Numerical Methods for First Order IVPs

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1 Introduction

We will we looking at numerical methods to approximate solutions to First Order IVPs, that is problems of the form

$$y' = f(t,y),\ a \le t \le b,\ y(t_0) = y_0.$$

Problems like this are called IVPs or Initial Value Problems. The ones that we are interested in for now are those made up of one ordinary derivative of some function y and $f(t,y)$ which is a function of both the independent and dependent variables. Numerical solutions to the aforementioned problem type are needed for much the same reason that numerical integration, zero finding and many others are needed. Either a solution in terms of elementary functions does not exist or an analytical solution is too troublesome or difficult to calculate. And, of course, another natural reason to why we need numerical solutions to IVPs, is because such problems frequently arise in the separate but related fields of physics and engineering. Both of which make extensive use of Calculus and numerical techniques of solving IVPs.

To illustrate the following numerical techniques, we will focus on a particular IVP which is

$$y' = -ty,\ 0 \le t \le 2,\ y(0) = 1.$$

And after using the numerical methods, we will compare the exact and numerical values at $t = 2$ with $n = 20, 40, 80, 160, 320$.

1.1 Applications - Physics

Before going directly to the numerical methods, we will first look at a couple of the applications of the numerical methods in Physics. To be specific, the applications will include typical examples of problems in Classical Mechanics.

Application 1 - Frictional Forces - The Ideal Case

$$v' = -\mu v,\ 0 \le t \le 10,\ v(0) = 2\text{m/s}$$

Application 2 - Frictional Forces - The More Realistic Case

$$v' = -\mu v + \nu v^3,\ 0 \le t \le 10,\ v(0) = 3.5\text{m/s}$$

In either case, the problem is that we are given a relationship between velocity, its derivative, and some constants that change given the medium that the object being modeled is travelling through. The difference in the two applications is that the first one is much simpler than the second. The is a trend that generally happens in physics. The more accurate your model, the more difficult or tedious it is to solve. With these examples specifically there do exist analytical solutions. However, for the purposes of modeling a given system, it is often more convenient to get a good approximation and use that instead.

2 Numerical Methods

Euler's Method:

An $O(h)$ method for solving IVPs using one point.

$w_{i+1} = w_i + hf(t_i, w_i),\ w_0 = y(t_0) = y_0$

Midpoint Method (RK2):

An $O(h^2)$ method for solving IVPs using one point.

$w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)),\ w_0 = y(t_0) = y_0$

Heun's Method:

An $O(h^2)$ method for solving IVPs using one point.

$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right),\ w_0 = y(t_0) = y_0$

Runge-Kutta Method of Order 4:

An $O(h^4)$ method for solving IVPs using one point.

$w_{i+1} = w_i + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right),\ w_0 = y(t_0) = y_0$

$K_1 = f(t_i, w_i)$

$K_2 = f(t_i + \frac{h}{2}, w_i + \frac{hK_1}{2})$

$K_3 = f(t_i + \frac{h}{2}, w_i + \frac{hK_2}{2})$

$K_4 = f(t_i + h, w_i + hK_3)$

Three-step Adams-Bashforth Predictor and Adams-Moulton Corrector Method:

A method using three points for solving IVPs. It requires an additional method to find w_1 and w_2.

$w_t = w_i + \frac{h}{12}\left(5f(t_{i-2}, w_{i-2}) - 16f(t_{i-1}, w_{i-1}) + 23f(t_i, w_i)\right),\ w_0 = y(t_0) = y_0, w_1, w_2$

$w_{i+1} = w_i + \frac{h}{24}\left(f(t_{i-2}, w_{i-2}) - 5f(t_{i-1}, w_{i-1}) + 19f(t_i, w_i) + 9f(t_{i+1}, w_t)\right)$

3 Numerical Results

3.1 Euler's Method

Exact Value: $y(2) = e^{-\frac{2^2}{2}} = 0.135335283237$.

Number of Time Steps	Approximate Value	Absolute Error
20	0.130399501820	4.935781416e-3
40	0.132979904856	2.355378381e-3
80	0.134183283678	1.151999558e-3
160	0.134765424541	5.69858696e-4
320	0.135051855338	2.83427898e-4

Exact Order of Accuracy is 1.

Estimated Order of Accuracy
1.067319664338422
1.031818677714121
1.015464029825690
1.007622457612335

We see that the estimated order of accuracy agrees with the theoretical order of accuracy.

3.2 Midpoint Method (RK2)

Exact Value: $y(2) = 0.135335283237$.

Number of Time Steps	Approximate Value	Absolute Error
20	0.135578109043	2.42825807e-4
40	0.135393889548	5.8606311e-5
80	0.135349660447	1.4377210e-5
160	0.135338842709	3.559473e-6
320	0.135336168723	8.85487e-7

Exact Order of Accuracy is 2.

Estimated Order of Accuracy
2.050793822007180
2.027272291430970
2.014048174691326
2.007120626116294

We see that the estimated order of accuracy agrees with the theoretical order of accuracy.

