Numerical Differentiation

Cody Maxwell

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1 Introduction

The conventional definition of a derivative is:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

if the limit exists, which is simply taking the slope of a secant line connecting two points are taking the limit as the two points approach each other.

However, a natural question would be, for a given h how close is the approximate value to the "true" value of the derivative. That is, can we find a bound on the error? If we apply Taylor polynomials with a remainder term, assuming the required continuity for f, we get:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{f\left(x\right)+hf'\left(x\right)+\frac{h^2}{2}f''\left(\xi\right)-f(x)}{h}\\ &= \frac{hf'\left(x\right)+\frac{h^2}{2}f''\left(\xi\right)}{h}\\ &= f'\left(x\right)+\frac{h}{2}f''\left(\xi\right)\end{aligned}$$

So we can say that:

$$E(h)=\left|f'(x)-\frac{f(x+h)-f(x)}{h}\right|=\left|\frac{h}{2}f''\left(\xi\right)\right|=\frac{h}{2}\left|f''(\xi)\right|$$

for some $\xi\in[x,x+h]$.

A method whose error depends linearly on the step size h is usually called a linear or order 1 method. And any function f where $f''=0\ \ \forall\xi\in[x,x+h]$, this method is exact, that is all linear functions. Of course this isn't a practical way to calculate the error bound, the main purpose of the bound is to show that $\lim_{h\to 0}E(h)=0$. In other words, the approximation works as intended.

But, this is only a linearly accurate method and is almost never used in practice. There are many other methods that are better than the one mentioned above. Before looking at these better methods, we should look at an alternative way of deriving methods of this type. The derivation extends naturally to those approximations of higher order.

2 Derivations

Consider the proposed equality:

$$f'(x) = Af(x+h) + Bf(x) + \tau$$

where A, B and τ are functions of h. τ is the symbol usually used for truncation error or the error resulting from truncated the Taylor polynomial. Using the same Taylor polynomial $f(x) + hf'(x) + \frac{h^2}{2}f''(\xi)$ we get

$$f'(x) = A\left(f(x) + hf'(x) + \frac{h^2}{2}f''(\xi)\right) + Bf(x) + \tau$$

And by rearranging the terms we get

$$(A+B)f(x) + (Ah-1)f'(x) + \left(A\frac{h^2}{2}f''(\xi) + \tau\right) = 0$$

This yields the system of equations"

$$\begin{aligned} A + B &= 0 \\ Ah &= 1 \end{aligned}$$

And the associated equation for the truncation error:

$$A\frac{h^2}{2}f''(\xi) + \tau = 0$$

Solving the equations tells us what we already know which is that

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2}f''(\xi)$$

but with one key difference, this derivation can be used to find more general methods.

With that derivation out of the way we can derive other methods.

Consider

$$f'(x) = Af(x+h) + Bf(x) + Cf(x-h) + \tau$$

Using the same process from earlier we get

$$f'(x) = A\left(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f''(\xi)\right) + Bf(x) + C\left(f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f''(\xi)\right)$$

And rearrange the terms into

$$(A+B+C)f(x) + (Ah - 1 - Ch)f'(x) + \left(A\frac{h^2}{2} + C\frac{h^2}{2}\right)f''(x) + \left(A\frac{h^3}{6}f'''(\xi) - C\frac{h^3}{6}f'''(\xi) + \tau\right) = 0$$

This is equivalent to the system of equations:

$$\begin{aligned} A + B + C &= 0 \\ Ah - Ch &= 1 \\ A\frac{h^2}{2} + C\frac{h^2}{2} &= 0 \end{aligned}$$

And truncation error equation:

$$A\frac{h^3}{6}f'''(\xi) - C\frac{h^3}{6}f'''(\xi) + \tau = 0$$

This tells us that the approximation scheme with error terms looks like:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6}f'''(\xi)$$

