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Traveling Salesman Problems with Profits

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Traveling salesman problems with profits (TSPs with profits) are a generalization of the traveling salesman problem (TSP), where it is not necessary to visit all vertices. A profit is associated with each vertex. The overall goal is the simultaneous optimization of the collected profit and the travel costs. These two optimization criteria appear either in the objective function or as a constraint. In this paper, a classification of TSPs with profits is proposed, and the existing literature is surveyed. Different classes of applications, modeling approaches, and exact or heuristic solution techniques are identified and compared. Conclusions emphasize the interest of this class of problems, with respect to applications as well as theoretical results.

Key words: vehicle routing; traveling salesman problem; selective TSP; weighted girth problem; prize-collecting TSP
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1. Introduction

The traveling salesman problem (TSP) and the vehicle routing problem (VRP) are among the most widely studied combinatorial optimization problems. Both problems, as well as their numerous extensions, deal with optimally visiting customers from a central depot. A very large number of papers and books deal with these problems (Gutin and Punnen 2002; Toth and Vigo 2001). Two usual characteristics of these problems are that every customer has to be serviced and that, consequently, no value is associated to the service. However, some variant problems propose to select customers depending on a profit value that is gained when the visit occurs. This feature gives rise to a number of problems that we gather together under the name of traveling salesman problems with profits (TSPs with profits), when a single vehicle is involved. More general problems in which several vehicles might be involved are called routing problems with profits. In this article we review the existing literature on these problems, with a focus on TSPs with profits, which have been more widely studied.

TSPs with profits may be seen as bicriteria TSPs with two opposite objectives, one pushing the salesman to travel (to collect profit) and the other inciting him to minimize travel costs (with the right to drop vertices). Viewed in this light, solving TSPs with profits should result in finding a noninferior solution set, i.e., a set of feasible solutions such that neither objective can be improved without deteriorating the other. Actually, most researchers interested in these problems address them in single-criterion versions. Thus, either the two objectives are weighted and combined linearly, or one of the objectives is constrained with a specified bound value. To our knowledge, the only attempts to solve the true bicriteria problem are by Keller (1985) and Keller and Goodchild (1988), who call it the multiobjective vending problem, but their approaches consist of sequentially solving single-criterion versions of the problem. One should note, however, that many articles deal with other kinds of multicriteria TSPs. Interested readers are referred to Ehrgott (2000) for a recent survey.

This paper is organized as follows. Section 2 defines the class of TSPs with profits, details the different generic problems that make up the class, and presents some complexity results and properties. Sections 3–7 focus on these generic problems. Section 3 describes their main applications up to now. Section 4 is devoted to formulations and structural properties. Sections 5, 6, and 7 respectively address exact, heuristic, and metaheuristic solution procedures. Sections 8 and 9 complete the survey, respectively with single-vehicle and multivehicle variant problems. Section 10 concludes the paper.

2. Definition and Properties

Let $G = (V, A)$ be a graph where $V = \{v_1, \ldots, v_n\}$ is a set of n vertices and A is a set of arcs (directed TSPs with profits) or edges (undirected TSPs with profits). Let a profit p_i be associated with each vertex $v_i \in V$ (with $p_1 = 0$) and a distance c_{ij} be associated with each arc or edge $(v_i, v_j) \in A$. In the following, we interpret vertices as customers and distances as travel costs. Vertex v_1 is interpreted as a depot. TSPs with profits consist of determining an elementary circuit (i.e., a circuit such that each vertex is visited at most once) that includes vertex v_1, with a concern in both the collected profit and the travel costs.

This definition of TSPs with profits does not enforce any condition on distances. However, unless the opposite is explicitly stated, we will assume that distances are positive and satisfy the triangle inequality. We will also assume that the graph is complete. Actually, our definition of TSPs with profits implies that the tour is elementary, which essentially means that profit is available only once at each vertex. Under this assumption, it is possible to complete the arc set, if necessary, by introducing new arcs corresponding to shortest paths between nonadjacent vertices. Note also that if some vertices (other than the depot) have a nonpositive profit, they can be excluded from the vertex set if the above assumptions are satisfied.

Three generic problems make up TSPs with profits, depending on the way the two objectives are addressed:

(1) Both objectives are combined in the objective function; the aim is to find a circuit that minimizes travel costs minus collected profit.

(2) The travel cost objective is stated as a constraint; the aim is to find a circuit that maximizes collected profit such that travel costs do not exceed a preset value $c_{\max}$.

(3) The profit objective is stated as a constraint; the aim is to find a circuit that minimizes travel costs and whose collected profit is not smaller than a preset value $p_{\min}$.

These problems have appeared under several names in the literature. In this article we will adopt the taxonomy that follows.

Problem 1 (with respect to the above numbering) has been defined as the profitable tour problem (PTP) by Dell'Amico et al. (1995). While this problem has seldom been studied as such in the literature, it is often encountered (with some additional constraints) as a subproblem in solution schemes based on column generation for a variety of routing problems. In such a subproblem, the profit values collected at vertices typically correspond to the dual values associated with the vertex-covering constraints of the master problem. In most cases, it is then solved as an elementary shortest path problem between two copies of the depot.

Problem 2 will be called the orienteering problem (OP), which is the name under which it is usually encountered in the literature. Note that OP is generally defined in terms of a path to be found between specified points rather than a circuit. This slight difference, however, is innocuous: In many applications, the two endpoints of the desired path do indeed coincide; in other cases, adding a dummy arc from the destination to the origin of the path makes the two problems equivalent. Other names under which the OP can be found are the selective TSP (Laporte and Martello 1990) and the maximum collection problem (Kataoka and Morito 1988).

Problem 3 will be called the prize-collecting TSP (PCTSP). It should be noted that in the original definition of the PCTSP by Balas (1989), penalty terms for unvisited vertices are also added to the objective function. Most of the authors who have worked on this problem, however, have null penalty terms in their applications. Problem 3 has also been studied as the quota TSP by Awerbuch et al. (1998).

Finally, our definition of TSPs with profits implies that the tour start and finish at a depot. Several problems have similar features except that any circuit is searched for in the graph. While it would be easy to enforce the visit of any particular vertex (just one would add a large value to its profit), the opposite is not true: Finding any circuit in a graph is a more general problem than finding a circuit linked to a given depot. In particular, it is not possible to simply introduce a dummy vertex representing the depot. In our definition and throughout this article, we restrict ourselves to problems with a depot. It should be noted, however, that one may solve a problem without a depot condition by repeatedly applying solution approaches devised for the case where the depot condition is enforced.

2.1. Complexity

It is intuitively clear that TSPs with profits belong to the class of NP-hard problems because they trivially belong to NP and because a TSP instance can be stated as a TSP with profits instance by defining very large profits on vertices. It needs to be formally proven, though.

For the PTP and the PCTSP, it is obvious that the TSP is a particular case of the problem (Fichetti and Toth 1988). For the OP, Golden et al. (1987) and Laporte and Martello (1990) have proposed proofs based on simple reductions.

Golden et al. (1987) use the recognition version of the TSP: Given a collection of vertices, is there a tour of length less than $c_{\max}$ through all the vertices? They consider any instance of the recognition version of the TSP, assign a profit of 1 to every vertex, set the time limit to $c_{\max}$, and choose a vertex to be the depot. They then claim that solving the OP for this instance will answer the question. Indeed, if the total score is equal to the number of vertices, the answer is yes; otherwise it is no. Laporte and Martello (1990) propose a similar proof, using the Hamiltonian circuit problem.

Note that some special instances are proved to be polynomial. Balas (1999) introduces ordering constraints for which the PCTSP becomes polynomially solvable. In the same way, Kabadi and Punnen (1996) extend results on polynomially solvable cases of the TSP to the PCTSP.

2.2. Transformation of the PTP into an Asymmetric TSP

An interesting property of the PTP is that its solution amounts to solving an instance of an asymmetric TSP on a transformed graph having $2n+1$ vertices (see Figure 1). Hence, this transformation scheme, proposed by Volgenant and Jonker (1987), allows one to solve the PTP with every solution procedure developed for the asymmetric TSP.

Let us call $G' = (V', A')$ the transformed graph. The depot is duplicated in a vertex v_{n+1}, with a straightforward update of the arc set. Vertices $v_{n+2}, \ldots, v_{2n+1}$ are added, which respectively duplicate vertices $v_2, \ldots, v_{n+1}$. Thus, $V' = V \cup \{v_{n+1}, \ldots, v_{2n+1}\}$. The existing arcs are considered with the same cost coefficients. Other arcs are added:

- $\{(v_{n+i}, v_i), 2 \le i \le n\}$, with cost coefficients set to p_i
- $\{(v_i, v_{n+i+1}), 2 \le i \le n\}$, with cost coefficients set to 0
- $\{(v_{n+i}, v_{n+i+1}), 2 \le i \le n\}$, with cost coefficients set to 0
- (v_{n+1}, v_{n+2}) and (v_{2n+1}, v_1), with cost coefficients set to 0.

