Integer Programming Formulation of Traveling Salesman Problems*

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It has been observed by many people that a striking number of quite diverse mathematical problems can be formulated as problems in integer programming, that is, linear programming problems in which some or all of the variables are required to assume integral values. This fact is rendered quite interesting by recent research on such problems, notably by R. E. Gomory [2, 3], which gives promise of yielding efficient computational techniques for their solution. The present paper provides yet another example of the versatility of integer programming as a mathematical modeling device by representing a generalization of the well-known "Traveling Salesman Problem" in integer programming terms. The authors have developed several such models, of which the one presented here is the most efficient in terms of generality, number of variables, and number of constraints. This model is due to the second author [4] and was presented briefly at the Symposium on Combinatorial Problems held at Princeton University, April 1960, sponsored by SIAM and IBM. The problem treated is:

(1) A salesman is required to visit each of n cities, indexed by $1, \cdots, n$. He leaves from a "base city" indexed by 0, visits each of the n other cities exactly once, and returns to city 0. During his travels he must return to 0 exactly t times, including his final return (here t may be allowed to vary), and he must visit no more than p cities in one tour. (By a tour we mean a succession of visits to cities without stopping at city 0.) It is required to find such an itinerary which minimizes the total distance traveled by the salesman.

Note that if t is fixed, then for the problem to have a solution we must have $tp \geqq n$. For $t = 1$, $p \geqq n$, we have the standard traveling salesman problem.

Let d_{ij} ($i \neq j = 0, 1, \cdots, n$) be the distance covered in traveling from city i to city j. The following integer programming problem will be shown to be

equivalent to (1):

(2) Minimize the linear form

$$\sum_{0 \le i \ne j \le n} d_{ij} x_{ij}$$

over the set determined by the relations

$$\sum_{\substack{i=0 \\ i \ne j}}^{n} x_{ij} = 1 \qquad\qquad (j = 1, \cdots, n)$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{n} x_{ij} = 1 \qquad\qquad (i = 1, \cdots, n)$$

$$u_i - u_j + p x_{ij} \le p - 1 \qquad\qquad (1 \le i \ne j \le n)$$

where the x_{ij} are non-negative integers and the u_i $(i = 1, \cdots, n)$ are arbitrary real numbers. (We shall see that it is permissible to restrict the u_i to be non-negative integers as well.)

If t is fixed it is necessary to add the additional relation:

$$\sum_{i=1}^{n} x_{i0} = t$$

Note that the constraints require that $x_{ij} = 0$ or 1, so that a natural correspondence between these two problems exists if the x_{ij} are interpreted as follows: The salesman proceeds from city i to city j if and only if $x_{ij} = 1$. Under this correspondence the form to be minimized in (2) is the total distance to be traveled by the salesman in (1), so the burden of proof is to show that the two feasible sets correspond; i.e., a feasible solution to (2) has x_{ij} which do define a legitimate itinerary in (1), and, conversely a legitimate itinerary in (1) defines x_{ij}, which, together with appropriate u_i, satisfy the constraints of (2).

Consider a feasible solution to (2).

The number of returns to city 0 is given by

$$\sum_{i=1}^{n} x_{i0}.$$

The constraints of the form

$$\sum x_{ij} = 1, \quad \text{all } x_{ij} \text{ non-negative integers,}$$

represent the conditions that each city (other than zero) is visited exactly once. The u_i play a role similar to node potentials in a network and the inequalities involving them serve to eliminate tours that do not begin and end at city 0 and tours that visit more than p cities. Consider any $x_{r_0 r_1} = 1$ $(r_1 \ne 0)$. There exists a unique r_2 such that $x_{r_1 r_2} = 1$. Unless $r_2 = 0$, there is a unique r_3 with $x_{r_2 r_3} = 1$. We proceed in this fashion until some $r_j = 0$. This must happen since the alternative is that at some point we reach an $r_k = r_j$, $j + 1 < k$. Since none of

