Student No. ....

Teacher: JH MN SE GS

2021

YEAR 12 TRIAL HSC EXAMINATION

ASCHAM SCHOOL

Mathematics Extension 1

Tuesday 20th July 2021

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided

Total marks: 70

SECTION I - (10 marks)

- Use the multiple-choice answer sheet for Questions 1-10
- Allow about 15 minutes for this section

SECTION II - (60 marks)

- This section consists of four questions each worth 15 marks
- All necessary working should be shown
- Start a new booklet for each question
- Allow about 1 hour and 45 minutes for this section

SECTION I (10 marks)

Attempt Questions 1 - 10
Use the multiple-choice answer sheet for Questions 1 - 10

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1. Which of the following is the solution to $\dfrac{x-2}{x} \leq 2$?

A. $x \in [-2, 0)$

B. $x \in [-2, 0]$

C. $x \in (-\infty, -2] \cup (0, \infty)$

D. $x \in (-\infty, -2] \cup [0, \infty)$

2. When a polynomial $P(x)$ is divided by $x - 2$ the remainder is 4 and when it is divided by $x + 3$ the remainder is 9. What is the remainder when it is divided by $(x-2)(x+3)$?

A. $x + 6$

B. $x - 6$

C. $-x + 6$

D. $-x - 6$

3. A bag contains 6 red, 8 green, 10 blue and 12 yellow balls. What is the minimum number of balls chosen randomly from the bag to ensure that we have 9 balls of the same colour?

A. 30

B. 31

C. 32

D. 33

4. What is the exact value of $\sin\frac{\pi}{8}\cos\frac{\pi}{8}$?

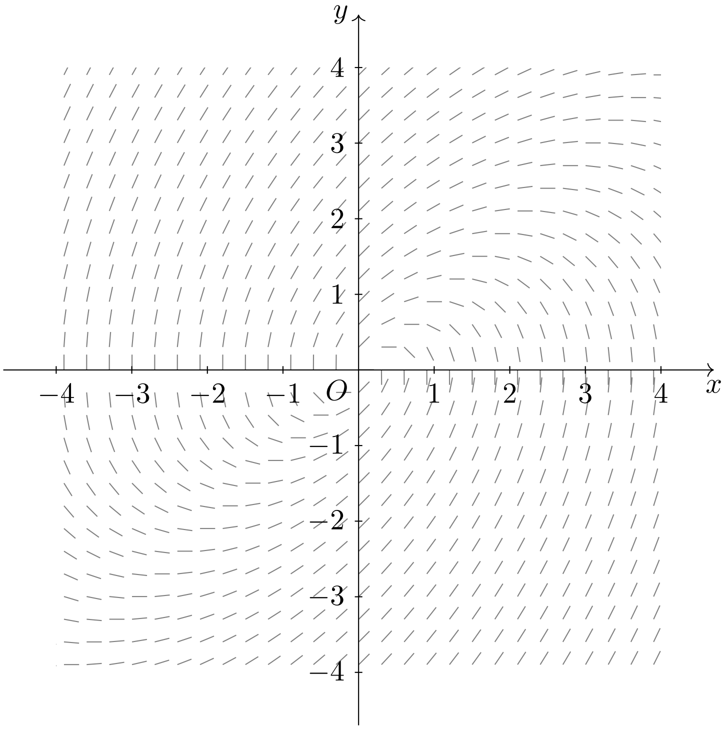
A. $2\sqrt{2}$

B. $\sqrt{2}$

C. $\dfrac{\sqrt{2}}{2}$

D. $\dfrac{\sqrt{2}}{4}$

5. The slope field for a first order differential equation is shown below.



Which of the following could be the differential equation represented?

A. $y' = \dfrac{y - x}{y}$

B. $y' = \dfrac{y + x}{y}$

C. $y' = \dfrac{x - y}{x}$

D. $y' = \dfrac{x + y}{x}$

6. Six girls and three boys are to be seated randomly around a circular table. What is the probability that the three boys are seated together?

A. $\dfrac{1}{12}$

B. $\dfrac{1}{24}$

C. $\dfrac{3}{28}$

D. $\dfrac{1}{56}$

7. It is known that the percentage of high school students that use public transport is 64%. Twenty random samples of 50 students are taken and the mean of these sample proportions of public transport using students is $\hat{p} = 0.585$. If five more random samples of 50 students were taken, which of the following is most likely to occur?

