

EMS702P Statistical Thinking and Applied Machine Learning

Week 6 – PBL Solutions

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Solution 1:

$$\mathbf{A}^T = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}$$

$$(\mathbf{A} + \mathbf{B})^{-1} = \frac{1}{0 \times 3 - 1 \times 4} \begin{bmatrix} 3 & -1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^T (\mathbf{A} + \mathbf{B})^{-1} = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} & -\frac{1}{4} \\ 3 & 0 \end{bmatrix}$$

Solution 2:

(1): Based on the measured data, the second-order regression model can be written into a matrix form as

$$\begin{bmatrix} 1.2 \\ 2.2 \\ 3.5 \\ 7.3 \\ 10.8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.5 & 0.25 \\ 1 & 1 \\ 1.5 & 2.25 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (1)$$

Denote

$$\mathbf{Y} = \begin{bmatrix} 1.2 \\ 2.2 \\ 3.5 \\ 7.3 \\ 10.8 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.2 \\ 2.5 \\ 6.3 \\ 9.8 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.25 \\ 1 & 1 \\ 1.5 & 2.25 \\ 2 & 4 \end{bmatrix}, \quad \boldsymbol{\Theta} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (2)$$

Based on the LS method, the parameters $\boldsymbol{\Theta}$ can be evaluated as

$$\boldsymbol{\Theta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} \quad (3)$$

(2): In equation (3),

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} 0 & 0.5 & 1 & 1.5 & 2 \\ 0 & 0.25 & 1 & 2.25 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.5 & 0.25 \\ 1 & 1 \\ 1.5 & 2.25 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7.5 & 12.5 \\ 12.5 & 22.125 \end{bmatrix} \quad (4)$$

$$\begin{aligned}
 (\mathbf{X}^T \mathbf{X})^{-1} &= \frac{1}{7.5 \times 22.125 - 12.5^2} \begin{bmatrix} 22.125 & -12.5 \\ -12.5 & 7.5 \end{bmatrix} \\
 &= \frac{1}{9.688} \begin{bmatrix} 22.125 & -12.5 \\ -12.5 & 7.5 \end{bmatrix} = \begin{bmatrix} 2.284 & -1.29 \\ -1.29 & 0.774 \end{bmatrix}
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T &= \begin{bmatrix} 2.284 & -1.29 \\ -1.29 & 0.774 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 1 & 1.5 & 2 \\ 0 & 0.25 & 1 & 2.25 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0.82 & 0.994 & 0.524 & -0.592 \\ 0 & -0.452 & -0.516 & -0.194 & 0.516 \end{bmatrix}
 \end{aligned} \tag{6}$$

