EMS702P Statistical Thinking and Applied Machine Learning

Weeks 4.1-4.3 – Introduction to Fuzzy Sets and Fuzzy Relations

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Introduction to Statistics and Machine Learning

Edition: v1.1

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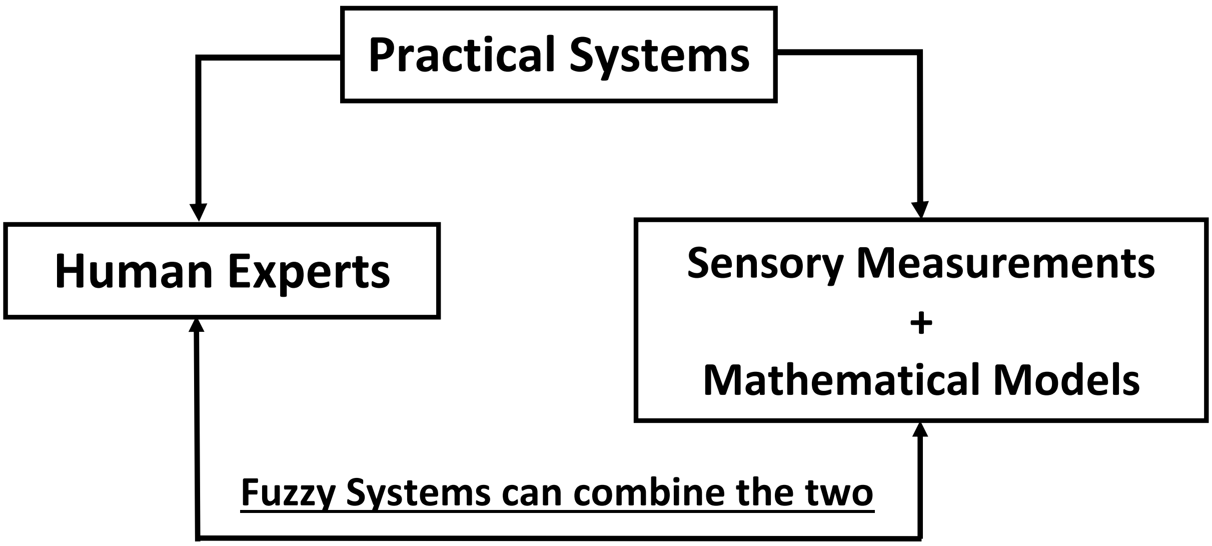
1 Introduction to Fuzzy Systems

There are two justifications for adopting fuzzy systems.

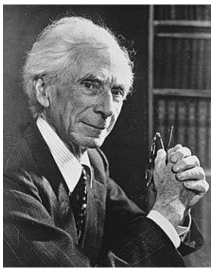
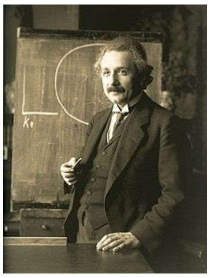
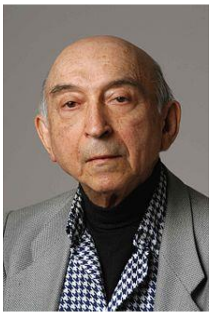
- Real world is too complicated to model and fuzziness is introduced to obtain a **reasonable** model.
- There is increasing dissatisfaction with quantitative approaches. Human knowledge is becoming more and more important. The fuzziness concept can formulate knowledge in a systematic manner and give it an engineering flavour.

A good engineering theory is one that makes use of **all** available information **effectively**.



Some great logicians, mathematicians, and scientists shared their views about the world and reality.

