Week 3 – PBL Solutions

1.

$\sum y_i = 611.0$, $\sum y_i^2 = 6227.34$ and $n = 60$. We estimate μ using the sample mean:

$$\bar{y} = \frac{\sum y_i}{n} = \frac{611.0}{60} = 10.1833 \text{ V}$$

We estimate σ^2 using the sample variance:

$$\begin{aligned} s^2 &= \frac{1}{n-1}\sum(y_i - \bar{y})^2 = \frac{1}{n-1}\left\{\sum y_i^2 - \frac{1}{n}\left[\sum y_i\right]^2\right\} \\ &= \frac{1}{59}\left\{6227.34 - \frac{1}{59}611.0^2\right\} = 0.090226 \end{aligned}$$

The estimated standard error of the mean is

$$\sqrt{\frac{s^2}{n}} = \sqrt{\frac{0.090226}{60}} = 0.03878 \text{ V}$$

The 99% confidence interval for μ is $\bar{y} \pm 2.58\sqrt{s^2/n}$. That is

$$10.08 < \mu < 10.28$$

2.

From the data we calculate $\sum y_i = 11598$ and $\sum y_i^2 = 6725744$ and we have $n = 20$. Hence

$$(n-1)s^2 = \sum(y_i - \bar{y})^2 = 6725744 - \frac{11598^2}{20} = 63.8$$

The number of degrees of freedom is $n - 1 = 19$. We know that

$$\chi^2_{0.975,19} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{0.025,19}$$

with probability 0.95. So a 95% confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{0.025,19}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{0.975,19}}$$

That is $\dfrac{63.8}{32.85} < \sigma^2 < \dfrac{63.8}{8.91}$ so $1.942 < \sigma^2 < 7.160$

This gives a 95% confidence interval for σ: $1.394 < \sigma < 2.676$

3.

There are $n-1=9$ degrees of freedom. Now

$$
\begin{aligned}
0.9 &= \mathsf{P}\left(\chi^2_{0.05,9} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{0.95,9}\right) \\
&= \mathsf{P}\left(\frac{\chi^2_{0.05,9}\sigma^2}{n-1} < S^2 < \frac{\chi^2_{0.95,9}\sigma^2}{n-1}\right) \\
&= \mathsf{P}\left(\frac{3.33 \times 3.0}{9} < S^2 < \frac{16.92 \times 3.0}{9}\right) = \mathsf{P}(1.11 < S^2 < 5.64)
\end{aligned}
$$

Hence $L = 1.11$ and $U = 5.64$.

4.

Given that the dispensing machine can over-fill or under-fill the containers, the null and alternative hypotheses are:

$$H_0: \quad \mu = 3 \qquad H_1: \quad \mu \neq 3$$

Since the sample size is large (≥ 30) and we can regard the population as infinite but with a known variance, we can calculate the relevant value of the test statistic Z by using the formula:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Hence, in this case:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{3.005 - 3}{0.015/\sqrt{40}} = 2.108$$

and since we are performing a two-tailed test at the 5% level of significance and have found that $|Z| > 1.96$, that is, Z is outside the range $[-1.96, 1.96]$, we must reject the null hypothesis and conclude that the machine is not operating acceptably and needs adjustment.

5.

The null and alternative hypotheses are:

$$H_0: \ \mu = 100 \qquad H_1: \ \mu < 100$$

Our test statistic is

$$T = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

In this case

$$
\begin{aligned}
T &= \frac{99.5 - 100.0}{\sqrt{18.49/33}} \\
&= -0.668
\end{aligned}
$$

and the number of degrees of freedom is $n - 1 = 33 - 1 = 32$. The table does not give values for 32 degrees of freedom but it does give values for 30 degrees of freedom and for 40 and the values for 32 must be in between. The lower 5% points for 30 and 40 degrees of freedom are -1.697 and -1.684 respectively. Clearly our observed value of -0.668 is not significant and we do not have sufficient evidence to reject the null hypothesis that $\mu = 100$.

6.

Let D be the difference between the throttle reaction times of the two turbochargers. We assume that the distribution of D is normal. Our null hypothesis is that μ_D, the mean of the population of differences, is zero. We must decide between the two hypotheses

$$H_0: \ \mu_D = 0 \qquad H_1: \ \mu_D \neq 0$$

The alternative hypothesis here indicates that we perform a two-tailed test.
Let $\bar{d}$ be the sample mean of the seven observed differences. Then

$$\bar{d} = \frac{\sum d}{7} = \frac{0.042}{7} = 0.006$$

The sample variance of the differences is

$$s_d^2 = \frac{\sum(d - \bar{d})^2}{n-1} = \frac{0.000214}{6} = 3.5667 \times 10^{-5}$$

The value of the test statistic is

$$|t| = \frac{|\bar{d} - 0|}{\sqrt{s_d^2/n}} = \frac{0.006}{\sqrt{3.5667 \times 10^{-5}/7}} = 2.658$$

The number of degrees of freedom is $7 - 1 = 6$ and the critical value from the table is 2.447. Since $2.658 > 2.447$ we reject H_0 at the 5% level and conclude that the evidence suggests that there is a difference in the throttle reaction times between the two turbochargers.