3 More General

$$f^n(x_0) = \sum_{i=1}^{k} c_i f(x_0 + h_i) + \tau$$

$$f^n(x_0) = c_1 f(x_0 + h_1) + c_2 f(x_0 + h_2) + c_3 f(x_0 + h_3) + \cdots + c_k f(x_0 + h_k) + \tau$$

$$\begin{aligned} f^n(x_0) = {} & c_1 f(x_0 + h_1) \\ & + c_2 f(x_0 + h_2) \\ & + c_3 f(x_0 + h_3) \\ & \cdots \\ & + c_k f(x_0 + h_k) \\ & + \tau \end{aligned}$$

$$\begin{aligned} f^n(x_0) = {} & c_1 \left(f\left(x_0\right) + h_1 f'\left(x_0\right) + \frac{h_1^2}{2} f''\left(x_0\right) + \frac{h_1^3}{6} f'''\left(x_0\right) + \cdots + \frac{h_1^n}{n!} f^n\left(x_0\right) + \cdots + \frac{h_1^k}{k!} f^k\left(\xi\right) \right) \\ & + c_2 \left(f\left(x_0\right) + h_2 f'\left(x_0\right) + \frac{h_2^2}{2} f''\left(x_0\right) + \frac{h_2^3}{6} f'''\left(x_0\right) + \cdots + \frac{h_2^n}{n!} f^n\left(x_0\right) + \cdots + \frac{h_2^k}{k!} f^k\left(\xi\right) \right) \\ & + c_3 \left(f\left(x_0\right) + h_3 f'\left(x_0\right) + \frac{h_3^2}{2} f''\left(x_0\right) + \frac{h_3^3}{6} f'''\left(x_0\right) + \cdots + \frac{h_3^n}{n!} f^n\left(x_0\right) + \cdots + \frac{h_3^k}{k!} f^k\left(\xi\right) \right) \\ & \cdots \\ & + c_k \left(f\left(x_0\right) + h_k f'\left(x_0\right) + \frac{h_k^2}{2} f''\left(x_0\right) + \frac{h_k^3}{6} f'''\left(x_0\right) + \cdots + \frac{h_k^n}{n!} f^n\left(x_0\right) + \cdots + \frac{h_k^k}{k!} f^k\left(\xi\right) \right) \\ & + \tau \end{aligned}$$

Note: $k \geq n + 1$

$$\begin{aligned} f^n(x_0) = {} & \left(c_1 + c_2 + c_3 + \cdots + c_n + \cdots + c_k\right) f\left(x_0\right) + \left(c_1 h_1 + c_2 h_2 + c_3 h_3 + \cdots + c_n h_n + \cdots + c_k h_k\right) f'\left(x_0\right) \\ & + \frac{1}{2} \left(c_1 h_1^2 + c_2 h_2^2 + c_3 h_3^2 + \cdots + c_n h_n^2 + \cdots + c_k h_k^2\right) f''\left(x_0\right) \\ & + \frac{1}{6} \left(c_1 h_1^3 + c_2 h_2^3 + c_3 h_3^3 + \cdots + c_n h_n^3 + \cdots + c_k h_k^3\right) f'''\left(x_0\right) \\ & + \cdots \\ & + \frac{1}{n!} \left(c_1 h_1^n + c_2 h_2^n + c_3 h_3^n + \cdots + c_n h_n^n + \cdots + c_k h_k^n\right) f^n\left(x_0\right) \\ & + \cdots \\ & + \frac{1}{k!} \left(c_1 h_1^k + c_2 h_2^k + c_3 h_3^k + \cdots + c_k h_k^k\right) f^k\left(\xi\right) \\ & + \tau \end{aligned}$$

$$
\begin{aligned}
c_1 + c_2 + c_3 + \cdots + c_n + \cdots + c_k &= 0 \\
c_1 h_1 + c_2 h_2 + c_3 h_3 + \cdots + c_n h_n + \cdots + c_k h_k &= 0 \\
\frac{1}{2}\left(c_1 h_1^2 + c_2 h_2^2 + c_3 h_3^2 + \cdots + c_n h_n^2 + \cdots + c_k h_k^2\right) &= 0 \\
\frac{1}{6}\left(c_1 h_1^3 + c_2 h_2^3 + c_3 h_3^3 + \cdots + c_n h_n^3 + \cdots + c_k h_k^3\right) &= 0 \\
&\cdots \\
\frac{1}{n!}\left(c_1 h_1^n + c_2 h_2^n + c_3 h_3^n + \cdots + c_n h_n^k + \cdots + c_k h_k^n\right) &= 1 \\
&\cdots \\
\frac{1}{k!}\left(c_1 h_1^k + c_2 h_2^k + c_3 h_3^k + \cdots + c_n h_n^k + \cdots + c_k h_k^k\right) &= -\tau
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\
h_1 & h_2 & h_3 & \cdots & h_n & \cdots & h_k \\
h_1^2 & h_2^2 & h_3^2 & \cdots & h_n^2 & \cdots & h_k^2 \\
h_1^3 & h_2^3 & h_3^3 & \cdots & h_n^3 & \cdots & h_k^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h_1^n & h_2^n & h_3^n & \cdots & h_n^n & \cdots & h_k^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h_1^{k-1} & h_2^{k-1} & h_3^{k-1} & \cdots & h_n^{k-1} & \cdots & h_k^{k-1}
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_n \\ \vdots \\ c_k \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ n! \\ \vdots \\ 0 \end{bmatrix}
$$

