

All of the theorems and definitions here are due to Warner and are for convenience, though not necessarily word for word, similar to a quote or paraphrase in any text. The proofs of theorems are not given, but discussed when not straightforward. The solutions to exercises are my own.

Chapter 1. Algberaic Structures

S4. Neutral Elements and Inverses

Definition. Let Δ be a composition on E. Then $e \in E$ is a neutral element (aka identity or unity element) for Δ if $e\Delta x = x\Delta e = x$ for all $x \in E$.
Theorem 4.1. There exists at most one neutral element for a composition Δ on E.
Definition. An element $y \in E$ satisfying $x\Delta y = e = y\Delta x$ is said to be an inverse of x for Δ. The text will use the notation x^* for the inverse of x for an associative operation.
Theorem 4.2. If Δ is an associative composition on E, an element $x \in E$ has at most one inverse. Note: Associativity implies uniqueness, not existence.
Theorem 4.3. If y is an inverse of x for a composition Δ on E, then x is an inverse of y for Δ.
Theorem 4.4. If x and y are invertible elements for an associative composition Δ on E, then $x\Delta y$ is invertible for Δ, and $(x\Delta y)^* = y^*\Delta x^*$.
Theorem 4.5. Let Δ be an associative composition on E, and let $x, y, z \in E$. Then:

1. If both x and y commute with z, then $x\Delta y$ also commutes with z..
2. If x commutes with y and y is invertible, then x commutes with y^*.
3. If x commutes with y and if both x and y are invertible, then x^* commutes with y^*.

Exercises for this section were skipped.

S5. Composites and Inverses of Functions

Definition. Let f be a function from E into F, and let g be a function from F into G. The **composite** of g and f is the function $g \circ f$ from E into G defined by $(g \circ f)(x) = g(f(x))$
Theorem 5.1. For functions $f : E \to F$, $g : F \to G$, and $h : G \to H$, $(f \circ g) \circ h = h \circ (g \circ f)$.
Definition. For any set E, the identity function on E is the function $I_E : E \to E$ defined by $x \to x$. Formally, $I_E = \{(x, x) : x \in E\}$, and is called the diagonal subset of $E \times E$.
Definition. A function f from E to F is injective if for all $u, x \in E$, if $u \neq x \Rightarrow f(u) \neq f(x)$. Similarly, $f(u) = f(x) \Rightarrow u = x$ by the contrapositive.
Definition. The inverse of a function f is the set $f^{\leftarrow}$ defined by $f^{\leftarrow} = \{(y, x) : (x, y) \in f\}$
Theorem 5.2. Let f be a function from E into F. Then f is injective if and only if $f^{\leftarrow}$ is a function.
Definition. A function f from E into F is a surjection onto F if F is the range of f.
Definition. A function f from E into F is a bijection if f is both an injection and a surjection.
Theorem 5.3. Let f be a function from E into F, and let g be a function from F into G. Then:

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.

Theorem 5.4. Let $f : E \to F$. If there exists a functions $g, h : F \to E$ such that $g \circ f = I_E$ and $f \circ h = I_f$, then f is a bijection and $g = h = f^{\leftarrow}$.
Theorem 5.5. If $f : E \to F$ is bijective, then $f^{\leftarrow}$ is bijective.
Theorem 5.6. If $f : E \to F$ and $g : F \to G$ are bijective, then $g \circ f$ is bijective and $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$.
Exercises for this section were skipped.

S6. Isomorphisms of Algebraic Structures

Definition. An algebraic structure with one composition (a magma/binar) is an ordered pair (E, Δ) where E is a nonempty set and where Δ is a composition on E. We may similarly define higher algebraic structures.
Definition. Let (E, Δ) and (F, ∇) be algebraic structures. An isomorphism from (E, Δ) onto (F, ∇) is a bijection $f : E \to F$ such that $f(x \Delta y) = f(x) \nabla f(y)$ for all $x, y \in E$. Similar definitions apply for higher structures. If such an isomorphism exists between two structures, they are said to be isomorphic. An isomorphism from a structure to itself is an automorphism, and is not necessarily trivial (e.g. rotating a dihedral group by some rotational symmetry)
Theorem 6.1. Let (E, Δ), (F, ∇), and $(G, \vee)$ be algebraic structures, and let $f : E \to F$ and $g : F \to G$ be bijections. Then:

1. Then identity function I_E is an automorphism of (E, Δ)
2. The bijection f is an isomorphism from (E, Δ) onto (F, ∇) if and only if $f^{\leftarrow}$ is an isomorphism from (F, ∇) onto (E, Δ).
3. If f is an isomorphism from (E, Δ) onto (F, ∇) and g is an isomorphism from (F, ∇) onto $(G, \vee)$, then $g \circ f$ is an isomorphism from (E, Δ) onto $(G, \vee)$.

Theorem 6.2. Let f be an isomorphism from (E, Δ) onto (F, ∇).

1. Associativity of Δ is equivalent to associativity of ∇
2. Commutativity of Δ is equivalent to commutativity of ∇
3. An element $e \in E$ is an identity of Δ iff $f(e)$ is an identity of ∇
4. An element $y \in E$ is the inverse of $x \in E$ for Δ iff $f(y)$ is an inverse of $f(x)$ for ∇

Theorem 6.3. (Transplanting Theorem) Let (E, Δ) be an algebraic structure, and let $f : E \to F$ be a bijection. Then there is only one composition ∇ on F such that f is an isomorphism from (E, Δ) onto (F, ∇), and it is defined as $x \nabla y = f(f^{\leftarrow}(x) \Delta f^{\leftarrow}(y))$ for all $x, y \in F$.
Exercises for this section were skipped.