In this graph, noncollected profit on a vertex v_i is seen as a penalty paid by visiting the vertex through arcs (v_{n+i}, v_i) and (v_i, v_{n+i+1}).

It must be noted that in this transformation, vertices $n+2$ to $2n$ have out-degrees equal to 2 and vertices $n+3$ to $2n+1$ have in-degrees equal to 2. As a consequence, vertices $n+2$ to $2n+1$ must belong to sequences of the form $\{v_{n+k} \to v_k \to v_{n+k+1}\}$ (in the case vertex v_k is not visited in the PTP tour) or $\{v_{n+k} \to v_{n+k+1}\}$ (in the case vertex v_k is visited in the PTP tour). In fact, every solution can be split into two parts: The first one corresponds to the "real" tour being performed and exists in a sequence of original vertices between vertices 1 and $n+1$; the second one accounts for the unvisited vertices that appear between duplicate vertices $n+2$ to $2n+1$.

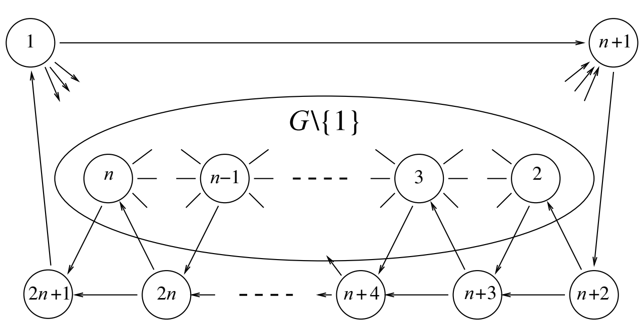

Figure 1 **Transformation of PTP into ATSP**

Solving the PTP thus amounts to solving the asymmetric TSP on a graph augmented with $n+1$ vertices and $3n-1$ arcs. Unfortunately, the structure of the instance is quite particular, and no computational results are provided by Volgenant and Jonker (1987) to assess the efficiency of the transformation. Note also that the transformation is not directly suitable for other TSPs with profits that introduce resource constraints.

3. Applications

TSPs with profits are encountered in many different situations. A first kind of applications is TSP applications where, for any logistic reason, it is not possible to satisfy all the demand. For instance, two standard applications of the classical TSP are the problem that actually faces a traveling salesman and the scheduling of jobs in a system where costs depend only on the consecutive jobs in the production sequence. In both cases, some constraints can enforce a selection. Gensch (1978) and Pekny et al. (1990) are interested in these problems in steel and chemical firm contexts, respectively.

A historical application of TSP with profits is orienteering events, introduced by Tsiligirides (1984). Orienteering is a competition that takes place in mountainous or forested areas. Competitors start from a control point and have to reach another control point within a prescribed time limit. Meantime, they can visit other control points and collect scores. The winner is the competitor ending with the maximum score. The optimal route is obtained by the solution of an OP.

Golden et al. (1984) propose applying the same modeling to a vehicle routing problem with an inventory component. A fleet of trucks must deliver fuel to a large number of customers on a daily basis. Each customer has a known tank capacity whose fuel level has to remain above a critical value. A first step of the solution procedure is to determine which customers to serve each day. A forecasted tank level for each customer results in a measure of emergency. Finding the set of selected customers then amounts to solving an OP.

Another historical application of TSPs with profits is the scheduling of daily operations of a steel rolling mill, as introduced by Balas and Martin (1985). The context of steel rolling mills gives rise to complex production scheduling problems. A rolling mill produces steel sheets from slabs by hot or cold rolling. From an inventory, schedulers have to choose a collection of slabs and order it so as to minimize some function of the sequence. It gives rise to a PCTSP with penalty terms in the objective function (i.e., a PCTSP as it was defined originally; see §2).

Some other applications of TSPs with profits can be encountered in the literature. Ramesh and Brown (1991) propose an application in control theory. Fischetti and Toth (1988) notice that TSPs with profits arise when a factory needs a given amount of product, which can be provided by a set of suppliers with given amounts and costs. For multivehicle extensions, examples of applications are the recruiting of athletes, fishery surveillance, aerial reconnaissance, or ship replenishment (see §9).

Finally, TSPs with profits sometimes appear as subproblems in solution procedures devoted to different kinds of problems. Göthe-Lundgren et al. (1995, 1996) address them in the context of vehicle routing cost-allocation problems. Being interested in a scheduling problem on m nonidentical machines with sequence dependent setup times, Helmberg (1999) faces a problem that he calls the m-cost asymmetric TSP. It consists of an asymmetric TSP with m salesmen, where the travel costs of these salesmen are distinct. He studies the one-machine subproblem, which turns out to be a TSP with profits. Finally, Noon et al. (1994) propose a heuristic procedure for the solution of the VRP, based on the iterative solution of TSPs with profits.

4. Integer Linear Programming Formulations and Structural Properties

Most of the results presented here originate from previous studies devoted to the TSP that have been adapted or extended to the context of TSPs with profits.

4.1. Formulations

4.1.1. Directed Case. We propose a possible formulation framework. Most of the formulations found in the literature are structured in accordance with this framework (e.g., Gensch 1978; Fischetti and Toth 1988; Balas 1989). We associate one binary variable x_{ij} to every arc $(v_i, v_j) \in A$, equal to 1 if and only if the corresponding arc is used in the solution, and one binary variable y_i to every vertex $v_i \in V$, equal to 1 if and only if the corresponding vertex is visited.

TSPs with profits all share a common set of constraints:

$$\sum_{v_j \in V\setminus\{v_i\}} x_{ij} = y_i \quad (v_i \in V), \tag{1}$$

$$\sum_{v_i \in V\setminus\{v_j\}} x_{ij} = y_j \quad (v_j \in V), \tag{2}$$

$$\text{subtour elimination constraints,} \tag{3}$$

$$y_1 = 1, \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad ((v_i, v_j) \in A), \tag{5}$$

$$y_i \in \{0, 1\} \quad (v_i \in V). \tag{6}$$

Constraints (1) and (2) are the so-called assignment constraints. Constraints (3) eliminate subtours that do not involve the depot. Different formulations for this last set of constraints are detailed in the next subsection. Depending on the problem, the objective function is different and a resource constraint may also have to be taken into account.

In the case of the PTP, the objective function is

$$\text{minimize} \sum_{(v_i, v_j) \in A} c_{ij} x_{ij} - \sum_{v_i \in V} p_i y_i, \tag{7}$$

subject to (1–6).

For the OP, the formulation is

$$\text{maximize} \sum_{v_i \in V} p_i y_i, \tag{8}$$

subject to (1–6) plus

$$\sum_{(v_i, v_j) \in A} c_{ij} x_{ij} \le c_{\max}. \tag{9}$$

For the PCTSP, the formulation is

$$\text{minimize} \sum_{(v_i, v_j) \in A} c_{ij} x_{ij}, \tag{10}$$

subject to (1–6) plus

$$\sum_{v_i \in V} p_i y_i \ge p_{\min}. \tag{11}$$

In the following, we refer to constraints of the form (9) as knapsack constraints and we refer to constraints of the form (11) as generalized covering constraints. For a structural study purpose, it is interesting to note that constraint (11) can be replaced by a knapsack constraint involving nonvisited vertices. Some slight changes on variable definition can also lead to similar formulations. For example, it is possible to remove variables y_i (e.g., Gensch 1978). Packing constraints then replace assignment constraints (1), and constraints (12) are added in place of constraints (2):

$$\sum_{v_i \in V} x_{ik} = \sum_{v_j \in V} x_{kj} \quad (v_k \in V). \tag{12}$$

At last, let us point out that a completely different formulation, involving permutation and flow variables, is proposed by Millar and Kiragu (1997) for the OP.

4.1.2. Undirected Case. Let us now consider the undirected case. For every vertex subset S, let $\delta(S)$ be the set of edges with one end in S and the other end in $V\backslash S$. To conform to usual notations, we use E to denote the set of edges within this section. The backbone of the model is then

$$\sum_{e\in\delta(\{v_i\})} x_e = 2y_i \quad (v_i \in V), \tag{13}$$

subtour elimination constraints, (14)

$$y_1 = 1, \tag{15}$$

$$x_e \in \{0, 1\} \quad (e \in E), \tag{16}$$

$$y_i \in \{0, 1\} \quad (v_i \in V). \tag{17}$$

We do not reformulate objective functions and additional constraints that are easy to determine from the directed formulations, for the PTP, the OP, and the PCTSP.