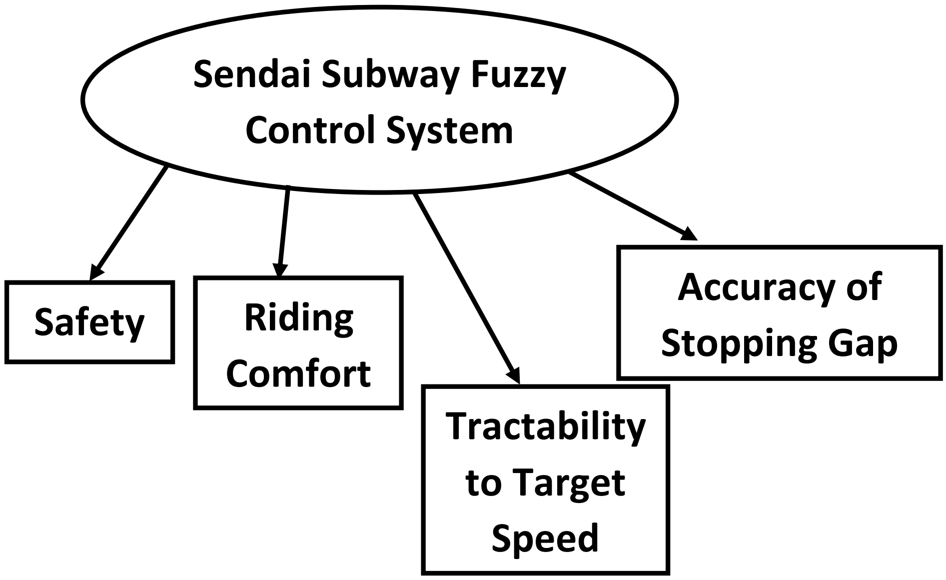
	**Charles Sanders Peirce** (1839 – 1914), American Philosopher, logician, mathematician and scientist. Laughed at the 'sheep and goat separators' who split the world into true and false. 'All that exists is continuous and such continuums govern knowledge.'

	**Bertrand Russell** (1872 – 1970), British mathematician, philosopher, logician, and public intellectual. 'Both vagueness and precision are features of language, not reality. Vagueness clearly is a matter of degree.'
	**Jan Lukasiewicz** (1878 – 1956), Polish logician and philosopher. Proposed a formal model of vagueness, a logic 'based on more values than TRUE or FALSE'. 1 stands for TRUE, 0 stands for FALSE, 1/2 stands for possible. The three-valued logic stayed just one step away from the multivalued fuzzy logic by Zadeh and can be considered as its closest relative.
	**Albert Einstein** (1879 – 1955), German-born theoretical physicist. 'So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.'
	Lofti Zadeh (1921 – 2017), Azerbaijan-born mathematician, computer scientist, electrical engineer, artificial intelligence researcher. Introduced fuzzy sets and logic theory. 'As the complexity of a system increases, our ability to make precise and significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics… A corollary principle may be stated succinctly as, 'The closer one looks at a real-world problem, the fuzzier becomes its solution.'

Some examples of applications of fuzzy systems include

- **Fuzzy washing machines:** the first major products to use fuzzy systems (Matsushita Electric Industrial Company in Japan in 1991). The fuzzy system, hence built, had 3 inputs (dirtiness, dirt type and load size), which were ‘measured’ using optical sensors, and one output (correct cycle).

- **Fuzzy control of subway train (Sendai Subway in Japan):** the most significant application of fuzzy systems to date, which is still operational and is still considered to be the most advanced subway system in the world. The following is a configuration of the control systems.



The controller was represented by two levels:

Level 1: Constant speed controller

Starts the train and Keeps speed below its safety limit

Level 2: Automatic stopping controller

Regulates speed limit to stop at target position

Here is a sample of the rules:

For Safety

IF *the speed of the train is approaching the limit speed*, **THEN** *select the maximum brake notch*

For Riding Comfort

IF *the speed of the train is in the allowed range*, **THEN** *do not change the control notch*