3.3 Heun's Method

Exact Value: $y(2) = 0.135335283237$.

Number of Time Steps	Approximate Value	Absolute Error
20	0.136317647528	9.82364291e-4
40	0.135570503958	2.35220722e-4
80	0.135392858327	5.7575090e-5
160	0.135349527438	1.4244201e-5
320	0.135338825853	3.542616e-6

Exact Order of Accuracy is 2.

Estimated Order of Accuracy
2.062242958205005
2.030498495087441
2.015070062371132
2.007487699962371

We see that the estimated order of accuracy agrees with the theoretical order of accuracy.

3.4 Runge-Kutta Method of Order 4

Exact Value: $y(2) = 0.135335283237$.

Number of Time Steps	Approximate Value	Absolute Error
20	0.135336623397	1.340160e-6
40	0.135335362669	7.9432e-8
80	0.135335288068	4.831e-9
160	0.135335283534	2.98e-10
320	0.135335283255	1.8e-11

Exact Order of Accuracy is 4.

Estimated Order of Accuracy
4.076541113228116
4.039326571157978
4.018937616969915
4.049243519019849

3.5 Three-step Predictor Corrector

Exact Value: $y(2) = 0.135335283237$.

Number of Time Steps	Approximate Value	Absolute Error
20	0.135318348360	1.6934877e-5
40	0.135334311745	9.71491e-7
80	0.135335225180	5.8057e-8
160	0.135335279690	3.547e-9
320	0.135335283018	2.19e-9

Exact Order of Accuracy is 4

Estimated Order of Accuracy
4.123653067595164
4.064658700476691
4.032798789611145
4.017596555334015

3.6 Graph of Errors

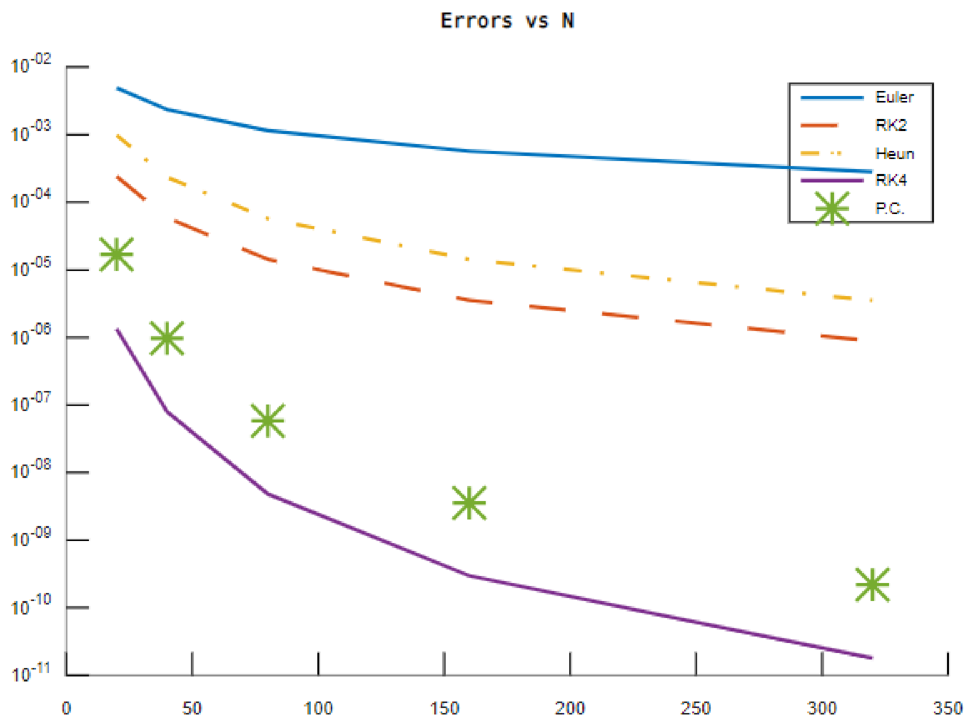

Figure 1: Errors against N

4 Conclusion and Discussion

In conclusion, it is obvious that numerical solutions to IVPs are needed since not only are the interesting to look at how their behavior changes as individual parts of the problem change the methods are also practical since IVPs show up in Physics and Engineering frequently. The two related problems in the applications section as well as several others, sometimes numerical solutions are the easiest way to get a workable solution. However, in the case of the problems looked at in the application section, there should be fairly straightforward analytical solutions. It is in the case where there is not an easy or obvious solution to the IVPs that the numerical methods discussed in this paper are especially useful. Earlier we looked at a specific IVP, it was

$$y' = -ty, \ 0 \leq t \leq 2, \, y(0) = 1.$$

we looked at approximations of its solution with the methods that we discussed and compared the error. It was an example problem meant to illustrate the error and order of convergence of the methods. Since the error building for large t, we looked at the largest t which is $t = 2$ and looked at the error and that point. This is probably one of the most applicable chapters of the course and the most interesting.