Note that this formulation forbids cycles of length 2. Fischetti et al. (1998) claim that one can assume, without loss of generality, that the optimal cycle contains at least three edges. Laporte and Martello (1990) prefer to widen the domain of variables x_e to $\{0, 1, 2\}$ for every $e \in E$ connected to the depot, in order to include cycles of length 2 in the model.

4.2. Polyhedral Study

A first important concern about TSP with profits polytopes is how to settle subtour elimination constraints (3). Let us begin with the directed case. Several formulations are described in the literature. They mostly are straightforward adaptations of existing formulations for the TSP, taking into account the presence of a subtour involving vertex v_1. Fischetti and Toth (1988) first propose—

$$\sum_{(v_i,v_j)\in\delta^-(S)} x_{ij} \geq y_k \quad (S \subset V\backslash\{v_1\}, v_k \in S) \tag{18}$$

or equivalently

$$\sum_{(v_i,v_j)\in\delta^+(S)} x_{ij} \geq y_k \quad (S \subset V\backslash\{v_1\}, v_k \in S), \tag{19}$$

where $\delta^-(S)$ is $\{(v_i, v_j) \in A\colon v_i \in V\backslash S, v_j \in S\}$ and $\delta^+(S)$ is $\{(v_i, v_j) \in A\colon v_i \in S, v_j \in V\backslash S\}$. These inequalities indicate that if vertex $v_k \in S$ is visited, an arc necessarily enters the set S and an arc necessarily leaves it. Using constraints (2), we know that

$$\sum_{v_j\in S} y_j = \sum_{\{v_i,v_j\}\subset S} x_{ij} + \sum_{(v_i,v_j)\in\delta^-(S)} x_{ij}.$$

Combined with (18), it is straightforward to establish that

$$\sum_{\{v_i,v_j\}\subset S} x_{ij} \leq \sum_{v_j\in S\backslash\{v_k\}} y_j \quad (S \subset V\backslash\{v_1\}, v_k \in S), \tag{20}$$

or the weaker inequalities

$$\sum_{\{v_i,v_j\}\subset S} x_{ij} \leq |S| - 1 \quad (S \subset V\backslash\{v_1\}). \tag{21}$$

By summing constraints (18) and (19), Laporte and Martello (1990) also deduce other weaker inequalities:

$$2\sum_{v_k\in S} y_k \leq |S|\left(\sum_{(v_i,v_j)\in\delta^-(S)} x_{ij} + \sum_{(v_i,v_j)\in\delta^+(S)} x_{ij}\right) \quad (S \subset V\backslash\{v_1\}, |S| \geq 2). \tag{22}$$

Balas (1989) studies more deeply subtour elimination constraints for TSPs with profits to reinforce these formulations. In particular, he determines several reinforcements and conditions under which they are facet defining. Note that Balas (1989) actually provides results for four polytopes: the PTP polytope, the PCTSP polytope, and the two related polytopes where the depot condition is removed.

In the undirected case, similar results may be found. For example, Bienstock et al. (1993) propose to formulate subtour elimination constraints as

$$\sum_{e\in\delta(S)} x_e \geq 2y_k \quad (S \subset V\backslash\{v_1\}, v_k \in S), \tag{23}$$

while Leifer and Rosenwein (1994) propose the following:

$$\sum_{e\subset S} x_e \leq |S| - 1 \quad (S \subset V\backslash\{v_1\}, 3 \leq |S| \leq n-2). \tag{24}$$

Finally, Kubo and Kasugai (1992) adapt to the undirected case some of the results proven by Balas (1989) in directed graphs.

Apart from these studies devoted to subtour elimination constraints, several papers report various results in connection with the polytopes of TSPs with profits (Balas 1989, 1995, 2002; Leifer and Rosenwein 1994; Gendreau et al. 1998a; Fischetti et al. 1998, 2002; Helmberg 1999). A full description of these results is, however, beyond the scope of this article.

It is important to remark that the polytopes of interest are included with other polytopes whose structural properties are highly relevant in this context. Indeed, most of the results in the articles mentioned above were first established in these other polytopes and then extended to the case of TSPs with profits. Hence, polytopes of TSPs with profits are included in the cycle polytope, i.e., the convex hull of the simple cycles of a graph, which has been widely studied (Coullard and Pulleyblank 1989; Kubo and Kasugai 1992; Bauer 1997; Helmberg 1999). The OP and the PCTSP polytopes are also included in the intersection of the cycle and the knapsack polytopes. Bauer et al. (1998) extend properties of both these polytopes to their intersection, in the undirected case.

5. Exact Solution Approaches

In this section, we present exact solution procedures for TSPs with profits. These approaches are branch-and-bound solution procedures that are often adapted from existing procedures devoted to the TSP, in order to take advantage or to take account of the new objective functions and constraints. Compared to the TSP, the novelties concern essentially the bounding schemes, which are thoroughly described below; branchings are not detailed.

5.1. The Assignment-Related Bound and Solution Procedures

A classical relaxation for the TSP consists in relaxing subtour elimination constraints (3). The resulting assignment problem can then be solved efficiently with a specialized algorithm. In the context of TSPs with profits, relaxing subtour elimination constraints is still an attractive approach. For this purpose, many authors (e.g., Gensch 1978; Fischetti and Toth 1988) propose changing the model to introduce nonvisited vertices as separate subtours of length 1 (self-loops), i.e., variables x_{ii} replace variables y_i (with $x_{ii} = 1 - y_i$). Constraints (1), (2), and (4)–(6) are then rewritten as

$$\sum_{v_j \in V} x_{ij} = 1 \quad (v_i \in V), \tag{25}$$

$$\sum_{v_i \in V} x_{ij} = 1 \quad (v_j \in V), \tag{26}$$

$$x_{11} = 0, \tag{27}$$

$$x_{ij} \in \{0, 1\} \quad ((v_i, v_j) \in A \cup \{(v_i, v_i), v_i \in V\}), \tag{28}$$

which also define an assignment problem. For the PTP, the solution of this assignment problem provides a bound that is discussed by Dell'Amico et al. (1995). However, TSPs with profits often involve a resource constraint, which is usually either a generalized covering constraint (PCTSP) or a knapsack constraint (OP). This resource constraint modifies the structure of the relaxation, which becomes a constrained assignment problem. Several approaches were developed to address this problem:

(1) Integrality constraints are relaxed and the linear program is solved with a standard algorithm. In fact, Fischetti and Toth (1988) and Pekny et al. (1990) prefer to compute the same bound by using a Lagrangian relaxation of the resource constraint and a specialized algorithm to solve the resulting assignment problem. We will subsequently refer to this bound as the LP bound.

(2) Gensch (1978) computes an optimal integer solution of the constrained assignment problem, which gives a bound tighter than the LP bound but which, of course, is more difficult to obtain. For this end, he relaxes the resource constraint in a Lagrangian fashion and solves the Lagrangian dual problem with a bisection method, but using the operator theory developed by Srinivasan and Thompson (1972a, 1972b) near the end to catch the optimal integer solution. In an equivalent attempt, Kataoka and Morito (1988) propose to solve optimally the relaxation with a branch-and-bound procedure. Note that this procedure is devoted to the solution of the OP and that it takes advantage of the special structure of the cost function.

(3) Another possibility is to compute the LP bound and strengthen it with valid inequalities (Leifer and Rosenwein 1994, Gendreau et al. 1998a, Fischetti et al. 1998).

All these bounding schemes permit very effective bounds for asymmetric TSPs with profits. As for the TSPs, they are less effective for symmetric instances and when the number of vertices is small. It is not surprising that the more efficient solution procedures are found in Gendreau et al. (1998a) and Fischetti et al. (1998), which quickly tighten the bounds with valid inequalities all along the search tree (in branch-and-cut procedures). Thus, these last two procedures permit one to solve symmetric instances of the OP involving up to 300 vertices in less than 3 hours on a Sun Sparc Station 1000 and instances involving up to 500 vertices in a few more hours on an HP Apollo 9000/720 computer, respectively. Other approaches permit one to solve instances of moderate size, especially when they are asymmetric. Note also that computation times are generally longer for intermediate values of the resource constraint, when both decision problems (selection and ordering of the vertices) are nontrivial.