A. $\hat{p}$ increases and $\sigma_{\hat{p}}$ increases

B. $\hat{p}$ increases and $\sigma_{\hat{p}}$ decreases

C. $\hat{p}$ decreases and $\sigma_{\hat{p}}$ increases

D. $\hat{p}$ decreases and $\sigma_{\hat{p}}$ decreases

8. What is the gradient of the tangent to the curve $y = \tan^{-1}\left(\frac{x}{2}\right)$ at $x = 1$?

A. $\frac{1}{3}$

B. $\frac{2}{5}$

C. $\frac{2}{3}$

D. $\frac{4}{5}$

9. What is the domain of the cartesian equation with parametric equations

$$x = \cos^2 t \quad \text{and} \quad y = 1 + \sin^2 t\,?$$

A. $[0, 1]$

B. $[-1, 0]$

C. $[-1, 1]$

D. $(-\infty, \infty)$

10. Which of the following is an equivalent expression for $\sin(\tan^{-1} x)$?

A. $\dfrac{x}{\sqrt{1-x^2}}$

B. $\dfrac{x}{\sqrt{1+x^2}}$

C. $\dfrac{1}{\sqrt{1-x^2}}$

D. $\dfrac{1}{\sqrt{1+x^2}}$

SECTION II (60 marks)

Attempt Questions 11 - 14
Answer each question in a new booklet. Extra booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new booklet

(a) A bag contains red and black counters in the ratio $7:3$. A counter is drawn, its colour recorded, and it is returned into the bag. This Bernoulli experiment is repeated 20 times. Let X be the random variable representing the number of red counters.

(i) Find the expected value $E(X)$. **1**

(ii) Find the variance $\mathrm{Var}(X)$. **1**

(iii) What is the probability that the number of red counters is the expected value? **1**

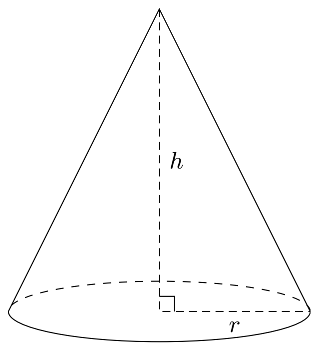
(b) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\cos(x-\alpha)$, where $R>0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **3**

(ii) Hence or otherwise solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \le x \le 2\pi$. **2**

(c) Find the general solution to the differential equation $\dfrac{dy}{dx} = x(y+1)$. **2**

(d) A committee of 5 is to be chosen from 5 men and 4 women. What is the probability that the committee contains three women? **2**

(e) Mineral ore is mined and stockpiled, forming a conical pile whose base diameter is equal to the height of the cone, i.e. $h = 2r$, as shown in the diagram below. **3**



If the ore is poured onto the pile at a rate of $5\,\mathrm{m}^3$ per minute, find the rate at which the height of the cone is increasing when it is 3 metres high? [Note: $V_{\text{cone}} = \dfrac{1}{3}\pi r^2 h$]

End of Question 11

Question 12 (15 marks) Start a new booklet

(a) Find the value of the constant term in the expansion of $(x+2)\left(2-\dfrac{3}{x}\right)^6$. **2**

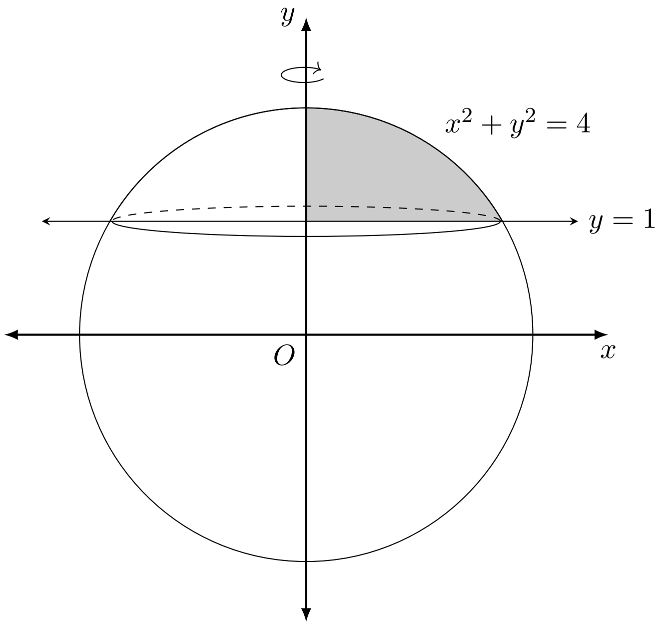
(b) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2 x\,dx$. **3**

(c) Use the substitution $u = x+3$ to evaluate $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x+3}}\,dx$. **3**

(d) Given the function $f(x) = x^2 - 1$, sketch $y^2 = f(x)$. **2**

(e) The percentage of male babies born is 51%. If a random sample of 250 babies is taken use the standard normal distribution tables provided to calculate the probability that there are more female babies in the sample. **3**

(f) Find the volume of the spherical cap formed when the region bounded by $x^2 + y^2 = 4$, the y-axis and the line $y = 1$ is rotated around the y-axis. **2**