7.

The null and alternative hypotheses are:

$$H_0: \ \mu_1 - \mu_2 = 0 \qquad H_1: \ \mu_1 - \mu_2 \neq 0$$

The null hypothesis represent the statement 'there is no difference in the tensile strengths of the two components.' The test statistic Z is calculated as:

$$\begin{aligned} Z &= \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \\ &= \frac{(90 - 88) - (0)}{\sqrt{\dfrac{2.3^2}{15} + \dfrac{2.2^2}{10}}} \\ &= \frac{2}{\sqrt{0.3527 + 0.484}} \\ &= 2.186 \end{aligned}$$

Since $2.186 > 1.96$ we conclude that, on the basis of the (limited) evidence available, there is a difference in tensile strength between the components tested. The manufacturer should carry out more comprehensive tests before making a final decision as to which component to use. The decision is a serious one with safety implications as well as economic implications. As well as carrying out more tests the manufacturer should consider the level of rejection of the null hypothesis, perhaps using 1% instead of 5%. Component 1 appears to be stronger but this may not be the case after more tests are carried out.

8. (a)

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The pooled estimate of the variance is

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9 \times 2.00 + 11 \times 2.72}{20} = 2.396$$

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{2.396}\sqrt{\frac{1}{10} + \frac{1}{12}}} = -1.267$$

From t-tables, the critical values with 20 degrees of freedom and a two-tailed test are ± 2.086. Since $-2.086 < -1.267 < 2.086$ we conclude that we cannot reject the null hypothesis in favour of the alternative. A 95% confidence interval for the difference between the mean currents is given by $\bar{x}_1 - \bar{x}_2 \pm 2.086 \times S_c\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$. The confidence interval is $-2.683 < \mu_1 - \mu_2 < 0.083$.

(b)

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{\frac{2.00}{10} + \frac{2.72}{12}}} = -\frac{1.3}{\sqrt{0.427}} = -1.990$$

The number of degrees of freedom is given by

$$\begin{aligned} \nu &= \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{n_1}\right)^2}{n_1 - 1} + \dfrac{\left(\dfrac{S_2^2}{n_2}\right)^2}{n_2 - 1}} \\ &= \frac{\left(\dfrac{2.00}{10} + \dfrac{2.72}{12}\right)^2}{\dfrac{(2.00/10)^2}{9} + \dfrac{(2.72/12)^2}{11}} = \frac{0.182}{0.0044 + 0.0047} \approx 20 \end{aligned}$$

From t-tables, the critical values (two-tailed test, 5% level of significance) are ± 2.086. Since $-2.086 < -1.990 < 2.086$ we conclude that there is insufficient evidence to reject the null hypothesis in favour of the alternative at the 5% level of significance.

9.

We assume that each data set is taken from separate continuous distributions. It can be shown that this ensures that the distribution of differences is then symmetric and continuous. In this case the median and mean are identical. We are testing to find any differences in the median miles per gallon figures for each injection system. The null and alternative hypotheses are:

$$H_0 : \mu_1 = \mu_1 \quad \text{or} \quad H_0 : \mu_{\text{differences}} = 0$$

$$H_1 : \mu_1 \neq \mu_2 \quad \text{or} \quad H_1 : \mu_{\text{differences}} \neq 0$$

We perform a two-tailed test.

The signed ranks are obtained as shown in the table below:

	Fuel Injection System				
Car	1	2	Differences	Sorted Abs	Signed
1	27.6	26.3	1.3	0.4	+1
2	29.4	31.0	−1.6	0.9	+2.5
3	29.5	28.2	1.3	0.9	+2.5
4	27.2	26.1	1.1	1.1	+5
5	25.8	27.6	−1.8	1.1	+5
6	26.9	25.8	1.1	1.1	−5
7	26.7	28.2	−1.5	1.3	+8
8	28.9	27.6	1.3	1.3	+8
9	27.3	26.9	0.4	1.3	+8
10	29.2	30.3	−1.1	1.5	−10
11	27.8	26.9	0.9	1.6	−11
12	29.2	28.3	0.9	1.8	−12

We now calculate the sums S_N, S_P and S in order to decide whether to reject the null hypothesis.

$$S_N = |-5 - 10 - 11 - 12| = 38$$

$$S_P = |1 + 2.5 + 2.5 + 5 + 5 + 8 + 8 + 8| = 40$$

$$S = \min(S_P, S_N) = \min(40, 38) = 38$$

From Table 1, the critical value at the 5% level of significance for a two-tailed test performed with a sample of 12 values is 13.

Since $13 < 38$ we conclude that we cannot reject the null hypothesis and that on the basis of the available evidence, the two injection systems do not differ significantly in respect of the fuel economy they offer.