Brief history of fuzzy technology

1965 Concept of fuzzy sets theory by Lofti Zadeh (USA)

1972 First working group on fuzzy systems in Japan by Toshiro Terano

1973 Paper about fuzzy algorithms by Zadeh (USA)

1974 Steam engine control by Ebrahim Mamdani (UK)

1977 First fuzzy expert system for loan applicant evaluation by Hans Zimmermann (Gemany)

1980 Cement Kiln control by F. L. Smidth & Co. – Lauritz P Holmblad (Denmark) – the first permanent industrial application

Fuzzy logic chess and backgammon program – Hans Berliner (USA)

1984 Water treatment (chemical injection) control (Japan)

Subway Sendai transportation system control (Japan)

1985 First fuzzy chip developed by Masaki Togai and Hiroyuke Watanabe in Bell Labs (USA)

1986 Fuzzy expert system for diagnosing illnesses in Omron (Japan)

1987 Container crank control, tunnel excavation, soldering robot, automated aircraft vehicle landing, Second IFSA conference in Tokyo

Togai InfraLogic Inc. – first fuzzy company in Irvine (USA)

1988 Kiln control by Yokogawa

First dedicated fuzzy company in Irvine (USA)

1989 Creation of Laboratory for International Fuzzy Engineering Research (ILFE) in Japan

1990 Fuzzy TV set by Sony (Japan), Fuzzy electronic eye by Fujitsu (Japan), Fuzzy Logic Systems Institute (FLSI) by Takeshi Yamakawa (Japan), Intelligent Systems Control Laboratory in Siemens (Germany)

1991 Fuzzy AI Promotion Centre (Japan)

Educational kit by Motorola (USA)

1992 Onwards the number of events, interventions and projects becomes too voluminous to mention them all

1.1 Fuzzy Sets

Traditionally a **set** is defined as a collection of objects of any kind, numbers, geometric points, chairs, pencils, etc. Usually, naming all members of the set will determine the set.

To indicate that an individual object u *is a member of a set* A*, we write* $u \in A$*. Whenever* u *is not an element of a set* A*, we write* $u \notin A$*.*

Any set is said to be a subset of a **Universal Set**. This latter contains all possible elements having the nature as well as the property which are under considerations.

The processes by which individuals from the universal set U are determined to be either members or non-members of a set, is known as **mapping** the universal set into the **determined set**. As a mapping, it can be defined by a function which is referred to as the **characteristic function** (or **Membership Function**).

Hence, given a set A, this membership function assigns a value $\mu_A(u)$ to each $u \in U$ which can either be *TRUE* or *FALSE*, i.e.

$$\mu_A(u) = \begin{cases} 1, & u \in A \\ 0, & u \notin A \end{cases}$$

Or

$$\mu_A : U \to \{0, 1\} \qquad (1)$$

Instead of (1), consider the following mapping:

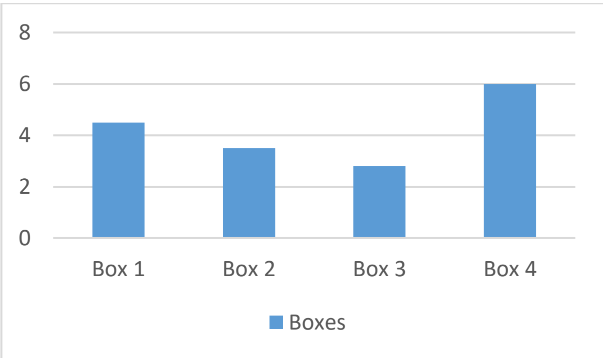
$$\mu_A : U \to [0, 1] \qquad (2)$$

Equation (2) implies that an element of the Universal set belongs to the set A with the determined membership degree which is assessed by a number between 0 and 1 (inclusive).