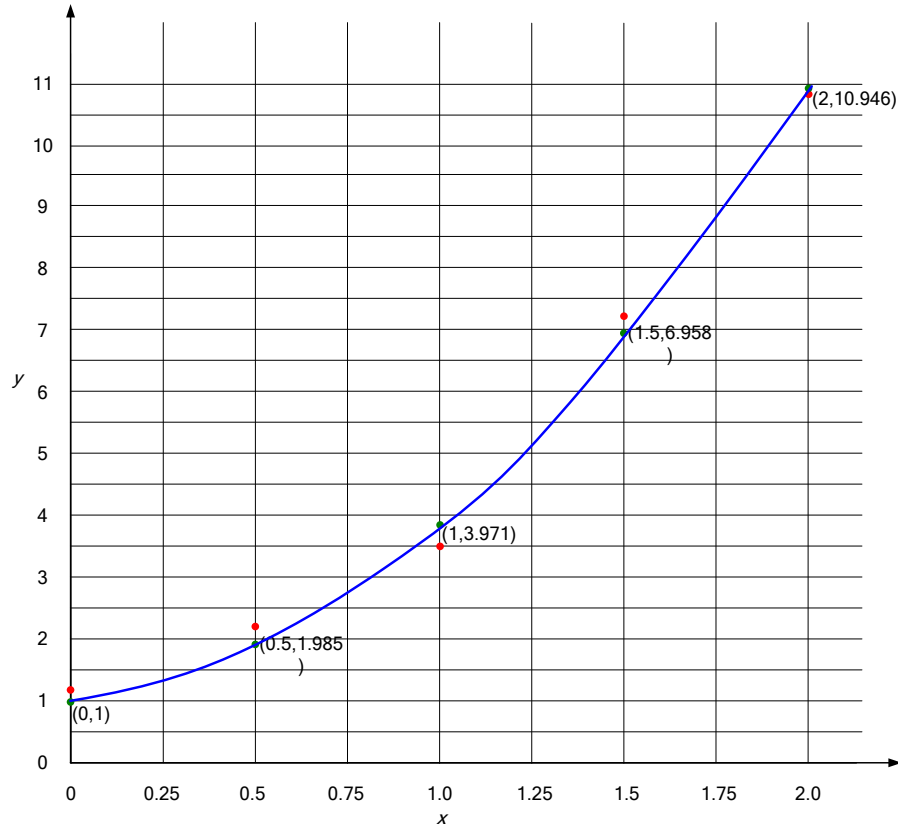
Therefore,

$$\Theta = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} = \begin{bmatrix} 0 & 0.82 & 0.994 & 0.524 & -0.592 \\ 0 & -0.452 & -0.516 & -0.194 & 0.516 \end{bmatrix} \begin{bmatrix} 0.2 \\ 1.2 \\ 2.5 \\ 6.3 \\ 9.8 \end{bmatrix} = \begin{bmatrix} 0.969 \\ 2.002 \end{bmatrix} \tag{7}$$

(3): The evaluated second-order regression model is

$$y = 1 + 0.969x + 2.002x^2 \tag{8}$$

Plot the regression model over $x \in [0, 2]$ on the coordinate system as below.



(4): The MSE of the regression model is evaluated as

$$\eta = \frac{1}{5} \left[(1.2-1)^2 + (2.2-1.985)^2 + (3.5-3.971)^2 + (7.3-6.958)^2 + (10.8-10.946)^2 \right] = 0.0892 \text{ (9)}$$

Solution 3:

$$\begin{cases} (x_1, x_2, y_1) = (0, 1, 1), (1, 0, 0), (0, -1, 0), (-1, 0, 0) \\ (x_1, x_2, y_2) = (0, 1, 0), (1, 0, 1), (0, -1, 0), (-1, 0, 0) \\ (x_1, x_2, y_3) = (0, 1, 0), (1, 0, 0), (0, -1, 1), (-1, 0, 0) \\ (x_1, x_2, y_4) = (0, 1, 0), (1, 0, 0), (0, -1, 0), (-1, 0, 1) \end{cases}$$

$$y_1 : \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1^{(1)} \\ w_2^{(1)} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} w_1^{(1)} \\ w_2^{(1)} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \times \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4-0} \times \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4} \times \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$y_1 = \frac{1}{2} x_2$$

$$y_2 : \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1^{(2)} \\ w_2^{(2)} \end{bmatrix} \Rightarrow$$

$$\begin{aligned}
\begin{bmatrix} w_1^{(2)} \\ w_2^{(2)} \end{bmatrix} &= \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \end{pmatrix}^{-1} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{pmatrix}^{-1} \times \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4-0} \times \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4} \times \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\end{aligned}$$

$$y_2 = \frac{1}{2}x_1$$

$$\begin{aligned}
y_3 : \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1^{(3)} \\ w_2^{(3)} \end{bmatrix} \Rightarrow \\
\begin{bmatrix} w_1^{(3)} \\ w_2^{(3)} \end{bmatrix} &= \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \end{pmatrix}^{-1} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{pmatrix}^{-1} \times \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{4-0} \times \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{4} \times \begin{bmatrix} 0 \\ -2 \end{bmatrix}
\end{aligned}$$

$$y_3 = -\frac{1}{2}x_2$$

$$y_4 : \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1^{(4)} \\ w_2^{(4)} \end{bmatrix} \Rightarrow$$

$$\begin{aligned}
\begin{bmatrix} w_1^{(4)} \\ w_2^{(4)} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4-0} \times \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{4} \times \begin{bmatrix} -2 \\ 0 \end{bmatrix}
\end{aligned}$$

$$y_4 = -\frac{1}{2}x_1$$

Boundaries:

$$\begin{aligned}
&\left. \begin{array}{l} y_1 = y_2 \\ y_3 = y_4 \end{array} \right\} \Rightarrow x_2 = x_1, \quad \left. \begin{array}{l} y_1 = y_4 \\ y_2 = y_3 \end{array} \right\} \Rightarrow x_1 = -x_2, \\
&y_1 = y_3 \Rightarrow x_2 = 0, \quad y_2 = y_4 \Rightarrow x_1 = 0
\end{aligned}$$