$$
c_1 h_1^k + c_2 h_2^k + c_3 h_3^k + \cdots + c_n h_n^k + \cdots + c_k h_k^k = -k!\tau
$$

For the method:

$$
f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\xi)
$$

$k = 3, n = 1, h_1 = 1, h_2 = 0, h_3 = -1$. The system is:

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

And the truncation error:

$$
\frac{h^3}{6}\left(c_1 - c_3\right) = -\tau
$$

This agrees with what was previously shown. And with this general representation, called a Vandermonde Matrix, we can try to find some results. For example the statement $k \geq n + 1$ makes more sense now that we can see that if that were not the case, then the linear system would have more variables than equations and would not have a unique solution.

And since this is a special matrix, we can see that

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\
h_1 & h_2 & h_3 & \cdots & h_n & \cdots & h_k \\
h_1^2 & h_2^2 & h_3^2 & \cdots & h_n^2 & \cdots & h_k^2 \\
h_1^3 & h_2^3 & h_3^3 & \cdots & h_n^3 & \cdots & h_k^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h_1^n & h_2^n & h_3^n & \cdots & h_n^n & \cdots & h_k^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h_1^{k-1} & h_2^{k-1} & h_3^{k-1} & \cdots & h_n^{k-1} & \cdots & h_k^{k-1}
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ \vdots \\ \vdots \\ c_k \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

has only the trivial solution, since the rows are independent.
That is, if $m \in \{1, ..., k-1\}$ and
$c_1 + c_2 h_m + c_3 h_m^2 + \cdots + c_n h_m^n + \cdots + c_k h_m^{k-1} = 0$ and for every $i, j \in \{1, ..., k-1\}$ if $i \neq j$ then $h_i \neq h_j$. This means that our polynomial of degree at most $k-1$ has k roots, which by the fundamental theorem of algebra, the polynomial must be zero, therefore, $c_1 = c_2, c_3 = \cdots = c_k = 0$. Therefore, given for every $i, j \in \{1, ..., k-1\}$ if $i \neq j$ then $h_i \neq h_j$ and $k \geq n+1$, the system for the derivative approximation has a unique solution.

To reiterate, an approximation scheme for derivatives of the form:

$$f^n(x_0) = \sum_{i=1}^{k} c_i f(x_0 + h_i) + \tau$$

$$f^n(x_0) = c_1 f(x_0 + h_1) + c_2 f(x_0 + h_2) + c_3 f(x_0 + h_3) + \cdots + c_k f(x_0 + h_k) + \tau$$

exists, provided $k \geq n+1$ and for every $i, j \in \{1, ..., k-1\}$ if $i \neq j$ then $h_i \neq h_j$, and the coefficients satisfy the linear system:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ h_1 & h_2 & h_3 & \cdots & h_n & \cdots & h_k \\ h_1^2 & h_2^2 & h_3^2 & \cdots & h_n^2 & \cdots & h_k^2 \\ h_1^3 & h_2^3 & h_3^3 & \cdots & h_n^3 & \cdots & h_k^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_1^n & h_2^n & h_3^n & \cdots & h_n^n & \cdots & h_k^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_1^{k-1} & h_2^{k-1} & h_3^{k-1} & \cdots & h_n^{k-1} & \cdots & h_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_n \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ n! \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1 h_1^k + c_2 h_2^k + c_3 h_3^k + \cdots + c_n h_n^k + \cdots + c_k h_k^k = -k! \tau$$

From this point, the required coefficients for the desired approximation can be systematically calculated directly or using an iterative method. In practice, the most popular methods are the three point centered differences for the first and second derivatives.