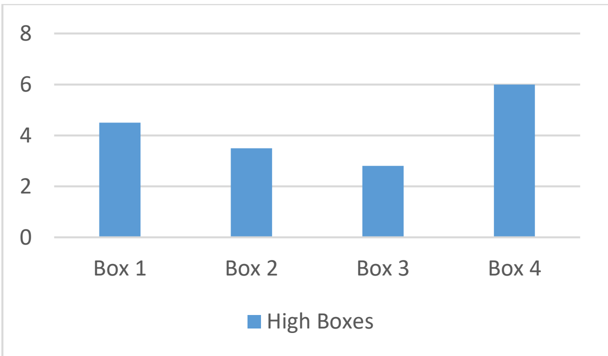
A fuzzy set in the universal set U is a set of ordered pairs of an element u and its membership degree $\mu_A(u)$ such as:

$$A = \{(u, \mu_A(u) | u \in U\} \qquad (3)$$

Example 1



Solution



Notes:

- A fuzzy set is given by its membership function $\mu_A(u)$. The value of this function determines if the element **belongs** to the fuzzy set and to **what extent (degree)**.
- We should not confuse the term 'certainty' with probability or likelihood. $\mu_A(u)$ is not a probability density function, i.e. $certainty \equiv Degree\ of\ Truth$.

The support $S(A)$ of a fuzzy set A is the crisp set of all the elements of the Universal Set for which the membership function value is **non-zero**, i.e.

$$S(A) = \{\, \mu_A(u) > 0 | u \in U\} \qquad (4)$$

The element of the Universal Set for which the membership function has the value of 0.5 is called a **crossover point**.

Note: the crossover point marks the point where the certainty of belonging becomes lower than the certainty of not belonging.

There are different **shapes of membership functions** which can be used in a particular fuzzy system.

Triangular

In the continuous case, we have

$$\mu_A(u) = \begin{cases} 0 & u < a_1 \\ \dfrac{u - a_1}{a_2 - a_1} & a_1 \le u \le a_2 \\ \dfrac{a_3 - u}{a_3 - a_2} & a_2 \le u \le a_3 \\ 0 & u \ge a_3 \end{cases}$$

Trapezoidal

In the continuous case, we have

$$\mu_A(u) = \begin{cases} 0 & u < a_1 \\ \dfrac{u - a_1}{a_2 - a_1} & a_1 \le u < a_2 \\ 1 & a_2 \le u < a_3 \\ \dfrac{a_4 - u}{a_4 - a_3} & a_3 \le u < a_4 \\ 0 & u \ge a_4 \end{cases}$$

Gaussian

$$\mu_A(u) = e^{-\left(\frac{u-m}{\sigma}\right)^2}$$

Question

Which membership function shape will at best represent fuzzy systems?

Answer

The **height of a fuzzy set** A, $hgt(A)$, is given by the maximum of the membership function over all $u \in U$.

As a summary, we have

1.2 Operations on Fuzzy Sets

The **complement** of a fuzzy set A has a membership function which is defined for all $u \in U$ by:

$$\overline{\mu_A}(u) = 1 - \mu_A(u) \qquad (5)$$

The **intersection** of two fuzzy sets $C = A \cap B$ is a fuzzy set with the following membership function:

$$\mu_{C=A\cap B}(u) = \begin{cases} \min\left(\mu_A(u), \mu_B(u)\right) \\ \mu_A(u) \cdot \mu_B(u) \end{cases} \qquad (6)$$

Note: the **product** operation introduces more smoothness in the data output than the **min** operation.

Example 2

Consider two fuzzy sets:

$$A = \{0/0, 0/1, 0.1/2, 0.3/3, 0.5/4, 0.7/5, 0.8/6, 0.9/7, 1/8\}$$

$$B = \{0/0, 0.1/1, 0.5/2, 0.8/3, 1/4, 0.8/5, 0.5/6, 0.1/7, 0/8\}$$

Using the min and product operations to work out $C = A \cap B$, and illustrate C graphically.

Solution

The **union** of two fuzzy sets $C = A \cup B$ is a fuzzy set with the following membership function:

$$\mu_{C=A\cup B}(u) = \begin{cases} \max\left(\mu_A(u), \mu_B(u)\right) \\ \mu_A(u) + \mu_B(u) - \mu_A(u) \cdot \mu_B(u) \end{cases} \qquad (7)$$

Example 3

Consider the same two fuzzy sets in Example 2, illustrate $C = A \cup B$ graphically.