End of Question 12

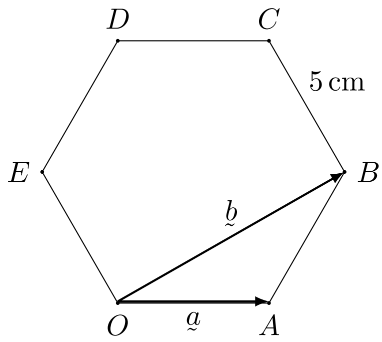
Question 13 (15 marks) Start a new booklet

(a) Find $f^{-1}(x)$ if $f(x) = \dfrac{x+1}{x-2}$, clearly indicating the domain and range of $f^{-1}(x)$. **3**

(b) Find the value of b for which **3**

$$\int_0^1 \frac{dx}{\sqrt{4-3x^2}} = \int_0^b \frac{dx}{\sqrt{1-x^2}}.$$

(c) A regular hexagon $OABCDE$ is drawn below. The sides of the hexagon are 5 cm.
Let $\overrightarrow{OA}$ be $\underset{\sim}{a}$ and $\overrightarrow{OB}$ be $\underset{\sim}{b}$.



(i) Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) Given that $\angle OAB = 120°$, show that $|\underset{\sim}{b}| = 5\sqrt{3}$ cm. **2**

(iii) Hence or otherwise find the exact area of the hexagon. **2**

(d) The mass in kilograms M of a chemical produced per hour in a chemical reaction is modelled by the differential equation **4**

$$\frac{dM}{dt} = 6 - \frac{M}{20}.$$

Initially there are 20 kilograms of the chemical. Solve the differential equation to find the time taken, to the nearest minute, for another 20 kilograms to be produced.

End of Question 13

Question 14 (15 marks) Start a new booklet

(a) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1} - 1.$$

(b) Points A and B have position vectors $\overrightarrow{OA} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB} = 2\underset{\sim}{j}$.

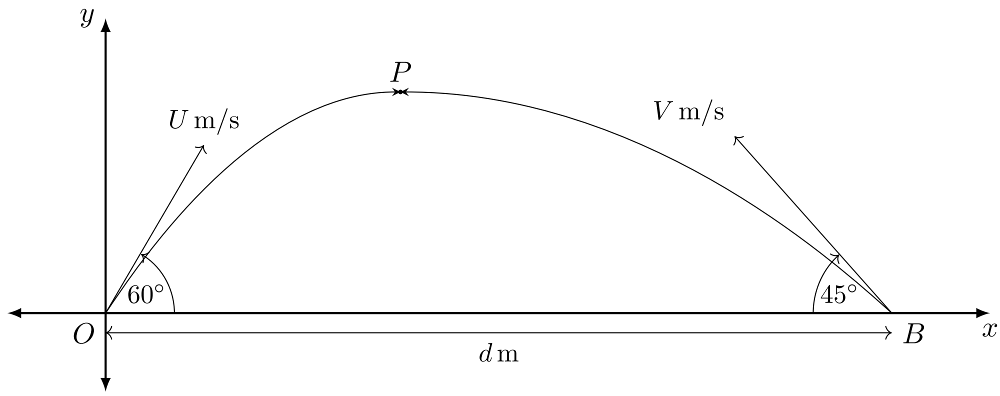
(i) Find $\overrightarrow{AB}$ and hence find the projection of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$. **2**

(ii) Let P be a point on $\overrightarrow{AB}$ such that $\overrightarrow{OP} \perp \overrightarrow{AB}$. Find $\overrightarrow{OP}$. **2**

(c) (i) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Let $x = \cos\theta$. Find three distinct roots of $8x^3 - 6x - 1 = 0$. **2**

(d) Two projectiles are fired at the same time to collide at a point P, when they are both at their maximum height. Projectile A is fired from the origin at a speed of U m/s and at an angle of 60° from the horizontal. Projectile B is fired from point B, d metres from O at a speed of V m/s and at an angle of 45° from the horizontal, as shown in the diagram below.



The displacement vector equations from O are as follows. You do NOT need to derive these.

$$\underset{\sim}{r}_A(t) = \left(\frac{U}{2}t\right)\underset{\sim}{i} + \left(\frac{U\sqrt{3}}{2}t - 5t^2\right)\underset{\sim}{j}$$

$$\underset{\sim}{r}_B(t) = \left(d - \frac{V}{\sqrt{2}}t\right)\underset{\sim}{i} + \left(\frac{V}{\sqrt{2}}t - 5t^2\right)\underset{\sim}{j}$$

(i) Show that the projectile from O reaches its maximum height after $t = \dfrac{U\sqrt{3}}{20}$ seconds. **1**

(ii) Show that for the collision to occur, the speed of the projectile from B must be **1**

$$V = \frac{U\sqrt{6}}{2}.$$

(iii) Find the distance d between the two projectiles in terms of U. **2**

End of Examination