5.2. Shortest Spanning 1-Tree Relaxation Style Bounds and Solution Procedures

The previous relaxation, based on the assignment substructure of TSPs with profits, is valid in both symmetric and asymmetric situations. However, its effectiveness decreases significantly in the first case (unless valid inequalities are added). A more adequate relaxation used for the symmetric TSP is to relax constraints (1) that enforce a single successor for each customer. The resulting shortest spanning 1-arborescence problem (shortest spanning arborescence plus the length of the remaining shortest arc incident to vertex v_1) can then be solved in polynomial time. In the context of TSPs with profits, this is still an attractive approach. For this purpose, Fischetti and Toth (1988) propose working in an augmented graph. A dummy vertex v_0 is introduced, to which nonvisited vertices are connected (arc variables x_{0i} replace variables y_i, with $x_{0i} = 1 - y_i$). Constraints (2)–(6) are then rewritten as

$$\sum_{v_i \in V \cup \{v_0\}} x_{ij} = 1 \quad (v_j \in V \cup \{v_0\}), \tag{29}$$

$$\text{subtour elimination constraints,} \tag{30}$$

$$x_{ij} \in \{0, 1\} \quad (v_i, v_j \in V \cup \{v_0\}), \tag{31}$$

which defines a shortest spanning 1-arborescence problem. This relaxation could be used for the computation of bounds for TSPs with profits. However, Fischetti and Toth (1988) point out that it would not be effective, since the subtree rooted at x_0 in the solution may have a depth greater than 1. They thus propose instead to relax constraint (1) in a Lagrangian fashion to obtain a tighter bound. Note that they are interested in the PCTSP and that they also consider a resource constraint that is relaxed in the same fashion. To improve the quality of the bound, they finally enforce an out-degree of 1 for vertex v_1, by adding a large value to the cost of every arc out of this vertex while computing the shortest spanning 1-arborescence. Ramesh et al. (1992) use a similar bounding scheme for the solution of the undirected OP.

Kataoka et al. (1998) propose using the same Lagrangian relaxation scheme for the solution of the OP, but they strengthen the relaxation with the help of valid inequalities. Indeed, they add constraints $x_{ij} + x_{0i} \leq 1$ for each arc $(v_i, v_j) \in A$ that are clearly valid. With these constraints, a vertex that is not visited (a descendent from vertex v_0) can no longer have any descendent vertex, which strongly enhances the quality of the relaxation. The relaxed problem then becomes a minimum directed 1-subtree problem and is NP-hard, but it is nevertheless possible to compute effective bounds for this.

Bounds based on the arborescence substructure of TSPs with profits clearly outperform those obtained from the ordinary assignment relaxation for symmetric instances. However, they do not enable the solution of instances as large as the branch-and-cut methods of §5.1.

Finally, let us mention that Dell'Amico et al. (1995) also suggest a bounding scheme indirectly based on the shortest spanning 1-arborescence for the PTP. Their suggestion is to first transform the problem into an asymmetric TSP (see §2.2) and then to apply a classical bounding scheme to this latter problem. Unfortunately, the computed bound is not very effective.

5.3. Lagrangian Decomposition Approach

To obtain a bound when a resource constraint is present, Göthe-Lundgren et al. (1995) propose another approach, based on Lagrangian decomposition. For this purpose, the resource constraint is duplicated and inserted within the subtour elimination constraints. Furthermore, separability is created by duplicating some variables. More precisely, y_i variables are duplicated into new variables z_i, and matching constraints $y_i = z_i$ (for every $v_i \in V$) are introduced and relaxed in a Lagrangian fashion. For fixed Lagrangian multipliers, two separate problems appear: an assignment problem involving x_i and z_i variables and a binary knapsack problem in y_i variables. The difference between the optimal solutions of these two problems provides a lower bound for the original problem. An optimization of the Lagrangian dual problem permits one to tighten this bound. Note that this scheme also permits are to derive good feasible solutions (by computing a TSP tour through the set of vertices selected in the knapsack subproblem). Computational results show that this bounding method outperforms the LP bound (see §5.1). Results are especially significant for symmetric instances. However, computing times are rather long and, even for symmetric instances, the duality gap is still too large to consider solving large instances.

5.4. Additive Approach

Besides the bounding schemes mentioned above in §5.1 and §5.2, Fischetti and Toth (1988) assess the effectiveness of using sequentially different bounds in a so-called additive approach. This approach amounts to finding new bounds in sequence, each iteration resulting in a bound and a residual instance used for the next iteration. This approach is evaluated with two different combinations:

- The LP bound of §5.1 is applied iteratively, with a shrinking of the residual graph at each iteration.
- The LP bound and a simple disjunction-based bound are applied in sequence. The principle of the disjunction is that a feasible solution either visits a distant vertex or is restricted to closer vertices.

These bounding schemes prove to be very effective for symmetric instances. For asymmetric instances, they only slightly outperform the LP bound, but computing times are almost as short. The two bounding schemes are embedded in a branch-and-bound procedure. Asymmetric and symmetric instances with up to 100 and 40 vertices, respectively, are solved in a few minutes on a Digital VAX 11/780.

5.5. The Knapsack Bound for the OP

Laporte and Martello (1990) propose a bounding approach specifically devoted to the OP. It takes advantage of the special cost function (8) of the problem and of the knapsack constraint (9). The bound is evaluated as the solution of a knapsack problem. It is valid in both the directed and undirected cases. To compute it, weights w_j are defined for every vertex $v_j \in V$. In the directed case, the definition is

$$w_j = \alpha \min_{i \neq j} c_{ij} + (1 - \alpha) \min_{k \neq j} c_{jk},$$

with $0 \leq \alpha \leq 1$ (the definition is easy to adapt for the undirected case). With this definition, the length of a route is not smaller than the sum of the weights of its vertices. Thus, the optimal solution of the 0-1 knapsack problem applied to vertices $v_j \in V\backslash\{v_1\}$ for maximizing the collected profit with a total weight limited to $c_{\max} - w_1$ is an upper bound for the OP. Note that the authors slightly tighten this bound using the fact that vertex v_1 necessarily belongs to the route.

The approach proves effective, as it allows one to solve instances involving up to 90 vertices in less than 100 seconds on a VAX 11/780 computer. Note, however, that such instances could only be solved when the knapsack constraint was very tight. Finally, note also that this bounding scheme could certainly be adapted for other TSPs with profits, as long as a resource constraint strongly limits the maximal number of visited vertices.

6. Classical Heuristic Solution Procedures

In this section, we first describe approximation algorithms with a performance guarantee that were developed for TSPs with profits. We then present the main principles that underlie heuristic procedures. Some examples of procedures based on these components are given, as well as procedures driven by other principles. The following section presents metaheuristic approaches that make use of these components in a more effective fashion.

6.1. Approximation Algorithms with a Performance Guarantee

Awerbuch et al. (1998) derive an approximation algorithm for the PCTSP from an approximation algorithm for the k-minimum-spanning-tree problem (k-MST problem). The k-MST problem consists of finding a tree of least weight (distance) that spans exactly k vertices on a given graph. For the PCTSP solution, the authors propose replacing each vertex $v_i \in V$ with p_i copies of itself at the same location and computing an approximate solution of the k-MST problem in this graph, with $k = p_{\min}$. It only remains to classically double the computed tree to obtain a tour. Thus, the main topic of the Awerbuch et al. paper is an approximation algorithm for the k-MST problem. Their approximation algorithm achieves a factor $O(\log^2(\min(n, p_{\min})))$ for the PCTSP. This field of investigation has seen a huge development in the past few years, and several other papers have recently improved the approximation ratio for the k-MST problem, and consequently for the PCTSP (see Blum et al. 1999; Arora and Karakostas 2000; Arora 1998).

Bienstock et al. (1993) propose a polynomial approximation algorithm having a factor of 5/2 performance guarantee for the undirected PTP. In the first step of their algorithm, the linear programming relaxation of the problem is solved using the ellipsoid method and vertices v_i with $y_i \geq 3/5$ are selected. A TSP heuristic with a worst case performance guarantee is then applied on this set of vertices. Goemans and Williamson (1995) improve the above performance guarantee and obtain a $(2 - 1/(n-1))$-approximation algorithm with a purely combinatorial approach.

6.2. Main Principles of Heuristic Procedures

Unlike the classical TSP, the visit of every customer is not compulsory in TSPs with profits. Hence, routes limited to the depot as well as routes visiting all vertices are candidates for the solution. These two extreme types of routes are respectively optimal for the travel cost objective and the profit objective (independently of the sequence). At the same time, each one of them can possibly yield a very bad value for the other objective. Thus, the purpose of heuristic procedures is to balance the quality of both objectives. For this purpose, four main operations may be used to transform a route:

- Adding a vertex to the route
- Deleting a vertex from the route
- Resequencing the route
- Replacing a vertex of the route with a vertex outside the route.