the r's are zero we have

$$u_{r_i} - u_{r_{i+1}} + px_{r_i r_{i+1}} \leqq p - 1$$

or

$$u_{r_i} - u_{r_{i+1}} \leqq -1.$$

Summing from $i = j$ to $k - 1$, we have

$$u_{r_j} - u_{r_k} = 0 \leqq j + 1 - k,$$

which is a contradiction. Thus all tours include city 0. It remains to observe that no tour is of length greater than p. Suppose such a tour exists, x_{0r_1}, $x_{r_1 r_2}$, $\cdots$, $x_{r_p r_{p+1}} = 1$ with all $r_i \neq 0$. Then, as before,

$$u_{r_1} - u_{r_{p+1}} \leqq -p$$

or

$$u_{r_{p+1}} - u_{r_1} \geqq p.$$

But we have

$$u_{r_{p+1}} - u_{r_1} + px_{r_{p+1} r_1} \leqq p - 1$$

or

$$u_{r_{p+1}} - u_{r_1} \leqq p(1 - x_{r_{p+1} r_1}) - 1 \leqq p - 1,$$

which is a contradiction.

Conversely, if the x_{ij} correspond to a legitimate itinerary, it is clear that the u_i can be adjusted so that $u_i = j$ if city i is the jth city visited in the tour which includes city i, for we then have

$$u_i - u_j = -1 \quad \text{if} \quad x_{ij} = 1,$$

and always

$$u_i - u_j \leqq p - 1.$$

The above integer program involves $n^2 + n$ constraints (if t is not fixed) in $n^2 + 2n$ variables. Since the inequality form of constraint is fundamental for integer programming calculations, one may eliminate $2n$ variables, say the x_{i0} and x_{0j}, by means of the equation constraints and produce an equivalent problem with $n^2 + n$ inequalities and n^2 variables.

The currently known integer programming procedures are sufficiently irregular in their behavior to cast doubt on the heuristic value of machine experiments with our model. However, it seems appropriate to report the results of the five machine experiments we have conducted so far. The solution procedure used was the all-integer algorithm of R. E. Gomory [3] without the ranking procedure he describes.

The first three experiments were simple model verification tests on a four-city standard traveling salesman problem with distance matrix

$$\begin{bmatrix} & \underline{20} & 23 & 4 \\ 30 & & \underline{7} & 27 \\ 25 & 5 & & \underline{25} \\ \underline{3} & 21 & 26 & \end{bmatrix}$$

The first experiment was with a model, now obsolete, using roughly twice as many constraints and variables as the current model (for this problem, 28 constraints in 21 variables). The machine was halted after 4000 pivot steps had failed to produce a solution.

The second experiment used the earlier model with the x_{i0} and x_{0j} eliminated, resulting in a 28-constraint, 15-variable problem. Here the machine produced the optimal solution in 41 pivot steps.

The third experiment used the current formulation with the x_{i0} and x_{0j} eliminated, yielding 13 constraints and 9 variables. The optimal solution was reached in 7 pivot steps.

The fourth and fifth experiments were used on a standard ten-city problem, due to Barachet, solved by Dantzig, Johnson and Fulkerson [1]. The current formulation was used, yielding 91 constraints in 81 variables. The fifth problem differed from the fourth only in that the ordering of the rows was altered to attempt to introduce more favorable pivot choices. In each case the machine was stopped after over 250 pivot steps had failed to produce the solution. In each case the last 100 pivot steps had failed to change the value of the objective function.

It seems hopeful that more efficient integer programming procedures now under development will yield a satisfactory algorithmic solution to the traveling salesman problem, when applied to this model. In any case, the model serves to illustrate how problems of this sort may be succinctly formulated in integer programming terms.

REFERENCES

1. DANTZIG, FULKERSON AND JOHNSON On a linear-programming, combinatorial approach to the traveling salesman problem. *JORSA*, 7 (1959), 1.
2. GOMORY, R. E. An algorithm for integer solutions to linear programs. Princeton—IBM Math. Research Project Techn. Report No. 1, 1958.
3. ——, All-integer integer programming algorithm IBM Research Center Report RC-189, 1960.
4. TUCKER, A. W On directed graphs and integer programs, Princeton—IBM Math. Research Project Techn. Report, 1960.