Introduction to Elementary Topology

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This shall be a brief introduction to elementary topology of the point-set variety, which parallels but does not coincide with the introduction to topology given in "Modern Analysis and Topology" by Howes. There will be more substance here than is presented by Howes, since an introduction to elementary topology is the chief matter of these notes. For a more thorough introduction to topology I recommend Topology by Munkres and perhaps General Topology by Kelley. Though the latter is older, I would not consider it outdated.

Letting X be a set, we define τ to be a subset of X such that it satisfies the following conditions:

1. $\varnothing \in \tau$
2. $X \in \tau$
3. For any family $(a_i)_{i\in I}$ where $a_i \in \tau$, $\bigcup\limits_{i\in I} a_i \in \tau$
4. For all $A, B \in \tau$, $A \cap B \in \tau$

The importance of the usage of families for condition 3 versus 4 is that 3 applies to non-finite families whereas the intersection of a non-finite family of open sets is not necessarily open, so 4 is expressed in such a way that it only implies that such holds for finite families of cardinality larger than 1 (for cardinality 0 it is supplemented by condition 2 and for cardinality 1 it is trivial). Really these four conditions can be shown to be equivalent to the following.

1 & 3. For any family $(a_i)_{i\in I}$ where $a_i \in \tau$, $\bigcup\limits_{i\in I} a_i \in \tau$

2 & 4. For any family $(a_i)_{i\in I}$ where $a_i \in \tau$ and $\operatorname{card}(I) < \operatorname{card}(\mathbb{N})$ (e.g. I is finite), $\bigcap\limits_{i\in I} a_i \in \tau$

Here condition 1 implies condition 1 of

We call X the set of points and τ the set of open sets, and (X, τ) is a topological space. τ is said to be a topology on X, where X is topologized by τ. We are given that τ is the set of open subsets, so a set $A \subseteq X$ is open iff $A \in \tau$. It would be natural to assume that we would say that A is closed iff $A \notin \tau$, but this is not so. To understand this we will formally define closed sets.

Definition 0.1 *A set A is said to be closed if its complement, $X \setminus A$, is an open set*

From the conditions of a topological space, we have that $X \in \tau$ and $X \setminus X = \varnothing \in \tau$, so we have that X is both open and closed, which is termed clopen. A similar argument can be made for $\varnothing$. For other examples of clopen sets, you don't have to look far. For the standard topology on $\mathbb{R}$, any interval of the form $(a, b]$ or $[a, b)$ is clopen.

Terminology 0.1 *For topologies σ and τ on a set X, if $\sigma \subset \tau$ then we say that σ is coarser than τ and that τ is finer than σ.*

Definition 0.2 A basis of a topology τ is a subcollection β of τ such that for all $A \in \tau$, $A = \bigcup_{i \in I} a_i$ where $(a_i \in \beta)_{i \in I}$ (e.g. all open sets can be constructed as the union of a family of sets in the basis).

The members of β are called **basic open sets**. A **sub-basis** for a topology τ in X is a subcollection σ of τ such that the finite intersections of members of σ form a basis for τ. In other words, for each $U \in \tau$ and $p \in U$ there are finitely members of σ, say $V_1, \ldots, V_n$ such that $p \in V_1 \cap \ldots \cap V_n \in U$.