In most cases, these operations lead to the improvement of one of the objectives at the expense of the other. An important question is how to manage these four operations to improve the quality of the solution, avoid local optima, prevent cycling.... All these points are discussed in the subsequent sections. For the moment, we present each one of these four operations. In the following, we note $\delta(p)$ and $\delta(c)$ the respective variations of profit and travel cost when a change is performed.

6.2.1. Adding a Vertex to the Route. The addition of a vertex $v_i \in V$ to the route yields an increase $\delta(p) = p_i$ of the quantity of collected profit, at the expense of increased travel costs. The best point to insert v_i corresponds to the vertex v_r for which the added cost $\delta(c)$ of visiting v_i instead of its current successor v_s, i.e., $c_{ri} + c_{is} - c_{rs}$, is minimum.

The most effective criterion for choosing v_i is seemingly to select the candidate vertex that maximizes $\delta(p)/\delta(c)$. Yet the inconvenience of this criterion is that it does not anticipate future modifications of the route. Hence, Golden et al. (1987, 1988) have proposed several other criteria with which this primary criterion can be combined:

- A distance to the center-of-gravity measure, with different definitions for the center of gravity (center of gravity of all the vertices weighted by their profits, center of gravity of the vertices of the current route, center of gravity of a previously known route weighted by the profits)

- A distance to the destination measure
- A density measure, given by $\sum_{v_j \in V} p_j e^{-\mu c_{ij}}$, taking into account the density of profit around the vertex v_i, where μ is a parameter chosen to reflect the scale of intervertex distances.

All these measures attempt to attract the route toward vertices that seem promising for future modifications. This idea is rather original and is not found in insertion heuristics for the TSP where all vertices have to be included in the route.

Similarly, note that Ramesh and Brown (1991) develop the idea of a double insertion and that Gendreau et al. (1998b) extend it to the insertion of clusters.

6.2.2. Deleting a Vertex from the Route. The deletion of a vertex $v_i \in V$ of the route leads to decreased travel costs, at the expense of decreased $\delta(p) = p_i$ collected profit. As for the insertion, the variation of travel costs is $\delta(c) = c_{ri} + c_{is} - c_{rs}$, where v_r and v_s are the neighbors of v_i in the route and a natural criterion is to select a vertex $v_i \in V$ that minimizes $\delta(p)/\delta(c)$.

Ramesh and Brown (1991) propose extending the principle and assessing the deletion of every vertex followed by some 2-opt interchanges. The vertex that minimizes the profit-to-savings ratio is deleted. The same idea could be applied for the insertion of vertices.

6.2.3. Resequencing the Route. Resequencing a route always leads to a better solution because it may decrease travel costs while leaving profit unchanged. It amounts to simply considering the set of vertices currently visited by the route and trying to shorten the length of the route through these vertices. Hence, we are exactly in the situation of an improvement procedure for the classical TSP.

The TSP solution procedures used by the researchers for TSPs with profits are the 2-opt procedure of Lin (1965, see Chao et al. 1996a), the 3-opt procedure of Lin (1965, see Ramesh and Brown 1991), and the GENIUS algorithm of Gendreau et al. (1992; see Gendreau et al. 1998b). Actually, the field of improvement procedures for the TSP has seen a lot of attention, and many effective procedures, mostly based on Lin's k-opt mechanism, are available, the question being how much time one is ready to spend for improvement. Nevertheless, note that Tsiligirides (1984) and Keller (1989) develop routines of their own.

6.2.4. Replacing a Vertex. Another possibility is to swap two vertices, one belonging to the route and one outside the route. Whatever the TSP with profits, it is a pure improvement procedure if one limits swapping to pairs of vertices such that $\delta(p) \geq 0$ and $\delta(c) \leq 0$. However, less-drastic conditions are sufficient to ensure improvement for each type of TSP with profit. For instance, one can consider substitutions that increase profit provided that the route remains feasible in the context of the OP (Tsiligirides 1984). Note that Keller (1989) extends this scheme by authorizing the deletion of two consecutive vertices and that Dell'Amico et al. (1998) propose deleting a chain.

6.3. Classical Heuristics

Many heuristic procedures are obtained with a clever combination of the components described above. However, mixing these components may induce some difficulties, in particular, cycling. Some procedures that we present below iteratively apply the basic steps described above, with a descent strategy:

- Greedy insertion procedures consist of iteratively inserting vertices in a route until no more vertices can be added (OP), the solution is feasible (PCTSP), or no improvement is possible (PTP). Initially, the route is restricted to a loop around the depot. These procedures can easily be combined with resequencing or vertex swapping to decrease the travel costs at some steps of the solution procedure or when insertions are no longer possible.
- The equivalent greedy deletion procedures are also possible. The initial route is then a TSP solution. For the PTP, note that insertion and deletion can be applied simultaneously, with the best candidate vertex for a modification being selected at each step, as proposed by Mittenthal and Noon (1992).
- Based on these ideas, Dell'Amico et al. (1998) present two heuristic procedures for the PCTSP. In the first one, the Lagrangian relaxation of the knapsack constraint and the solution of the resulting PTP instances yield an initial route. Insertion is then used to attain feasibility of the route. The route is improved afterward with two procedures, extension and collapse, that are applied iteratively until no further improvement is possible. Extension applies insertion as long as insertions are over a computed average ratio. Collapse carries out the replacement of a chain by a single vertex. The second heuristic uses the same components, but in a different order. In particular, the extension and collapse procedures are applied during the optimization of the Lagrangian dual. This second heuristic proves to be more effective, especially when the resource constraint is naturally satisfied, since, in this case, the Lagrangian optimization often directly provides a feasible solution.

Besides the previous heuristics, other approaches have been developed for the solution of TSPs with profits:

- Path-extension procedures: These construction procedures involve extending a path until no more vertices can be added (OP). They are faster but less effective than the equivalent greedy-insertion procedures. However, because of their efficiency, a randomized behavior can be introduced by selecting one of the most promising vertices in a probabilistic fashion (Tsiligirides 1984). It enables one to repeat the procedure many times and to select the best solution. Even so, the myopic behavior of the extension is quite detrimental, and this procedure does not provide very good solutions.

- Sweep-based procedure: Tsiligirides (1984) proposes a solution approach based upon the sweep algorithm (Wren and Holliday 1972) for the TSP. The geographic area is divided into sectors determined by two concentric circles and an arc of given length. Sectors are changed by varying the two radii of the circles and rotating the arcs. Routes are built up within each of the sectors. Many cases are examined and the best route is selected. This heuristic is compared with the stochastic path extension procedure described above. For similar computing times, results are clearly inferior for the sweep approach.
- Partitioning-based procedure: Unlike previous methods, Chao et al. (1996a) do not focus on a single route, but try to improve the best route among a set of feasible routes. Vertices are partitioned in a set of feasible routes, where the best route is emphasized. Two local search procedures are described. In the so-called two-point exchange procedure, each vertex of the best route is considered in sequence. It is possible to move the vertex to another route while in return a vertex of this route moves to the best route. In the so-called one-point movement procedure, every vertex of every route is considered in sequence. This time, it is just possible to move it to another route. In both procedures and for each considered vertex, moves are made as soon as the highest profit among the set of routes increases. An important point is that the best route might change during the process.

7. Metaheuristic Procedures

The combination of the components described in §6.2 can be very efficient, provided that the research is not trapped into local optima and that cycling is avoided. Metaheuristics provide solutions for bypassing these difficulties. Furthermore, solution procedures for TSPs with profits might allow one to slightly violate resource constraints. Indeed, with the bicriteria outlook of the problems, resource constraints correspond to criteria and the overall goal pursued is to improve these criteria. Also, feasibility is always easy to recover, for example, using insertion or deletion.

Several types of metaheuristics have been applied to TSPs with profits and are described below.

7.1. Tabu Search

Ramesh and Brown (1991) propose a heuristic procedure for the OP that consists of four phases:

(1) Insertion phase: A route is built up by a series of double insertions, followed by single insertions until no more insertion is possible. The travel cost constraint might be violated as the limit considered for the insertions is fixed to $\lambda_1 c_{\max}$, with $\lambda_1 \geq 1$.

(2) Improvement phase: The 2-opt improvement procedure is carried out for each pair of edges in the tour. If the improvement has gone beyond a given threshold λ_2, the 3-opt procedure is also carried out. If the total improvement has gone beyond another threshold λ_3, the algorithm returns to Phase 1 (with a reduction of λ_1 when $\lambda_1 > 1$).

(3) Deletion phase: A vertex is deleted. The algorithm returns to Phase 1.

A stopping criterion based on the improvement between two successive iterations and the total number of iterations is included in the previous phases. When this criterion is met, a final insertion phase is performed (Phase 4). Due to the deletion phase, the algorithm may cycle. A tabu list is used to store the routes computed in order to avoid it. Note, however, that the authors do not present their algorithm under the tabu search label.