Solution

1.3 Extension Principles

The extension principles (one of the most important result of fuzzy sets theory) enable us to get a fuzzy model for a variable if we know the fuzzy model for another variable and the functional relationship between them. The extension principles allow one to calculate the output of a fuzzy system.

Extension Principal 1: if 'A' is a fuzzy set in the universe 'U' and 'f' is a mapping from 'U' to the universe of discourse 'Y' such that $y = f(u)$, then the extension principal allows to define a fuzzy set 'B' in 'Y' such that

$$B = f(A) = \{(y, \mu_B(y))\,|\, y = f(u), u \in U\} \qquad (8)$$

Where,

$$\mu_B(y) = \begin{cases} sup\mu_A(u) & if\ f^{-1}(y) \neq 0 \\ 0 & if\ f^{-1}(y) = 0 \end{cases} \qquad (9)$$

Example 4

Let $A = \{0.3/-2, 0.4/-1, 0.8/0, 1/1, 0.7/2\}$, and $y = f(u) = u^2$. Find a fuzzy set B in Y.

Solution

Extension principal 1 deals with 1 input. When we have more than one input, we will need **Extension Principal 2**.

Extension Principal 2: let 'X' be a Cartesian product of the universes '$U_1, \dots, U_r$' such that $X = U_1 * U_2 * \dots * U_r$ and '$A_1, \dots, A_r$' are fuzzy sets in these universes. 'f' is a mapping from 'X' to the universe of discourse 'Y', where $y = f(u_1, \dots, u_r)$. The extension principal allows to define the fuzzy set 'B' in 'Y' such that:

$$B = f(A) = \{(y, \mu_B(y))|y = f(u_1, \dots, u_r), (u_1, \dots, u_r) \in X\} \qquad (10)$$

Where,

$\mu_B(y)$

$$= \begin{cases} \sup \min \{\mu_{A_1}(u_1), \dots, \mu_{A_r}(u_r)\} \ (u_1, \dots, u_r) \in f^{-1}(y) & if\ f^{-1}(y) \neq 0 \\ 0 & if\ f^{-1}(y) = 0 \end{cases} \quad (11)$$

Example 5

Consider the case of two universes of discourse:

$U_1 = U_2 = \{1, 2, 3, \dots, 10\}$

Define $A = \{\frac{0.6}{1}, \frac{1}{2}, \frac{0.8}{3}\}$ and $C = \{\frac{0.8}{2}, \frac{1}{3}, \frac{0.6}{4}\}$ and a mapping $y = f((u_1, (u_2) = u_1 \cdot u_2$. Find a fuzzy set B in Y.

Solution

1.4 Fuzzification

The process, which allows one to convert a numeric input into a fuzzy input, is called fuzzification and there are two ways to achieve that:

Fuzzification maps a crisp input $u_i \in U$ into a fuzzy set A_{u_i} in U such that:

- A_{u_i} is a fuzzy Singleton.

$$\mu_{A_{u_i}}(u) = \begin{cases} 1 & if\ u = u_i \\ 0 & otherwise \end{cases} \qquad (12)$$

- A_{u_i} is a fuzzy set (triangular, trapezoidal, Gaussian MF) such that:

$$\mu_{A_{u_i}}(u) = \begin{cases} 1 & if\ u = u_i \\ decreases\ from\ 1\ as\ u\ moves & from\ u_i \end{cases} \qquad (13)$$

Note: Singleton fuzzification is generally used in implementations where there is no noise

1.5 Defuzzification

The conversion of a fuzzy quantity into a crisp one is done via a defuzzification operation. Several defuzzification methods do exist and here we will review the most popular three.