4 Richardson Extrapolation

The motivation for Richardson Extrapolation is, given different the approximations for different methods, can you use them to create an additional approximation that is better than the ones you started with.
Consider the method:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\xi)$$

Now suppose we have the two approximations $D_h(x_0)$ and $D_{h/2}(x_0)$, where the are calculate with the spacings h and $\frac{h}{2}$ at x_0 respectively.
We also know that

$$f'(x_0) = D_h(x_0) - \frac{h^2}{6} f'''(\xi)$$

$$f'(x_0) = D_{h/2}(x_0) - \frac{h^2}{24} f'''(\xi)$$

Since $D_h(x_0)$ and $D_{h/2}(x_0)$ are fixed values, we can use these two equations to solve for $f'(x_0)$ in terms of $D_h(x_0)$ and $D_{h/2}(x_0)$. The result is:

$$f'(x_0) = D_{h/2}(x_0) + \frac{D_{h/2}(x_0) - D_h(x_0)}{3}$$

It is worth noting that the resulting approximation scheme is third order whereas the two two methods that its based on were order 2. So given two or more approximations of a derivative with known schemes and spacing we can create a linear combination that is generally even more accurate.

5 Multi-variable

The above statement about ordinary derivatives also clearly holds for partial derivatives, its just a change in notation.
Consider the mixed partial derivative

$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y}.$$

For the sake of notation we let $f(x,y) = f_{i,j}$, $f(x+h, y+k) = f_{i+1,j+1}$, etc. We will also use big Oh notation, that is if the error is proportional to h we write $+O(h)$ on the method. So we have:

$$f_x = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + O(h^2)$$

$$f_y = \frac{f_{i,j+1} - f_{i,j-1}}{2k} + O(k^2)$$

And

$$f_{xy} \approx \frac{(f_y)_{i+1,j} - (f_y)_{i-1,j}}{2h}$$

$$f_{xy} \approx \frac{\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2k} - \frac{f_{i-1,j+1} - f_{i-1,j-1}}{2k}}{2h}$$

$$f_{xy} \approx \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$$

6 Finding Order

For approximating derivatives its helpful to know the order of the method, but finding it for any given method can be difficult or time consuming for an arbitrary method. Using Taylor Expansions can help find the entire error term, but if we only want the order there is a simpler way.

Starting with ordinary derivative schemes like the order 2 approximation:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

We know it is order 2 from a previous derivation, but an interesting fact is that its exact for quadratics. This trivially follows since its error term is proportional to the functions third derivative, but suppose we don't know that in advance.

One could start to see how the approximation works with some test functions that we already have simple derivative formulas for, like polynomials. For a linear function, $f(x) = ax + b$, the formula gives:

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{a(x+h) + b - (a(x-h) + b)}{2h} = \frac{(a-a)x + 2ah + (b-a)}{2h} = \frac{2ah}{2h} = a,$$

which is the exact value. One could also use the formula

$$\frac{f(x+h) - f(x)}{h}$$

also gives the exact value of the derivative. But what about a quadratic? If $f(x) = ax^2 + bx + c$, the formulas give:

$$\frac{a(x+h)^2 + b(x+h) + c - a(x-h)^2 - b(x-h) - c}{2h} = \frac{a[(x+h)^2 - (x-h)^2] + 2bh}{2h} = \frac{a(2x)(2h) + 2bh}{2h} = 2ax+b$$

and

$$\frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h} = \frac{a((x+h)^2 - x^2) + bh}{h} = \frac{a(2x+h)h + bh}{h} = 2ax + b + ah.$$

The first one gets the derivative exact again, but the second formula has this extra 'ah' term sticking around. This is the motivation for the "order" of a method, its a quality of the error term that says how fast it goes to zero as h tends to zero. This can be given as a limit of ratios directly for the polynomial we used:

$$\lim_{h\to 0} \frac{\left|f'(x) - \frac{f(x+h)-f(x)}{h}\right|}{h} = \lim_{h\to 0} \frac{\left|2ax + b - (2ax + b + ah)\right|}{h} = \lim_{h\to 0} \frac{\left|-ah\right|}{h} = a$$

This is essentially saying that the error is going to zero at the rate of h or linearly. This is one reason why we say the the approximation is order h. To see the same of the method we said was order 2, we need to look at a cubic polynomial. And since every method we're going to look at is linear in the same sense as the derivatives where $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$, we will use $f(x) = x^3$ as the next test function. So applying the method yields:

$$\begin{aligned} f'(x) \approx \frac{(x+h)^3 - (x-h)^3}{2h} &= \frac{2h((x+h)^2 + (x+h)(x-h)) + (x-h)^2)}{2h} = (x+h)^2 + (x+h)(x-h)) + (x-h)^2 \\ &= x^2 + 2xh + h^2 + x^2 - h^2 + x^2 - 2xh + h^2 = 3x^2 + h^2 \\ &= 3x^2 + h^2 \end{aligned}$$