Fischetti et al. (1998) adapt and simplify this heuristic to use it inside a branch-and-cut procedure. They also integrate as a first step an insertion system based on the value of arc variables in their current relaxed solution. Gendreau et al. (1998a) also propose two related heuristics inside a branch-and-cut procedure. These heuristics perform efficiently and effectively but do not introduce any major new concept.

Gendreau et al. (1998b) propose another tabu search procedure. Moves consist either in deleting part of the route or inserting a vertex cluster (clusters of size one are considered). Gendreau et al. introduce a parameter reflecting the relative importance attributed to travel cost and profit of a route, to balance insertion and deletion. At every iteration, it is updated to favor insertion or deletion whether the route is feasible or not. This local search scheme is embedded in a tabu search procedure to avoid cycling. When a vertex is removed from the route, it is assigned a tabu status for a randomly selected number of iterations. Furthermore, an insertion procedure and the improvement algorithm GENIUS are regularly called for within the procedure. This tabu search procedure is tested on randomly generated instances involving up to 300 vertices; it always yields optimal or near-optimal solutions (with a gap typically less than 1%). Computation times never go beyond a couple of minutes on a Sun Sparcstation 1000. Even if no result indicates any direct comparison with other heuristic procedures, it clearly emerges among the most effective and efficient approaches.

7.2. Evolutive Procedure with a Learning Measure

Golden et al. (1988) propose a procedure that iteratively constructs routes with a stochastic insertion procedure. A learning measure is defined and interacts with other criteria for the selection of vertices. It evaluates the amount of presence of a vertex in the best routes generated during the previous iterations of the algorithm. More precisely, the learning measure associated with a vertex $v_i \in V$ is the average route score for routes that include v_i against the total average route score. This algorithm is not mentioned to be a metaheuristic, but the principle of the learning measure is reminiscent of the ideas behind ant algorithms (which were not formulated at that time). Note also that Liang and Smith (2001) recently proposed a standard ant colony algorithm, hybridized with local search, for the OP.

7.3. Deterministic Annealing

Chao et al. (1996a) develop a solution procedure based on the record-to-record approach (Dueck 1993). The algorithm is driven by a partition of the vertices, such as explained in §6.3. To obtain an initial solution, routes are first constructed by a nearest neighbor insertion procedure. When a route is constructed, another route is built up through the remaining vertices, and so on until every vertex is covered. The process is repeated several times, one of the vertices farthest from the depot being first inserted in the first route each time. The procedure keeps the set of routes that contains the route with the highest score.

The record-to-record approach is then applied. This approach allows one to deteriorate the solution, as long as the deterioration incurred is not larger than a given percentage (10% here) of the current solution score. The move operators described in §6.3 and a 2-opt procedure are successively used. To diversify the approach, each 10 iterations—or when no more improvement is obtained—a set of vertices (one the first time, two the second time…) is shifted from the optimal route to other routes. After 10 executions of this diversification scheme, the procedure stops and returns the best solution generated so far.

7.4. Genetic Algorithm

Tasgetiren and Smith (2000) develop a genetic algorithm for the OP. In this algorithm, a chromosome is a sequence of visited vertices. A classic order-based crossover operator is used to generate the offspring. Mutations are obtained with a combination of basic operations similar to the ones described in §6.2. Nonfeasible solutions are accepted, with a penalty based on distance from feasibility.

This algorithm provides results comparable to other metaheuristics (except tabu search), but with a longer computing time.

7.5. Neural Network Approach

Wang et al. (1995) apply a continuous Hopfield neural network method for solving the OP. The neural network is conceptualized as a matrix, with each row corresponding to a vertex and each column corresponding to a position in the path. A last column is added to determine selected vertices and calculate the total score of the path.

The energy function is a fourth-order convex function. It consists of six terms that penalize columns with more than one activated vertex, solutions with fewer than n vertices, and solutions exceeding the travel cost limit, that encourage to start and end at the depot, that match up the last column with the first n ones, and that try to maximize the total score. Even if the model supports solutions with exactly n vertices, it will work efficiently for solutions with fewer than n visited vertices, since solutions with consecutive visits of a same vertex are authorized with no penalty.

The link weights are not constant, each weight being the second partial derivative of the energy function with respect to the state. The model also includes a route-insertion procedure and a 2-opt improvement routine that are essential factors in the success of the method.

This approach proves to be competitive with the solution procedures of Ramesh and Brown (1991), Chao et al. (1996a), and Golden et al. (1988).

8. Extensions

Up to this section, we have focused on solution strategies for the three defined generic TSPs with profits. However, many papers deal with extensions of these problems, including some new features. The most significant modification consists of introducing a fleet of vehicles instead of a single one. We then speak of routing problems with profits, which we detail in §9. In this section, we focus on single-vehicle extensions of TSPs with profits.

8.1. Resource Constrained TSPs with Profits

A first kind of extension involves adding a single resource constraint. The solution approaches are then very similar to the ones described in previous sections, and we do not detail them. Note, however, that the extension of the PTP with a generalized covering constraint, which is equivalent to the original definition of the PCTSP by Balas (1989), has been tackled by several authors.

A more general extension adds several resource constraints. Classical resource constraints for routing problems are the so-called time windows constraints, which imply to respect some schedule for the visit of customers. Kantor and Rosenwein (1992) introduce them for the OP. Basic heuristic components described in §6.2 are easily adapted to these new constraints.

Note, however, that when the time windows are very tight, the feasible space is limited and it appears more difficult to avoid getting trapped in local optima; Kantor and Rosenwein suggest using a heuristic enumeration scheme in such a situation.

Feillet et al. (2003) study the elementary shortest path problem with resource constraints, which can be interpreted as a generalization of the PTP. They solve it using dynamic programming. Finally, Gendreau et al. (1998a) introduce compulsory vertices, which define a compromise between TSPs with profits and TSP.

8.2. TSPs with Profits with Complex Cost Functions

Golden et al. (1986) introduce a relaxation of the OP where a violation of the travel cost limit induces a penalty proportional to the violation. This means that the travel cost criterion only influences the objective function when travel costs exceed the limit (which would correspond to a delay, for example). For this feature, selection criteria that are used in basic heuristic components of §6.2 should be changed, depending on whether the current situation is an OP (low values of travel cost consumption) or a PTP (values near to or higher than the limit). Golden et al. also note that the penalty depends only on the surplus value, and they propose evaluating different solutions of the OP with varying fixed surplus. Caramia et al. (1999) study a similar problem in the context of tourist flow organization and propose a simple tabu search algorithm.

Another adaptation of TSPs with profits that may occur in a competitive environment concerns the decrease of profit values with time. For instance, profits might represent sales and decline, with several competing salespeople operating in the same area. This adaptation might also be relevant when profits correspond to services with a value depending on delays (e.g., for an equipment repair company). This extension is introduced by Brideau and Cavalier (1994) and by Erkut and Zhang (1996), who propose path extension heuristics. A first possibility simply consists of dynamically adapting the value of profits during the construction of the solution (Brideau and Cavalier 1994). However, it fails to capture an important characteristic of the problem. Indeed, each consumption of a unit of travel cost results in the decrease of the value of profits for all the vertices that remain to be visited. Hence, it is interesting to visit vertices as quickly as possible. Erkut and Zhang (1996) integrate into the vertex selection measure a value that favors vertices whose profit would largely decrease if they are visited later. This approach finds near-optimal solutions on small instances.

Johnston (1999) studies more deeply the problem of a competitive environment, where carriers compete in the same area for serving customers. He address it from a game theory standpoint.

Another possible objective function is when travel costs are time dependent. It is introduced by Fomin and Lingas (2002), who provide a $(2+\epsilon)$-approximation algorithm.

Finally, Golden et al. (1997) and Wang et al. (1996) generalize TSPs with profits to a situation where profits are evaluated with respect to several attributes (natural beauty, historical significance, cultural and educational attractions, and business opportunities) and the objective function is nonlinear.

8.3. TSPs With Profits Without a Depot

As previously indicated, problems that only differ from TSPs with profits by the relaxation of the depot condition are beyond the scope of this article. Even so, we think it useful to mention several references concerning these problems. First note that, when the depot condition is dropped out, the PTP is known as weighted girth problem, circuit problem, or subtour problem. Kubo and Kasugai (1992) develop several mixed-integer programming formulations and bounding schemes. Bauer et al. (1998) introduce the knapsack constrained circuit problem and develop a branch-and-cut algorithm in the case of unit consumption of the resource (cardinality constrained circuit problem). Note also that the Awerbuch et al. (1998) approximation algorithm can be adapted to the OP when the depot condition is suppressed.