Max-Membership method

u_c is chosen such that

$$\forall u \in U, \mu_A(u_c) \geq \mu_A(u) \qquad (14)$$

Centroid of Gravity (Area) (COG) method

u_c is chosen such that it represents the centre of the above shaded area, i.e.

$$\mu_c = \begin{cases} \dfrac{\int \mu_A(u) \cdot u \cdot du}{\int \mu_A(u) \cdot du} & in\ the\ continuous\ case \\ \dfrac{\sum \mu_A(u) \cdot u}{\sum \mu_A(u)} & in\ the\ discrete\ case \end{cases} \qquad (15)$$

Mean-of-Maxima (MOM) method

This method consists of taking the mean level of all maxima within the fuzzy membership shape.

Example 6

Which method proved better as far as designing fuzzy systems is concerned?

Solution

Example 7

Consider the following fuzzy set

Find the defuzzified crisp values using the COG and MOM methods.

Solution

2 Fuzzy Relations

Having described operations between fuzzy sets, we need to look at how we can represent linguistic statements mathematically. In fact, many application problem descriptions include fuzzy relations. A fuzzy system is usually represented by statements or rules of the following form:

$$\textbf{IF}\ \underbrace{speed}_{A}\ \text{is slow}\ \textbf{then}\ \underbrace{pressure}_{B}\ \text{should be high}$$

$$\textbf{IF}\ A\ \textbf{then}\ B\ or\ A \rightarrow B$$

A fuzzy relation of the form $A \rightarrow B$ is denoted R.

2.1 Cartesian Product

A fuzzy relation R is defined as a relationship of two fuzzy sets $A \in U$, $B \in V$ and it is a subset on the Cartesian Product $U * V$. R will be characterised by the membership function $\mu_R(u, v), u \in U, v \in V$ such that:

$$R = A * B = \sum \frac{\mu_R(u, v)}{u, v} = \begin{cases} \sum \min\left(\mu_A(u), \mu_B(v)\right) \\ \sum \mu_A(u) \cdot \mu_B(u) \end{cases} \tag{16}$$

Note:

- The sum does not represent a mathematical operation but shows rather all possible combinations of all elements of both universes of discourse.
- R is also called the relational matrix.

Example 8

Assume a fuzzy relation of the type: $A \rightarrow B$ with

$A = \{\frac{1}{1}, \frac{0.8}{2}, \frac{0.6}{3}, \frac{0.5}{4}\}$ and $B = \{\frac{0.5}{1}, \frac{1}{2}, \frac{0.3}{3}, \frac{0}{4}\}$

Find the relational matrix for the above fuzzy statement using the 'min' operation.

Solution

2.2 Compositional Rule of Inference

If R is a fuzzy relation in $U * V$ and A is a fuzzy set in U then the fuzzy set B in V is given by:

$$B = A \circ R$$

B is inferred from A using the relational matrix R which defines the mapping between U and V, and the operation $\circ$ is defined as the 'max-min' operation.

Example 9

Given a fuzzy mapping f between U and V via the following relational matrix:

$$R = \begin{bmatrix} 1 & 0.8 & 0.1 \\ 0.8 & 0.6 & 0.3 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Find the fuzzy output B in V if a fuzzy input A is U, which is defined as follows:

$A = \{\frac{0.9}{1}, \frac{0.4}{2}, \frac{0}{3}\}$. Assume that B is defined in the same universe of discourse as A.

Solution

3 Further Readings

[1] Mendel, J.M., 1995. Fuzzy logic systems for engineering: a tutorial. *Proceedings of the IEEE*, *83*(3), pp.345-377.

[2] Garrido, A., 2012. A brief history of fuzzy logic. *BRAIN. Broad Research in Artificial Intelligence and Neuroscience*, *3*(1), pp.71-77.

[3] Seising, R., 2013. On fuzzy sets and the precisiation of meaning—An interview with Prof. Dr. Lotfi A. Zadeh. *Archives Philosophy History Soft Computing*, *1*, pp.1-18.

[4] Bonissone, P.P., 2018. Obituary for Lotfi A. Zadeh [In Memoriam]. *IEEE Computational Intelligence Magazine*, *13*(1), pp.13-22.