Here we can see that, for the test function:

$$\lim_{h\to 0} \frac{\left|f'(x) - \frac{f(x+h)-f(x-h)}{2h}\right|}{h} = \lim_{h\to 0} \frac{\left|3x^2 - 3x^2 - h^2\right|}{h} = \lim_{h\to 0} h = 0,$$

but also

$$\lim_{h\to 0} \frac{\left|f'(x) - \frac{f(x+h)-f(x-h)}{2h}\right|}{h^2} = 1.$$

Again, this is the motivation for the notion of order, the rate at which the error for the above difference quotient goes to zero as fast as h^2. This means, as we said before, the scheme is order 2 or quadratic.

More generally for some derivative approximation $D_h(x)$, we say the method is order α where α is the largest number such that the following limit exists and is finite.

$$\lim_{h\to 0} \frac{\left|f'(x) - D_h(x)\right|}{h^\alpha}$$

Also this definition does not necessarily have to be only for derivative schemes. It can just as easily be modified for convergence sequences or differential equations.

7 Finding Order - Multi-variable

In the case of a single variable, we use polynomials of the variable to measure an ordinary derivatives schemes order. The same is true for partial derivatives, in particular, mixed partial derivatives, since partial derivatives like $\frac{\partial^2 f}{\partial x^2}$ are essentially the same as an ordinary derivative due to the other variables being held constant. However, the polynomials in more than one variable has a different notion of degree than single variable polynomials. Degree for these polynomials is the highest "total" degree of the polynomial. For example, $x^2 + y$ is degree 2, but so is $xy + 1$.

This is relevant since when derivative schemes are chosen it is common to choose different uniform spacing for your x-axis, y-axis, etc. This why in the multi-variable derivative approximations we have h and k. Since we want our test functions to have all the possible terms of a particular order, an easy way to get all possible combinations of terms with a particular degree, call it n, is to use:

$$f(x, y) = (x + y)^n$$

With this we can analyze our formula for $\frac{\partial^2 f}{\partial x \partial y}$, recall that:

$$\frac{\partial^2 f}{\partial x \partial y} \approx \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$$

plugging in $(x+y)^n$ for $f(x,y)$ yields:

$$\frac{(x+h+y+k)^n - (x+h+y-k)^n - (x-h+y+k)^n + (x-h+y-k)^n}{4hk}.$$

The following results for n = 1 to 5 are below:

n	Approximate Value
1	0
2	2
3	$6(x+y)$
4	$12(x+y)^2 + 4(h^2+k^2)$
5	$20(x+y)^3 + (20x+20y)(h^2+k^2)$

Here the error term appears to be proportional to $h^2 + k^2$. It looks like if $\frac{\partial^4 f}{\partial x^2 \partial y^2} = 0$, then the formula is exact. This is a simple differential equation that we can solve to test the hypothesis. The solution is:

$$xa(y) + yb(x) + c(x) + d(y).$$

Every function whose derivative was calculated exactly matches this form. However, unlike in the single variable case, we cannot describe the functions that the formula calculates exactly as all power functions of some simple type. Partial derivatives make this collection of functions much larger.

Even still we can essentially experiment and find the order of the approximation. In the same way as we have before, we say the approximation for $\frac{\partial^2 f}{\partial x \partial y}$ is $O(h^2 + k^2)$.

As a closing remark on the order of the error term for derivative approximation methods, their useful on non-polynomial functions vary. Every differentiable function is "locally linear", but the step size h or h and k required to get a sufficiently close approximation may be prohibitively small. The other higher order methods are better since a curve at a point is generally better approximated by a parabola than a line and even better by a cubic. Keep this in mind for functions that aren't quite as well behaved as the standard sums and products of $1, x, x^2, \cdots, e^x, \sin(x), \cos(x), etc.$

An example of a not well behaved function is

$$\frac{1}{\pi} \frac{M}{M^2 x^2 + 1},$$

which granted, it does look contrived, however it is essentially a Gaussian pulse, common in signal processing, where the parameter M dictates the amplitude. And for any M, its Taylor series converges for $|x| < \frac{1}{M}$. Its functions like this as well as others that make careful error analysis necessary. Below is the function for $M = 100\pi$.