8.4. Arc Routing TSPs with Profits

Deitch and Ladany (2000) are interested in the determination of an attractive bus route in a tourist region. With this goal, they generalize the TSPs with profits by introducing profits on arcs (denoting scenic route segments). For the solution, the authors use a transformation scheme that results in a node routing TSP with profits. One can assume that algorithms specifically devised for the arc-routing setting would be more effective and could advantageously be investigated, as in other arc-routing contexts.

8.5. More Elusive Extensions

Besides these straightforward extensions, one can find several more elusive problems that derive from TSPs with profits.

The problems faced by carriers for the transportation of freight are recurrent. Bookbinder and Sural (1999) consider the situation of a carrier that makes deliveries and also has a number of optional backhauls. The objective is to find the lowest cost tour by selecting the most profitable backhauls. For this purpose, they study the related polyhedron, tightening the subtour elimination constraints and developing valid inequalities.

Diaby and Ramesh (1995) introduce a problem that arises in the distribution of commodities. A single vehicle delivers customers along a tour starting and ending at the depot, while respecting capacity and time limit constraints. A given fee is paid for unvisited customers, who must be served by an outside carrier. The problem is an extension of the PTP with an additional time limit and a capacity constraint. Actually, the problem also presents a dynamic feature. A trip is initiated each day, for a two-working-day duration. Hence, the visit of a costumer can either be scheduled inside a vehicle trip or inside the vehicle trip starting the day after (or never with a penalty). A rolling horizon approach is used to take this characteristic into account. The problem is solved using a branch-and-bound strategy. The lower bounding scheme relies on the linear programming relaxation of the problem, strengthened with valid inequalities. This approach is used in the context of an industrial chemical company. Diaby and Ramesh report impressive savings illustrating the operational efficiency of the procedure compared to the original system used in the company.

Another complex application of TSPs with profits is found in the steel industry, for the planning of rolling mills. A simplified version of the problem was described in §3. However, in reality, schedulers face a problem involving other resources and complex constraints. Lopez et al. (1998) propose a detailed description of the industrial context and describe a fast and effective heuristic based on tabu search that addresses the problem in its whole complexity. Lopez et al. also review the related literature. In particular, they describe a study by Cowling (1995), who also proposes a heuristic based on tabu search for a problem that might be seen as a prize-collecting VRP with side constraints.

Besides the previous variants, it is interesting to mention that one of the engineering design problems at NASA has been formulated as a version of the OP and has been posed as a design challenge problem by the American Institute of Aeronautics and Astronautics. It consists of an OP with several side constraints (e.g., precedence constraints) and stochastic components. Many attempts were devised for solving the challenge. Unfortunately, the challenge problem introduces too many features to offer new insights into TSPs with profits.

9. Multivehicle Routing Problems with Profits

As in the TSP context, many realistic situations involve several vehicles. The special case of two vehicles is studied by Nocito (1993), who proposes heuristic solution procedures. It is in the field of military applications that the first attempts to deal with real-life, multivehicle routing problems with profits appear. Moser (1990) proposes a heuristic and an exact solution procedure for a tactical aerial reconnaissance problem that takes time windows into account. Wu (1992) and Dunn (1992) face another military application. Logistics ships are used in a dispersed battle group to travel long distances between ships and replenish them. When operational requirements limit the amount of available time, replenished ships are selected depending on a combat value. Dunn (1992) proposes a dynamic programming approach running quickly enough to be useful for scheduling underway replenishment in operational situations.

As it happens in the TSP context, some heuristics can effectively be extended to the multivehicle situation. Chao et al. (1996b) propose a straightforward extension of their heuristic developed for the OP (see §7.3). Millar and Kiragu (1997) also extend their formulation and their branch-and-bound solution procedure (see §§4 and 5.5) for the multivehicle OP. They consider both time (travel cost) limits for each vehicle and for the total time over all vehicle tours. The context is a fishery patrol problem. A deterrent value (depending on fishing factors and historical patrol activity for the zone) is associated with each fishing zone, and patrol routes are computed so as to maximize the deterrent value of the patrols while respecting time limits.

Besides the previous papers, one can find some other attempts to solve multivehicle routing problems with profits. Butt and Cavalier (1994) are concerned with recruiting athletes. A college recruiter has to recruit football players from high schools in the area in a given number of days. Each day, he has a limited amount of time to visit the chosen high schools and come back home. A reward is assigned to each high school based on its recruiting potential. The objective is to visit a set of high schools maximizing the recruiting potential. Butt and Cavalier propose a greedy algorithm for solving this problem. Butt and Ryan (1999) face the same problem and propose a branch-and-price solution procedure. Gueguen (1999) also proposes branch-and-price solution procedures for the so-called selective VRP with time windows and prize-collecting VRP with time windows, where vertices stand for customers and where vehicle capacity, service times, and time windows for customers are introduced. These two branch-and-price approaches, which appear to be very different, deserve to be described in depth. Before that discussion, we mention a last application of multivehicle routing problems with profits.

This application is addressed by Feillet (2001) and concerns the tactical planning of freight transportation between plants in the car industry. Freight

Table 1 Multivehicle Routing Problems with Profits

Reference	Context	Solution approach
Moser (1990)	aerial reconnaissance	heuristic + exact solution procedure
Wu (1992)	underway replenishment	
Dunn (1992)	underway replenishment	dynamic programming
Nocito (1993)		heuristics
Butt and Cavalier (1994)	recruiting athletes	greedy heuristic
Chao et al. (1996b)	team orienteering	heuristic
Millar and Kiragu (1997)	fisheries patrol	branch and bound
Butt and Ryan (1999)	recruiting athletes	branch and price
Gueguen (1999)		branch and price
Feillet (2001)	freight transportation	branch and price

movements are planned in advance, but two transportation methods with different costs can be used to realize them. The more expensive method consists of one-way journeys; the other consists of circuits. Feillet starts the optimization with the inferior solution of using one-way journeys only. Revising this initial choice for some freight movements then amounts to solving a multivehicle PTP without a depot. This problem is tackled with a branch-and-price algorithm, and exact solutions of real-life instances with up to 20 plants are obtained in a few seconds.

Table 1 summarizes attempts for solving multivehicle routing problems with profits.

9.1. Branch and Price Procedures

Butt and Ryan (1999) and Gueguen (1999) adapt the classical set-partitioning formulation used in column-generation solution procedures for constrained VRPs to different routing problems with profits. They formulate the same model:

$$\text{minimize} \sum_{k\in\Omega} c_k x_k \tag{32}$$

subject to

$$\sum_{k\in\Omega} a_{ik} x_k \leq 1 \quad (i \in \Omega), \tag{33}$$

$$\sum_{k\in\Omega} x_k \leq m, \tag{34}$$

$$x_k \in \{0, 1\} \quad (k \in \Omega), \tag{35}$$

where Ω is the set of feasible routes, index k represents the routes, c_k is the cost of route k, and a_{ik} is equal to 1 if route k visits customer i and 0 otherwise. Binary variables x_r indicate whether route r is part of the solution. The subproblem is to find feasible routes having a negative cost, where profit values are updated using dual variable values. Butt and Ryan's and Gueguen's approaches are very different for solving this subproblem.

To start with, feasible route sets differ. For Butt and Ryan (1999), the fleet of vehicles is heterogeneous (the time limit depends on the type of vehicle) and a service time for the vertices is taken into account. The cost of a route is given by the sum of the profits of visited vertices (multivehicle OP). For Gueguen (1999), a vehicle capacity constraint, time windows, and service times for the clients are introduced. Both the multivehicle OP and the multivehicle PCTSP are addressed. Note that the multivehicle extension of the OP considers a travel cost limit for each vehicle, while the multivehicle extension of the PCTSP enforces a minimal quantity of profit to collect using all the vehicles.

9.1.1. Butt and Ryan's Subproblem Solution Approach. Butt and Ryan begin to rank vertices according to their updated profits. Vertices with negative profits are put aside, as they cannot improve the value of a route. The principle of the algorithm is to scan the set of vertex subsets containing the depot and to solve a TSP on a subset when its profit is high enough. The set of vertex subsets is scanned in lexicographic order. When a TSP solution goes over the time limit, the child subsets are not considered. Subsets S such that $|S| > 2$ are also checked for feasibility to avoid assessing many long, infeasible routes when profits are low. The enumeration procedure stops either when it is completed, a number of good routes have been found, or all children of a subset $\{v_i\}$ have been scanned and at least one good route has been found. TSPs are solved with an existing exact procedure. An important point is that each time the feasibility problem is solved for a subset of vertices, the result is kept and used in subsequent iterations. The weak point of this approach is that it is very specific to the OP situation.

9.1.2. Gueguen's Subproblem Solution Approach. Gueguen (1999) tackles the subproblem as an elementary shortest path problem with resource constraints. He claims that, unlike what happens in the case of the classical VRP with time windows, the elementary restriction is necessary here, as visiting a customer many times would give rise to repeated profits. A dynamic programming procedure, which extends dynamic programming procedures developed for the nonelementary shortest path problem with resource constraints (Feillet et al. 2003), is used to solve the subproblem.

9.1.3. Search Tree. In both approaches, the column generation scheme is embedded into a branch-and-price procedure. Gueguen (1999) branches on arcs. An arc (v_i, v_j) with a fractional value in the relaxed solution is either imposed or excluded. Butt and Ryan (1999) branch on a vertex pair (v_i, v_j). Either both vertices are imposed or just one of them is accepted.

Butt and Ryan (1999) manage to solve instances involving up to 100 customers and 15 vehicles in a few hours. With the time windows constraining the set of feasible routes, Gueguen (1999) manages to solve instances involving up to 100 customers and 10 vehicles in less than a hour.

10. Conclusion

In this article we have proposed a classification and a survey of a class of problems we called TSPs with profits. This class includes monocriterion versions of a bicriteria extension of the TSP, which consists of finding a tour on a subset of customers to visit, to maximize a profit measure while minimizing travel costs. Three generic problems make up the class, depending on the way the two objectives are addressed. The PTP consists of finding a circuit that minimizes travel costs minus collected profit. The OP seeks to find a circuit that maximizes collected profit such that travel costs do not exceed a preset value $c_{\max}$. The PCTSP aims to find a circuit that minimizes travel costs and whose collected profit is not smaller than a preset value $p_{\min}$. Many papers deal with this general class of problems, but the lack of a clearly defined classification is obvious. Our goal was to emphasize the differences and common features of the different versions, and then to clarify the connections with related problems.

TSPs with profits occur in a wide variety of situations, including realistic traveling salesman problems, job scheduling, freight transportation, or, indirectly, as a subproblem in solution approaches dedicated to other problems. Our second goal was to confirm the necessity to tackle this class of problem. Tables 2 and 3 summarize all these applications.

Our third goal was to bring together all the attempts dedicated to each of these problems. Authors have considered them at different levels. Indeed, this class is closely related to the TSP but has a very specific structure. This new structure has been the core subject of several papers studying structural properties or bounding schemes based on linear programming formulations (e.g., Fischetti and Toth 1988; Balas 1989; Fischetti et al. 1998; Gendreau et al. 1998a). At a completely different level, authors have taken this specific structure into account when including novel features in heuristic procedures otherwise widely adapted from classical TSP heuristics, such as insertion procedures (e.g., Mittenthal and Noon 1992). Other researchers have been more interested in assessing more recent or technical methods for this kind of problem (e.g., Pekny et al. 1990). We believe all these approaches are praiseworthy because one needs simple heuristic algorithms, easy to implement and fast, as well as more advanced ones, complex and more time-consuming, but more effective. All these approaches are summarized in Tables 4, 5, and 6, respectively, for exact, heuristic, and metaheuristic procedures.

Even if they are not direct extensions of TSPs with profits, we have to mention the traveling purchaser problem and the generalized traveling salesman problem, which are two other extensions of the TSP in which customers have to be selected. The traveling purchaser problem involves a set of commodities. Each vertex (standing for a market) holds commodities, each with its own associated purchase cost. The objective is to find a tour (including the depot) that enables one to purchase all commodities so as to minimize the total travel and purchase cost. Readers are referred to Voss (1996), Singh and van Oudheusden (1997), Pearn and Chien (1998) or Boctor et al. (2003) for more information. In the generalized traveling salesman problem, vertices are components of clusters and the aim is to find a minimum cost circuit visiting at least one vertex of each cluster. Recent investigations are found in Fischetti et al. (1997) or

Table 2 Using TSPs with Profits for Solving Real-Life Applications

Application	Reference
Traveling salesman scheduling	Gensch (1978)
Orienteering	Tsiligirides (1984)
Inventory routing	Golden et al. (1984)
Steel rolling mill scheduling	Balas and Martin (1985)
	Cowling (1995)
	Lopez et al. (1998)
Job scheduling	Pekny et al. (1990)
	Pekny and Miller (1990)
Control theory	Ramesh and Brown (1991)
Carriers transportation problems	Diaby and Ramesh (1995)
	Bookbinder and Sural (1999)
	Feillet (2001)

Table 3 Using TSPs with Profits as Subproblems

Problem	Reference
VRP	Noon et al. (1994)
	Bauer et al. (1998)
Cost-allocation problem	Göthe-Lundgren et al. (1995)
m-Cost ATSP	Helmberg (1999)
VRP with time windows	Feillet et al. (2003)

Table 4 Exact Solution Procedures for TSPs with Profits

		Number of vertices	
Reference	Technique	Symmetric	Asymmetric
Gensch (1978)	branch and bound	30	
Kataoka and Morito (1988)	branch and bound		10
Fischetti and Toth (1988)	branch and bound	40	100
Laporte and Martello (1990)	branch and bound	70	90
Pekny et al. (1990)	branch and bound		200
Ramesh et al. (1992)	branch and bound	150	150
Millar and Kiragu (1997)	branch and bound	15	15
Gendreau et al. (1998a)	branch and cut	300	
Fischetti et al. (1998)	branch and cut	500	

Table 5 Classical Heuristic Solution Procedures for TSPs with Profits

Reference	Principle
Tsiligirides (1984)	Path extension, resequencing
Tsiligirides (1984)	Sweep algorithm, resequencing
Golden et al. (1987)	Insertion, resequencing
Keller (1989)	Path extension, resequencing
Laporte and Martello (1990)	Path extension
Laporte and Martello (1990)	Insertion
Mittenthal and Noon (1992)	Insertion, deletion
Bienstock et al. (1993)	Approximation algorithm with performance guarantee
Goemans and Williamson (1995)	Approximation algorithm with performance guarantee
Göthe-Lundgren et al. (1995)	Lagrangian heuristic
Awerbuch et al. (1998)	Approximation algorithm with performance guarantee
Dell'Amico et al. (1998)	Insertion, resequencing, substitution
Fischetti et al. (1998)	Adaptation of Ramesh and Brown (1991)
Gendreau et al. (1998a)	Adaptations of Ramesh and Brown (1991)
Gendreau et al. (1998b)	Route insertion + improvement

Renaud and Boctor (1998). The multivehicle version of this last problem has also been studied in Ghiani and Improta (2000). Finally, far-removed extensions of the TSP for which customers are selected can be found under such names as the shortest covering path problem (Current et al. 1994), the covering tour problem (Gendreau et al. 1997), or the traveling circus problem (ReVelle and Laporte 1993). More details about these can be found in Labbé et al. (1998). Finally, we have to mention a slightly different but related problem, where the aim is to determine the shortest circuit (or path) in a graph where some specified vertices have to be visited exactly once while the others are visited at most once (see Laporte et al. 1984; Volgenant and Jonker 1987). All these problems merge the questions of sequencing a tour and selecting vertices. However, they are not guided by two concurrent objectives, as are TSPs with profits, which explains why we do not integrate them into the class.

In conclusion, our opinion is that in the field of TSPs, the condition imposing the visit of every customer is not always relevant, especially in real-life contexts. However, people are often inclined to tackle the optimization of vehicles visiting the complete set of vertices, possibly after having restricted it. Thus, many two-phase algorithms that first restrict the set of customers and then solve a TSP can be found in the literature. Yet, results of TSPs with profits show that it is not necessarily Utopian to try to aggregate vertex selection and travel cost optimization. Indeed, it does not imply special difficulties for modeling and tackling the problem. Furthermore, the literature is full of solution algorithms that are very efficient and effective (even if they are still rare for the multivehicle versions). Hence, seeing that many real-life problems fit within the framework we have defined, seeing the need for still more realistic models, and seeing the modern computing abilities and the efficient algorithmic solutions, we hope to see many practical situations modeled as TSPs with profits in the future. We hope this survey to be helpful for this purpose and for the design of new, efficient solution procedures.

Table 6 Metaheuristic Procedures for TSPs with Profits

Reference	Technique
Golden et al. (1988)	Evolutive algorithm
Ramesh and Brown (1991)	Tabu search
Wang et al. (1995)	Neural networks
Chao et al. (1996a)	Deterministic annealing
Gendreau et al. (1998b)	Tabu search
Tasgetiren and Smith (2000)	Genetic algorithm

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